TW 115

Reprinted from JOURNAL OF APPROXIMATION THEORY Vol. 4, No. 4, December 1971

Printed in Belgium

On Absolute Convergence of Jacobi Series

H. BAVINCK

Mathematical Centre, Amsterdam, The Netherlands

Communicated by Oved Shisha

Received October 22, 1969

1. INTRODUCTION

This paper answers a question concerning the expansion of functions in an absolutely convergent series of Jacobi polynomials. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$(1-x)^\alpha(1+x)^\beta \qquad (\alpha > -1, \beta > -1).$$

They satisfy the relation

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha}(1+x)^{n+\beta}\} \tag{1.1}$$

(see Szegö [5, Section 4.3]), usually called Rodrigues's formula. The orthogonality property is given by

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx = h_n(\alpha,\beta)\, \delta_{m,n} \tag{1.2}$$

with

$$h_n(\alpha,\beta) = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\,\Gamma(n+\alpha+\beta+1)}, \tag{1.3}$$

$\delta_{m,n} = 0$ if $m \neq n$ and $\delta_{m,n} = 1$ if $m = n$.

With a function $f(x)$ we can associate a series:

$$f(x) \sim \sum_{k=0}^{\infty} a_k P_k^{(\alpha,\beta)}(x), \tag{1.4}$$

where

$$a_k = (h_k(\alpha,\beta))^{-1} \int_{-1}^{1} f(x)\, P_k^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx, \tag{1.5}$$

provided that the integral in (1.5) exists for all k. The coefficients a_k are then called the Fourier coefficients of $f(x)$.

DEFINITION. *A function $f(x)$ is said to be in the class $A(\alpha, \beta)$ if $\sum_{k=0}^{\infty} | a_k | \, | P_k^{(\alpha,\beta)}(x)|$ converges uniformly on the interval $-1 \leqslant x \leqslant 1$, where a_k are the Fourier coefficients of $f(x)$.*

It is a well-known fact (see Szegö [5, Section 7.32]), that the Jacobi polynomials reach the maximum of their absolute value on the interval $[-1, 1]$ at $x = 1$, provided that $\alpha \geqslant \beta$ and $\alpha \geqslant -\frac{1}{2}$. Since

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k + \alpha + 1)}{k!\, \Gamma(\alpha + 1)} = O(k^\alpha),$$

it follows that a necessary and sufficient condition for $f(x)$ to be in $A(\alpha, \beta)$ $(\alpha \geqslant \beta, \alpha \geqslant -\frac{1}{2})$ is

$$\sum_{k=0}^{\infty} | a_k | \, k^\alpha < \infty. \tag{1.6}$$

We shall study the question: for which values of γ and δ does

$$f(x) \in A(\alpha, \beta) \qquad \text{imply} \quad f(x) \in A(\gamma, \delta); \tag{A}$$

where $\alpha \geqslant \beta$ and $\alpha \geqslant -\frac{1}{2}$?

In the following it will always be assumed that $\alpha \geqslant \max(\beta, -\frac{1}{2})$, $\beta > -1$.

2. THEOREMS

There is a unique way of expressing the polynomials $P_k^{(\alpha,\beta)}(x)$ in terms of the polynomials $P_j^{(\gamma,\delta)}(x)$, $j = 0, 1, 2,..., k$:

$$P_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} c_{jk}(\alpha, \beta; \gamma, \delta)\, P_j^{(\gamma,\delta)}(x). \tag{2.1}$$

The coefficients $c_{jk}(\alpha, \beta; \gamma, \delta)$ are defined to be 0 if $j > k$. Rivlin and Wilson [4] have proved the following:

THEOREM 1. *If $\gamma \geqslant \delta$, $\gamma \geqslant -\frac{1}{2}$ and $c_{jk}(\alpha, \beta; \gamma, \delta) \geqslant 0$ for all j and k, then relation* (A) *holds.*

Proof. Let $f(x) \in A(\alpha, \beta)$. Then

$$\sum_{k=0}^{\infty} | a_k | \, P_k^{(\alpha,\beta)}(1) < \infty.$$

where the a_k are given by (1.5). We now consider the expansion

$$f(x) \sim \sum_{j=0}^{\infty} b_j P_j^{(\gamma,\delta)}(x).$$

Then

$$\begin{aligned} b_j &= (h_j(\gamma, \delta))^{-1} \int_{-1}^{1} f(x)\, P_j^{(\gamma,\delta)}(x)(1-x)^{\gamma}(1+x)^{\delta}\, dx \\ &= (h_j(\gamma, \delta))^{-1} \int_{-1}^{1} \left\{ \sum_{k=0}^{\infty} a_k P_k^{(\alpha,\beta)}(x) \right\} P_j^{(\gamma,\delta)}(x)(1-x)^{\gamma}(1+x)^{\delta}\, dx \\ &= \sum_{k=0}^{\infty} a_k \left\{ (h_j(\gamma, \delta))^{-1} \int_{-1}^{1} P_k^{(\alpha,\beta)}(x)\, P_j^{(\gamma,\delta)}(x)(1-x)^{\gamma}(1+x)^{\delta}\, dx \right\} \\ &= \sum_{k=j}^{\infty} a_k c_{jk}(\alpha, \beta; \gamma, \delta). \end{aligned}$$

The term-by-term integration is justified by the uniform convergence. Since $\gamma \geqslant \delta$ and $\gamma \geqslant -\frac{1}{2}$, we know that

$$\max_{-1 \leqslant x \leqslant 1} | P_j^{(\gamma,\delta)}(x)| = P_j^{(\gamma,\delta)}(1), \qquad j = 0, 1, 2, \ldots.$$

Thus it remains to show that the sequence

$$F_m = \sum_{j=0}^{m} | b_j | P_j^{(\gamma,\delta)}(1)$$

is bounded.

