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On probability distributions arising from points on a graph

by

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1. Introduction

Given a set of n points, numbered 1,...,n, and a n x n matrix M, with elements m_{ij}, satisfying

(1.1) $m_{ij} = m_{ji}$ $(i \neq j)$,

(1.2) $m_{ii} = 0$,

(1.3) for each i $m_{ij} \neq 0$ for at least one j, and

(1.4) $0 \leq m_{ij} < \infty$.

In the special case that all m_{ij} are integers, the set of points and the matrix M can be interpreted as a finite multigraph (cf. C. BERGE (1958), D. KOENIG (1936)), where the number of joins between point i and j is equal to m_{ij}. If $m_{ij} = 0$, this means in this case that there is no join between i and j. Assumption (1.2) states that there are no loops. Assumption (1.3) implies that no point is isolated.

From the n points two samples are taken. We shall consider two cases.

Case I "non free sampling": from the points 1,.. ,n r_1 and r_2 points are chosen at random without replacement $(r_1+r_2 \leq n)$. The r_1 points will be denoted as black (B) points, the r_2 points as white (W) ones, while finally the $n-r_1-r_2$ remaining points are the red (R) ones.

Case II "free sampling": n independent trials are performed, each trial resulting in the event B with probability p_1, in the event W with probability p_2, and in the event R with probability $1-p_1-p_2$. Point number i is alotted the colour indicated by the outcome of the i-th trial.

Consider the random variables $\underline{x}_{ij}^{(W)}$, $\underline{x}_{ij}^{(B)}$, $\underline{y}_{ij}$ (i,j=1,.. ,n) defined by

$$\underline{x}_{ii}^{(W)} = 0 \qquad \text{spr } 0,$$

$$\underline{x}_{ii}^{(B)} = 0 \qquad \text{spr } 0,$$

$$\underline{y}_{ii} = 0 \qquad \text{spr } 0,$$

and for $i \neq j$

$$\underline{x}_{ij}^{(W)} = \begin{cases} 1 & \text{if point i and j are both white} \\ 0 & \text{if not,} \end{cases}$$

$$\underline{x}_{ij}^{(B)} = \begin{cases} 1 & \text{if point i and j are both black} \\ 0 & \text{if not,} \end{cases}$$

$$\underline{y}_{ij} = \begin{cases} 1 & \text{if one of the points i and j is black and the other is white} \\ 0 & \text{if not.} \end{cases}$$

Obviously

$$\underline{x}_{ij}^{(W)} = \underline{x}_{ji}^{(W)},$$

$$\underline{x}_{ij}^{(B)} = \underline{x}_{ji}^{(B)},$$

and $\underline{y}_{ij} = \underline{y}_{ji}$.

Define

$$\underline{x}_W = \sum_{i,j} m_{ij} \underline{x}_{ij}^{(W)},$$

(1.5) $$\underline{x}_B = \sum_{i,j} m_{ij} \underline{x}_{ij}^{(B)},$$

$$\underline{y} = \sum_{ij} m_{ij} \underline{y}_{ij},$$

We also introduce a set of random variables $\underline{z}_{ij}$, taking the values 0 or 1, ($\underline{z}_{ii}$ = 0 spr 0), and a random variable

$$\underline{z} = \sum_{ij} m_{ij} \underline{z}_{ij} .$$

We attribute to $\underline{z}_{ij}$ and to $\underline{z}$ all properties that $\underline{x}_{ij}^{(B)}$, $\underline{x}_{ij}^{(W)}$, $\underline{y}_{ij}$ and $\underline{x}_W$, $\underline{x}_B$ and $\underline{y}$ have in common. Thus $\underline{z}$ is a generalization of $\underline{x}_W$, $\underline{x}_B$ and $\underline{y}$.

In the following we shall give results on the stochastic properties of $\underline{x}_W$, $\underline{x}_B$, $\underline{y}$ and $\underline{z}$. The proofs will be given in a forthcoming thesis.

