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MATHEMATICS

AN EXAMPLE ILLUSTRATING THE POSSIBILITIES OF RENEWAL THEORY AND WAITING-TIME THEORY FOR MARKOV-DEPENDENT ARRIVAL-INTERVALS [1)]

BY

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1. *Introduction, basic definitions*

In waiting-time theory so far one has considered almost exclusively independent arrival-intervals. It is not hard to find a reason: the difficulty of the problem increases considerably if the independence assumption is dropped. Still, one would like to have at least some idea of the influence of dependence on known results. In his thesis the author obtained a number of theorems, both of a theoretical and a more practical nature, concerning Markov-dependent arrival-intervals (or renewal-intervals) and the corresponding waiting-time theory. In this paper just one reasonably simple (and hence rather attractive) example is considered, from which it will be seen that it is indeed possible to work with dependent variables and to obtain even then practically useful formulae. The results can easily be generalized, but that is not done here. Related theorems can be found in RUNNENBURG (1960) and RUNNENBURG (1961).

We consider a waiting-system with one counter, where the queue-discipline is "first come, first served". At time 0 a customer enters for service at the counter, which is then busy finishing some work, which will take a further time $\mathbf{w}_0$ [2)]. Hence $\mathbf{w}_0$ is the waiting-time of the 0th customer. Let new customers arrive at moments $\mathbf{t}_1, \mathbf{t}_2, \ldots$ with

$$(1.1)\qquad \mathbf{y}_n \overset{\text{def}}{=} \mathbf{t}_{n+1} - \mathbf{t}_n \qquad n = 0, 1, 2, \ldots \qquad (\mathbf{t}_0 \overset{\text{def}}{=} 0)$$

denoting the length of the arrival-interval (or renewal-interval) between the nth and $(n+1)$st customer (or renewal). We assume that the $\mathbf{y}_n$ are (simple) Markov-dependent random variables, the only possible values (or states) of the $\mathbf{y}_n$ being the integers $1, 2, \ldots, r$, where r is *finite*. The transition probabilities are stationary, i.e.

$$(1.2)\qquad p_{ij} \overset{\text{def}}{=} P\{\mathbf{y}_{n+1} = j \mid \mathbf{y}_n = i\}$$

1) Report S 285 of the Mathematical Centre, Amsterdam, presented to the Troisième Congrès International de Télétrafic, Paris 1961.

2) Random variables are printed in bold type.

is independent of n. We further assume

$$p_{ij} = \frac{a_i a_j + b_i b_j}{a_i a + b_i b}, \tag{1.3}$$

with a_i and b_j complex numbers, which satisfy [1])

$$\begin{cases} a = \sum\limits_i a_i, \\ b = \sum\limits_i b_i, \\ a^2 + b^2 = 1, \\ a_i a + b_i b > 0, \\ 0 \leqslant a_i a_j + b_i b_j \leqslant 1. \end{cases} \tag{1.4}$$

We call this particular chain the Markov chain M_1. The a_i and b_j are assumed to be *complex* numbers to obtain a larger set of transition probabilities. One can easily prove that for $r=2$ all possible Markov matrices with two states are obtained in this way.

Assumption (1.3) may seem rather odd, but is inspired by the following considerations. We wish to be able to handle the matrix with entries p_{ij} and its powers, without too much computational difficulties. The situation is analogous to that in the theory of integral equations. There a function $k(x, y)$ of two variables x and y, called the kernel of the integral equation, is replaced by a finite sum of products

$$k^*(x, y) = \sum_{\mu=1}^{m} f_\mu(x)\, g_\mu(y) \tag{1.5}$$

to simplify the equation and develop a theory. The present problem contains a "kernel" p_{ij}, function of two "variables" i and j. One way of simplifying the "general" p_{ij} would be to assume from the outset,that for all i and j

$$p_{ij} = \sum_{\mu=1}^{2} f_\mu(i)\, g_\mu(j). \tag{1.6}$$

We prefer to start with the stationary absolute probability distribution of the vector $(\mathbf{y}_n, \mathbf{y}_{n+1})$. We assume that the bivariate probability distribution of that vector is given by

$$P\{\mathbf{y}_n = i,\ \mathbf{y}_{n+1} = j\} = a_i a_j + b_i b_j, \tag{1.7}$$

which is symmetric [2]) in i and j. If now we compute $P\{\mathbf{y}_n = i\}$, we find

$$P\{\mathbf{y}_n = i\} = a_i a + b_i b, \tag{1.8}$$

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[1]) If the range of summation is not indicated, the set of integers $\{1, 2, \ldots, r\}$ is meant.

[2]) Hence an irreducible aperiodic chain M_1 with $r = 2$ is always symmetric.

where $a = \sum_i a_i$ and $b = \sum_i b_i$. Hence

$$p_{ij} = \frac{a_i a_j + b_i b_j}{a_i a + b_i b}. \tag{1.9}$$

In order that (1.7) and (1.8) describe probability distributions and (1.9) a set of transition probabilities, the a's and b's must satisfy the conditions (1.4).

We consider only irreducible aperiodic Markov chains M_1. It will be seen that a necessary and sufficient condition to that effect is

$$|c| < 1, \tag{1.10}$$

where

$$c \overset{\text{def}}{=} \sum_i \frac{(a_i b - b_i a)^2}{a_i a + b_i b}. \tag{1.11}$$

In section 2 we derive some theorems for the $\mathbf{y}_n$, which simplify to classic results of renewal theory if the $\mathbf{y}_n$ are assumed independent. In that section we use only simple theorems from KEMENY and SNELL (1960) in our proofs and need not bother with more general theory.

In section 3 the waiting-time problem is treated. Here use is made of the general method discussed in RUNNENBURG (1960), chapter IV. The service-times $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \ldots$ of the 0th, 1st, 2nd, ... customer are assumed to be independent, identically distributed random variables, with an exponential distribution

$$P\{\mathbf{s}_n \leqslant s\} = 1 - e^{-\mu s} \qquad (s \geqslant 0),$$

where μ is a positive constant. The service-times are independent of the arrival-intervals.

Finally in section 4 we compare the results obtained in sections 2 and 3 with well-known theorems for independent $\mathbf{y}_n$.

