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A Special Class of Jacobi Series and Some Applications

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In this paper, series of the form

$$F(\cos\theta) \sim \sum_{n=2}^{\infty} \frac{\omega_n^{(\alpha,\beta)}}{n^\gamma(\log n)^\delta} R_n^{(\alpha,\beta)}(\cos\theta)$$

are considered, where $R_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial normalized to be 1 at $x = 1$ and

$$\omega_n^{(\alpha,\beta)} = \frac{(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\,\Gamma(n+1)\,\Gamma(\alpha+1)\,\Gamma(\alpha+1)} = 0(n^{2\alpha+1}).$$

It turns out, that for $0 < \theta \leqslant \pi$, the function $F(\cos\theta)$ is continuous. As $\theta \to 0^+$, its behavior is given by

$$F(\cos\theta) \simeq \frac{\Gamma(\alpha+1-\gamma/2)}{\Gamma(\gamma/2)\,\Gamma(\alpha+1)} \left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-2} (\log\theta^{-1})^{-\delta},$$

if $0 < \gamma < 2\alpha + 2$. If $\gamma = 0$ the results are slightly different. Next, fractional integration and differentiation for Jacobi series are introduced and the above results are used to show that the classical theorems on fractional integration and differentiation can be carried over. As an application sufficient conditions are given for $F(\cos\theta)$ to have a uniformly convergent or an absolutely convergent Fourier–Jacobi series. This paper will be followed by another one in which the results of this paper are used to deal with approximation theorems for Jacobi series.

1. A Convolution Structure for Jacobi Series

We shall consider some classes of complex-valued functions on the interval $[0, \pi]$. By C is denoted the Banach space of continuous functions on $[0, \pi]$ with the norm

$$\| f \|_C = \sup_{0 \leqslant \theta \leqslant \pi} | f(\cos\theta)| .$$

We write L^p, $1 \leqslant p < \infty$, for the Banach space of functions f on $[0, \pi]$, the p-th power of which is integrable with respect to the weight function

$$\rho^{(\alpha,\beta)}(\theta) = \left(\sin \frac{\theta}{2}\right)^{2\alpha+1} \left(\cos \frac{\theta}{2}\right)^{2\beta+1}, \tag{1.1}$$

with the norm

$$\|f\|_p = \left[\int_0^\pi |f(\cos\theta)|^p \rho^{(\alpha,\beta)}(\theta)\, d\theta\right]^{1/p}.$$

We define L^∞ to be the Banach space of functions f on $[0, \pi]$ which are essentially bounded, with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{0\leqslant\theta\leqslant\pi} |f(\cos\theta)|\,.$$

We call M the space of all regular, finite Borel measures μ on $[0, \pi]$ with the norm

$$\|\mu\|_M = \int_0^\pi |\, d\mu(\cos\theta)|\,.$$

We use the notation

$$(f, g) = \int_0^\pi f(\cos\theta)\, \overline{g(\cos\theta)}\, \rho^{(\alpha,\beta)}(\theta)\, d\theta.$$

The functions

$$R_n^{(\alpha,\beta)}(\cos\theta) = \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)},$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of order (α, β) and degree n, provide a complete orthogonal system on $[0, \pi]$ with respect to the weight function $\rho^{(\alpha,\beta)}(\theta)$ defined by (1.1). If $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, they satisfy the following relation [14, (7.32.2)]:

$$\sup_{0\leqslant\theta\leqslant\pi} |\, R_n^{(\alpha,\beta)}(\cos\theta)| = R_n^{(\alpha,\beta)}(1) = 1. \tag{1.2}$$

In the rest of this paper we shall always assume that $\alpha \geqslant \beta \geqslant -\frac{1}{2}$.

The functions $R_n^{(\alpha,\beta)}(\cos\theta)$ are the eigenfunctions of the differential operator

$$P\left(\frac{d}{d\theta}\right) = -\{\rho^{(\alpha,\beta)}(\theta)\}^{-1} \frac{d}{d\theta}\left\{\rho^{(\alpha,\beta)}(\theta) \frac{d}{d\theta}\right\} \tag{1.3}$$

with the boundary conditions

$$\frac{dR_n^{(\alpha,\beta)}(\cos\theta)}{d\theta} = 0, \qquad \theta = 0, \quad \theta = \pi. \tag{1.4}$$

The eigenvalues are $\lambda_n = n(n + \alpha + \beta + 1)$. The differential operator $P(d/d\theta)$ is self-adjoint with respect to the scalar product defined above,

$$(Pf, g) = (f, Pg).$$

By $A_\theta = A_\theta{}^p$ we shall denote the corresponding realization of P in L^p. The domain of $A_\theta{}^p$ is denoted $D^p = D(A_\theta{}^p)$.

With a function f, belonging to one of the spaces L^p, $1 \leqslant p \leqslant \infty$, or C, we associate the Fourier–Jacobi expansion

$$f(\cos\theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta), \tag{1.5}$$

where

$$\begin{aligned} \omega_n^{(\alpha,\beta)} &= \left(\int_0^\pi \{R_n^{(\alpha,\beta)}(\cos\theta)\}^2 \rho^{(\alpha,\beta)}(\theta)\, d\theta\right)^{-1} \\ &= \frac{(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\,\Gamma(n+1)\,\Gamma(\alpha+1)\,\Gamma(\alpha+1)} \\ &= 0(n^{2\alpha+1}), \end{aligned} \tag{1.6}$$

and

$$f^{\wedge}(n) = \int_0^\pi f(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)\, d\theta, \qquad n = 0, 1, 2,\dots . \tag{1.7}$$

For Jacobi series a generalized translation operator and a convolution structure have been developed by Askey and Wainger [4]. Recently, Gasper [9] has pointed out, that the important kernel that occurs in their work, is positive. In fact, he proves that for $0 < \theta, \phi, \psi < \pi$, $n = 0, 1, 2,\dots$,

$$\begin{aligned} &R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\phi) \\ &\qquad = \int_0^\pi R_n^{(\alpha,\beta)}(\cos\psi)\, K(\cos\theta, \cos\phi, \cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi, \end{aligned} \tag{1.8}$$

with

$$K(\cos\theta, \cos\phi, \cos\psi) \geqslant 0.$$

The generalized translate of a function $f(\cos\theta) \in L^1$, which has a Fourier–Jacobi expansion (1.5), can be defined by

$$\begin{aligned} f(\cos\theta, \cos\phi) &= \int_0^\pi f(\cos\psi)\, K(\cos\theta, \cos\phi, \cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi \\ &\sim \sum_{n=0}^{\infty} f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\phi). \end{aligned} \tag{1.9}$$

We shall also use the notation

$$T_\phi f(\cos\theta) = f(\cos\theta, \cos\phi).$$

Using Hölder's inequality, it follows from (1.8), in the case $n = 0$ and (1.9), that

$$\| T_\phi f \|_p \leqslant \| f \|_p, \qquad 1 \leqslant p \leqslant \infty. \tag{1.10}$$

As

$$\lim_{\phi \to 0^+} T_\phi f = f \quad \text{in} \quad L^p \qquad (1 \leqslant p < \infty)$$

holds for a dense set of functions, the polynomials, it follows from (1.10) and Helly's theorem [14, Theorem 1.6] that

$$\lim_{\phi \to 0^+} \| T_\phi f - f \|_p = 0, \qquad 1 \leqslant p < \infty. \tag{1.11}$$

$T_\phi f(\cos\theta)$ is a generalized translate in the sense used by Löfström and Peetre [12]. In their paper they make the connection between a generalized translation operator and a differential operator of the form (1.3) with boundary conditions (1.4). They show that for functions $f \in D^p(A_\theta)$, that is, for functions f such that

$$A_\theta f(\cos\theta) \sim \sum_{n=1}^{\infty} f^\wedge(n)\, n(n + \alpha + \beta + 1)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

is in L^p, the following estimate holds:

$$\| T_\phi f - f \|_p \leqslant C\phi^2 \| A_\theta f \|_p, \qquad 1 \leqslant p \leqslant \infty. \tag{1.12}$$

C is a constant independent of f.

For $f_1, f_2 \in L^1$, the convolution $f_1 * f_2$ will be defined by

$$\begin{aligned} (f_1 * f_2)(\cos\theta) &= \int_0^\pi \int_0^\pi f_1(\cos\phi) f_2(\cos\psi)\, K(\cos\theta, \cos\phi, \cos\psi)\, \rho^{(\alpha,\beta)}(\phi)\, \rho^{(\alpha,\beta)}(\psi)\, d\phi\, d\psi \\ &= \int_0^\pi f_1(\cos\theta, \cos\psi) f_2(\cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi. \end{aligned} \tag{1.13}$$

The convolution (1.13) satisfies the following properties:

LEMMA 1.1. *Let* $f_1, f_2, f_3 \in L^1$. *Then* $f_1 * f_2 \in L^1$ *and*

(i) $f_1 * f_2 = f_2 * f_1$,

(ii) $f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$,

(iii) $(f_1 * f_2)\hat{}\,(n) = f_1\hat{}(n) f_2\hat{}(n)$,

(iv) *If* $g \in L^p$, $1 \leqslant p \leqslant \infty$, *then* $f_1 * g \in L^p$ *and*

$$\| f_1 * g \|_p \leqslant \| f_1 \|_1 \| g \|_p .$$

In the following we shall study a special class of Jacobi series, such as

$$F(\cos\theta) \sim \sum_{n=2}^{\infty} n^{-\gamma} (\log n)^{\delta}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Series of this kind have shown to be useful in various branches of Fourier analysis. For trigonometric series they are investigated in Zygmund [17, Chap. V] and they furnish important counterexamples. In k dimensions trigonometric series of this form are treated by Wainger [15] and several applications are given. In our investigation we shall follow the line of Ref. [5], where Askey and Wainger deal with ultraspherical series of the same type. As they point out in the ultraspherical case, series of this kind supply counterexamples to the transplantation theorem for Jacobi series [2, Theorem 1] in the cases $p = 1$ and $p = \infty$. In the last section, we use this special class of Jacobi series to define fractional integration and differentiation. We shall prove that the usual properties of fractional integration and differentiation remain valid.