2. Previous work on the subject

P.A.P. MORAN (1948) considers a "statistical map", equivalent to our graph for $m_{ij} = 0$ or 1, where the points are chosen by "free" and "non free" sampling. He gives for both cases the first and second moments of the number of black-black joins (thus for x_B) and the third and fourth moment for the case of free sampling. He proves the asymptotic normality of x_B and y (free sampling) for a rectangular twodimensional lattice, where there are joins between neighbouring points in the direction of both axis (cf. also P.A.P. MORAN (1947)).

There exists a large number of papers on the subject by P.V. KRISHNA IYER (1948-1953). He only deals with rectangular lattices, where neighbouring points are joined in the direction of both axis, but also diagonal joins are considered in a number of his papers. The results of KRISHNA IYER are mostly on the first four moments or cumulants, and statements about asymptotic normality.

A report by VAN EEDEN and BLOEMENA (1959) contains a number of exact results for rectangular lattices (non-free samples). The present report is an outgrowth of the last mentioned paper, which arose from a study of the distribution of a statistic, obtained in a psychological test.

Some older papers on the subject are by H. TODD (1940) and D.J. FINNEY (1947).

3. Some graphtheoretical notions

Consider a set S of points and a subset U of the set of all possible joins between these points. The combination (S,U) is usually called a graph. For a detailed treatment of theory of graphs, we refer to D. KOENIG (1936) and C. BERGE (1958).

In this report the word "graph" is used in two different ways. In section 1 it has been shown that in the case that m_{ij} are non negative integers, the set of points from which the samples are taken, and the matrix M can be interpreted as a graph. In the following we use the word "graph" in a different situation.

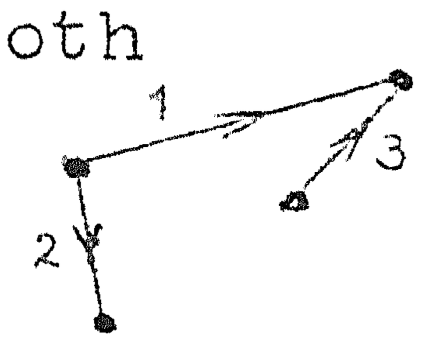
For the purposes of the following sections we use the word "graph" to denote a set of k oriented joins, labelled $J_1, \ldots, J_k$, between ℓ $(2 \leq \ell \leq 2k)$ points, such that no points are isolated (are not connected to at least one other point), and loops do not occur. Multiple joins are admitted.

A point to which join J_i is connected will be called the second point of J_i if the orientation of the join is towards the point; if not, it will be called the first point of J_i.

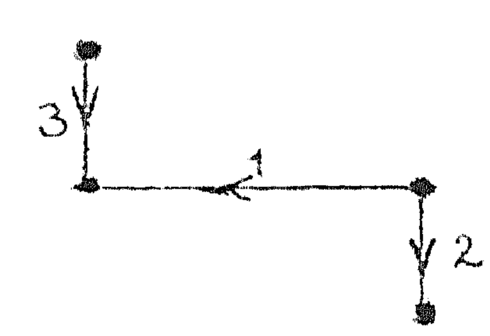
To each graph there corresponds a symmetrical 2k x 2k matrix A, consisting of k^2 2 x 2 block-matrices A_{ij} $(i,j = 1,\ldots,k)$, with elements 0 if i=j and for $i \neq j$ and $\mu, \lambda = 1,2$:

$$a_{i\mu,j\lambda} = \begin{cases} 1 \text{ if the } \mu\text{-th point of } J_i \text{ coincides with} \\ \quad \text{the } \lambda\text{-th point of } J_j. \\ 0 \text{ if not.} \end{cases}$$

All graphs having the same matrix A are considered as equivalent. E.g. both



and



have as matrix

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

and are therefore equivalent.

Two graphs that are not equivalent will be called distinct.