2. *Renewal theory for Markov-dependent renewal-intervals*

If $b = 0$ in (1.4), then $a^2 = 1$ and so $a = 1$ or $a = -1$. By changing the sign of all a_i's (if necessary), we obtain $a = 1$ and $a_i > 0$ for all i. If $b \neq 0$, we replace a_i and b_i by $\bar{a}_i$ and $\bar{b}_i$, defined by

$$\begin{cases} \bar{a}_i \overset{\text{def}}{=} a_i a + b_i b, \\ \bar{b}_i \overset{\text{def}}{=} -a_i b + b_i a. \end{cases} \tag{2.1}$$

Because $\sum_i \bar{b}_i = -ab + ba = 0$ and $\sum_i \bar{a}_i = a^2 + b^2 = 1$, we again arrive at $\bar{b} = 0$ and $\bar{a}^2 = 1$. It is thus no restriction to assume as we now do (instead of (1.3)) that the transition matrix of the Markov chain M_1 has entries

$$p_{ij} \overset{\text{def}}{=} a_j + \frac{b_i}{a_i} b_j, \tag{2.2}$$

with (corresponding to (1.4))

$$\left\{\begin{aligned} &\sum_j a_j = 1, \\ &\sum_j b_j = 0, \\ &a_j > 0, \\ 0 \leqslant\ &a_j + \frac{a_i}{b_i} b_j \leqslant 1, \end{aligned}\right. \tag{2.3}$$

where the b_i must be either all real or all imaginary.

In order to obtain the eigenvalues of the transition matrix, we compute $\det(p_{ij} - z\delta_{ij})$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. It is not hard to verify, that

$$\det(p_{ij} - z\delta_{ij}) = (-z)^{r-2}(1-z)(c-z), \tag{2.4}$$

where c is given by (1.11), which now reads

$$c \stackrel{\text{def}}{=} \sum_i \frac{b_i^2}{a_i}. \tag{2.5}$$

Our assumption (1.10), i.e. $|c| < 1$, is hence necessary and sufficient to make the matrix p_{ij} irreducible and aperiodic, cf. FELLER (1950). We only need the eigenvalue theory to explain why the restriction (1.10) is imposed, it will not be used.

Next we consider

$$p_{ij}(z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} p_{ij}^{(n)} z^{n-1} \quad \text{for} \quad |z| < 1, \tag{2.6}$$

where $p_{ij}^{(n)}$ denotes the probability of reaching state j from state i in n steps. By solving the system of linear equations

$$p_{ij}(z) = p_{ij} + z \sum_k p_{ik} p_{kj}(z), \tag{2.7}$$

we easily find

$$p_{ij}(z) = \frac{a_j}{1-z} + \frac{b_i}{a_i} \frac{b_j}{1-cz}. \tag{2.8}$$

We thus have

$$\lim_{n\to\infty} p_{ij}^{(n)} = a_j, \tag{2.9}$$

showing that p_{ij} is a regular transition matrix in the sense of Kemeny and Snell (all entries of $p_{ij}^{(n)}$ are positive for some finite n), while moreover

$$\lim_{z\to 1} \left\{ p_{ij}(z) - \frac{a_j}{1-z} \right\} = \frac{b_i}{a_i} \frac{b_j}{1-c}. \tag{2.10}$$

Kemeny and Snell define a matrix Z (with entries z_{ij}) for each regular transition matrix P (with entries p_{ij}) by

$$z_{ij} \overset{\text{def}}{=} \delta_{ij} + \sum_{n=1}^{\infty} \{p_{ij}^{(n)} - a_j\}, \tag{2.11}$$

where $a_j \overset{\text{def}}{=} \lim_{n\to\infty} p_{ij}{}^{(n)}$. This conforms with our notation, because of (2.9). They prove, that z_{ij} has finite entries and use the matrix Z to obtain a number of interesting relations. We use the corollary to their theorem 4.6.1 (page 86): If $f(\mathbf{y}_k)$ is a function of the state $\mathbf{y}_k$ entered at the kth step in a regular Markov chain, with $f(\mathbf{y}_k) = f_i$ if $\mathbf{y}_k = i$, then

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{var} \sum_{k=1}^{n} f(\mathbf{y}_k) = \sum_i \sum_j f_i c_{ij} f_j, \tag{2.12}$$

independent of the $q_0(i)$ (the initial absolute probability distribution), where [1])

$$c_{ij} \overset{\text{def}}{=} a_i z_{ij} + a_j z_{ji} - a_i \delta_{ij} - a_i a_j. \tag{2.13}$$

From (2.10) we have for the chain M_1

$$z_{ij} = \delta_{ij} + \frac{b_i}{a_i} \frac{b_j}{1-c}. \tag{2.14}$$

If we take

$$f_i \overset{\text{def}}{=} i, \tag{2.15}$$

then

$$\sum_{k=1}^{n} f(\mathbf{y}_k) = \sum_{k=1}^{n} \mathbf{y}_k, \tag{2.16}$$

from which we find with (2.12), (2.13) and (2.14) after some reductions

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{var} \sum_{k=1}^{n} \mathbf{y}_k = \sigma(a)^2 + 2\,\frac{\mu_1(b)^2}{1-c}, \tag{2.17}$$

where we have used the abbreviations

$$\left\{\begin{aligned} \mu_1(a) &\overset{\text{def}}{=} \sum_i i\, a_i, \\ \mu_1(b) &\overset{\text{def}}{=} \sum_i i\, b_i, \\ \mu_2(a) &\overset{\text{def}}{=} \sum_i i^2 a_i, \\ \sigma(a)^2 &\overset{\text{def}}{=} \mu_2(a) - \mu_1(a)^2. \end{aligned}\right. \tag{2.18}$$

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[1]) Our δ_{ij} is the well-known Kronecker delta and differs from the d_{ij} used by Kemeny and Snell. Their definition of c_{ij} is incorrect.

To obtain renewal theorems for the Markov chain M_1, we introduce a new Markov chain M_2, with states described by a vector (i, j) with $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant i$. If $\mathbf{l}_n$ denotes the number of the last customer having moment of arrival $\leqslant n$, then the state after the nth step (in the figure: length of $\mathbf{y}_{\mathbf{l}_n}$ and height at time n) is given by (i, j), where $i = t_{l+1} - t_l$ and $j = t_{l+1} - n$ if $\mathbf{l}_n = l$, $\mathbf{t}_l = t_l$ and $\mathbf{t}_{l+1} = t_{l+1}$. The Markov chain M_2 has transition probabilities

$$p_{ij;\,kl} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } k = i,\ l = j - 1, \\ p_{ik} & \text{if } j = 1,\ k = l, \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

