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Reprinted from Proceedings, Series A, 65, No. 4 and Indag. Math., 24, No. 4, 1962

MATHEMATICS

A NOTE ON THE SUMMATION OF SOME SERIES OF BESSEL FUNCTIONS

BY

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(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of May 26, 1962)

1. RUTGERS [1] has derived explicit expressions for the sum of series of the kind

$$S_k(r) = \sum_{n=0}^{\infty} (\nu + 2n)^k I_{\nu+2n}(r), \tag{1.1}$$

where k is a non-negative integer. In a recent paper CARLITZ [2] considered these series from a different point of view. Here it will be shown that such series may be summed in a much more simple and elementary way. From Bessel's differential equation it follows at once that

$$\left\{\left(r\frac{\partial}{\partial r}\right)^2 - r^2\right\} I_\mu(r) = \mu^2 I_\mu(r). \tag{1.2}$$

Hence the following recurrence relation exists

$$S_{k+2}(r) = \left\{\left(r\frac{\partial}{\partial r}\right)^2 - r^2\right\} S_k(r). \tag{1.3}$$

Therefore we need only consider the cases $k=0$ and $k=1$.

In the special case $\nu=0$ we may start from the well-known Fourier expansion

$$e^{r\cos\varphi} = \sum_{n=0}^{\infty} \varepsilon_n \cos n\varphi \, I_n(r). \tag{1.4}$$

Expanding both sides in rising powers of φ it follows at once that the sum of the series

$$\sum_{n=0}^{\infty} \varepsilon_n n^{2k} I_n(r) \tag{1.5}$$

is determined by the coefficient of φ^{2k} in the expansion of $\exp(r\cos\varphi) = \exp\{r(1-\frac{1}{2}\varphi^2 + \ldots)\}$ which is obviously of the form $\exp(r\cdot\varphi_k(r))$ where $\varphi_k(r)$ is a polynomial of degree k. A recurrence relation between successive $\varphi_k(r)$ could be obtained by noting that both sides of (1.4) satisfy the Helmholtz equation $(\Delta - 1)f = 0$ where Δ is the Laplacian in polar coordinates (r, φ). However, it is simpler to use (1.4) for $\varphi = 0$ which gives the well-known result

$$\sum_{n=0}^{\infty} \varepsilon_n I_n(r) = e^r, \tag{1.6}$$

and then to apply the recurrence relation (1.3).

This gives

$$\sum_{n=0}^{\infty} \varepsilon_n\, n^{2k}\, I_n(r) = \left\{\left(r\frac{\partial}{\partial r}\right)^2 - r^2\right\}^k \cdot e^r. \tag{1.7}$$

Since

$$\left\{\left(r\frac{\partial}{\partial r}\right)^2 - r^2\right\} \cdot e^r\,\varphi(r) = e^r\left\{r^2\frac{\partial^2}{\partial r^2} + (2r^2+r)\frac{\partial}{\partial r} + r\right\} \cdot \varphi(r), \tag{1.8}$$

we may also write [1])

$$\sum_{n=0}^{\infty} \varepsilon_n\, n^{2k}\, I_n(r) = e^r\left\{r^2\frac{\partial^2}{\partial r^2} + (2r^2+r)\frac{\partial}{\partial r} + r\right\}^k \cdot 1. \tag{1.9}$$

By making the substitution $\varphi = \frac{1}{2}\pi$ in (1.4) we obtain

$$\sum_{n=0}^{\infty} (-1)^n\, \varepsilon_{2n}\, I_{2n}(r) = 1, \tag{1.10}$$

and next [2])

$$\sum_{n=0}^{\infty} (-1)^n\, \varepsilon_{2n}\, (2n)^{2k}\, I_{2n}(r) = \left\{\left(r\frac{\partial}{\partial r}\right)^2 - r^2\right\}^k \cdot 1. \tag{1.11}$$

If (1.4) is differentiated with respect to φ we obtain when substituting $\varphi = \frac{1}{2}\pi$

$$\sum_{n=0}^{\infty} (-1)^n\, \varepsilon_{2n+1}(2n+1)\, I_{2n+1}(r) = r, \tag{1.12}$$

and next [3])

$$\sum_{n=0}^{\infty} (-1)^n\, \varepsilon_{2n+1}\, (2n+1)^{2k+1}\, I_{2n+1}(r) = \left\{\left(r\frac{\partial}{\partial r}\right)^2 - r^2\right\}^k \cdot r. \tag{1.13}$$