The k x k matrix with elements

$$b_{ij} = \sum_{\mu=0}^{1} \sum_{\lambda=0}^{1} a_{i\mu,j\lambda}$$

will be called the <u>configurationmatrix</u>. Here b_{ij} $(i \neq j)$ indicates the number of common endpoints of join J_i and J_j, thus b_{ij} is either 0, 1 or 2. For i=j $b_{ij}=0$. The orientation of the joins does not influence the configuration of the graph.

Consider two graphs G_1 and G_2, each based on ℓ points and k (labelled and oriented) joins. If G_1 and G_2 are not equivalent, but a permutationmatrix P exists such that for the configuration-matrices B_1 and B_2 the relation

$$B_1 = P\, B_2\, P'$$

holds, we shall say that G_1 and G_2 have the same configuration. If two graphs G_1 and G_2 have the same configuration, this means that permuting the numbers of the joins and/or reversing the orientation of some joins of configuration G_1 can make G_1 equivalent to G_2.

A graph $G = (S,U)$ is called connected if from every point $i \in S$ one can reach any other point of S by travelling along the joins of the set U, neglecting the orientation of the joins. A graph which is not connected, can be decomposed in a number of connected components. This decomposition is unique (cf. D. KONIG, 1936, p.15). A configuration-matrix of a not connected graph (if necessary after premultiplication with a permutationmatrix P, and postmultiplication with P`) is a logical sum of the configuration-matrices of each of the connected components.

A connected graph with k joins has at most (k+1) points. It has at least two points. For ℓ satisfying

$$z \leq \ell \leq k + 1,$$

there exist finitely many, say $q_{k,\ell}$, different configurations corresponding to connected graphs, based on k joins and ℓ points. Let $C_{k,\ell}^{(\alpha)}$ be the α-th one ($\alpha = 1,\ldots,q_{k,\ell}$). The configuration of a graph having h connected components ($1 \leq h \leq [\frac{\ell}{2}]$) can now be indicated symbolically by

$$\sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)},$$

if the i-th connected component has a configuration $C_{k_i,\ell_i}^{(\alpha_i)}$. If among the h conncected components g_j have the same configuration $C_{k_j,\ell_j}^{(\alpha_j)}$, we may also write $\sum_{j=1}^{s} g_j\, C_{k_j,\ell_j}^{(\alpha_j)}$ as the symbol of the configuration of the graph.

By means of the operator N(...), operating on the symbol of a configuration, we indicate the number of distinct graphs, having this configurations. It can be proved that

$$(3.1)\quad N\Big(\sum_{j=1}^{s} g_j\, C_{k_j,\ell_j}^{(\alpha_j)}\Big) = k!\,\prod_{1}^{s} \frac{1}{g_j!}\left\{\frac{N(C_{k_j,\ell_j}^{(\alpha_j)})}{k_j!}\right\}^{g_j}$$

The calculation of $N(C_{k_j,\ell_j}^{(\alpha_j)})$ proceeds by means of recurrence relations.

4. A general expression for the moments

In order to calculate the k-th moment of $\underline{z}$, we have to consider products

$$(4.1)\qquad \underline{z}_{\nu_{1,1},\nu_{1,2}}\,\underline{z}_{\nu_{2,1},\nu_{2,2}}\cdots\underline{z}_{\nu_{k,1},\nu_{k,2}}\ ,$$

where $\nu_{1,1},\ldots,\nu_{k,2}$ are integers from the range $1,\ldots,n$. If for some $j=1,\ldots,k$

$$\nu_{j,1} = \nu_{j,2}$$

(4.1) is zero a.c. by assumption . We shall therefore only consider the case

$$(4.2)\qquad \nu_{j,1} \neq \nu_{j,2}$$

for all $j = 1,\ldots,k$.

Assume that among $\nu_{1,1},\ldots,\nu_{k,2}$, say ℓ different integers occur $(2\leq \ell \leq 2k)$. We shall denote these by $\lambda_1,\ldots,\lambda_\ell$.