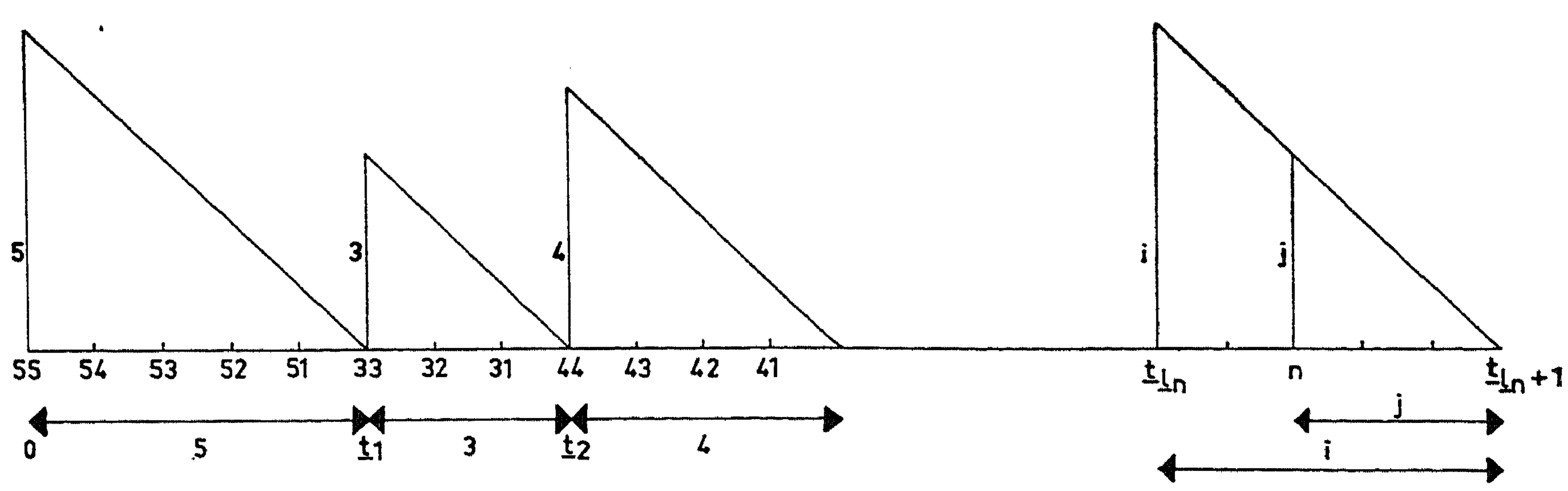

Fig. 1

Let $q_n(i, j)$ denote the absolute probability of entering state (i, j) at the nth step for the chain M_2, given the initial probability distribution $q_0(i, j)$ (with $q_0(i, j) = 0$ for $i \neq j$, because we assumed that an arrival occurs at $\mathbf{t}_0 = 0$). We may ask for the probability $U_n(i)$, that time n is the moment of arrival of some customer, under the condition that the chain starts at time 0 in state (i, i). Now

$$U_n(i) = \sum_j p^{(n)}_{ii;\,jj}, \tag{2.20}$$

where $p^{(n)}_{ij;\,kl}$ denotes the probability of reaching state (k, l) at the nth step (in the chain M_2) starting from state (i, j). The absolute probability U_n of having an arrival occur at time n is given by

$$U_n = \sum_i q_0(i, i)\ U_n(i). \tag{2.21}$$

Take

$$p_{ij;\,kl}(z) \overset{\text{def}}{=} \sum_{n=1}^{\infty} p^{(n)}_{ij;\,kl}\, z^{n-1} \qquad \text{for } |z| < 1. \tag{2.22}$$

We may again write down a system of linear equations like (2.7). Its solution is not as easily obtained as before. However, now we only need $p_{ii;jj}(z)$. Because

$$p_{ii;\,kk}(z) = z^{i-1} p_{ik} + z^i \sum_j p_{ij}\, p_{jj;\,kk}(z), \tag{2.23}$$

we have by (2.2), if we use the abbreviations

$$\text{(2.24)} \qquad \begin{cases} \alpha_k(z) \stackrel{\text{def}}{=} \sum_j a_j\, p_{jj;\,kk}(z), \\ \beta_k(z) \stackrel{\text{def}}{=} \sum_j b_j\, p_{jj;\,kk}(z), \end{cases}$$

and

$$\text{(2.25)} \qquad \begin{cases} A(z) \stackrel{\text{def}}{=} \sum_j a_j z^j, \\ B(z) \stackrel{\text{def}}{=} \sum_j b_j z^j, \\ C(z) \stackrel{\text{def}}{=} \sum_j \dfrac{b_j{}^2}{a_j}\, z^j, \end{cases}$$

that

$$\text{(2.26)} \qquad \begin{cases} \alpha_k(z) = a_k z^{-1}A(z) + b_k z^{-1}B(z) + \alpha_k(z)A(z) + \beta_k(z)B(z), \\ \beta_k(z) = a_k z^{-1}B(z) + b_k z^{-1}C(z) + \alpha_k(z)B(z) + \beta_k(z)C(z). \end{cases}$$

Solving these equations for $\alpha_k(z)$ and $\beta_k(z)$, we obtain

$$\text{(2.27)} \qquad \begin{cases} z\alpha_k(z) = \dfrac{a_k[A(z)\{1-C(z)\}+B(z)^2]+b_k B(z)}{\{1-A(z)\}\{1-C(z)\}-B(z)^2}, \\[2ex] z\beta_k(z) = \dfrac{a_k B(z)+b_k[C(z)\{1-A(z)\}+B(z)^2]}{\{1-A(z)\}\{1-C(z)\}-B(z)^2}. \end{cases}$$

Because

$$\text{(2.28)} \qquad p_{ii;\,kk}(z) = z^{i-1}a_k + z^{i-1}\frac{b_i}{a_i}\, b_k + z^i\alpha_k(z) + z^i\frac{b_i}{a_i}\,\beta_k(z),$$

we have if the chain M_2 is regular

$$\text{(2.29)} \qquad \lim_{n\to\infty} p^{(n)}_{ii;\,kk} = \lim_{z\to 1}\,(1-z)\,p_{ii;\,kk}(z) = \lim_{z\to 1}\,(1-z)\alpha_k(z) = \frac{a_k}{\mu_1(a)}.$$

Now the chain M_2 is certainly irreducible (i.e. every state can be reached in a finite number of steps from any state) because the same is true for the chain M_1. Hence (2.29) holds if the chain M_2 is aperiodic. This can be *verified* in any particular case by finding the greatest common divisor g of those n for which $p^{(n)}_{ii;\,ii} > 0$ for some conveniently chosen i. If $g=1$, the chain M_2 is aperiodic, otherwise it is periodic with period g, all states having the same period. The chain M_2 is certainly aperiodic if $p_{11;11} > 0$, i.e. if

$$\text{(2.30)} \qquad a_1{}^2 + b_1{}^2 > 0.$$

Because of (2.21), (2.20) and (2.29) we have for aperiodic M_2

$$\text{(2.31)} \qquad \lim_{n\to\infty} U_n = \sum_i \sum_j q_0(i,\,i) \lim_{n\to\infty} p^{(n)}_{ii;\,jj} = \sum_i \sum_j q_0(i,\,i)\,\frac{a_j}{\mu_1(a)} = \frac{1}{\mu_1(a)}.$$