Using the fact that $c_{jk}(\alpha, \beta; \gamma, \delta) \geqslant 0$ for all j and k, we obtain

$$\begin{aligned} F_m &= \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \left| \sum_{k=j}^{\infty} a_k c_{jk}(\alpha, \beta; \gamma, \delta) \right| \\ &\leqslant \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \sum_{k=j}^{\infty} | a_k |\, c_{jk}(\alpha, \beta; \gamma, \delta) \\ &\leqslant \sum_{k=0}^{\infty} | a_k | \sum_{j=0}^{m} c_{jk}(\alpha, \beta; \gamma, \delta)\, P_j^{(\gamma,\delta)}(1) \\ &\leqslant \sum_{k=0}^{\infty} | a_k |\, P_k^{(\alpha,\beta)}(1) < \infty. \end{aligned}$$

Q.E.D.

It is known (see Askey [1]) that the positivity condition for $c_{jk}(\alpha, \beta; \gamma, \delta)$ is satisfied in the following cases (see Fig. 1):

(i) $\beta = \delta$ and $\alpha > \gamma$, $\gamma \geqslant \delta$,

(ii) $\alpha = \beta$, $\gamma = \delta$, and $\alpha > \gamma$,

(iii) $\alpha = \gamma$, $\beta = \delta - n$ (n a positive integer), $\gamma \geqslant \delta$.

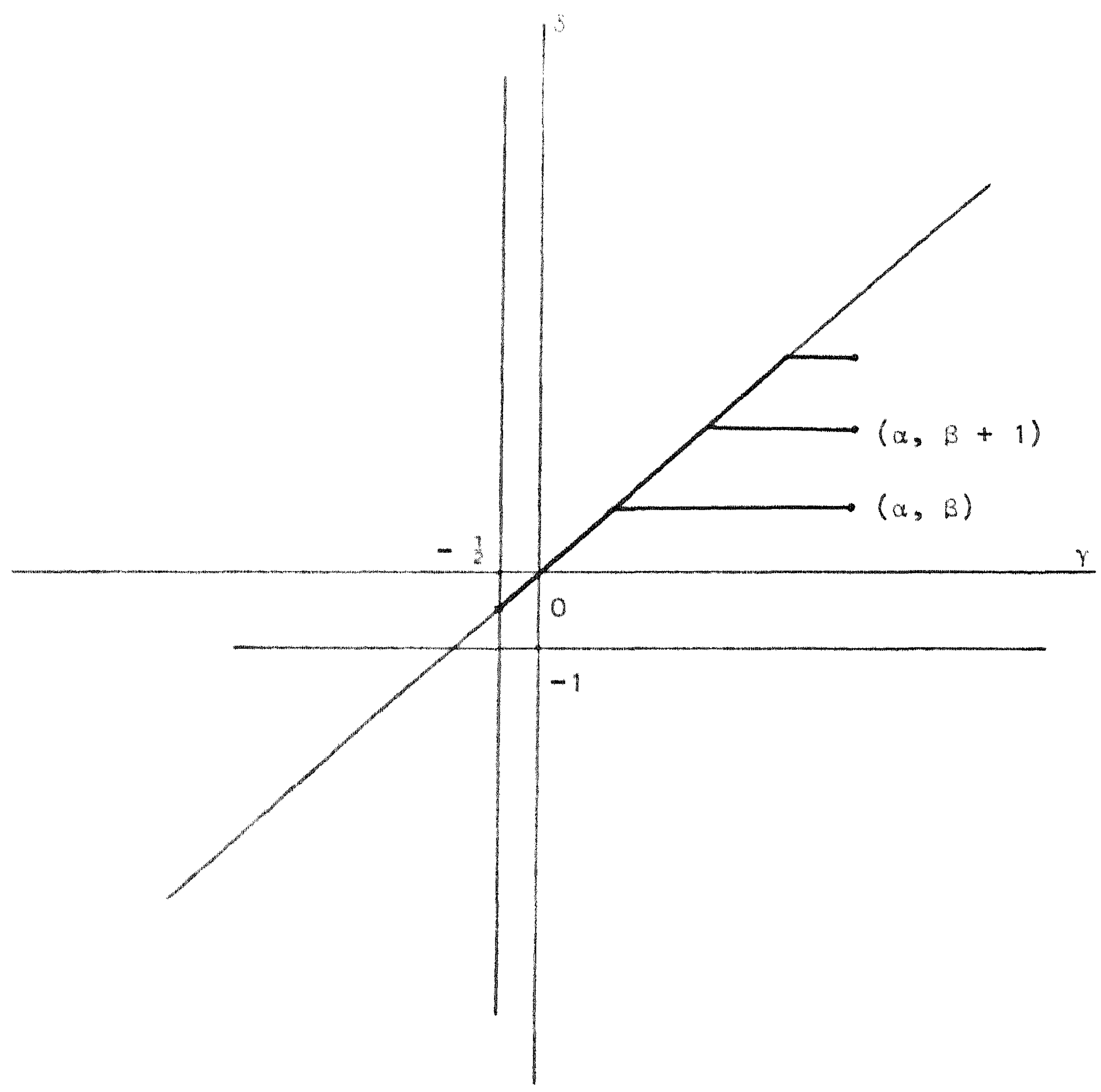

FIGURE 1.