Notation. We shall use O and o in the usual manner. We write $F(x) \simeq G(x)$ as x tends to a, to mean

$$\lim_{x \to a} \frac{F(x)}{G(x)} = 1.$$

2. SUMMATION BY PARTS AND A CRITERION FOR FOURIER–JACOBI SERIES

In this section we develop a method of summation by parts, which depends strongly on the Christoffel–Darboux formula for Jacobi polynomials [14, (4.5.3)].

As an application we shall prove a simple sufficient condition for a series

$$\sum_{n=1}^{\infty} a(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

to be a Fourier–Jacobi series of some L^1 function.

LEMMA 2.1. *Let $a(n)$ be a function defined on the nonnegative integers. Let*

$$H(N, \cos\theta) = \sum_{n=0}^{N} a(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Then

$$H(N, \cos\theta) = \sum_{n=0}^{N} \Delta' a(n) \frac{(\alpha+1)}{(2n+\alpha+\beta+2)}\, \omega_n^{(\alpha+1,\beta)} R_n^{(\alpha+1,\beta)}(\cos\theta), \tag{2.1}$$

where, if $d(n)$ is a sequence of numbers,

$$\Delta' d(n) = \Delta d(n) = d(n) - d(n+1), \qquad n = 0, 1,..., N-1,$$
$$\Delta' d(N) = d(N).$$

If, in particular, $a(n) = O(\exp - \epsilon n)$ $(\epsilon > 0)$, we have

$$\begin{aligned} H(\cos\theta) &= \sum_{n=0}^{\infty} a(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) \\ &= \sum_{n=0}^{\infty} \Delta a(n) \frac{(\alpha+1)}{(2n+\alpha+\beta+2)}\, \omega_n^{(\alpha+1,\beta)} R_n^{(\alpha+1,\beta)}(\cos\theta). \end{aligned} \tag{2.2}$$

Proof. The essence of the proof is the application of the Christoffel–Darboux formula for Jacobi polynomials [14, (4.5.3)] and summation by parts:

$$\begin{aligned} H(N, \cos\theta) &= \sum_{n=0}^{N} \Delta' a(n) \sum_{k=0}^{n} \omega_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta) \\ &= \sum_{n=0}^{N} \Delta' a(n) \frac{(\alpha+1)}{(2n+\alpha+\beta+2)}\, \omega_n^{(\alpha+1,\beta)} R_n^{(\alpha+1,\beta)}(\cos\theta). \end{aligned}$$

(2.2) follows by taking the limit as $N \to \infty$, since $\omega_n^{(\alpha,\beta)}$ does not grow faster than a polynomial in n and $|R_n^{(\alpha,\beta)}(\cos\theta)|$ is bounded by 1 (see (1.2)).

We shall need a lemma, which deals with the repeated application of Lemma 2.1. We shall state the results in terms of derivatives rather than finite differences:

LEMMA 2.2. *Let ν be any positive integer and let $a(t)$ be a function of a real variable t possessing ν continuous derivatives. Assume that*

$$\left| \frac{d^j a(t)}{dt^j} \right| = O(\exp - \epsilon t), \qquad j = 0, 1,..., \nu,$$

and any $\epsilon > 0$. *Define*

$$H(\cos\theta) = \sum_{n=1}^{\infty} a(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Then we may write

$$H(\cos\theta) = (-1)^{\nu} \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)} \sum_{n=1}^{\infty} q_\nu(n)\, \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta) + E_1(\cos\theta), \tag{2.3}$$

where

$$q_1(t) = \frac{1}{2t+\alpha+\beta+2} \frac{d}{dt} a(t),$$

$$q_k(t) = \frac{1}{2t+\alpha+\beta+k+1} \frac{d}{dt} q_{k-1}(t), \qquad k \geqslant 2.$$

Also,

$$H(\cos\theta) = \sum_{j=0}^{\nu-1} c(j,\nu) \sum_{n=1}^{\infty} n^{-\nu-j} \left\{ \frac{d^{\nu-j}}{dt^{\nu-j}} a(t) \right\}_{t=n} \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta) + E_2(\cos\theta). \tag{2.4}$$

Here $c(j,\nu)$ *are numbers.*
For $i = 1, 2$,

$$E_i(\cos\theta) = \sum_{j=0}^{\nu-1} d_i(j,\nu) \sum_{n=1}^{\infty} n^{-\nu-j} \gamma_{n,\nu-j+1} \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta),$$

where $d_i(j,\nu)$ *are numbers and*

$$|\gamma_{n,j}| \leqslant \max_{n \leqslant t \leqslant n+a_\nu} \left| \frac{d^j a(t)}{dt^j} \right|, \tag{2.5}$$

a_ν *is some integer depending only on* ν.

Proof. We start with Eq. (2.2) and then apply Lemma 2.1 again. We repeat the process ν times in all and then we use the mean-value theorem to replace differences by derivatives.

THEOREM 2.3. *Let* ν *be an integer* $> \alpha + \frac{3}{2}$. *Assume* $a(t)$ *is continuous on* $[0, \infty)$ *and that* $a(t)$ *approaches zero as* $t \to \infty$. *Furthermore, assume* $a(t)$ *has* $\nu + 1$ *continuous derivatives on* $[0, \infty)$ *and let* $\gamma_{n,j}$ *be defined as in* (2.5). *Finally, suppose*

$$\sum_{n=1}^{\infty} n^{j-1} \gamma_{n,j} < \infty, \qquad j = 1, 2, \ldots, \nu.$$

Then there exists a function $F(\cos\theta) \in L^1$ *with*

$$a(n) = F^\wedge(n).$$

Proof. Let

$$F_\epsilon(\cos\theta) = \sum_{n=1}^{\infty} e^{-\epsilon n} a(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

By Lemma 2.2, Eq. (2.4), (with $e^{-\epsilon n}a(n)$ instead of $a(n)$), and (1.6),

$$F_\epsilon(\cos\theta) = \sum_{k,j,m} \sum_{n=1}^{\infty} c_{k,j,m,n} e^{-\epsilon n} n^{\nu-j+2\alpha+1} \epsilon^m \left\{\frac{d^k a(t)}{dt^k}\right\}_{t=n} R_n^{(\alpha+\nu,\beta)}(\cos\theta)$$

$$+ E_1(\cos\theta).$$

The coefficients $c_{k,j,m,n}$ are bounded for fixed ν and the summation is extended over nonnegative integers k, j, and m, such that $k + j + m = \nu$, and, what is very important, at least one of m and k is $\geqslant 1$. The remainder term $E_1(\cos\theta)$ has the same form, but here $k + j + m > \nu$. It can be handled in the same way as the main term. Let $S_\epsilon(\cos\theta)$ denote those terms with $k \geqslant 1$. We use the trivial estimate

$$\epsilon^m \exp(-\epsilon n) = O(n^{-m}). \tag{2.6}$$

Hence, since $k + j + m = \nu$,

$$|S_\epsilon(\cos\theta)| = O\left\{\sum_{k,j,m} \sum_{n=1}^{\infty} n^{k+2\alpha+1} \gamma_{n,k}\, |R_n^{(\alpha+\nu,\beta)}(\cos\theta)|\right\}$$

with $k \geqslant 1$.

We need the following estimate [14, (7.32.6)]:

$$|R_n^{(\alpha,\beta)}(\cos\theta)| = O(\theta^{-\alpha-\frac{1}{2}}(\pi-\theta)^{-\beta-\frac{1}{2}}\, n^{-\frac{1}{2}-\alpha}), \qquad 0 < \theta < \pi. \tag{2.7}$$

Using (1.2) and (2.7) we find [14, (7.34.1)]

$$\begin{aligned} \int_0^\pi |R_n^{(\alpha+\nu,\beta)}(\cos\theta)|\, \rho^{(\alpha,\beta)}(\theta)\, d\theta &= O(n^{-2\alpha-2}), && \nu > \alpha + \tfrac{3}{2}, \\ &= O(n^{-\frac{1}{2}-\alpha-\nu}\log n), && \nu = \alpha + \tfrac{3}{2}, \\ &= O(n^{-\frac{1}{2}-\alpha-\nu}), && \nu < \alpha + \tfrac{3}{2}. \end{aligned} \tag{2.8}$$

Thus, by hypothesis,

$$\int_0^\pi |S_\epsilon(\cos\theta)|\, \rho^{(\alpha,\beta)}(\theta)\, d\theta = O\left(\sum_{k,j,m} n^{k-1}\gamma_{n,k}\right) < K < \infty.$$

K does not depend on ϵ. Therefore, $S_\epsilon(\cos\theta)\,\rho^{(\alpha,\beta)}(\theta)$ converges weakly to a measure $\in M$ [11, p. 175]. Moreover, using (2.7), we see that $S_\epsilon(\cos\theta)\,\rho^{(\alpha,\beta)}(\theta)$ converges pointwise as $\epsilon \to 0^+$ for $0 < \theta \leqslant \pi$ and uniformly in any compact subinterval of $(0, \pi]$. The term $(\pi - \theta)^{-\beta-\frac{1}{2}}$ in (2.7) is compensated by the factor $(\cos\theta/2)^{2\beta+1}$ of $\rho^{(\alpha,\beta)}(\theta)$. The factor $(\sin\theta/2)^{2\alpha+1}$, however, cannot compensate the singularity at 0 completely because we have here a factor $\theta^{-\alpha-\nu-\frac{1}{2}}$ due to the summation by parts.