If r is replaced by ir we obtain expansions containing Bessel functions of the first kind. E.g. from (1.11) we may derive

$$\sum_{n=0}^{\infty} \varepsilon_{2n}\, (2n)^{2k}\, J_{2n}(r) = \left\{\left(r\frac{\partial}{\partial r}\right)^2 + r^2\right\}^k \cdot 1. \tag{1.14}$$

2. More generally we now consider the series

$$\sum_{n=0}^{\infty} c_n(\nu+n)^k\, I_{\nu+n}(r) \tag{2.1}$$

where again k is a non-negative integer and where the coefficients c_n are given by the power series expansion

$$f(t) = \sum_{n=0}^{\infty} c_n\, t^n. \tag{2.2}$$

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1) Cf. Carlitz l.c. formula (3.4).
2) Ib. formula (6.2).
3) Ib. formula (6.1).

It is sufficient to consider only the case $k=0$. Since

$$\mu I_\mu(r) = r\, I_{\mu-1}(r) - r\, I_\mu'(r) \tag{2.3}$$

the case $k=1$ may be reduced to the previous one. For larger values of k we may use the recurrence relation (1.3). We shall write

$$S(r,\nu,f(t)) = \sum_{n=0}^{\infty} c_n\, I_{\nu+n}(r). \tag{2.4}$$

Using Sommerfeld's integral expression

$$I_\mu(r) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty-\pi\mathrm{i}}^{-\infty+\pi\mathrm{i}} \mathrm{e}^{r \operatorname{ch} w + \mu w}\, dw \tag{2.5}$$

we obtain without difficulty

$$S(r,\nu,f) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty-\pi\mathrm{i}}^{-\infty+\pi\mathrm{i}} \mathrm{e}^{r \operatorname{ch} w + \nu w} f(\mathrm{e}^w)\, dw. \tag{2.6}$$

The right-hand side of (2.6) is obviously reducible to a Bessel function for the following particular choice

$$\mathrm{e}^{\nu w} f(\mathrm{e}^w) = \frac{d}{dw} \frac{\mathrm{e}^{\nu w}}{1-\mathrm{e}^{-2w}}, \tag{2.7}$$

i.e. when

$$f(t) = t^{1-\nu} \left(\frac{t^\nu}{1-t^2} \right)' = \sum_{n=0}^{\infty} (\nu+2n)\, t^{2n}. \tag{2.8}$$

After partial integration it follows that

$$S(r,\nu,f) = \tfrac{1}{2} r\, I_{\nu-1}(r). \tag{2.9}$$

Applying (1.3) we obtain [1])

$$2 \sum_{n=0}^{\infty} (\nu+2n)^{2k+1} I_{\nu+2n}(r) = \left\{ \left(r \frac{\partial}{\partial r} \right)^2 - r^2 \right\}^k \cdot r\, I_\nu(r). \tag{2.10}$$

The same technique enables us to find simple expressions for series such as

$$\sum_{n=0}^{\infty} \varepsilon_n\, I_{\nu+n}(r). \tag{2.11}$$

In this case we have $f(t) = (1+t)/(1-t)$ so that

$$S\left(r,\nu,\frac{1+t}{1-t}\right) = \frac{-1}{2\pi \mathrm{i}} \int_{-\infty-\pi\mathrm{i}}^{-\infty+\pi\mathrm{i}} \mathrm{e}^{r \operatorname{ch} w + \nu w} \operatorname{cth} \tfrac{1}{2} w\, dw. \tag{2.12}$$

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[1]) Ib. formula (5.3).

A simple calculation shows that

$$\left(\frac{\partial}{\partial r} - 1\right) S = \frac{\nu}{r} I_\nu(r) \tag{2.13}$$

so that

$$S = \nu e^r \int_0^r e^{-\varrho} \varrho^{-1} I_\nu(\varrho)\, d\varrho. \tag{2.14}$$

The derivation of similar results of this kind may be left to the reader.

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