Next we consider $\mathbf{l}_n$, the number of the last customer having a moment of arrival $\leqslant n$. Let $(\mathbf{i}_n, \mathbf{j}_n)$ be the random vector, denoting the state of the chain M_2 at time n. If now $f^*(\mathbf{i}_k, \mathbf{j}_k)$ is a function of the state at time k, with $f^*(\mathbf{i}_k, \mathbf{j}_k) = f_{ij}$ if $\mathbf{i}_k = i$ and $\mathbf{j}_k = j$, then again the relation (2.12) of Kemeny and Snell may be applied. We take in particular

$$f_{ij} \stackrel{\text{def}}{=} \delta_{ij}, \tag{2.32}$$

for then

$$\mathbf{l}_n = \sum_{k=1}^{n} f^*(\mathbf{i}_k, \mathbf{j}_k). \tag{2.33}$$

Hence we immediately have

$$\mathscr{E}\mathbf{l}_n = \sum_{k=1}^{n} \mathscr{E}f^*(\mathbf{i}_k, \mathbf{j}_k) = \sum_{k=1}^{n} \sum_i \sum_j q_0(i, i)\, p^{(k)}_{ii;\,jj} \tag{2.34}$$

and so, whether M_2 is aperiodic or not,

$$\lim_{n\to\infty} \frac{1}{n}\, \mathscr{E}\mathbf{l}_n = \sum_i \sum_j q_0(i, i) \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p^{(k)}_{ii;\,jj} = \sum_i \sum_j q_0(i, i)\, \frac{a_j}{\mu_1(a)} = \frac{1}{\mu_1(a)}, \tag{2.35}$$

the value of the Cesàro-limit following from (2.29) (it always *exists* as is well-known).

Instead of (2.11) we now write

$$z_{ij;\,kl} \stackrel{\text{def}}{=} \delta_{ik}\delta_{jl} + \sum_{n=1}^{\infty} (p^{(n)}_{ij;\,kl} - a_{kl}), \tag{2.36}$$

where

$$a_{kl} \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p^{(k)}_{ij;\,kl} \tag{2.37}$$

(cf. Kemeny and Snell (1960), page 102 for this generalization). It is here again irrelevant whether the chain M_2 is aperiodic or not. We have to replace (2.12) and (2.13) by

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{var} \sum_{k=1}^{n} f^*(\mathbf{i}_k, \mathbf{j}_k) = \sum_i \sum_j \sum_k \sum_l f_{ij}\, c_{ij;\,kl}\, f_{kl} \tag{2.38}$$

and

$$c_{ij;\,kl} = a_{ij} z_{ij;\,kl} + a_{kl} z_{kl;\,ij} - a_{ij}\delta_{ik}\delta_{jl} - a_{ij}a_{kl}, \tag{2.39}$$

where as before the limit does not depend on the initial probability distribution, i.e. on the $q_0(i, j)$.

We need only use

$$a_{ll} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p^{(k)}_{ii;\,ll} = \frac{a_l}{\mu_1(a)}, \tag{2.40}$$

a consequence of (2.29). For by (2.32)

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{var} \mathbf{l}_n = \sum_i \sum_j \sum_k \sum_l \delta_{ij}\, c_{ij;\,kl}\, \delta_{kl} = \sum_i \sum_k c_{ii;\,kk} \tag{2.41}$$

and $c_{ii;kk}$ can be found from $z_{ii;kk}$. If now we substitute (2.27) in (2.28) and use the result together with (2.40) to calculate (2.36), there results

$$(2.42)\quad \left\{\begin{aligned} z_{ii;kk} = \delta_{ik} + \frac{b_i}{a_i}\left\{-a_k \frac{\mu_1(b)}{\mu_1(a)(1-c)} + b_k \frac{1}{1-c}\right\} + \\ + a_k \left\{\frac{\mu_2(a)+\mu_1(a)}{2\mu_1(a)^2} + \frac{\mu_1(b)^2}{\mu_1(a)^2(1-c)} - \frac{i}{\mu_1(a)}\right\} - b_k \frac{\mu_1(b)}{\mu_1(a)(1-c)}. \end{aligned}\right.$$

If now we use (2.38), we obtain

$$(2.43)\qquad \lim_{n\to\infty} \frac{1}{n} \operatorname{var} \mathbf{l}_n = \frac{1}{\mu_1(a)^3}\left\{\sigma(a)^2 + 2\,\frac{\mu_1(b)^2}{1-c}\right\}.$$

3. *The waiting-time problem*

From theorem 2.3.1 in RUNNENBURG (1960) it follows that under the present assumptions, if the chain M_1 is irreducible and aperiodic (i.e. if the condition $|c|<1$ is imposed) and if

$$(3.1)\qquad \mathscr{E}\mathbf{s} < \mu_1(a),$$

then we have for $\mathbf{w}_n$, the waiting-time of the nth customer,

$$(3.2)\qquad \lim_{n\to\infty} P\{\mathbf{w}_n \leqslant w\} = F(w),$$

where $F(w)$ is a distribution function. One can extend the theory and show, that

$$(3.3)\qquad \lim_{n\to\infty} P\{\mathbf{y}_n = k,\, \mathbf{w}_{n+1} \leqslant w\} = a_k F_k(w),$$

where $F_k(w)$ is a distribution function for each k with $1 \leqslant k \leqslant r$ and

$$(3.4)\qquad F(w) = \sum_k a_k F_k(w).$$

Moreover, the initial situation is irrelevant, i.e.

$$(3.5)\qquad \lim_{n\to\infty} P\{\mathbf{y}_n = k,\, \mathbf{w}_{n+1} \leqslant w \,|\, \mathbf{y}_0 = i,\, \mathbf{w}_1 = w_1\} = a_k F_k(w),$$

independent of the value of i and w_1.