To each product (4.1) there corresponds a graph: let each subscript of (4.1) corresponds to a point. If two subscripts are equal, they correspond to the same point, thus there are ℓ points in all. Let the first subscript $\nu_{j,1}$, of $\underline{z}_{\nu_{j,1},\nu_{j,2}}$ $(j=1,\ldots,k)$ correspond to the first point of a join, and the second subscript to the second point. Now we have obtained a graph with k oriented joins and ℓ points. Loops do not occur because of (4.2). Multiple joins between points correspond to powers of z. Let the graph corresponding to (4.1) have the configuration $\sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)}$, then in the following sections we show that for $\underline{z}_{ij}$ for $i \neq j$ holds:

For each $k = 1,\ldots,$ the expectation of (4.1) does not depend on the actual values of $\nu_{1,1},\ldots,\nu_{k,2}$, but only on the configuration $\sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)}$.

We can therefore use the following notation for the expectation of (4.1):

$$E(\underline{z}^{(1)}\ldots\underline{z}^{(k)}\,|\,\sum_{i=1}^{h} c_{k_i,\ell_i}^{(\alpha_i)}).$$

In calculating the k-th moment of $\underline{z}$, we also need ℓ-fold sums of the type

$$(4.3)\qquad \underbrace{\sum_{\theta_1=1}^{n}\ldots\sum_{\theta_\ell=1}^{n}}_{D^*} m_{\theta_{\mu_{1,1}},\theta_{\mu_{2,2}}}\; m_{\theta_{\mu_{2,1}},\theta_{\mu_{2,2}}}\ldots m_{\theta_{\mu_{k,1}},\theta_{\mu_{k,2}}},$$

where among the subscripts of the θ's in the general term of the sum all integers $1,\ldots,\ell$ occur. The summation is restricted by a condition D^*, yet to be specified.

If for some $j = 1,\ldots,k$

$$\mu_{j,1} = \mu_{j,2},$$

the sum is equal to zero by (1.2). We therefore consider only the case

$$(4.4)\qquad \mu_{j,1} \neq \mu_{j,2},$$

for $j = 1,\ldots,k$.

In the same way used to construct a graph corresponding to the subscripts in (4.1), we now construct a graph corresponding to the subscripts of the θ's. Let each subscript $\mu_{1,1},\ldots\mu_{k,2}$ correspond to a point. If two are equal they correspond to the same point, thus there are ℓ points in all.

Let the subscript $\mu_{j,1}$ of $m_{\theta_{\mu_{j,1}},\theta_{\mu_{j,2}}}$ $(j = 1,\ldots,k)$ correspond to the first point of a join, and the subscript $\mu_{j,2}$ to the second point. Now we have obtained a graph with k oriented joins and ℓ points. Loops do not occur because of (4.4). Let the graph have the configuration $\sum_{i=1}^{h} c_{k_i,\ell_i}^{(\alpha_i)}$, then the following notation for (4.3) will be used:

1) if there is no restriction D^*:

$$\sum\{m^{(1)}\ldots m^{(k)}\,|\,\sum_{i=1}^{h} c_{k_i,\ell_i}^{(\alpha_i)}\},$$

2) if restriction D^* is that in the summation indices $\theta_1,\dots,\theta_\ell$ are all unequal:

$$\sum{}^{*}\left\{ m^{(1)} \dots m^{(k)} \middle| \sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)} \right\} .$$

Now one can write

$$(4.5) \qquad E\underline{z}^k = \sum_{\ell=2}^{2k} \sum_{h=1}^{[\frac{\ell}{2}]} \sum{}''_{\lfloor \sum k_i = k,\ \sum \ell_i = \ell \rfloor} N\left(\sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)}\right)$$

$$E\left(\underline{z}^{(1)} \dots \underline{z}^{(k)} \middle| \sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)}\right) \sum{}^{*}\left\{ m^{(1)} \dots m^{(k)} \middle| \sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)} \right\} ,$$

where $\sum{}''_{\lfloor \sum_{i=1}^{h} k_i = k,\ \sum_{i=1}^{h} \ell_i = \ell \rfloor}$ means a summation over all configurations satisfying the indicated restrictions $\sum_{i=1}^{k} k_i = k$, and $\sum_{i=1}^{h} \ell_i = \ell$.