We put

$$T_\epsilon(\cos\theta) = F_\epsilon(\cos\theta) - S_\epsilon(\cos\theta) - E_1(\cos\theta).$$

In this series expansion of $T_\epsilon(\cos\theta)$ the terms with $k = 0$ and $m \geqslant 1$ are left.

$$T_\epsilon(\cos\theta) = \sum_{j=0}^{\nu} c_j \sum_{n=1}^{\infty} e^{-\epsilon n} n^{\nu-j+2\alpha+1} \epsilon^{\nu-j} a(n)\, R_n^{(\alpha+\nu,\beta)}(\cos\theta).$$

Using a slightly modified form of estimate (2.6), we obtain

$$|\, T_\epsilon(\cos\theta)| = O\left(\sum_{n=1}^{\infty} \epsilon e^{-(\epsilon/2)n} n^{2\alpha+2}\, |\, a(n)|\, |\, R_n^{(\alpha+\nu,\beta)}(\cos\theta)|\right).$$

Since $a(n)$ is bounded and (2.8) holds, we have

$$\int_0^\pi |\, T_\epsilon(\cos\theta)|\, \rho^{(\alpha,\beta)}(\theta)\, d\theta = O\left(\epsilon \sum_{n=1}^{\infty} e^{-(\epsilon/2)n}\right) = O(1).$$

Thus we may conclude, that $F_\epsilon(\cos\theta)\,\rho^{(\alpha,\beta)}(\theta)$ converges weakly to a measure $\mu \in M$ as $\epsilon \to 0^+$. Moreover, $F_\epsilon(\cos\theta)\,\rho^{(\alpha,\beta)}(\theta)$ converges uniformly on any compact subinterval of $(0, \pi]$. This implies that the singular part of μ is concentrated at 0 and therefore is a δ function at 0. We wish to show that μ is actually absolutely continuous that is, that μ has no singular part. Let $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous and μ_s is a δ function at 0. We have

$$\left| \int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\, d\mu_a \right| \leqslant \int_0^{\epsilon/n} |\, R_n^{(\alpha,\beta)}(\cos\theta)|\, |\, d\mu_a\, | + \int_{\epsilon/n}^{\pi} |\, R_n^{(\alpha,\beta)}(\cos\theta)|\, |\, d\mu_a\, | = o(1).$$

From (1.2), it follows that, if μ_s is not zero,

$$\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\, d\mu_s \qquad \text{is not } o(1) \text{ as } n \to \infty.$$

On the other hand,

$$\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\,d\mu = \lim_{\epsilon\to 0^+}\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\,F_\epsilon(\cos\theta)\,\rho^{(\alpha,\beta)}(\theta)\,d\theta$$
$$= a(n) = o(1).$$

This is a contradiction, unless μ_s is zero.

Let $H(\cos\theta)$ be the derivative of μ and take

$$F(\cos\theta) = H(\cos\theta)\,\{\rho^{(\alpha,\beta)}(\theta)\}^{-1}.$$

Then $F(\cos\theta)$ is in L^1. Also $F_\epsilon(\cos\theta)$ tends to $F(\cos\theta)$ weakly (in L^1). Therefore

$$a(n) = \int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\,F(\cos\theta)\,\rho^{(\alpha,\beta)}(\theta)\,d\theta = F^\wedge(n).$$

This finishes the proof.

THEOREM 2.4. *The Fourier–Jacobi expansion*, (1.5) *with* (1.6) *and* (1.7), *of a function* $f \in L^1$ *is Abel summable to* f *in the* L^1 *norm.*

Proof. The theorem is trivial for a dense set of functions in L^1, the polynomials, because their expansion (1.5) has only a finite number of terms. On the other hand, the Abel–Poisson kernel

$$K_r(\cos\theta, \cos\phi) = \sum_{n=0}^{\infty} r^n \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)\,R_n^{(\alpha,\beta)}(\cos\phi)$$

has been calculated explicitly by Bailey [6, p. 102] as an F_4.

From this representation the nonnegativity is obvious. The theorem then follows from Helly's theorem.

3. ELEMENTARY PROPERTIES OF SLOWLY VARYING FUNCTIONS

We begin with two definitions:

DEFINITION 3.1. We shall say that a function $b(t)$ is slowly varying if $b(t)$ satisfies the following conditions:

(i) $b(t)$ is a C^∞ function on $(0, \infty)$;

(ii) For any $\delta > 0$, there is a $t_0 > 0$ such that $t^\delta\,|\,b(t)|$ is increasing for $t > t_0$.

(iii) For any $\delta > 0$, there is a $t_1 > 0$ such that $t^{-\delta}\,|\,b(t)|$ is decreasing for $t > t_1$.

llowing a more restricted class of functions is needed. We shall use
ion:

$$h_0(t) = b(t),$$

$$h_n(t) = t \frac{d}{dt} h_{n-1}(t), \qquad n = 1, 2,\dots .$$

sed in the next definition.

TION 3.2. A slowly varying function $b(t)$ is said to belong to the f all its associated $h_n(t)$ are slowly varying.
on examples of functions of the class S are

$$10), \qquad \log \log^{a}(t + 100) \qquad \text{and} \qquad \log^{a}(t + 10) \log \log^{c}(t + 100)$$

are arbitrary numbers). A large class of slowly varying functions rdy's L-functions is given in the appendix to Wainger [15].
all need to use the following properties of slowly varying functions:

3.3. *Let $b(t)$ be slowly varying. Then*

(t) is either nonpositive or nonnegative for sufficiently large t.
$b'(t)| = o(t^{-1} \mid b(t)|)$ as $t \to \infty$.

See Ref. [15, Lemma 1].

3.4. *Let $b(t)$ be a slowly varying function. Let ξ_1 and ξ_2 be positive with $\xi_1 < \xi_2$. Then*

$$\max_{\xi_1/R \leqslant t \leqslant \xi_2/R} \left| b(t) - b\left(\frac{1}{R}\right) \right| = o\left(\left| b\left(\frac{1}{R}\right) \right|\right)$$

)+.

See Ref. [15, Lemma 2].

3.5. *Let $b(t)$ be in the class S. Then for $n \geqslant 1$,*

$$\frac{d^n b(t)}{dt^n} = t^{-n} \sum_j \beta_j h_j(t).$$

re numbers. The $h_j(t)$ are the slowly varying functions associated with sum is extended over a finite range of summation, and the value $j = 0$ occur.

See Ref. [5, Lemma 9].

LEMMA 3.6. *Let $b(t)$ be in the class S and let $b(t) \to 0$ as $t \to \infty$. Then*

$$\sum_{n=1}^{\infty} | b'(n)| < \infty.$$

Proof. See Ref. [5, p. 214].

LEMMA 3.7. *Let $b(t)$ be slowly varying. Then*

(i) $| b(kt)| \simeq | b(t)|$ *for every fixed $k > 0$ and uniformly in every interval*

$$\gamma \leqslant k \leqslant \frac{1}{\gamma}, \qquad 0 < \gamma < 1.$$

(ii) *If we write*

$$B(t) = \int_1^t \tau^{-1} | b(\tau)| \, d\tau, \qquad B^*(t) = \sum_{n=1}^{[t]} n^{-1} | b(n)| ,$$

and $B(t) \neq O(1)$, $t \to \infty$, then

$$| b(t)| = o(B(t)) \qquad \text{and} \qquad B(t) \simeq B^*(t).$$

Proof. See Ref. [17, p. 188].

4. BEHAVIOR OF A SPECIAL CLASS OF JACOBI SERIES

The main goal in this section is to study the behavior near $\theta = 0$ of a Jacobi series of the form

$$\sum_{n=1}^{\infty} b(n) \, n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos \theta),$$

with $\gamma \geqslant 0$. $b(n)$ is a slowly varying function of the class S (see Definition 3.2). Theorem 4.5 treats the case $0 < \gamma < 2\alpha + 2$. In Theorem 4.6 we shall investigate the case $\gamma = 0$. Theorem 4.7 will deal with the case $\gamma \geqslant 2\alpha + 2$. Finally, a remark will be made concerning the case $\gamma < 0$.

We need the following lemmas:

LEMMA 4.1. *Let $\gamma > \alpha + \frac{1}{2}$. For $0 < \theta < \pi$,*

$$F(\cos \theta) = \lim_{N \to \infty} \sum_{n=1}^{N} n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos \theta)$$

exists. Also, $F(\cos\theta)$ *is continuous for* $0 < \theta < \pi$. *If* $\gamma > 2a + 2$, $F(\cos\theta)$ *is continuous for* $0 \leqslant \theta \leqslant \pi$. *As* $\theta \to 0^+$,

$$F(\cos\theta) = \frac{\Gamma\left(\alpha + 1 - \frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma(\alpha+1)}\left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-2} + E(\cos\theta),$$

$$E(\cos\theta) = O\left\{\left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-1}\right\}, \qquad \alpha + \frac{1}{2} < \gamma \leqslant 2\alpha + 1.$$

If $\gamma > 2\alpha + 1$, *then* $E(\cos\theta)$ *is continuous for* $0 \leqslant \theta \leqslant \pi$ *and can be written in the form*

$$E(\cos\theta) = C + O\left\{\left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-1}\right\},$$

where C *is a constant.*

Proof. As is easily derived from the formula [7, (10.20.3)]

$$\begin{aligned}&\frac{\Gamma\left(\alpha + 1 - \frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma(\alpha+1)}\left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-2}\\ &\quad = \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)\,\Gamma\left(n+\alpha+1-\frac{\gamma}{2}\right)}{\Gamma(n+\alpha+1)\,\Gamma\left(n+\beta+\frac{\gamma}{2}+1\right)}\,\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma > \alpha + \tfrac{1}{2}.\end{aligned} \tag{4.1}$$

For any positive j there exists a set of numbers λ_k, $k = 1, 2, \ldots, j$, such that

$$\begin{aligned}&\frac{\Gamma(n+\beta+1)\,\Gamma\left(n+\alpha+1-\frac{\gamma}{2}\right)}{\Gamma(n+\alpha+1)\,\Gamma\left(n+\beta+\frac{\gamma}{2}+1\right)}\\ &\quad = \frac{1}{n^{\gamma}} + \sum_{k=1}^{j} \lambda_k \frac{\Gamma(n+\beta+1)\,\Gamma\left(n+\alpha-\frac{\gamma+k}{2}+1\right)}{\Gamma(n+\alpha+1)\,\Gamma\left(n+\beta+\frac{\gamma+k}{2}+1\right)} + O(n^{-\gamma-j-1}).\end{aligned}$$

If we choose j sufficiently large, Lemma 4.1 follows immediately from (4.1) and the estimate (1.2).