For the particular chain M_1 considered here we shall obtain the Laplace–Stieltjes transform of the $F_k(w)$. To this end consider

$$(3.6)\quad C(i, w_1; k, \xi; z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} z^n \int_{0-}^{\infty} e^{-\xi w}\, d_w\, P\{\mathbf{y}_n = k,\, \mathbf{w}_{n+1} \leqslant w | \mathbf{y}_0 = i,\, \mathbf{w}_1 = w_1\}$$

for $|z|<1$, where $\operatorname{Re}\xi > 0$. If we consider only real z, we have

$$(3.7)\quad \lim_{z\uparrow 1}\, (1-z) \sum_{n=0}^{\infty} z^n P\{\mathbf{y}_n = k,\, \mathbf{w}_{n+1} \leqslant w \,|\, \mathbf{y}_0 = i,\, \mathbf{w}_1 = w_1\} = a_k F_k(w)$$

and hence

$$(3.8)\qquad a_k \check{F}_k(\xi) \stackrel{\text{def}}{=} a_k \int_{0-}^{\infty} e^{-\xi w}\, dF_k(w) = \lim_{z\uparrow 1}\, (1-z)\, C(i, w_1; k, \xi; z).$$

It is convenient to introduce

$$\check{C}(i, \omega; k, \xi; z) \overset{\text{def}}{=} \int_0^\infty e^{-\omega w_1}\, d_{w_1} C(i, w_1; k, \xi; z) \tag{3.9}$$

for $\operatorname{Re} \omega \geqslant 0$, where we take

$$P\{\mathbf{y}_n = k,\ \mathbf{w}_{n+1} \leqslant w | \mathbf{y}_0 = i,\ \mathbf{w}_1 = w_1\} = P\{\mathbf{y}_n = k,\ \mathbf{w}_{n+1} \leqslant w | \mathbf{y}_0 = i,\ \mathbf{w}_1 = 0\} \tag{3.10}$$

for $w_1 < 0$.

As

$$\check{C}(i, 0; k, \xi; z) = -C(i, 0; k, \xi; z), \tag{3.11}$$

we also have

$$\lim_{z \uparrow 1} (1-z)\, \check{C}(i, 0; k, \xi; z) = -a_k \check{F}_k(\xi), \tag{3.12}$$

from which $F_k(w)$ can be obtained by inverting the Laplace–Stieltjes transform.

In order to find a set of equations from which $\check{C}(i, \omega; k, \xi; z)$ can be obtained, we use the fact that

$$\left\{\begin{aligned} &P\{\mathbf{y}_{n+1} = k,\ \mathbf{w}_{n+2} \leqslant w \,|\, \mathbf{y}_0 = i,\ \mathbf{w}_1 = w_1\} = \\ &= \sum_j p_{ij} \int_0^\infty P\{\mathbf{y}_{n+1} = k,\ \mathbf{w}_{n+2} \leqslant w \,|\, \mathbf{y}_1 = j,\ \mathbf{w}_2 = w_1 + s - j\}\, \mu e^{-\mu s}\, ds = \\ &= \sum_j p_{ij} \int_0^\infty P\{\mathbf{y}_n = k,\ \mathbf{w}_{n+1} \leqslant w \,|\, \mathbf{y}_0 = j,\ \mathbf{w}_1 = w_1 + s - j\}\, \mu e^{-\mu s}\, ds. \end{aligned}\right. \tag{3.13}$$

If now we take the Laplace–Stieltjes transform with respect to w (with parameter ξ), multiply the resulting equation by z^{n+1} and sum over n from 0 to ∞, we obtain for $w_1 \geqslant 0$

$$\left\{\begin{aligned} &C(i, w_1; k, \xi; z) = \\ &= C(i, w_1; k, \xi; 0) + z \sum_j p_{ij} \int_0^\infty C(j, w_1 + s - j; k, \xi; z)\, \mu e^{-\mu s}\, ds. \end{aligned}\right. \tag{3.14}$$

Next we apply the Laplace–Stieltjes transformation with respect to w_1 (with parameter ω). This leads to (cf. Runnenburg (1960), pages 108 and 119 for an indirect derivation)

$$\left\{\begin{aligned} &\check{C}(i, \omega; k, \xi; z) = \check{C}(i, \omega; k, \xi; 0) + \\ &\qquad + \frac{\mu z}{\mu - \omega} \sum_j p_{ij} \{e^{-j\omega}\, \check{C}(j, \omega; k, \xi; z) - e^{-j\mu}\, \check{C}(j, \mu; k, \xi; z)\}. \end{aligned}\right. \tag{3.15}$$

This system of linear equations for $\check{C}(i, \omega; k, \xi; z)$ we have to solve in order to obtain $\check{F}_k(\xi)$.

For $w_1 \geqslant 0$ we find

$$C(i, w_1; k, \xi; 0) = \int_{0-}^\infty e^{-\xi w}\, d_w P\{\mathbf{y}_0 = k,\ \mathbf{w}_1 \leqslant w \,|\, \mathbf{y}_0 = i,\ \mathbf{w}_1 = w_1\} = \delta_{ik}\, e^{-\xi w_1}, \tag{3.16}$$

and hence

$$(3.17)\qquad \check{C}(i, \omega; k, \xi; 0) = \delta_{ik} \int_0^\infty e^{-\omega w_1}\, de^{-\xi w_1} = -\frac{\xi}{\xi+\omega}\, \delta_{ik}.$$

We rewrite (3.15) and substitute (3.17)

$$(3.18)\qquad \begin{cases} \check{C}(i, \omega; k, \xi; z) - \dfrac{\mu z}{\mu-\omega} \sum_j p_{ij}\, e^{-j\omega}\, C(j, \omega; k, \xi; z) = \\ \qquad = -\dfrac{\xi}{\xi+\omega}\, \delta_{ik} - \dfrac{\mu z}{\mu-\omega} \sum_j p_{ij}\, e^{-j\mu}\, \check{C}(j, \mu; k, \xi; z). \end{cases}$$

Here we have for fixed ω, k, ξ and z a system of r linear equations for the r functions $\check{C}(i, \omega; k, \xi; z)$, provided the $\check{C}(i, \mu; k, \xi; z)$ are known.

We further introduce for $|z|<1$, $\operatorname{Re}\omega \geqslant 0$

$$(3.19)\qquad \begin{cases} \alpha_k(\omega; z) \stackrel{\text{def}}{=} \alpha(\omega; k, \xi; z) \stackrel{\text{def}}{=} \sum_j a_j\, e^{-j\omega}\, \check{C}(j, \omega; k, \xi; z) \\ \beta_k(\omega; z) \stackrel{\text{def}}{=} \beta(\omega; k, \xi; z) \stackrel{\text{def}}{=} \sum_j b_j\, e^{-j\omega}\, \check{C}(j, \omega; k, \xi; z) \end{cases}$$

and rewrite (3.18) by means of (2.2) in terms of $\alpha_k(\omega; z)$ and $\beta_k(\omega; z)$

$$(3.20)\qquad \begin{cases} \check{C}(i, \omega; k, \xi; z) - \dfrac{\mu z}{\mu-\omega} \left\{\alpha_k(\omega; z) + \dfrac{b_i}{a_i}\, \beta_k(\omega; z)\right\} = \\ \qquad = -\dfrac{\xi}{\xi+\omega}\, \delta_{ik} - \dfrac{\mu z}{\mu-\omega} \left\{\alpha_k(\mu; z) + \dfrac{b_i}{a_i}\, \beta_k(\mu; z)\right\}. \end{cases}$$