We define: $m_{i+} \overset{\text{def}}{=} \sum_j m_{ij}$

and $m_{++} \overset{\text{def}}{=} \sum_i m_{i+}$.

5. The moments of $\underline{x}_W$ and $\underline{x}_B$

The moments of $\underline{x}$ can be found by replacing r_1 or p_1 in the corresponding formulae for $\underline{x}_B$ by r_2 or p_2 respectively.

a) non-free sampling

$$(5.1) \qquad E\left(\underline{x}_B^{(1)} \dots \underline{x}_B^{(k)} \middle| \sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)}\right) = \frac{\binom{r_1}{\ell}}{\binom{n}{\ell}} ,$$

where $\sum_{i=1}^{h} \ell_i = \ell$.

From (4.5) we find after some simplifications

$$E\,\underline{x}_B = \frac{r_1(r_1-1)}{n(n-1)}\, m_{++} ,$$

$$\sigma^2 = E\,\underline{x}_B^2 - (E\,\underline{x}_B)^2 = 4\,\frac{r_1(r_1-1)(r_1-2)(n-r_1)}{n(n-1)(n-2)(n-3)}\sum_i (m_{i+} - \frac{1}{n}m_{++})^2$$

$$+\frac{2r_1(r_1-1)(n-r_1)(n-r_1-1)}{n^2(n-1)^2(n-2)(n-3)}\left\{n(n-1)\sum_{ij} m_{ij}{}^2 - m_{++}^2\right\}.$$

If m_{i+} does not depend on i, the first term of σ^2 is equal to zero.

b) free sampling

$$(5.2)\qquad E(\underline{x}_B^{(1)}\ldots\underline{x}_B^{(k)} \mid \sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)}) = p^{\ell},$$

so

$$E\,\underline{x}_B = p_1^2 \sum_{ij} m_{ij},$$

$$\sigma^2 = 2p_1^2(1-p_1)^2 \sum_{ij} m_{ij}^2 + 4p^3(1-p)\sum_i m_{i+}^2$$

6. The moments of y

In order to calculate

$$E\{\underline{y}^{(1)}\ldots\underline{y}^{(k)} \mid \sum_{i=1}^{h} C_{k_i,\ell_i}^{(\alpha_i)}\}$$

we first take a point P_i of the i-th connected component (i=1,.. ,h) as a reference point. Colour P_i white, next all points connected by a join to P_i are coloured black, then all points connected to these black points are coloured white. If in repeating this procedure one arrives at a point which has already been given one colour, but should be coloured by the just-mentioned rule in the other colour as well, then we conclude that the i-th connected component is not bichromatic.
If no such situation arises one arrives at a stage, where all points have been allotted a colour, viz τ_i points are white and $\ell_i - \tau_i$ black; we then say that the i-th connected component is bichromatic. The fact wether a graph is bichromatic or not does

not depend on the choice of the initial point.

Define

$$(6.1)\quad B\Big(\sum_{i=1}^{h} c_{k_i,\ell_i}^{(\alpha_i)}\Big) = \begin{cases} 1 & \text{if all connected components of } \sum_{i=1}^{h} c_{k_i,\ell_i}^{(\alpha_i)} \text{ are bichromatic} \\ 0 & \text{if not.} \end{cases}$$

a) <u>non free sampling</u>

$$(6.2)\quad E\Big\{\underline{y}^{(1)}\ldots\underline{y}^{(k)} \,\Big|\, \sum_{i=1}^{h} c_{k_i,\ell_i}^{\alpha_i}\Big\} = B\Big(\sum_{i=1}^{h} c_{k_i,\ell_i}^{(\alpha_i)}\Big)\,\frac{(n-\ell)!}{n!}\sum_{i=1}^{h}\sum_{\rho_i=0}^{1}$$

$$\frac{r_1!}{(r_1-\sum_i(1-\rho_i)\,\tau_i-\sum_i\rho_i(\ell_i-\tau_i))!}\;\frac{r_2!}{(r_2-\sum_i\rho_i\tau_i-\sum_i(\ell_i-\tau_i)(1-\rho_i))!}.$$