LEMMA 4.2. *Let ω and Ω be fixed numbers. Then*

(i) *If $\gamma < 2\alpha + 2$ and θ tends to 0^+,*

$$\sum_{n=1}^{[\omega\theta^{-1}]} n^{-\gamma}\omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)| = O\{(\theta^{-1}\omega)^{2\alpha+2-\gamma}\}. \tag{4.2}$$

(ii) *If $\gamma > \alpha + \frac{3}{2}$ and θ tends to 0^+,*

$$\sum_{n=[\Omega\theta^{-1}]}^{\infty} n^{-\gamma}\omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)| = O\{\theta^{-(2\alpha+2-\gamma)}\Omega^{-\gamma+\alpha+\frac{3}{2}}\}. \tag{4.3}$$

The O's do not depend on ω or Ω.

Proof. (4.2) follows from (1.2) by application of

$$\sum_{n=1}^{N} n^p = O(N^{p+1}), \qquad p > -1.$$

(4.3) follows from (2.7), using

$$\sum_{n=N}^{\infty} n^p = O(N^{p+1}), \qquad p < -1.$$

LEMMA 4.3. *Let ω and Ω be fixed numbers such that $\omega < 1 < \Omega$. Assume $b(t)$ is slowly varying:*

(i) *If $\gamma < 2\alpha + 2$, choose $\delta < 2\alpha + 2 - \gamma$. Then, as $\theta \to 0^+$,*

$$\sum_{n=1}^{[\omega\theta^{-1}]} n^{-\gamma} \mid b(n)| \, \omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)| = O\{| b(\theta^{-1})| \, \theta^{\gamma-2\alpha-2}\omega^{2\alpha+2-\gamma-\delta}\}. \tag{4.4}$$

(ii) *If $\gamma > \alpha + \frac{3}{2}$, choose $\delta < \gamma - \alpha - \frac{3}{2}$. Then, as $\theta \to 0^+$,*

$$\sum_{n=[\Omega\theta^{-1}]}^{\infty} n^{-\gamma} \mid b(n)| \, \omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)| = O\{| b(\theta^{-1})| \, \theta^{\gamma-2\alpha-2}\Omega^{-\gamma+\alpha+\frac{3}{2}+\delta}\}. \tag{4.5}$$

Proof. (i) Choose $\delta < 2\alpha + 2 - \gamma$ and let m be an integer so large, that $t^\delta \mid b(t)|$ is increasing for $t \geqslant m$. (Such m exists, see Section 3.) Let θ be so close to 0 that $[\omega\theta^{-1}] \geqslant m$. Then, using (4.2),

$$\begin{aligned}
&\sum_{n=1}^{[\omega\theta^{-1}]} n^{-\gamma} \mid b(n)| \, \omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)| \\
&\quad = O(1) + \sum_{n=1}^{[\omega\theta^{-1}]} n^{-\gamma-\delta} n^{\delta} \mid b(n)| \, \omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)| \\
&\quad = O(1) + \theta^{-\delta} \mid b(\theta^{-1})| \sum_{n=m}^{[\omega\theta^{-1}]} n^{-\gamma-\delta}\omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)| \\
&\quad = O\{| b(\theta^{-1})| \, \theta^{\gamma-2\alpha-2}\omega^{2\alpha+2-\gamma-\delta}\}.
\end{aligned}$$

(ii) Choose $\delta < \gamma - \alpha - \frac{3}{2}$ and let m be an integer so large, that $t^{-\delta} \mid b(t)\mid$ is decreasing for $t \geqslant m$. (Such m exists, see Section 3.) Let θ be so close to 0 that $[\Omega\theta^{-1}] \geqslant m$. Then application of (4.3) leads to

$$\sum_{n=[\Omega\theta^{-1}]}^{\infty} n^{-\gamma} \mid b(n)\mid \omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)\mid$$

$$= \sum_{n=[\Omega\theta^{-1}]}^{\infty} n^{-\gamma+\delta} n^{-\delta} \mid b(n)\mid \omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)\mid$$

$$= O\{\mid b(\theta^{-1})\mid \theta^{\gamma-2\alpha-2}\Omega^{-\gamma+\alpha+\frac{3}{2}+\delta}\}.$$

LEMMA 4.4. *Assume $b(t)$ is slowly varying. Let*

$$F(\cos\theta) = \sum_{n=1}^{\infty} n^{-\gamma} b(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta), \tag{4.6}$$

with

$$\alpha + \tfrac{3}{2} < \gamma < 2\alpha + 2.$$

The sum (4.6) *converges absolutely and uniformly in any compact subinterval of* $(0, \pi)$.

As $\theta \to 0^+$,

$$F(\cos\theta) = \frac{\Gamma\left(\alpha + 1 - \frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma(\alpha+1)} \left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-2} b(\theta^{-1}) + E(\theta), \tag{4.7}$$

where

$$E(\theta) = o\{\mid b(\theta^{-1})\mid \theta^{\gamma-2\alpha-2}\} + O(1). \tag{4.8}$$

Proof. The fact that the series (4.6) converges uniformly and absolutely in any compact subinterval of $(0, \pi)$ follows from the estimate (2.7). Let ω and Ω be fixed (but arbitrary) numbers $\omega < 1 < \Omega$. Then

$$F(\cos\theta) = b(\theta^{-1}) \sum_{n=1}^{\infty} n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) + E(\theta),$$

$$E(\theta) = E_1(\theta) + E_2(\theta) + E_3(\theta) + E_4(\theta) + E_5(\theta),$$

$$E_1(\theta) = \sum_{n=[\omega\theta^{-1}]}^{[\Omega\theta^{-1}]} \{b(n) - b(\theta^{-1})\}\, n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

$$E_2(\theta) = -\, b(\theta^{-1}) \sum_{n=[\Omega\theta^{-1}]}^{\infty} n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

$$E_3(\theta) = - b(\theta^{-1}) \sum_{n=1}^{[\omega\theta^{-1}]} n^{-\gamma}\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

$$E_4(\theta) = \sum_{n=[\Omega\theta^{-1}]}^{\infty} b(n)\, n^{-\gamma}\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

$$E_5(\theta) = \sum_{n=1}^{[\omega\theta^{-1}]} b(n)\, n^{-\gamma}\omega_n^{(\alpha\ \beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Now, by Lemma 4.1, it suffices to show that the terms E_1 to E_5 satisfy (4.8).

Consider first E_1, if we choose $\epsilon > 0$, then by Lemma 3.4 and Lemma 4.2 it follows that

$$\begin{aligned} | E_1(\theta)| &\leqslant \max_{\omega\theta^{-1}\leqslant n \leqslant \Omega\theta^{-1}} | b(n) - b(\theta^{-1})| \sum_{n=1}^{\Omega\theta^{-1}} n^{-\gamma}\omega_n^{(\alpha,\beta)}\ | R_n^{(\alpha,\beta)}(\cos\theta)| \\ &\leqslant \epsilon\ | b(\theta^{-1})|\ O(\theta^{\gamma-2\alpha-2}\Omega^{2\alpha+2-\gamma}) \\ &= o\{| b(\theta^{-1})|\ \theta^{\gamma-2\alpha-2}\}. \end{aligned}$$

E_2 and E_3 may be estimated by Lemma 4.2, E_4 and E_5 by Lemma 4.3. We observe that each of these terms is

$$O\{| b(\theta^{-1})|\ \theta^{\gamma-2\alpha-2}(\Omega^{-\gamma+\alpha+\frac{3}{2}+\delta} + \omega^{2\alpha+2-\gamma-\delta}) + 1\},$$

where O is independent of ω, Ω, and θ and $\delta < \min(\gamma - \alpha - \frac{3}{2}, 2\alpha + 2 - \gamma)$. The desired conclusion now follows by taking ω sufficiently small and Ω sufficiently large.

THEOREM 4.5. *Let $b(t)$ be in S and let $0 < \gamma < 2\alpha + 2$. For $\epsilon > 0$ define*

$$F_\epsilon(\cos\theta) = \sum_{n=1}^{\infty} b(n)\, n^{-\gamma} e^{-\epsilon n} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta). \tag{4.9}$$

Then, for $\theta \neq 0$,

$$F(\cos\theta) = \lim_{\epsilon\to 0^+} F_\epsilon(\cos\theta)$$

exists in the pointwise sense. Also, $F(\cos\theta)$ is continuous on $0 < \theta \leqslant \pi$. As $\theta \to 0^+$,

$$F(\cos\theta) = \frac{\Gamma\left(\alpha + 1 - \frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma(\alpha+1)}\, b(\theta^{-1}) \left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-2} + E(\theta), \tag{4.10}$$

$$E(\theta) = o\{b(\theta^{-1})\,\theta^{\gamma-2\alpha-2}\} + O(1).$$

Finally, $F(\cos\theta) \in L^1$ and

$$F(\cos\theta) \sim \sum_{n=1}^{\infty} b(n)\, n^{-\gamma}\omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta). \tag{4.11}$$

Proof. We are going to apply Lemma 2.2 with $a(t) = b(t)\, t^{-\gamma} e^{-\epsilon t}$. We take the integer ν so large that $\gamma + \nu > \alpha + \frac{5}{2}$.