From (3.20) we find for $\alpha_k(\omega; z)$ and $\beta_k(\omega; z)$

$$(3.21)\qquad \begin{cases} \left\{1 - \dfrac{\mu z}{\mu-\omega} A(e^{-\omega})\right\} \{\alpha_k(\omega; z) - \alpha_k(\mu; z)\} - \dfrac{\mu z}{\mu-\omega} B(e^{-\omega}) \{\beta_k(\omega; z) - \beta_k(\mu; z)\} = \\ \qquad = -\dfrac{\xi}{\xi+\omega}\, a_k\, e^{-k\omega} - \alpha_k(\mu; z) \\ -\dfrac{\mu z}{\mu-\omega} B(e^{-\omega}) \{\alpha_k(\omega; z) - \alpha_k(\mu; z)\} + \left\{1 - \dfrac{\mu z}{\mu-\omega} C(e^{-\omega})\right\} \{\beta_k(\omega; z) - \beta_k(\mu; z)\} = \\ \qquad = -\dfrac{\xi}{\xi+\omega}\, b_k\, e^{-k\omega} - \beta_k(\mu; z) \end{cases}$$

with $A(z)$, $B(z)$ and $C(z)$ as in (2.25). The determinant of the system (3.21) is

$$(3.22)\qquad D(\omega; z) \stackrel{\text{def}}{=} \left\{1 - \frac{\mu z}{\mu-\omega} A(e^{-\omega})\right\} \left\{1 - \frac{\mu z}{\mu-\omega} C(e^{-\omega})\right\} - \left\{\frac{\mu z}{\mu-\omega} B(e^{-\omega})\right\}^2.$$

By Schwarz's inequality we have

$$(3.23)\qquad A(e^{-\omega})\, C(e^{-\omega}) \neq \{B(e^{-\omega})\}^2$$

unless $b_i = \lambda a_i$ for some constant λ for all i. But then $0 = \sum_i b_i = \lambda$ and hence $b_i = 0$ for all i. This case we exclude from the following discussion, i.e. we have (3.23) and hence μ is a pole of $D(\omega; z)$ of order two.

In order to find the zero's of $D(\omega; z)$ in the region $\mathrm{Re}\,\omega > 0$, we need the fact that both

$$(3.24) \qquad \mu - \omega - \mu z\, A(e^{-\omega})$$

under the condition (3.1) and

$$(3.25) \qquad \mu - \omega - \mu z\, C(e^{-\omega})$$

under the condition (1.10) have exactly one zero in $\mathrm{Re}\,\omega > 0$ for each z with $0 \leqslant z \leqslant 1$. If $\omega_a(z)$ is the zero of (3.24) and $\omega_c(z)$ the zero of (3.25), then for $0 < z \leqslant 1$

$$(3.26) \qquad 0 < \omega_a(z) < \mu$$

$$(3.27) \qquad \begin{cases} 0 < \omega_c(z) < \mu & \text{if all } b_j \text{ are real,} \\ \mu < \omega_c(z) & \text{if all } b_j \text{ are imaginary.} \end{cases}$$

This can be proved with Rouché's theorem: "If two functions $f(\omega)$ and $g(\omega)$ are analytic for all $\omega \in G$ (where G is a domain in the complex plane with the curve K as boundary, K being a simple closed contour contained in G) and satisfy $|f(\omega)| < |g(\omega)|$ for all $\omega \in K$, then $f(\omega) + g(\omega)$ and $g(\omega)$ have the same number of zero's inside K". For K we take a contour consisting of a segment of a line

$$(3.28) \qquad \{\omega \,|\, \mathrm{Re}\,\omega = \sigma \text{ and } -R \leqslant \mathrm{Im}\,\omega \leqslant R\}$$

together with a segment of a circle

$$(3.29) \qquad \{\omega \,|\, \mathrm{Re}\,\omega > \sigma \text{ and } |\omega - \sigma| = R\},$$

where the two positive constants σ and R are conveniently chosen. We take $g(\omega) = \omega - \mu$ and $f(\omega) = \mu z\, A(e^{-\omega})$ (to deal with (3.24)) and $f(\omega) = \mu z\, C(e^{-\omega})$ (for (3.25)). The roots $\omega_a(z)$ and $\omega_c(z)$ are found to be real (and to satisfy (3.26) and (3.27)) by inspection of the graph of the functions $f(\omega) + g(\omega)$ for real ω.

Next we apply Rouché's theorem to

$$(3.30) \qquad \begin{cases} f(\omega) = -\{\mu z\, B(e^{-\omega})\}^2 \\ g(\omega) = \{\mu - \omega - \mu z\, A(e^{-\omega})\}\{\mu - \omega - \mu z\, C(e^{-\omega})\}, \end{cases}$$

where again z is a constant and $0 < z \leqslant 1$. We use the same contour K. For $\mathrm{Re}\,\omega = \sigma$ and a sufficiently small constant $\sigma > 0$

$$(3.31) \qquad \begin{cases} \left|\left\{1 - \dfrac{\mu z}{\mu - \omega} A(e^{-\omega})\right\}\left\{1 - \dfrac{\mu z}{\mu - \omega} C(e^{-\omega})\right\}\right| \geqslant \left\{1 - \dfrac{\mu}{\mu - \sigma} A(e^{-\sigma})\right\}\left\{1 - \dfrac{\mu}{\mu - \sigma} C(e^{-\sigma})\right\} = \\ \qquad = \{1 - |c|\}\left\{\mu_1(a) - \dfrac{1}{\mu}\right\}\sigma + 0(\sigma^2) \end{cases}$$

and

$$(3.32) \qquad \left|\frac{\mu z}{\mu - \omega} B(e^{-\omega})\right|^2 \leqslant \left\{\frac{\mu}{\mu - \sigma}\,|B(e^{-\sigma})|\right\}^2 = |\mu_1(b)|^2\, \sigma^2 + 0(\sigma^3).$$

Hence we have $|f(\omega)|<|g(\omega)|$ for *all* ω with $\mathrm{Re}\,\omega=\sigma$ because of (1.10) and (3.1) for small σ. For sufficiently large R (large enough to have μ and $\omega_c(z)$ inside K) we also have $|f(\omega)|<|g(\omega)|$, because $A(e^{-\omega})$, $B(e^{-\omega})$ and $C(e^{-\omega})$ are bounded functions for $\mathrm{Re}\,\omega\geqslant 0$ and so $f(\omega)$ is a bounded function, while $g(\omega)$ behaves like ω^2.