We shall prove now, that relation (A) holds in the following cases:

(i) $\alpha = \gamma$, $\beta < \delta$, $\gamma \geqslant \delta$,

(ii) $\alpha = \gamma + \mu$, $\beta = \delta + \mu$, $\mu > 0$, $\gamma \geqslant \max(\delta, -\frac{1}{2})$, $\delta > -1$.

THEOREM 2. *If* $\gamma = \alpha$ *and* $\delta = \beta + \mu$, *where* $\mu > 0$ *and* $\gamma \geqslant \delta$, *then relation* (A) *holds.*

Proof. Following the proof of Theorem 1, it remains to show that the sequence

$$F_m = \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \left| \sum_{k=j}^{\infty} a_k c_{jk}(\alpha, \beta; \gamma, \delta) \right|$$

is bounded.

We now have

$$F_m \leqslant \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \sum_{k=j}^{\infty} | a_k | | c_{jk}(\alpha, \beta; \gamma, \delta) |$$

$$\leqslant \sum_{k=0}^{\infty} | a_k | \sum_{j=0}^{m} | c_{jk}(\alpha, \beta; \gamma, \delta) | P_j^{(\gamma,\delta)}(1).$$

As

$$P_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} c_{jk}(\alpha, \beta; \alpha, \beta + \mu) P_j^{(\alpha,\beta+\mu)}(x),$$

it follows from the identity

$$P_k^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) \qquad \text{(see Szegö [5, Section 4.1])}$$

that

$$P_k^{(\beta,\alpha)}(x) = \sum_{j=0}^{k} (-1)^{k-j} c_{jk}(\alpha, \beta; \alpha, \beta + \mu) P_j^{(\beta+\mu,\alpha)}(x).$$

In Section 9.4 of Szegö [5] the following relation is derived:

$$P_k^{(\beta,\alpha)}(x) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(-\mu)\,\Gamma(k + \alpha + \beta + 1)} \times \sum_{j=0}^{k} \frac{\left\{ \begin{matrix} \Gamma(k + j + \alpha + \beta + 1)\,\Gamma(k - j - \mu) \\ \times \Gamma(j + \alpha + \beta + \mu + 1)(2j + \alpha + \beta + \mu + 1) \end{matrix} \right\}}{\Gamma(k + j + \alpha + \beta + \mu + 2)\,\Gamma(k - j + 1)\,\Gamma(j + \alpha + 1)} \times P_j^{(\beta+\mu,\alpha)}(x).$$

Hence

$$F_m \leqslant \sum_{k=j}^{\infty} | a_k | \sum_{j=0}^{k} \left| \frac{\left\{ \begin{matrix} \Gamma(k + \alpha + 1)\,\Gamma(k + j + \alpha + \beta + 1)\,\Gamma(k - j - \mu) \\ \times \Gamma(j + \alpha + \beta + \mu + 1)(2j + \alpha + \beta + \mu + 1) \end{matrix} \right\}}{\left\{ \begin{matrix} \Gamma(-\mu)\,\Gamma(k + \alpha + \beta + 1)\,\Gamma(k + j + \alpha + \beta + \mu + 2) \\ \times \Gamma(k - j + 1)\,\Gamma(j + \alpha + 1) \end{matrix} \right\}} \right| \times P_j^{(\alpha,\beta+\mu)}(1).$$

Since $\Gamma(k + \alpha)/\Gamma(k) = O(k^\alpha)$, we can estimate the order of magnitude of F_m.

$$F_m \leqslant c \sum_{k=0}^{\infty} | a_k | k^{-\beta} \sum_{j=0}^{k} (k + j)^{-\mu-1}(k - j)^{-\mu-1} j^{\alpha+\beta+\mu+1}$$

$$\leqslant c \sum_{k=0}^{\infty} | a_k | k^{-\beta-\mu-1} \left(\sum_{j=0}^{[k/2]} k^{-\mu-1} j^{\alpha+\beta+\mu+1} + \sum_{j=[k/2]+1}^{k} k^{\alpha+\beta+\mu+1}(k - j)^{-\mu-1} \right)$$

$$\leqslant c \sum_{k=0}^{\infty} | a_k | k^{\alpha} < \infty.$$

THEOREM 3. *If* $\gamma = \alpha - \mu$ *and* $\delta = \beta - \mu$, *where* $\mu > 0$ *and* $\gamma \geq \max(\delta, -\frac{1}{2})$, $\delta > -1$, *then relation* (A) *holds.*

Proof. It suffices to show that

$$\sum_{j=0}^{k} |c_{jk}(\alpha, \beta; \alpha - \mu, \beta - \mu)| \, P_j^{(\alpha-\mu,\beta-\mu)}(1) = O(k^{\alpha}).$$