So

$$E\,\underline{y} = \frac{2r_1r_2}{n(n-1)}\sum_{ij} m_{ij},$$

$$\sigma^2 = \sum_i\Big(\sum_j m_{ij} - \frac{\sum_{ij} m_{ij}}{n}\Big)^2\Big\{\frac{r_1r_2(r_2-1)}{n(n-1)(n-2)} + \frac{r_2r_1(r_1-1)}{n(n-1)(n-2)} - \frac{4r_1r_2(r_1-1)(r_2-1)}{n(n-1)(n-2)(n-3)}\Big\}$$

$$+ 4\sum_{ij} m_{ij}^2\Big\{\frac{r_1r_2}{n(n-1)} - \frac{r_1r_2(r_2-1)}{n(n-1)(n-2)} - \frac{r_2r_1(r_1-1)}{n(n-1)(n-2)} + \frac{2r_1r_2(r_1-1)(r_2-1)}{n(n-1)(n-2)(n-3)}\Big\}$$

$$+ 4\Big(\sum_{ij} m_{ij}\Big)^2\,\frac{r_1r_2n(r_1+r_2+3)-r_1r_2(2r_1r_2+r_1+r_2+2)}{n^2(n-1)^2(n-2)(n-3)}.$$

b) <u>free sampling</u>

$$(6.3)\quad E\Big(\underline{y}^{(1)}\ldots\underline{y}^{(k)} \,\Big|\, \sum_{i=1}^{h} c_{k_i,\ell_i}^{(\alpha_i)}\Big) =$$

$$= B\Big(\sum_{i=1}^{h} c_{k_i,\ell_i}^{(\alpha_i)}\Big)\prod_{1}^{h}\big(p_1^{\tau_i}\,p_2^{\ell_i-\tau_i} + p_1^{\ell_i-\tau_i}\,p_2^{\tau_i}\big),$$

thus

$$E\,\underline{y} = 2p_1p_2\sum_{ij} m_{ij},$$

$$\sigma^2 = 4p_1p_2(p_1+p_2-4p_1p_2)\sum_{ijk} m_{ij}m_{ik} + 4p_1p_2(1-p_1-p_2+2p_1p_2)\sum_{ij} m_{ij}^2.$$

7. Tendency towards the normal distribution

The following theorems can be proved.

Theorem 7.1

i If r_1 and n tend to infinity such that

$$\lim \frac{r_1}{n} = \delta_1, \qquad 0 < \delta_1 < 1,$$

and if for all i

$$m_{i+} < C,$$

where C is a constant independent of i and n, then in the non-free sampling case the distribution of

$$\{\underline{x}_B - E\,\underline{x}_B\}\ \sigma(\underline{x}_B)^{-1}$$

tends to the standard normal one. $E\,\underline{x}_B$ and $\sigma(\underline{x}_B)^2$ have been given in section 5.

ii The same result holds for $\underline{x}_w$ if r_1 is replaced by r_2.

iii If the assumptions of part i of this theorem are satisfied and if moreover

$$\lim \frac{r_2}{n} = \delta_2, \qquad 0 < \delta_2 \leq 1 - \delta,$$

then in the non-free sampling case the distribution of

$$(\underline{y} - E\underline{y})\ \sigma(\underline{y})^{-1}$$

tends to the standard normal one. $E\underline{y}$ and $\sigma^2(y)$ have been given in section 6.