The number of zero's of $f(\omega)+g(\omega)$ inside K is two, say $\omega_1(z)$ and $\omega_2(z)$ (for each z with $0<z\leqslant 1$), because $g(\omega)$ has zero's $\omega_a(z)$ and $\omega_c(z)$. To avoid complications (equal roots) we assume

$$B(e^{-\omega_1(z)})\,B(e^{-\omega_2(z)})\neq 0 \qquad \text{for } 0\leqslant 1-z<\delta, \tag{3.33}$$

where δ is a positive constant.

For real $\omega\geqslant 0$ and $0\leqslant 1-z<\delta$ we then find

$$D(\omega;z)>0 \qquad \text{for } \omega=\sigma, \tag{3.34}$$

$$\begin{cases} D(\omega^*(z);z)<0 & \text{for } \omega^*(z)=\omega_a(z) \text{ and } \omega_c(z), \text{ if all } b_j \text{ are real,}\\ D(\mu;z)<0 & \text{if all } b_j \text{ are imaginary,}\end{cases} \tag{3.35}$$

$$D(\omega;z)>0 \qquad \text{for large } \omega>0. \tag{3.36}$$

The second inequality in (3.35) is a consequence of Schwarz's inequality.

Hence $D(\omega;z)=0$ has for $0\leqslant 1-z<\delta$ two real roots $\omega_1(z)$ and $\omega_2(z)$ with

$$\begin{cases} 0<\omega_1(z)<\min\,(\omega_a(z),\,\omega_c(z))\leqslant\max\,(\omega_a(z),\,\omega_c(z))<\omega_2(z)<\mu \\ \hfill \text{if all } b_j \text{ are real,}\\ 0<\omega_a(z)<\omega_1(z)<\mu<\omega_2(z)<\omega_c(z) \qquad \text{if all } b_j \text{ are imaginary.}\end{cases} \tag{3.37}$$

From (3.37) we see that $\omega_1(z)<\omega_2(z)$ for $0\leqslant 1-z<\delta$. Of course $\omega_1(z)$ and $\omega_2(z)$ are analytic functions of z.

From (3.21) we find

$$\left\{\begin{aligned} &D(\omega;z)\,\{\alpha_k(\omega;z)-\alpha_k(\mu;z)\}=\\ &=-\frac{\xi}{\xi+\omega}\left[a_k e^{-k\omega}\left\{1-\frac{\mu z}{\mu-\omega}C(e^{-\omega})\right\}+b_k e^{-k\omega}\frac{\mu z}{\mu-\omega}B(e^{-\omega})\right]+\\ &\qquad -\alpha_k(\mu;z)\left\{1-\frac{\mu z}{\mu-\omega}C(e^{-\omega})\right\}-\beta_k(\mu;z)\frac{\mu z}{\mu-\omega}B(e^{-\omega})\\ &D(\omega;z)\,\{\beta_k(\omega;z)-\beta_k(\mu;z)\}=\\ &=-\frac{\xi}{\xi+\omega}\left[a_k e^{-k\omega}\frac{\mu z}{\mu-\omega}B(e^{-\omega})+b_k e^{-k\omega}\left\{1-\frac{\mu z}{\mu-\omega}A(e^{-\omega})\right\}\right]+\\ &\qquad -\alpha_k(\mu;z)\frac{\mu z}{\mu-\omega}B(e^{-\omega})-\beta_k(\mu;z)\left\{1-\frac{\mu z}{\mu-\omega}A(e^{-\omega})\right\}.\end{aligned}\right. \tag{3.38}$$

Take $\omega=0$ in the first equation of (3.38), then we find with (3.22)

$$(1-z)(1-cz)\,\{\alpha_k(0;z)-\alpha_k(\mu;z)\}=-\,a_k(1-cz)-\alpha_k(\mu;z)\,(1-cz) \tag{3.39}$$

and so by (3.12)

$$\alpha_k(\mu) \overset{\text{def}}{=} \lim_{z\uparrow 1} \alpha_k(\mu;z) \tag{3.40}$$

exists, because

$$\lim_{z\uparrow 1} \alpha_k(\mu;z) = \lim_{z\uparrow 1} -\frac{(1-z)\,\alpha_k(0;z)+a_k}{z} = a_k\check{F}_k(\xi)-a_k. \tag{3.41}$$

Next take $\omega=\omega_1(z)$ in the first equation of (3.38). This yields for $z\uparrow 1$ the existence of

$$\beta_k(\mu) \overset{\text{def}}{=} \lim_{z\uparrow 1} \beta_k(\mu;z). \tag{3.42}$$

Now substitute $\omega=\omega_\nu(z)$ (where ν is either 1 or 2) in the second equation of (3.38). If

$$B_k(\omega;z) \overset{\text{def}}{=} a_k e^{-k\omega}\frac{\mu z}{\mu-\omega}B(e^{-\omega})+b_k e^{-k\omega}\left\{1-\frac{\mu z}{\mu-\omega}A(e^{-\omega})\right\}, \tag{3.43}$$

then the result may be written

$$\left\{\begin{aligned}&\frac{\mu z}{\mu-\omega_\nu(z)}B(e^{-\omega_\nu(z)})\,\alpha_k(\mu;z)+\left\{1-\frac{\mu z}{\mu-\omega_\nu(z)}A(e^{-\omega_\nu(z)})\right\}\beta_k(\mu;z)=\\&\qquad\qquad=-\frac{\xi}{\xi+\omega_\nu(z)}B_k(\omega_\nu(z);z).\end{aligned}\right. \tag{3.44}$$

To prove that the determinant of the system (3.44), i.e.