Substituting the values of $c_{jk}(\alpha, \beta; \alpha - \mu, \beta - \mu)$, we obtain

$$\sum_{j=0}^{k} P_j^{(\alpha-\mu,\beta-\mu)}(1)(h_j(\alpha - \mu, \beta - \mu))^{-1}$$

$$\times \left| \int_{-1}^{1} P_k^{(\alpha,\beta)}(x) \, P_j^{(\alpha-\mu,\beta-\mu)}(x)(1 - x)^{\alpha-\mu}(1 + x)^{\beta-\mu} \, dx \right|$$

$$= \left(\sum_{j=0}^{k} \frac{\Gamma(j + \alpha + \beta - 2\mu + 1)(2j + \alpha + \beta - 2\mu + 1)}{\Gamma(\alpha - \mu + 1) \, \Gamma(j + \beta - \mu + 1)} \right)$$

$$\times \left| \int_{0}^{\pi} P_k^{(\alpha,\beta)}(\cos\theta) \, P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta) \left(\sin\frac{\theta}{2}\right)^{2\alpha-2\mu+1} \left(\cos\frac{\theta}{2}\right)^{2\beta-2\mu+1} d\theta \right|.$$

We will take the liberty of omitting lower order terms in k when they are inessential.

We shall take the integral over $[0, \pi/2]$ only. The interval $[\pi/2, \pi]$ can be handled similarly. It suffices to show that

$$\left(\sum_{j=0}^{k} j^{\alpha-\mu+1} \right)$$

$$\times \left| \int_{0}^{\pi/2} \left(\sin\frac{\theta}{2}\right)^{2\alpha-2\mu+1} \left(\cos\frac{\theta}{2}\right)^{2\beta-2\mu+1} P_k^{(\alpha,\beta)}(\cos\theta) \, P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta) \, d\theta \right|$$

$$= O(k^{\alpha}).$$

We need the following estimates for Jacobi polynomials and Bessel functions:

$$| P_n^{(\alpha,\beta)}(\cos\theta)| \leq An^{\alpha}, \qquad 0 \leq \theta \leq \frac{\pi}{2}, \tag{2.2}$$

(Szegö [5, 7.32.6]),

$$| P_n^{(\alpha,\beta)}(\cos\theta)| \leq An^{-1/2}\theta^{-\alpha-1/2}, \qquad 0 \leq \theta \leq \frac{\pi}{2}, \tag{2.3}$$

$$| J_\alpha(x)| \leq Ax^{\alpha}, \qquad 0 \leq x \leq 1, \qquad \text{(Szegö [5, 1.71.10])}, \tag{2.4}$$

$$| J_\alpha(x)| \leq Ax^{-1/2}, \qquad x \geq 1, \qquad \text{(Szegö [5, 1.71.11])}, \tag{2.5}$$

$$J_\alpha(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}), \qquad \text{(Szegö 5, 1.71.7])}. \tag{2.6}$$

We shall also need the Sonine integral

$$\int_0^\infty \frac{J_\mu(at)\,J_\nu(bt)}{b^\nu t^{\mu-\nu-1}}\,dt = \frac{(a^2-b^2)^{\mu-\nu-1}}{2^{\mu-\nu-1}a^\mu\Gamma(\mu-\nu)}, \quad a > b \text{ (Watson [6, Section 13.46])} \tag{2.7}$$

and Hilb's formula

$$\left(\sin\frac{\theta}{2}\right)^\alpha \left(\cos\frac{\theta}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos\theta) = N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_\alpha(N\theta)$$
$$+\begin{cases}\theta^{1/2}0(n^{-3/2}), & \text{if } cn^{-1} \leqslant \theta \leqslant \pi-\epsilon,\\ \theta^{\alpha+2}0(n^\alpha), & \text{if } 0 < \theta < cn^{-1},\end{cases}$$

where $N = n + (\alpha+\beta+1)/2$; c and ϵ are fixed positive numbers [5, 8.21.17].

We follow the method used by Askey and Wainger [2], and therefore wish to replace

$$2^{1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-\mu+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-\mu+1/2} P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta)$$

by $\theta^{1/2}J_{\alpha-\mu}(J\theta)$, $J = j + (\alpha+\beta-2\mu+1)/2$, using Hilb's formula (2.8).

We have then to consider

$$I = \sum_{j=0}^{k} j^{\alpha-\mu+1}\left|\int_0^{\pi/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-\mu+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-\mu+1/2} P_k^{(\alpha,\beta)}(\cos\theta)\right.$$
$$\times\left\{2^{1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-\mu+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta-\mu+1/2} P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta)\right.$$
$$\left.\left.-\frac{J^{-\alpha+\mu}\Gamma(j+\alpha-\mu+1)}{\Gamma(j+1)}\,\theta^{1/2}J_{\alpha-\mu}(J\theta)\right\}d\theta\right|.$$

Setting $I = I_1 + I_2$, where, in I_1, the range of integration is $[1/k, \pi/2]$ and in I_2, $[0, 1/k]$, and using some of the estimates mentioned above, we get

$$\begin{aligned} I_1 &= O\left(\sum_{j=0}^{k} j^{\alpha-\mu+1}\int_{1/k}^{\pi/2} k^{-1/2}\theta^{-\alpha-1/2}\theta j^{-3/2}\theta^{\alpha-\mu+1/2}\,d\theta\right)\\ &= O\left(k^{\alpha-\mu}\int_{1/k}^{\pi/2}\theta^{1-\mu}\,d\theta\right)\\ &= O(k^{\alpha-\mu}(c + k^{\mu-2} + \delta_{\mu,2}\log k))\\ &= O(k^\alpha).\\ I_2 &= O\left(\sum_{j=0}^{k} j^{\alpha-\mu+1}\int_0^{1/k} k^\alpha\theta k^{-3/2}\theta^{\alpha-\mu+1/2}\,d\theta\right)\\ &= O\left(k^{2\alpha-\mu+1/2}\int_0^{1/k}\theta^{\alpha-\mu+3/2}\,d\theta\right)\\ &= O(k^{\alpha-2}).\end{aligned}$$

The process of replacing the other Jacobi polynomial by the appropriate Bessel function is similar.