Theorem 7.2

i If n tends to infinity and p_1 to a limit $p_1^*(0 < p_1^* < 1)$, and if for all i

$$m_{i+} < C,$$

where C is a constant independent of i and n, then in the free sampling case the distribution of

$$\{\underline{x}_B - E\,\underline{x}_B\}\,\sigma(\underline{x}_B)^{-1}$$

tends to the standard normal one. $E\,\underline{x}_B$ and $\sigma^2(\underline{x}_B)$ have been given in section 5.

ii The same result holds for $\underline{x}_W$ if p_1 is replaced by p_2.

iii If the assumption of part i holds, and if also p_2 tends to a limit $p_2^*(0 < p_2^* < 1-p_1^*)$ then in the free sampling case the distribution of

$$(\underline{y} - E\underline{y})\, \sigma(\underline{y})^{-1}$$

tends to the standard normal one. $E\underline{y}$ and $\sigma^*(\underline{y})$ have been given in section 6.

8. Tendency towards the compound Poisson-distribution

Theorem 8.1

i If r_1 and n tend to infinity such that

$$\lim \frac{r_1}{n} = 0,$$

$$\lim \frac{r_1^2}{n^2} m_{++} = 2\lambda, \qquad 0 < \lambda < \infty,$$

and for all α and $k \geq 2$,

$$(8.1) \qquad \lim \left(\frac{r_1}{n}\right)^{k+1} \Sigma \left\{ m^{(1)} \ldots m^{(k)} \mid C_{k,k+1}^{(\alpha)} \right\} = 0,$$

and if for all i and j

$$m_{ij} < C_1,$$

where C_1 does not depend on i, j and n, then the distribution of $\frac{1}{2}\underline{x}_B$ tends to a compound Poisson distribution with moment generating function

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} \lim E(\tfrac{1}{2}\underline{x}_B)^k = \exp\left\{\lambda \sum_{i=1}^{\infty} m_i^* \frac{z^i}{i!}\right\},$$

where

$$m_h^* = \lim \frac{\sum_{i,j} m_{ij}^h}{m_{++}}, \qquad h = 1,2,\ldots$$

Assumption (13.1) is satisfied if for all i $m_{i+} \leq C_2$, where C_2 does not depend on i and

$$\lim \frac{r_1}{n} C_2 = 0,$$

thus e.g. when C_2 is a constant not dependent on n.

ii The same result holds it $\underline{x}_B$ and r_1 is replaced by $\underline{x}_W$ and r_2 respectively.

iii If in part ii and iii $\frac{r_1}{n}$ is replaced by p_1, and $\frac{r_2}{n}$ by p_2, then the corresponding results for the free sampling case is obtained.

Theorem 13.2

i If r_1, r_2 and n tend to infinity, such that

$$\lim \frac{r_1}{n} = 0,$$

$$\lim \frac{r_2}{n} = 0,$$

$$\lim \frac{r_1 r_2}{n^2} m_{++} = \lambda , \qquad 0 < \lambda < \infty,$$

and for all α and $k = 2, \ldots .$

$$(13.2) \qquad \lim \frac{r_1^j r_2^{k-j+1}}{n^{k+1}} \sum \{ m^{(1)} \ldots m^{(k)} | C^{(\alpha)}_{k,k+1} \} = 0, \quad j=1, \ldots, k,$$

and if for all i and j

$$m_{ij} < C_1,$$

where C_1 does not depend on i, j and n, then in the non-free sampling case the distribution of $\frac{1}{2}\underline{y}$ tends to a compound Poisson-distribution with

$$\sum_{k=0}^{\infty} \frac{Z^k}{k!} \lim E(\tfrac{1}{2}\underline{y})^k = \exp\left\{ \lambda \sum_{i=1}^{\infty} \frac{m_i^* Z^i}{i!} \right\},$$

with

$$m_h^* = \lim \frac{\sum_{ij} m_{ij}^h}{m_{++}} , \qquad h = 1,2 \ldots .$$

Assumption (13.2) is satisfied if for all i $m_{i+} \leq C_2$, where C_2 does not depend on i and

$$\lim \frac{r_1}{n} C_2 = 0$$

$$\lim \frac{r_2}{n} C_2 = 0,$$

thus e.g. when C_2 is a constant not dependent on n.

ii If in part i $\frac{r_1}{n}$ is replaced by p_1, and $\frac{r_2}{n}$ by p_2, the corresponding result for the free sampling case is obtained.

9. References

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