$$\left\{\begin{aligned}&\Delta(z)\overset{\text{def}}{=}\frac{\mu z}{\mu-\omega_1(z)}B(e^{-\omega_1(z)})\left\{1-\frac{\mu z}{\mu-\omega_2(z)}A(e^{-\omega_2(z)})\right\}+\\&\qquad-\frac{\mu z}{\mu-\omega_2(z)}B(e^{-\omega_2(z)})\left\{1-\frac{\mu z}{\mu-\omega_1(z)}A(e^{-\omega_1(z)})\right\}\end{aligned}\right. \tag{3.45}$$

is not equal to zero for any z with $0\leqslant 1-z<\delta$, we sum in (3.44) over all values of k. The right-hand side then becomes

$$-\frac{\xi}{\xi+\omega_\nu(z)}B(e^{-\omega_\nu(z)}) \tag{3.46}$$

and so for any z with $0\leqslant 1-z<\delta$ the resulting system of equations for $\sum_k \alpha_k(\mu;z)$ and $\sum_k \beta_k(\mu;z)$, which is *not* contradictory, can never consist of two equations which can be obtained one from the other by multiplying with a constant which does *not* depend on ξ. But then

$$\Delta(z)\neq 0 \qquad \text{for } 0\leqslant 1-z<\delta. \tag{3.47}$$

We thus find from (3.44)

$$(3.48)\quad \begin{cases} \Delta(z)\,\alpha_k(\mu;z) = -\dfrac{\xi}{\xi+\omega_1(z)}\,B_k(\omega_1(z);z)\left\{1-\dfrac{\mu z}{\mu-\omega_2(z)}\,A(e^{-\omega_2(z)})\right\}+ \\ \qquad\qquad +\dfrac{\xi}{\xi+\omega_2(z)}\,B_k(\omega_2(z);z)\left\{1-\dfrac{\mu z}{\mu-\omega_1(z)}\,A(e^{-\omega_1(z)})\right\} \\ \Delta(z)\,\beta_k(\mu;z) = +\dfrac{\xi}{\xi+\omega_1(z)}\,B_k(\omega_1(z);z)\,\dfrac{\mu z}{\mu-\omega_2(z)}\,B(e^{-\omega_2(z)})+ \\ \qquad\qquad -\dfrac{\xi}{\xi+\omega_2(z)}\,B_k(\omega_2(z);z)\,\dfrac{\mu z}{\mu-\omega_1(z)}\,B(e^{-\omega_1(z)}) \end{cases}$$

and so we can easily calculate both the $\alpha_k(\mu)$ and the $\beta_k(\mu)$. From (3.21) we obtain on substituting $\omega=0$

$$(3.49)\quad \begin{cases} (1-z)\,\{\alpha_k(0;z)-\alpha_k(\mu;z)\} = -\,a_k-\alpha_k(\mu;z) \\ (1-cz)\,\{\beta_k(0;z)-\beta_k(\mu;z)\} = -\,b_k-\beta_k(\mu;z) \end{cases}$$

and so by (3.20)

$$(3.50)\quad \check{C}(i,0;k,\xi;z) = -\,\delta_{ik}+z\,[\{\alpha_k(0;z)-\alpha_k(\mu;z)\}+\frac{b_i}{a_i}\,\{\beta_k(0;z)-\beta_k(\mu;z\}].$$

Hence by (3.12), (3.50), (3.49) and (3.48), writing ω_1 for $\omega_1(1)$ and ω_2 for $\omega_2(1)$,

$$(3.51)\quad \begin{cases} a_k\check{F}_k(\xi) = -\lim\limits_{z\uparrow 1}\,(1-z)\,\check{C}(i,0;k,\xi;z) = \lim\limits_{z\uparrow 1}\,\{a_k+\alpha_k(\mu;z)\} = \\ = a_k+\alpha_k(\mu) = a_k+\dfrac{1}{\Delta(1)}\left[-\dfrac{\xi}{\xi+\omega_1}\,B_k(\omega_1;1)\left\{1-\dfrac{\mu}{\mu-\omega_2}A(e^{-\omega_2})\right\}+\right. \\ \qquad\qquad \left.+\dfrac{\xi}{\xi+\omega_2}\,B_k(\omega_2;1)\left\{1-\dfrac{\mu}{\mu-\omega_1}\,A(e^{-\omega_1})\right\}\right], \end{cases}$$

from which we further obtain by (3.4) and (3.43)

$$(3.52)\quad \begin{cases} \check{F}(\xi) = 1-\dfrac{\xi}{\xi+\omega_1}\,\dfrac{B(e^{-\omega_1})}{\Delta(1)}\left\{1-\dfrac{\mu}{\mu-\omega_2}\,A(e^{-\omega_2})\right\}+ \\ \qquad\qquad +\dfrac{\xi}{\xi+\omega_2}\,\dfrac{B(e^{-\omega_2})}{\Delta(1)}\left\{1-\dfrac{\mu}{\mu-\omega_1}\,A(e^{-\omega_1})\right\}. \end{cases}$$

The real numbers ω_1, ω_2, $A(e^{-\omega_1})$, $A(e^{-\omega_2})$, $B(e^{-\omega_1})$ and $B(e^{-\omega_2})$, which are the only functions occurring in (3.52), are obtained by solving the equation $D(\omega;1)=0$ for ω_1 and ω_2. We could have shortened the derivation of (3.52), but then the far more useful (3.51) would not have been obtained.

4. *Comparison with independent* $\mathbf{y}_n$

In sections 2 and 3 a number of results have been obtained, which can be compared with well-known theorems, derived for *independent* random variables $\mathbf{y}_n$.

The related results can be obtained in section 2 by *specialization* of the formulae. For independent $\mathbf{y}_n$ we specify the probability distribution by

$$P\{\mathbf{y}_n = i\} = a_i, \tag{4.1}$$

which means that we take $b_i = 0$ for all i. In all formulae from (2.2) onwards this specialization can be made. We thus obtain: a trivial result in (2.17), a renewal theorem due to Feller (and proved by him in the case of infinitely many states, here possible values of $\mathbf{y}_n$) in (2.31), a weaker version of this theorem in (2.35) and another renewal theorem, also due to Feller (and also proved by him for infinitely many states) in (2.43). Quite unexpectedly we find from (2.17) and (2.43)

$$\lim_{n\to\infty} \frac{\operatorname{var} \mathbf{l}_n}{\operatorname{var} \sum_{k=1}^{n} \mathbf{y}_k} = \frac{1}{\mu_1(a)^3}, \tag{4.2}$$

for dependent as well as for independent $\mathbf{y}_n$. It would be very interesting to know whether this result is even more generally true [1]). One can expect the relations proved in section 2 to be true for infinite r, i.e. infinitely many states, still with p_{ij} as specified in (2.2) and (2.3).

The related results for section 3 can also be obtained by specialization, i.e. by substitution of $b_i = 0$ for all i. There is here a slight difficulty: the substitution of $B(e^{-\omega}) = 0$, $C(e^{-\omega}) = 0$ leads to the kind of special difficulties we did not wish to bother with, e.g. in (3.23) the inequality sign must be replaced by an equality sign. It is not hard to obtain the solution in this particular case. We find

$$\check{F}(\xi) = \frac{\mu+\xi}{\mu} \frac{\omega_1}{\omega_1+\xi}, \tag{4.3}$$

where ω_1 is the only root with $\operatorname{Re} \omega_1 > 0$ of

$$\mu - \omega - \mu A(e^{-\omega}) = 0. \tag{4.4}$$

This is also a well-known result and can be found in Pollaczek (1957), page 85. Our derivation for this special case hardly differs from his.

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