Thus we are led to investigate

$$L = \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{\pi/2} \left(\sin\frac{\theta}{2}\right)^{-\mu} \left(\cos\frac{\theta}{2}\right)^{-\mu} \theta J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right|$$

where $K = k + (\alpha + \beta + 1)/2$. We want to replace $(\sin\theta/2)^{-\mu}(\cos\theta/2)^{-\mu}$ by $\theta^{-\mu}$. It is easily seen that $(\sin\theta/2)^{-\mu}(\cos\theta/2)^{-\mu} = (\theta/2)^{-\mu} G(\theta)$, where $G(0) = 1$, $G(\theta)$ is bounded and $1 - G(\theta) = O(\theta^2)$. Thus we have to consider

$$E = \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{\pi/2} \theta^{1-\mu}(1 - G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right|.$$

We set $E = E_1 + E_2$, where in E_1 the range of integration is $[0, 1/k]$, and in E_2, $[1/k, \pi/2]$.

Applying some of the estimates mentioned above, we get

$$\begin{aligned} E_1 &= \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{1/k} \theta^{1-\mu}(1 - G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right| \\ &= O\left(\sum_{j=0}^{k} j^{\alpha-\mu+1} j^{\alpha-\mu} k^\alpha \int_0^{1/k} \theta^{2\alpha-\mu+3-\mu}\, d\theta \right) \\ &= O(k^{\alpha-2}). \end{aligned}$$

Using the asymptotic formula for Bessel functions and the error term, we obtain, for $\mu < 1$,

$$\begin{aligned} E_2 &= \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_{1/k}^{\pi/2} \theta^{1-\mu}(1 - G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right| \\ &= O\left(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \left| \int_{1/k}^{\pi/2} \theta^{-\mu}(1 - G(\theta))\, e^{i(J\pm K)\theta}\, d\theta \right| \right) \\ &\quad + O\left(k^{-3/2} \sum_{j=0}^{k} j^{\alpha-\mu-1/2} \int_{1/k}^{\pi/2} \theta^{-\mu}\, d\theta \right) \\ &= O\left(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \frac{1}{K \pm J} \right) + O(k^{\alpha-\mu-1} + k^{\alpha-2}) \\ &= O(k^{\alpha-\mu}) + O\left(k^{-1/2} \sum_{j=0}^{[k/2]} \frac{j^{\alpha-\mu+1/2}}{k-j} + k^{-1/2} \sum_{j=[k/2]+1}^{k} \frac{j^{\alpha-\mu+1/2}}{k-j} \right) \\ &= O(k^{\alpha-\mu}) + O(k^{\alpha-\mu}) + O(k^{\alpha-\mu} \log k) \\ &= O(k^\alpha). \end{aligned}$$

The case $\mu \geqslant 1$ is easily handled:

$$\begin{aligned} E_2 &= O\left(\sum_{j=1}^{k} j^{\alpha-\mu+1} \left| \int_{1/k}^{\pi/2} \theta^{3-\mu} j^{-1/2} k^{-1/2} \theta^{-1} \, d\theta \right| \right) \\ &= \begin{cases} O(k^{\alpha-\mu+1}(c + k^{\mu-3})), & \mu \neq 3, \\ O(k^{\alpha-2} \log k), & \mu = 3, \end{cases} \\ &= O(k^{\alpha}). \end{aligned}$$

Finally, we want to replace the range of integration $[0, \pi/2]$ by $[0, \infty)$. Therefore we investigate

$$\sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_{\pi/2}^{\infty} \theta^{1-\mu} J_{\alpha-\mu}(J\theta)\, J_{\alpha}(K\theta) \, d\theta \right| = A_1 + A_2$$

by using (2.6). Here A_1 contains the main terms and A_2 all the error terms.

$$\begin{aligned} A_1 &= O\left(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \left| \int_{\pi/2}^{\infty} \theta^{-\mu} e^{i(K \pm J)\theta} \, d\theta \right| \right) \\ &= O\left(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} (k \pm j)^{-1} \right) \\ &= O(k^{\alpha-\mu} \log k). \\ A_2 &= O\left(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu-1/2} \int_{\pi/2}^{\infty} \theta^{-\mu-1} \, d\theta \right) = O(k^{\alpha-\mu}). \end{aligned}$$

Up to an error term that we have estimated, we may write for L,

$$\sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_{0}^{\infty} \theta^{1-\mu} J_{\alpha-\mu}(J\theta) J_{\alpha}(K\theta) \, d\theta \right|.$$