We obtain

$$F_\epsilon(\cos\theta) = M_\epsilon(\theta) + E_{\epsilon,1}(\theta) + E_{\epsilon,2}(\theta) + E_{\epsilon,3}(\theta).$$

The terms $M_\epsilon(\theta)$, $E_{\epsilon,1}(\theta)$, and $E_{\epsilon,2}(\theta)$ all come from the main term of Eq. (2.3), which in the present context is

$$\frac{1}{t}\frac{d}{dt}\left\{\frac{1}{t}\frac{d}{dt}\left\{\frac{1}{t}\frac{d}{dt}\left\{\cdots\frac{1}{t}\frac{d}{dt}\{b(t)\, t^{-\gamma}e^{-\epsilon t}\}\cdots\right\}\right\}\right\}.$$

The main term M_ϵ arises from taking derivatives only on powers of t. $E_{\epsilon,1}$ consists of the remaining contribution of terms not involving derivatives of $e^{-\epsilon t}$. $E_{\epsilon,2}$ is made up by terms in which at least one derivative is taken on $e^{-\epsilon t}$. $E_{\epsilon,3}$ corresponds to the remainder term $E_1(\cos\theta)$ in formula (2.3):

$$\begin{aligned} M_\epsilon(\theta) &= \frac{(-1)^\nu}{2^\nu}\frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty} b(n)\, e^{-\epsilon n}\left[\frac{1}{t}\frac{d}{dt}\left\{\cdots\frac{1}{t}\frac{d}{dt}(t^{-\gamma})\right\}\right]_{t=n} \\ &\quad\cdot \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta) \\ &= \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)}\frac{\Gamma\left(\frac{\gamma}{2}+\nu\right)}{\Gamma\left(\frac{\gamma}{2}\right)}\sum_{n=1}^{\infty} b(n)\, e^{-\epsilon n} n^{-\gamma-2\nu}\omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta). \end{aligned}$$

As $\gamma + \nu > \alpha + \frac{5}{2}$, it follows immediately from (2.7) that the series $M_\epsilon(\theta)$ converges uniformly in ϵ in any closed subinterval of $(0, \pi)$. $M_\epsilon(\theta)$ with $\epsilon = 0$ is a series of the type treated in Lemma 4.4. Hence, applying Lemma 4.4 and using the regularity of Abel summability, we find that

$$M(\theta) = \lim_{\epsilon\to 0} M_\epsilon(\theta)$$

exists and is continuous for $0 < \theta < \pi$. Moreover, as $\theta \to 0^+$,

$$\begin{aligned} M(\theta) &= \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)}\frac{\Gamma\left(\frac{\gamma}{2}+\nu\right)}{\Gamma\left(\frac{\gamma}{2}\right)}\frac{\Gamma\left(\alpha+1-\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\nu\right)\Gamma(\alpha+\nu+1)} \\ &\quad\times\left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-2} b(\theta^{-1}) + E_4 \\ &= \frac{\Gamma\left(\alpha+1-\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma(\alpha+1)}\, b(\theta^{-1})\left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-2} + E_4 \end{aligned}$$

and

$$E_4 = o\{b(\theta^{-1})\,\theta^{\gamma-2\alpha-2}\} + O(1).$$

We now investigate $E_{\epsilon,1}(\theta)$,

$$E_{\epsilon,1}(\theta) = \sum_{n=1}^{\infty} e^{-\epsilon n} \left(\sum_{j=1}^{\nu} c_j n^{-\gamma-2\nu+j} \frac{d^j b(t)}{dt^j} \bigg|_{t=n} \right) \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta),$$

where c_j are numbers, independent of n.

As $b(t)$ is assumed to belong to S, we may apply Lemma 3.5 and write

$$\frac{d^j b(t)}{dt^j} \bigg|_{t=n} = n^{-j} \sum_{k=1}^{j} \beta_k h_k(n),$$

where β_k are numbers and $h_k(t)$ are slowly varying. Thus, we obtain

$$E_{\epsilon,1}(\theta) = \sum_{j=1}^{\nu} \sum_{k=1}^{j} d(j,k) \sum_{n=1}^{\infty} e^{-\epsilon n} n^{-\gamma-2\nu} h_k(n)\, \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta),$$

$d_{j,k}$ are numbers.

It follows that $E_{\epsilon,1}(\theta)$ consists of a finite linear combination of series, which converge uniformly in ϵ in any closed subinterval of $(0, \pi)$. Moreover, from Lemma 3.3 and Lemma 4.4 it is easily seen that

$$E_1(\theta) = \lim_{\epsilon \to 0^+} E_{\epsilon,1}(\theta)$$

exists and is continuous for $0 < \theta < \pi$.

Also, as $\theta \to 0^+$,

$$E_1(\theta) = o\{b(\theta^{-1})\, \theta^{\gamma-2\alpha-2}\} + O(1).$$

Next we consider $E_{2,\epsilon}(\theta)$,

$$E_{\epsilon,2}(\theta) = \sum_{j,k,m} g_{j,k,m} \sum_{n=1}^{\infty} e^{-\epsilon n} \epsilon^m n^{-\gamma-\nu-j} b^{(k)}(n)\, \omega_n^{(\alpha+\nu,\beta)} R_n^{(\alpha+\nu,\beta)}(\cos\theta).$$

Here $g_{j,k,m}$ are numbers. The first summation is over nonnegative values of j, k, m with $m \geqslant 1$ and $j + k + m = \nu$.

Application of (2.6) and (2.7) yields

$$E_{\epsilon,2}(\theta) = \epsilon\theta^{-\alpha-\nu-\frac{1}{2}}(\pi - \theta)^{-\beta-\frac{1}{2}}\, O\left(\sum_{j,k,m} \sum_{n=1}^{\infty} n^{-\gamma-\nu-j-m+1+\alpha+\nu+\frac{1}{2}} \mid b^{(k)}(n) \mid \right)$$

$$= \epsilon\theta^{-\alpha-\nu-\frac{1}{2}}(\pi - \theta)^{-\beta-\frac{1}{2}}\, O\left(\sum_{n=1}^{\infty} n^{-\gamma-\nu+\frac{3}{2}+\alpha} \mid b(n) \mid \right).$$

Thus $E_{\epsilon,2}$ converges uniformly in ϵ in any closed subinterval of $(0, \pi)$. Moreover, we see that for $0 < \theta < \pi$, $E_{\epsilon,2} \to 0$ as $\epsilon \to 0^+$, since $\gamma + \nu > \alpha + \frac{5}{2}$ and $|b(n)| = O(n^\delta)$ for any $\delta > 0$.

Finally, we consider $E_{\epsilon,3}(\theta)$ which contains terms similar to those of M_ϵ, $E_{\epsilon,1}$, and $E_{\epsilon,2}$, except that here $m + j + k = \nu + 1$ instead of ν. Hence, if we apply to $E_{\epsilon,3}$ reasoning similar to that of the previous terms, we find that $E_{\epsilon,3}$ is a series, which converges uniformly in ϵ in any closed subinterval of $(0, \pi)$. Also, we find

$$E_3(\theta) = \lim_{\epsilon \to 0^+} E_{\epsilon,3}(\theta)$$

exists and is continuous for $0 < \theta < \pi$.

Furthermore, as $\theta \to 0^+$,

$$E_3(\theta) = o\{b(\theta^{-1})\, \theta^{\nu-2\alpha-2}\} + O(1).$$

We examine the behavior of $F(\cos\theta)$ near $\theta = \pi$.

It suffices to show that $F_\epsilon(\cos\theta)$ converges uniformly to $F(\cos\theta)$ as $\epsilon \to 0^+$ for θ sufficiently close to π. For $\theta = \pi$ the convergence follows from the well-known relation for Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

and Theorem 7 of Wainger [15], with $x = \pi/2$.

We use the Bateman integral [3, formula 3.4]

$$(1+x)^\beta \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)} = \frac{\Gamma(\beta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})} \int_{-1}^{x} (1+y)^{-\frac{1}{2}} \frac{P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(y)}{P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(-1)} (x-y)^{\beta-\frac{1}{2}}\, dy,$$

or, writing

$$x = 2u^2 - 1, \qquad y = 2z^2 - 1,$$

$$\frac{u^{2\beta}}{\Gamma(n+\beta+1)} P_n^{(\alpha,\beta)}(2u^2-1) = \frac{2}{\Gamma(\beta+\frac{1}{2})\,\Gamma(n+\frac{1}{2})} \int_0^u P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(2z^2-1)\,(u^2-z^2)^{\beta-\frac{1}{2}}\, dz.$$

Thus, applying Szegö [14, (4.1.5)],

$$P_n^{(\alpha,\beta)}(2u^2-1) = \frac{2u^{-2\beta}\Gamma(n+\beta+1)\,\Gamma(n+\alpha+\beta+\frac{3}{2})\,\Gamma(2n+1)}{\Gamma(\beta+\frac{1}{2})\,\Gamma(n+\frac{1}{2})\,\Gamma(2n+\alpha+\beta+\frac{3}{2})\,\Gamma(n+1)} \cdot \int_0^u P_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})}(z)\,(u^2-z^2)^{\beta-\frac{1}{2}}\, dz.$$

We investigate $F_\epsilon(\cos\theta)$ near π. If we put $\cos\theta = 2u^2 - 1$, we have to study $F_\epsilon(2u^2 - 1)$ with u in the neighborhood of 0. After some calculations we obtain

$$F_\epsilon(2u^2 - 1) = \frac{\Gamma(\alpha+\beta+\frac{3}{2})}{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)\,\Gamma(\beta+\frac{1}{2})}\, u^{-2\beta} \int_0^u (u^2 - z^2)^{\beta-\frac{1}{2}}$$
$$\times \left(\sum_{n=1}^{\infty} b(n)\, n^{-\gamma} e^{-\epsilon n} \cdot \omega_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})} R_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})}(z) \right) dz.$$

In the first part of this theorem we have shown, that the series in the integrand converges uniformly in ϵ in any closed subinterval of $(-1, 1)$ and that its limit as $\epsilon \to 0^+$ exists and is continuous. Indeed, if $\sum a_n R_n^{(\alpha,\alpha)}(x)$ and $\sum a_n R_n^{(\alpha,\alpha)}(-x)$ are continuous functions of x near $x = 0$, then so is their sum $\sum a_{2n} R_{2n}^{(\alpha,\alpha)}(x)$, which is a series of the kind used in the integrand. By the dominated convergence theorem, $F_\epsilon(2u^2 - 1)$ converges pointwise to a limit as $\epsilon \to 0^+$, at least if u is sufficiently small.