Using Sonine's integral (2.7), this leads to

$$\begin{aligned} &\sum_{j=0}^{k} j^{\alpha-\mu+1} \frac{2^{1-\mu} J^{\alpha-\mu} (K^2 - J^2)^{\mu-1}}{K^{\alpha} \Gamma(\mu)} \\ &\quad = O\left(k^{-\alpha} \sum_{j=0}^{k} j^{2\alpha-2\mu+1} (k+j)^{\mu-1} (k-j)^{\mu-1} \right) \\ &\quad = O\left(k^{-\alpha+\mu-1} \left\{ \sum_{j=0}^{[k/2]} j^{2\alpha-2\mu+1} (k-j)^{\mu-1} + \sum_{j=[k/2]+1}^{k} j^{2\alpha-2\mu+1} (k-j)^{\mu-1} \right\} \right) \\ &\quad = O(k^{\alpha}). \end{aligned}$$

Combining all the estimates, we have shown that

$$\sum_{j=0}^{k} | c_{jk}(\alpha, \beta; \alpha - \mu, \beta - \mu) | \, P_j^{(\alpha-\mu,\beta-\mu)}(1) = O(k^{\alpha}),$$

which proves Theorem 3.

3. Results

Combining Theorems 1, 2 and 3, we see that for all (γ, δ) in the shaded region of Fig. 2, relation (A) holds. We shall show now by means of examples that that region is exactly the set of all (γ, δ) with $\gamma \geq -\frac{1}{2}$, for which (A) holds.

Consider, first, the function $(1 + x)^{\mu}$, $\mu > 0$. Its Fourier coefficients are

$$a_n = h_n(\alpha, \beta))^{-1} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)(1 - x)^{\alpha}(1 + x)^{\beta+\mu} \, dx.$$

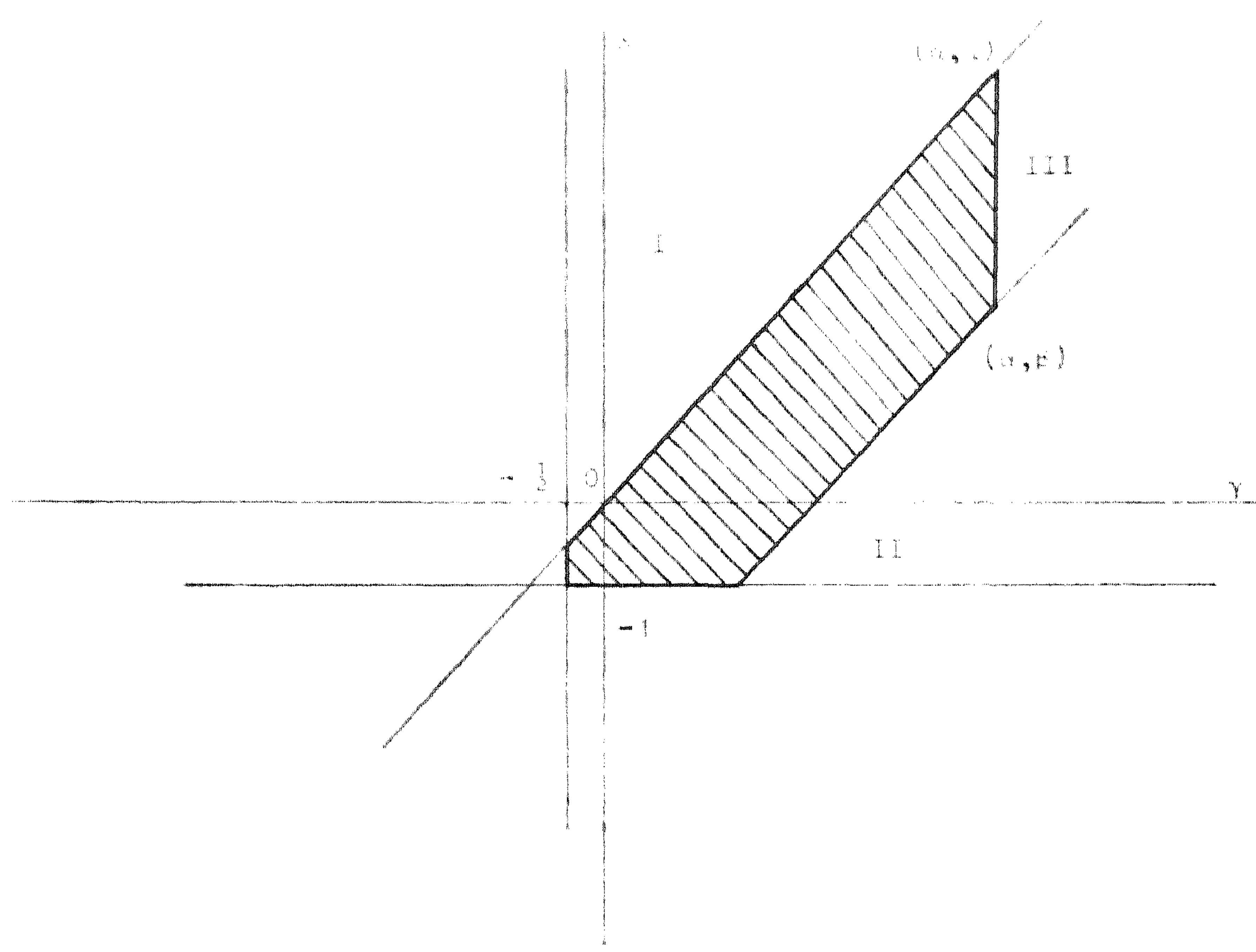

Figure 2.

Using Rodrigues's formula (1.1) and integrating by parts, we have

$$\begin{aligned} a_n &= \frac{(-1)^n}{2^n n!\, h_n(\alpha,\beta)} \int_{-1}^{1} (1+x)^\mu \left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}\, dx \\ &= \frac{\Gamma(\mu+1)}{2^n n!\, h_n(\alpha,\beta)\, \Gamma(\mu-n+1)} \int_{-1}^{1} (1-x)^{n+\alpha}(1+x)^{\beta+\mu}\, dx \\ &= (-1)^{n+1} \frac{2^\mu}{\pi} \Gamma(\mu+1) \sin \mu\pi\, \Gamma(\beta+\mu+1)(2n+\alpha+\beta+1) \\ &\quad \times \frac{\Gamma(n+\alpha+\beta+1)\, \Gamma(n-\mu)}{\Gamma(n+\alpha+\beta+\mu+2)\, \Gamma(n+\beta+1)}. \end{aligned} \tag{3.1}$$

Thus

$$| a_n | = O(n^{-\beta-2\mu-1}).$$

It follows that $(1+x)^\mu \in A(\alpha, \beta)$ if $\alpha - \beta < 2\mu$.

From (3.1) it is easily derived that the function $(1+x)^\mu$, with $(\alpha-\beta)/2 < \mu < (\gamma-\delta)/2$, μ not an integer, belongs to $A(\alpha, \beta)$ but not to $A(\gamma, \delta)$. Thus we have found a function for which relation (A) fails in region II of Fig. 2.

In the same way we can calculate the Fourier coefficients of the function $(1-x)^\mu$ and obtain

$$| a_n | = O(n^{-\alpha-2\mu-1}).$$

It follows that $(1-x)^\mu \in A(\alpha, \beta)$ if $\mu > 0$.

But if $\delta > \gamma$, the maximum of the absolute value of the Jacobi polynomials is assumed at $x = -1$ and $P_n^{(\gamma,\delta)}(-1) = O(n^\delta)$. If $\delta > \gamma$, the function $(1-x)^\mu$, with $0 < \mu < (\delta-\gamma)/2$, μ not an integer, belongs to $A(\alpha, \beta)$ but not to $A(\gamma, \delta)$. Thus, (A) is not valid in region I of Fig. 2.

In order to decide whether relation (A) holds in region III, we study the function $| x |^\mu$. Here

$$\begin{aligned} a_n &= (h_n(\alpha,\beta))^{-1} \int_{-1}^{1} | x |^\mu P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx \\ &= (h_n(\alpha,\beta))^{-1} \left\{ \int_0^1 x^\mu P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx \right. \\ &\quad \left. + (-1)^n \int_0^1 x^\mu P_n^{(\beta,\alpha)}(x)(1-x)^\beta(1+x)^\alpha\, dx \right\}. \end{aligned}$$

If Re $\mu > n - 1$, we can use Rodrigues's formula and integrate by parts. We obtain

$$\begin{aligned} a_n = {} & \frac{(2n+\alpha+\beta+1)\,\Gamma(\mu+1)\,\Gamma(n+\alpha+\beta+1)}{2^{n+\alpha+\beta+1}\Gamma(n+\beta+1)\,\Gamma(\alpha+\mu+2)} \\ & \times {}_2F_1(\mu-n+1, -\beta-n; \alpha+\mu+2; -1) \\ & + (-1)^n \frac{(2n+\alpha+\beta+1)\,\Gamma(\mu+1)\,\Gamma(n+\alpha+\beta+1)}{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(\beta+\mu+2)} \\ & \times {}_2F_1(\mu-n+1, -\alpha-n; \beta+\mu+2; -1). \end{aligned} \tag{3.2}$$

The hypergeometric series ${}_2F_1$ $(a, b; c; -1)$ is absolutely convergent if $\mathrm{Re}(a + b - c) < 0$, which means here $-\alpha - \beta - 2n - 1 < 0$. This is always satisfied (if $n \geqslant 1$). In this case ${}_2F_1(a, b; c; -1)$ is an analytic function of the parameters a, b and c. Since for Re $\mu > n - 1$, a_n is given by (3.2), it follows by analytic continuation that (3.2) holds for all μ with Re $\mu > -1$. Using the simple relation

$$\begin{aligned} {}_2F_1(a, b; c; z) &= (1-z)^{-b}\,{}_2F_1\left(b, c-a; c; \frac{z}{z-1}\right) \\ &= (1-z)^{-b}\,{}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right) \end{aligned}$$

[3, Section 3.8, (4)], a_n can be written in the following way:

$$\begin{aligned} a_n = {} & \frac{(2n+\alpha+\beta+1)\,\Gamma(\mu+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+1}\Gamma(n+\beta+1)\,\Gamma(\alpha+\mu+2)} \\ & \times {}_2F_1(\alpha+n+1, -\beta-n; \alpha+\mu+2; \tfrac{1}{2}) \\ & + (-1)^n \frac{(2n+\alpha+\beta+1)\,\Gamma(\mu+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\beta+1}\Gamma(n+\alpha+1)\,\Gamma(\beta+\mu+2)} \\ & \times {}_2F_1(\beta+n+1, -\alpha-n; \beta+\mu+2; \tfrac{1}{2}). \end{aligned}$$

An asymptotic expansion of the hypergeometric function in this case, for large n, has been given by Watson [7].