Moreover,

$$F(2u^2 - 1) = O\left(u^{-2\beta} \int_0^u c(z)\,(u^2 - z^2)^{\beta-\frac{1}{2}}\, dz\right),$$

where $c(z)$ is continuous near $z = 0$. And the convergence is uniform, since

$$\left| u^{-2\beta} \int_0^u (u^2 - z^2)^{\beta-\frac{1}{2}}\, dz \right| = O(1)$$

near $u = 0$.

To finish the proof we need to show that

$$F(\cos\theta) \sim \sum_{n=1}^{\infty} b(n)\, n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

By Theorem 2.3, the sum at the right side is a Fourier–Jacobi series of a function $G(\cos\theta) \in L^1$ and from Theorem 2.4 it follows that

$$G(\cos\theta) = \lim_{\epsilon\to 0^+} F_\epsilon(\cos\theta) = F(\cos\theta) \qquad \text{a.e.}$$

THEOREM 4.6. *Let*

$$F_\epsilon(\cos\theta) = \sum_{n=1}^{\infty} b(n)\, e^{-\epsilon n} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

with $b(t)$ *in* S *and* $\epsilon > 0$.

Then, for $\theta \neq 0$,

$$F(\cos\theta) = \lim_{\epsilon\to 0^+} F_\epsilon(\cos\theta)$$

exists in the pointwise sense. Moreover, $F(\cos\theta)$ *is continuous for* $0 < \theta \leqslant \pi$. *As* $\theta \to 0^+$,

$$F(\cos\theta) \simeq k\left(\sin\frac{\theta}{2}\right)^{-2\alpha-3} b'(\theta^{-1}),$$

provided $b'(t)$ *is not zero for all large* t. k *is a number* $\neq 0$. *Finally,* $F \in L^1$ *if and only if* $b(t) \to 0$ *as* $t \to \infty$. *If* $b(t) \to 0$ *as* $t \to \infty$, *then*

$$F(\cos\theta) \sim \sum_{n=1}^{\infty} b(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Proof. The proof of Theorem 4.6 is essentially the same as the proof of Theorem 4.5. As in Theorem 4.5, the proof of the first part is reduced to Lemma 4.4 by Lemma 2.2, where we take $a(t) = e^{-\epsilon t} b(t)$. The fact that $a(t)$ contains no power of t accounts for the different conclusion of Theorems 4.5 and 4.6. For the second part of the theorem we apply Theorem 2.3, which is possible in view of the Lemmas 3.5 and 3.6.

THEOREM 4.7. *Let* $b(t)$ *be in* S *and let* $\gamma \geqslant 2\alpha + 2$. *For* $\epsilon > 0$, *define*

$$F_\epsilon(\cos\theta) = \sum_{n=1}^{\infty} b(n)\, n^{-\gamma} e^{-\epsilon n} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Then,

$$F(\cos\theta) = \lim_{\epsilon\to 0^+} F_\epsilon(\cos\theta)$$

exists in the pointwise sense and $F(\cos\theta)$ *is continuous for* $0 < \theta \leqslant \pi$. *Furthermore,* $F \in L^1$ *and*

$$F(\cos\theta) \sim \sum_{n=1}^{\infty} b(n)\, n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

Let

$$B(x) = \int_1^x b(t)\, t^{-1}\, dt.$$

As $\theta \to 0^+$,

(i) *if* $\gamma = 2\alpha + 2$ *and* $\int_1^\infty |\, b(t)|\, t^{-1}\, dt = \infty$, *then*

$$F(\cos\theta) = \frac{2}{\{\Gamma(\alpha+1)\}^2} B(\theta^{-1}) + O(|\, b(\theta^{-1})|);$$

(ii) *if* $\gamma > 2\alpha + 2$ *or if* $\gamma = 2\alpha + 2$ *and* $\int_1^\infty |\, b(t)|\, t^{-1}\, dt < \infty$, *then* $\lim_{\theta\to 0^+} F(\cos\theta)$ *exists and thus* $F(\cos\theta)$ *is continuous on* $0 \leqslant \theta \leqslant \pi$.

Proof. Everything except (i) and (ii) follows as in the proof of Theorem 4.5. The proof of (ii) is trivial, since the hypothesis implies that

$$\sum_{n=1}^{\infty} b(n)\, n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

converges uniformly in view of (1.2) and (1.6).

So we only need to prove (i). By Lemma 4.3, Eq. (4.5),

$$\sum_{n=[\theta^{-1}]}^{\infty} n^{-2\alpha-2} \mid b(n)\mid \omega_n^{(\alpha,\beta)} \mid R_n^{(\alpha,\beta)}(\cos\theta)\mid = O\{\mid b(\theta^{-1})\mid\}.$$

Now we put

$$\sum_{n=1}^{[\theta^{-1}]} n^{-2\alpha-2} b(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) = A_1(\theta) + A_2(\theta),$$

where

$$A_1(\theta) = \sum_{n=1}^{[\theta^{-1}]} n^{-2\alpha-2} b(n)\, \omega_n^{(\alpha,\beta)} = \frac{2}{\{\Gamma(\alpha+1)\}^2} \sum_{n=1}^{[\theta^{-1}]} n^{-1} b(n) + O(1)$$

and

$$\mid A_2(\theta)\mid \leqslant \sum_{n=1}^{[\theta^{-1}]} n^{-2\alpha-2} \mid b(n)\mid \omega_n^{(\alpha,\beta)} \mid 1 - R_n^{(\alpha,\beta)}(\cos\theta)\mid .$$

Since

$$\mid R_n^{(\alpha,\beta)}(1) - R_n^{(\alpha,\beta)}(\cos\theta)\mid = O\left\{(1-\cos\theta) \max_{-1\leqslant x\leqslant 1} \left| \frac{d}{dx} R_n^{(\alpha,\beta)}(x) \right|\right\}$$

and by Szegö [14, (7.32.10)], we have

$$\left| \frac{d}{dx} R_n^{(\alpha,\beta)}(x) \right| = O(n^2),$$

it follows by the analysis of Lemma 4.3 that

$$\mid A_2(\theta)\mid = O\left\{\theta^2 \sum_{n=1}^{[\theta^{-1}]} n \mid b(n)\mid\right\} = O\{\mid b(\theta^{-1})\mid\}.$$

Hence we have

$$F(\cos\theta) = \frac{2}{\{\Gamma(\alpha+1)\}^2} \sum_{n=1}^{[\theta^{-1}]} n^{-1} b(n) + O\{\mid b(\theta^{-1})\mid\}.$$

Now, according to Lemma 3.7, $| b(t)| = o(B(t))$ as $t \to \infty$ and

$$B(t) \simeq \sum_{n=1}^{[t]} b(n)\, n^{-1},$$

which gives us the proof of (i).

Remarks. (a) Theorems 4.5 and 4.6 yield more information in the special case $b(t) = 1$. Let

$$F(\cos\theta) \sim \sum_{n=1}^{\infty} n^{-\gamma} \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

with $\gamma > 0$.

Then

$$F(\cos\theta) = \sum_{j=0}^{k} \beta_j \left(\sin\frac{\theta}{2}\right)^{\gamma-2\alpha-2+j} + \mu \log|\theta^{-1}| + E(\theta). \qquad (4.12)$$

$E(\theta)$ is at least $O(1)$ and has at least $\gamma - 2\alpha - 2 + k$ continuous derivatives. The β_j and μ are numbers. μ is zero unless $\gamma + j = 2\alpha + 2$ for some integer j, $0 \leqslant j \leqslant k$.

(b) An important part of Theorem 4.5 remains valid, when $F_\epsilon(\cos\theta)$ is defined by (4.9) with $\gamma < 0$, $\epsilon > 0$ and $b(t)$ in S. In fact, for $\theta \neq 0$,

$$\lim_{\epsilon\to 0^+} F_\epsilon(\cos\theta) = F(\cos\theta)$$

exists in the pointwise sense and $F(\cos\theta)$ is continuous for $0 < \theta \leqslant \pi$. As $\theta \to 0^+$, the asymptotic formula (4.10) still holds in the case $\gamma \neq -2k$, $k = 0, 1, 2,\ldots$. For $\gamma = -2k$, a slightly different formula can be obtained, analogous to Theorem 4.6. In the case $\gamma < 0$, the series (4.11) does not satisfy the conditions of Theorem 2.3 and therefore we can no longer conclude, that it is the Fourier–Jacobi series associated with $F(\cos\theta)$. $F(\cos\theta)$ is not an L^1 function but a distribution.

As application of the differential operator $P(d/d\theta)$, defined by (1.3), on $F(\cos\theta)$ results in multiplying the Fourier–Jacobi coefficients by $n(n + \alpha + \beta + 1)$, it becomes clear, that for $\theta \neq 0$, $F(\cos\theta)$ is infinitely differentiable by means of the operator P.

5. Fractional Integration and Differentiation

Let f be an L^1 function with Fourier–Jacobi expansion

$$f(\cos\theta) \sim \sum_{n=0}^{\infty} f^\wedge(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

We consider the Jacobi series

$$\omega_0^{(\alpha,\beta)} + \sum_{n=1}^{\infty} [n(n+\alpha+\beta+1)]^{-\sigma/2}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

where $\sigma > 0$.

Noticing that

$$[n(n+\alpha+\beta+1)]^{-\sigma/2} = n^{-\sigma} + \sum_{j=1}^{[2\alpha+2-\sigma]} c_j n^{-\sigma-j} + o(n^{-(2\alpha+2)})$$

for certain numbers c_j, Theorem 4.5 shows, that there exists an L^1 function $g_\sigma(\cos\theta)$, such that

$$g_\sigma(\cos\theta) \sim \omega_0^{(\alpha,\beta)} + \sum_{n=1}^{\infty} [n(n+\alpha+\beta+1)]^{-\sigma/2}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) \tag{5.1}$$

and

$$| g_\sigma(\cos\theta)| = O(\theta^{\sigma-2\alpha-2}), \qquad 0 < \sigma < 2\alpha + 2. \tag{5.2}$$

We now define the fractional integral of order θ of the function f by the convolution of $f(\cos\theta)$ with $g_\sigma(\cos\theta)$, which is (see Section 1)

$$I_\sigma f(\cos\theta) = \int_0^\pi f(\cos\theta, \cos\phi)\, g_\sigma(\cos\phi)\, \rho^{(\alpha,\beta)}(\phi)\, d\phi. \tag{5.3}$$

It follows that $I_\sigma f(\cos\theta)$ is in L^1 and

$$\begin{aligned} I_\sigma f(\cos\theta) &\sim f\hat{}(0)\, \omega_0^{(\alpha,\beta)} \\ &\quad + \sum_{n=1}^{\infty} f\hat{}(n)\, [n(n+\alpha+\beta+1)]^{-\sigma/2}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta). \end{aligned} \tag{5.4}$$

It is clear that this fractional integration satisfies the semigroup property

$$I_{\sigma_2}(I_{\sigma_1} f(\cos\theta)) = I_{\sigma_1+\sigma_2} f(\cos\theta). \tag{5.5}$$

Let L_*^p be the class of functions $h(\cos\theta)$, which belong to L^p and have mean value zero. This is equivalent to the vanishing of the first Fourier–Jacobi coefficient $h\hat{}(0) = 0$. For each $h \in L_*^p$ the operator I_2 is the inverse operator of the differential operator $P(d/d\theta)$, defined by (1.3) and with realization A_θ in L^p.

This means that

$$A_\theta I_2 h(\cos\theta) = h(\cos\theta).$$

Many of the classical theorems for fractional integration (see Ref. [17, Chap. XII] can be carried over. This will be done in this section.

We first introduce Lipschitz classes.

DEFINITION 5.1. Let f be a function in L^p, $1 \leqslant p \leqslant \infty$. If $0 \leqslant \tau \leqslant 2$, we say that f is in $\operatorname{Lip}(\tau, p)$ if

$$\sup_{0<\psi<\phi} \| \Delta_\psi f \|_p = \sup_{0<\psi<\varphi} \| T_\psi f - f \|_p = O(\phi^\tau).$$

Here, $T_\phi f$ denotes the generalized translate of f, defined in Section 1. If $\tau > 2$, we say that f is in $\operatorname{Lip}(\tau, p)$ if the following two conditions are satisfied:

(i) f is in the domain $D_p(A_\theta^m)$ for $0 \leqslant m \leqslant [\tau/2]$;
(ii) $A_\theta^k f$ is in $\operatorname{Lip}(\tau - 2[\tau/2], p)$ for $k = [\tau/2]$.

THEOREM 5.2. *Let $0 < \sigma < 2$ and $0 < \tau < 2$ and suppose $f \in \operatorname{Lip}(\tau, p)$, $1 \leqslant p \leqslant \infty$. Then, if $\sigma + \tau < 2$,*

$$I_\sigma f \in \operatorname{Lip}(\sigma + \tau, p).$$

Proof. We need the following estimates:

$$| g_\sigma(\cos \theta)| = O(\theta^{\sigma-2\alpha-2}), \qquad \sigma < 2\alpha + 2, \quad \sigma \neq -2k, \quad k = 0, 1, 2, \ldots . \tag{5.6}$$

For $\sigma > 0$, this is estimate (5.2). For $\sigma < 0$, it is a consequence of remark (b) following Theorem 4.7. If the fractional integration is of an order $\geqslant 2\alpha + 2$, σ must be broken up into k parts

$$\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_k , \qquad \sigma_j < 2\alpha + 2, \quad 1 \leqslant j \leqslant k$$

and the semigroup property (5.5) can be applied. $P(d/d\theta)\, g_\sigma(\cos \theta)$ exists for $\theta \neq 0$ and the remainder term of the Taylor series can be estimated [12] by

$$\begin{aligned} \sup_{0<\theta\leqslant\pi} | T_\phi g_\sigma(\cos \theta) - g_\sigma(\cos \theta)| &\leqslant C\phi^2 \sup_{0<\theta\leqslant\pi} \left| P\left(\frac{d}{d\theta}\right) g_\sigma(\cos \theta) \right| \\ &= O(\phi^2 \theta^{\sigma-2\alpha-4}). \end{aligned} \tag{5.7}$$

Suppose $f \in \operatorname{Lip}(\tau, p)$, then

$$\begin{aligned} I_\sigma f(\cos \theta) &= \int_0^\pi T_t f(\cos \theta)\, g_\sigma(\cos t)\, \rho^{(\alpha,\beta)}(t)\, dt \\ &= \int_0^\pi \{T_t f(\cos \theta) - f(\cos \theta)\}\, g_\sigma(\cos t)\, \rho^{(\alpha,\beta)}(t)\, dt + f(\cos \theta), \\ T_\phi I_\sigma f(\cos \theta) &= \int_0^\pi T_t f(\cos \theta)\; T_\phi g_\sigma(\cos t)\, \rho^{(\alpha,\beta)}(t)\, dt \\ &= \int_0^\pi \{T_t f(\cos \theta) - f(\cos \theta))\; T_\phi g_\sigma(\cos t)\, \rho^{(\alpha,\beta)}(t)\, dt + f(\cos \theta). \end{aligned}$$

Thus, we have

$$T_\phi I_\sigma f(\cos\theta) - I_\sigma f(\cos\theta) \\ = \int_0^\pi \{T_t f(\cos\theta) - f(\cos\theta)\}\{T_\phi g_\sigma(\cos t) - g_\sigma(\cos t)\}\,\rho^{(\alpha,\beta)}(t)\,dt. \tag{5.8}$$

Using the notation

$$\Delta_\phi f(\cos\theta) = T_\phi f(\cos\theta) - f(\cos\theta),$$

we have

$$\|\Delta_\phi I_\sigma f(\cos\theta)\|_p = \left\| \int_0^\pi \Delta_t f(\cos\theta)\,\Delta_\phi g_\sigma(\cos t)\,\rho^{(\alpha,\beta)}(t)\,dt \right\|_p \\ \leqslant \left\| \int_0^\phi \right\|_p + \left\| \int_\phi^\pi \right\|_p = A + B. \tag{5.9}$$

In the integral A we shall estimate $\Delta_\phi g_\sigma(\cos t)$ by means of (5.6). This can be done because of (1.10). In B, $\Delta_\phi g_\sigma(\cos t)$ is estimated by (5.7). We shall give the proof for $1 < p < \infty$. Here we use Hölder's inequality. The proof in the cases $p = 1$ and $p = \infty$ is easier. We write $p' = p/(p-1)$. For some $\lambda > 0$,

$$\begin{aligned} A^p &= \int_0^\pi \left| \int_0^\phi \Delta_t f(\cos\theta)\,\Delta_\phi g_\sigma(\cos t)\,\rho^{(\alpha,\beta)}(t)\,dt \right|^p \rho^{(\alpha,\beta)}(\theta)\,d\theta \\ &\leqslant \int_0^\pi \left[\int_0^\phi (t^{-1+(\lambda(p-1)+1)/p})^{p'}\,dt \right]^{p/p'} \\ &\quad \times \left[\int_0^\phi |\Delta_t f(\cos\theta)|^p\,|\Delta_\phi g_\sigma(\cos t)|^p\,t^{(2\alpha+2)p-\lambda(p-1)-1}\,dt \right] \rho^{(\alpha,\beta)}(\theta)\,d\theta \\ &= O\left(\phi^{\lambda(p-1)} \int_0^\phi |\Delta_\phi g_\sigma(\cos t)|^p\,t^{(2\alpha+2)p-\lambda(p-1)-1}\,\|\Delta_t f\|_p^p\,dt \right) \\ &= O\left(\phi^{\lambda(p-1)} \int_0^\phi t^{\sigma p-(2\alpha+2)p+(2\alpha+2)p-\lambda(p-1)-1+\tau p}\,dt \right) \\ &= O\left(\phi^{\lambda(p-1)} \int_0^\phi t^{\sigma p+\tau p-\lambda(p-1)-1}\,dt \right) \\ &= O(\phi^{\sigma p+\tau p}), \qquad \text{if} \qquad \lambda < (\sigma+\tau)\,p'. \end{aligned}$$

Also, for some $\mu > 0$,

$$\begin{aligned} B^p &= \int_0^\pi \left| \int_\phi^\pi \Delta_t f(\cos\theta)\,\Delta_\phi g_\sigma(\cos t)\,\rho^{(\alpha,\beta)}(t)\,dt \right|^p \rho^{(\alpha,\beta)}(\theta)\,d\theta \\ &\leqslant \int_0^\pi \left[\int_\phi^\pi (t^{-1-(\mu(p-1)-1)/p})^{p'}\,dt \right]^{p/p'} \\ &\quad \times \left[\int_\phi^\pi |\Delta_t f(\cos\theta)|^p\,|\Delta_\phi g_\sigma(\cos t)|^p\,t^{(2\alpha+2)p+\mu(p-1)-1}\,dt \right] \rho^{(\alpha,\beta)}(\theta)\,d\theta \end{aligned}$$

$$= O\left(\phi^{-\mu(p-1)} \int_\phi^\pi |\Delta_\phi g_\sigma(\cos t)|^p \, t^{(2\alpha+2)p+\mu(p-1)-1} \, \|\Delta_t f\|_p^p \, dt\right)$$

$$= O\left(\phi^{-\mu(p-1)+2p} \int_\phi^\infty t^{\sigma p-(2\alpha+4)p+(2\alpha+2)p+\mu(p-1)-1+\tau p} \, dt\right)$$

$$= O\left(\phi^{-\mu(p-1)+2p} \int_\phi^\infty t^{(\sigma+\tau-2)p+\mu(p-1)-1} \, dt\right)$$

$$= O(\phi^{\sigma p+\tau p}), \qquad \text{if} \qquad \mu < (2-\sigma-\tau)\, p'.$$

This proves Theorem 5.2.