The leading term is

$$\begin{aligned} {}_2F_1\left(a+n, b-n; c; \frac{1-z}{2}\right) \sim {} & \frac{2^{a+b-1}\Gamma(1-b+n)\,\Gamma(c)(1+e^{-\zeta})^{c-a-b-1/2}}{(n\pi)^{1/2}\Gamma(c-b+n)(1-e^{-\zeta})^{c-1/2}} \\ & \times \{e^{(n-b)\zeta} + \exp[\pm i\pi(c-\tfrac{1}{2})]\, e^{-(n+a)\zeta}\} \end{aligned}$$

where ζ is defined by $z = \cosh \zeta$ and $\operatorname{Re} \zeta > 0$, $-\pi < \operatorname{Im} \zeta \leq \pi$. The upper (lower) sign is taken if $\operatorname{Im} z > (<) 0$. In the case in which $z - 1$ is real and negative it is supposed that z attains its value by a limiting process which then determines if $\arg(z - 1)$ is π or $-\pi$. The discontinuity in the formula is only apparent; if z crosses the real axis between ± 1, account has to be taken of the discontinuity in the value of $\operatorname{Im} \zeta$. Therefore,

$$
\begin{aligned}
| a_n | &= O\left(\frac{n^{\alpha+1}\Gamma(n+\beta+1)}{n^{1/2}\Gamma(n+\alpha+\beta+\mu+2)} + \frac{n^{\beta+1}\Gamma(n+\alpha+1)}{n^{1/2}\Gamma(n+\alpha+\beta+\mu+2)}\right) \\
&= O(n^{-\mu-1/2}). \qquad (3.3)
\end{aligned}
$$

Thus, in the case that $\mu > \alpha + \frac{1}{2}$, the function $| x |^\mu$ belongs to $A(\alpha, \beta)$.

In the ultraspherical case ($\alpha = \beta$), the Fourier coefficients can easily be calculated. We have

$$a_n = (h_n(\alpha, \alpha))^{-1} \int_{-1}^{1} | x |^\mu P_n^{(\alpha,\alpha)}(x)(1 - x^2)^\alpha \, dx.$$

Because $| x |^\mu$ is an even function, the Fourier coefficients vanish for odd n. Application of a well-known formula for ultraspherical polynomials (see Szegö [5, 4.1.5]) yields

$$
\begin{aligned}
a_{2n} &= \frac{2n!\, \Gamma(2n+\alpha+1)}{h_{2n}(\alpha, \alpha)(2n)!\, \Gamma(n+\alpha+1)} \int_0^1 P_n^{(\alpha,-1/2)}(y)(1-y)^\alpha(1+y)^{(\mu-1)/2} \, dy \\
&= \frac{(-1)^n(4n+2\alpha+1)\, \Gamma(2n+2\alpha+1)\, \Gamma(\mu+1) \sin(\mu/2)\pi \Gamma(n-(\mu/2))}{2^{2\alpha+\mu+1}\Gamma(2n+\alpha+1)\, \Gamma(n+\alpha+(\mu/2)+\frac{3}{2})\pi^{1/2}}. \qquad (3.4)
\end{aligned}
$$

From (3.3) and (3.4) it follows that if $\gamma > \alpha$, the function $| x |^\mu$, with $\alpha + \frac{1}{2} < \mu < \gamma + \frac{1}{2}$, μ not an even integer, belongs to $A(\alpha, \beta)$ but not to $A(\gamma, \gamma)$. Combined with Theorem 2, this leads to the conclusion that relation (A) cannot hold in region III of Fig. 2.

Thus the shaded region in Fig. 2 is exactly the set (if $\gamma \geqslant -\frac{1}{2}$) where relation (A) holds.

By using the identity $P_n^{(\alpha,\beta)}(x) = (-1)^n P^{(\beta,\alpha)}(-x)$, similar results can be obtained when $\alpha < \beta$.

Acknowledgment

The author is much indebted to Professor R. A. Askey for setting the problem and for many helpful suggestions for its solution.

REFERENCES

1. R. ASKEY, Orthogonal polynomials and positivity, *in* "Studies in Applied Mathematics", (D. Ludwig and F. W. J. Olver, Eds.), Vol. 6, pp. 64–85, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1970,
2. R. ASKEY AND ST. WAINGER, A dual convolution structure for Jacobi polynomials, *in* "Orthogonal Expansions and Their Continuous Analogues," (D. T. Haimo, Ed.), pp. 25–36, Southern Illinois Univ. Press, Carbondale and Edwardsville, 1968.
3. Y. L. LUKE, "The Special Functions and Their Approximations," Academic Press, New York, 1969.
4. T. J. RIVLIN AND M. W. WILSON, An optimal property of Chebyshev expansions, *J. Approximation Theory* **2** (1969), 312–317.
5. G. SZEGÖ, "Orthogonal polynomials," *Amer. Math. Soc. Colloq. Publ.*, Vol. 23, 3rd. ed., Providence, RI, 1967.
6. G. N. WATSON, "A Treatise on the Theory of Bessel Functions," Cambridge University Press, London, 1966.
7. G. N. WATSON, Asymptotic expansions of hypergeometric functions, *Trans. Cambridge Phil. Soc.* **22** (1918), 277–308.

Printed in Belgium by the St. Catherine Press Ltd., Tempelhof 37, Bruges