Remark. Theorem 5.2 remains valid for all positive values of σ and τ, except in the cases where σ, τ or $\sigma + \tau$ are even integers. Let $\sigma = 2k + \sigma_1$ and $\tau = 2m + \tau_1$ (k, m integer $\geqslant 0$, $0 < \sigma_1 < 2$, $0 < \tau_1 < 2$). If $\sigma_1 + \tau_1 < 2$, we apply the differential operator Δ_θ, $k + m$ times in (5.8) and show that the result is $O(\phi^{\sigma_1+\tau_1})$. If $\sigma_1 + \tau_1 > 2$, we apply the differential operator Δ_θ, $k + m + 1$ times in (5.9) and show that the result is $O(\phi^{\sigma_1+\tau_1-2})$.

THEOREM 5.3. *Suppose* $f \in L^p$, $1 \leqslant p \leqslant \infty$. *Then*

(i) $I_\sigma f \in \mathrm{Lip}(\sigma, p)$, *if* $0 < \sigma < 2$;

(ii) $I_\sigma f \in \mathrm{Lip}\left(\sigma - \dfrac{2\alpha+2}{p}, \infty\right)$, *if* $\dfrac{2\alpha+2}{p} < \sigma < 2 + \dfrac{2\alpha+2}{p}$.

Proof. The proof (i) is a word for word copy of the proof of Theorem 5.2, where in (5.9) instead of $\Delta_t f(\cos\theta)$, we write $T_t f(\cos\theta)$, which by (1.10) is bounded in L^p.

In the proof of (ii) we again use Hölder's inequality:

$$\|\Delta_\phi I_\sigma f(\cos\theta)\|_\infty = \left\| \int_0^\pi T_\theta f(\cos t)\, \Delta_\phi g_\sigma(\cos t)\, \rho^{(\alpha,\beta)}(t)\, dt \right\|_\infty$$

$$\leqslant \|T_\theta f\|_p \, \|\Delta_\phi g_\sigma(\cos t)\|_{p'}, \tag{5.10}$$

where

$$p' = \frac{p}{p-1}.$$

By (1.10) it suffices to show that the last factor is $O(\phi^{\sigma-(2\alpha+2)/p})$,

$$\|\Delta_\phi g_\sigma(\cos t)\|_{p'} = \left(\int_0^\pi |\Delta_\phi g_\sigma(\cos t)|^{p'} \rho^{(\alpha,\beta)}(t)\, dt\right)^{1/p'}$$

$$\leqslant \left(\int_0^\phi\right)^{1/p'} + \left(\int_\phi^\pi\right)^{1/p'} = A + B;$$

$$A^{p'} \leqslant 2^{p'} \int_0^\phi |g_\sigma(\cos t)|^{p'} \rho^{(\alpha,\beta)}(t)\, dt$$

$$= O\left(\int_0^\phi t^{(\sigma-2\alpha-2)p'+2\alpha+1}\,dt\right)$$

$$= O(\phi^{(\sigma-2\alpha-2)p'+2\alpha+2}), \quad \text{if} \quad (\sigma - 2\alpha - 2)p' + 2\alpha + 1 > -1;$$

$$B^{p'} = O\left(\phi^{2p'}\int_\phi^\infty t^{(\sigma-2\alpha-4)p'+2\alpha+1}\,dt\right)$$

$$= O(\phi^{(\sigma-2\alpha-2)p'+2\alpha+2}), \quad \text{if} \quad (\sigma - 2\alpha - 4)p' + 2\alpha + 1 < -1.$$

So the last factor in (5.10) is $O(\phi^{\sigma-(2\alpha+2)/p})$. The inequalities, used in estimating A and B are equivalent to the hypothesis

$$\frac{2\alpha+2}{p} < \sigma < 2 + \frac{2\alpha+2}{p}.$$

THEOREM 5.4. *Let $f \in L^p$, $1 < p < \infty$, and let $0 < \sigma < (2\alpha+2)/p$. Then* $I_\sigma f \in L^r$, where

$$\frac{1}{r} = \frac{1}{p} - \frac{\sigma}{2\alpha+2}.$$

Proof. This theorem is a consequence of our Theorem 4.5 and Theorem 2.6 of O'Neil [13]. For the notation we refer to O'Neil's paper.

We first need to calculate $g_\sigma^{**}(\cos\theta)$. We define the set

$$E_y = \{\theta : |g_\sigma(\cos\theta)| > y\}$$

and define $g_\sigma^*(\cos\theta)$ as the inverse function of

$$m(g_\sigma(\cos\theta), y) = \text{meas}(E_y).$$

In view of (5.6) we essentially have

$$E_y = \{\theta : \theta > y^{1/(\sigma-2\alpha-2)}\}$$

and

$$\text{meas}(E_y) = \int_{y^{1/(\sigma-2\alpha-2)}}^{\pi} \rho^{(\alpha,\beta)}(\theta)\,d\theta = O(y^{(2\alpha+2)/(\sigma-2\alpha-2)}).$$

So the inverse function

$$g_\sigma^*(\cos\theta) = O(\theta^{(\sigma-2\alpha-2)/(2\alpha+2)})$$

and

$$g_\sigma^{**}(\cos\theta) = \frac{1}{\theta}\int_0^\theta g_\sigma^*(\cos\phi)\,d\phi = O(\theta^{(\sigma-2\alpha-2)/(2\alpha+2)}).$$

We use the norm

$$\| g_\sigma(\cos\theta)\|_{q,\infty} = \sup_{\theta>0} \theta^{1/q} g^{**}(\cos\theta)$$

and it follows that $g_\sigma(\cos\theta) \in L((2\alpha+2)/(2\alpha+2-\sigma), \infty)$. O'Neil's Theorem 2.6 now states that, if

$$f \in L(p,p) = L^p \qquad \text{and} \qquad g_\sigma \in L((2\alpha+2)/(2\alpha+2-\sigma), \infty)$$

with the condition $1/p + (2\alpha+2-\sigma)/(2\alpha+2) > 1$, then $I_\sigma f \in L(r,s)$, where $1/r = 1/p - \sigma/(2\alpha+2)$ and any number $s \geqslant p$. If we choose $s = r$, Theorem 5.4 is proved.

We now define the fractional derivative of order σ by

$$D_\sigma f(\cos\theta) = A_\theta I_{2-\sigma} f(\cos\theta).$$

THEOREM 5.5. *Let* $0 < \sigma < \tau < 2$ *and suppose* $f \in \mathrm{Lip}(\tau, p)$, $1 \leqslant p \leqslant \infty$. *Then* $D_\sigma f \in \mathrm{Lip}(\tau - \sigma, p)$.

Proof. The proof is an immediate consequence of remark to Theorem 5.2 and Definition 5.1. Again, Theorem 5.5 can be extended to all positive values of σ and τ, which satisfy $0 < \sigma < \tau$, σ, $\tau - \sigma \neq$ even integer.

A. *An application*

As an application, we give sufficient conditions for $f(\cos\theta)$ to have a uniformly convergent or an absolutely convergent Fourier–Jacobi series. The N-th partial sum $S_N(\cos\theta)$ of the series (1.5) can be written as the convolution of $D_\sigma f(\cos\theta)$, for some σ, with the kernel $g_\sigma{}^N(\cos\theta)$, where

$$g_\sigma{}^N(\cos\theta) = \omega_0^{(\alpha,\beta)} + \sum_{n=1}^{N} [n(n+\alpha+\beta+1)]^{-\sigma/2}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta). \qquad (5.11)$$

If there exists a σ_1, such that $D_{\sigma_1} f(\cos\theta) \in L^\infty$ and $\| g_{\sigma_1}^N(\cos\theta)\|_1$ is uniformly bounded, it follows from Lemma 1.1(iv) that

$$\| S_N(\cos\theta)\|_\infty \leqslant \| g_{\sigma_1}^N(\cos\theta)\|_1 \, \| D_{\sigma_1} f(\cos\theta)\|_\infty ,$$

which implies that $f(\cos\theta)$ has a uniformly convergent Fourier–Jacobi series. Summing (5.11) by parts once by means of Lemma 2.1 and applying (2.8), we obtain that

$$\| g_\sigma{}^N(\cos\theta)\|_1 < M, \qquad \text{if} \qquad \sigma > \alpha + \tfrac{1}{2}.$$

From Theorem 5.5 we can conclude that for some $\sigma_1 > \alpha + \frac{1}{2}$, $D_{\sigma_1} f \in L^\infty$, if $f \in \mathrm{Lip}(\alpha + \frac{1}{2} + \epsilon, \infty)$.

Therefore, $f(\cos\theta)$ has a uniformly convergent Fourier–Jacobi series if

$$f \in \mathrm{Lip}(\alpha + \tfrac{1}{2} + \epsilon, \infty).$$

Let $g_\sigma(\cos\theta)$ be defined by (5.1). If there exists a σ_2, such that $D_{\sigma_2} f(\cos\theta) \in L^2$ and $g_{\sigma_2}(\cos\theta) \in L^2$, it follows from the Cauchy–Schwarz inequality, that $f(\cos\theta)$ has an absolutely convergent Fourier–Jacobi series.

From (5.6) it is not hard to derive that $g_{\sigma_2} \in L^2$, if $\sigma_2 > \alpha + 1$. Theorem 5.5 now implies that $f(\cos\theta)$ has an absolutely convergent Fourier–Jacobi series, if $f \in \mathrm{Lip}(\alpha + 1 + \epsilon, 2)$. This argument is due to Weyl [16].

We have to mention, that these results are not best possible, but almost best possible, whereas the proofs are very simple. Best possible results concerning uniform convergence are given by Agahanov and Natanson [1] (or by the much older results of Gronwall [10] for Legendre polynomials). For slightly better results on absolute convergence we refer to the paper of Ganser [8].

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