Seminar notes on compactification and dimension in metric spaces

Collected by J. de Groot

We are investigating the following two related questions. (1) What are internal necessary and sufficient conditions on the separable metric space M such that M can be compactified by adjoining an n-dimensional set? (2) Can one obtain a fruitful generalization of dimension by replacing the empty set by some other class of spaces in the definition? Here we give a summary of results to date. In most cases, details have been omitted. The intention is to outline what has been done, and to point the way to further problems in this area.

ALL SPACES CONSIDERED ARE SEPARABLE AND METRIZABLE

I. DEFINITIONS. Much of our work is based on the notion of "n-compactness", which we define here.

1.1. Definition. A compact space is called -1-compact. M is said to be $\leq$ n-compact (notation: cmp M $\leq$ n) if every point in M has arbitrarily small neighbourhoods whose boundaries are $\leq$ n-1-compact.

1.2. Definition. Let p be a point in the space M. Then the compactness of M at p (notation: cmp_p M) is the smallest non-negative integer n such that p has arbitrarily small neighbourhoods whose boundaries are $\leq$ n-1-compact.

II. MAIN CONJECTURE. Much of our work has been concerned with the attemt to prove or disprove the following conjecture, or some modification of it (see VII).

2.1. Conjecture. If cmp M = n, then M can be compactified by an n-dimensional set. (The converse is true - see 3.2.)

III. PROPERTIES OF n-COMPACTNESS, WITH EXAMPLES.

3.1. Proposition. For every M, cmp M $\leq$ dim M. (Hence, e.g., cmp(M $\times$ N) $\leq$ dim M + dim N)

3.2. Theorem. Let M be compact, and let A $\subset$ M, with dim A $\leq$ n. Then cmp(M $\setminus$ A) $\leq$ n.

3.3. Proposition. A closed subspace of an $\leq$ n-compact space is $\leq$ n-compact.

An open subspace of an n-compact space (n $\geq$ 0) is $\leq$ n-compact.

3.4. Proposition. If cmp M=n, then M has a countable base consisting of open sets whose boundaries are $\leq$ n-1-compact.

3.5. Theorem. If $M' \subset M$, and $\mathrm{cmp}_p\ M' \leq n$, then p has arbitrarily small neighbourhoods $U \subset M$ such that $\mathrm{cmp}\ [(\mathrm{bdry}_M(U) \cap M')] \leq n-1$.

3.6. Theorem. If cmp M=n, then for each $-1 \leq k \leq n$, there is a closed subset F_k of M such that cmp $F_k = k$.

3.7. Example. Let M_n be I^{n+1} with an open face removed. We have shown that cmp $M_1=1$, cmp $M_2=2$. It is conjectured that cmp $M_n=n$. See 5.3.

We proceed in 3.8 - 3.13 to establish the existence of n-compact spaces for all n.

3.8. Definition. A space M is said to be <u>totally imperfect</u> if M contains no uncountable compact subset.

3.9. Theorem. Let $S^n = \{x \text{ in } E^{n-1} : \|x\| = 1\}$. Then S^n can be decomposed into two disjoint totally imperfect sets, which may be constructed so that each is the reflection in the origin of the other.

3.10. Theorem. If the compact, n-dimensional space M is a union of two disjoint totally imperfect sets, M_1, M_2, then cmp M_1 is n or n-1.

3.10a Een puntforme n-dim. ruimte heeft cmp ≥ n-1

3.11. Theorem. For each k such that $-1 \leq k \leq n-1$, E^n contains a set F_k such that cmp $F_k = k$. E^n contains no sets of higher cmp.

3.12. Theorem. If M is extremely disconnected (i.e., if every quasi-component of M is a point) and dim $M \geq 1$, then dim M = cmp M. (See 5.1(c)).

3.13. Remark. Since there are extremely disconnected spaces of each dimension, 3.12 tells us that for each n, there is a space M such that dim M = cmp M = n.

3.14. Problem. If "point" in 1.1 is replaced by "compact set", the class of 0-compact spaces remains the same. Is this true for n > 0? (See also VII)

3.15. Remark. If the answer to 3.14 is "no", then Conjecture 2.1 is false. Th 3.2 geldt ook als "point" in def 1.1 vervangen wordt door „compact set". Zie ook 8.7.2

IV. 0-COMPACT SPACES. We gather together here some results concerning spaces M for which cmp M=0. This is the first (and best explored) special case of our theory.

4.1. Definition. A 0-compact space is often called <u>rim-compact</u> (or semicompact).

4.2. Example. Every locally compact space is rim-compact, but the converse is false, as the following example shows. Remove the sequence $\{1/n\}$ from the closed unit interval. The resulting space is clearly not locally compact, but it is rim-compact (cf 3.2).

4.3. Proposition. Every 0-dimensional space is rim-compact.

4.4. Theorem. If M is rim-compact, then M can be compactified by a 0-dimensional set. (See 2.1.)

4.5. Theorem. If cmp M = 0, then any two disjoint closed subsets in M can be separated by a closed locally compact set. (See also VII)

4.6. Theorem. Let $M = \bigcup_i M_i$, where, for each i, M_i is closed in M, cmp $M_i \leq 0$, and $M_i \cap M_j$ $(i \neq j)$ is locally compact. Suppose further that the collection $\{M_i\}$ is locally finite. Then cmp $M \leq 0$. (See also VI.)

V. STATUS OF CONJECTURE 2.1.

5.1. Conjecture 2.1 has been proved correct in the following cases.

(a) n = 0 (4.4)

(b) dim M = cmp M

(c) dim M ≤ 1 (4.4 and (b)) (e) M een deelruimte v d E^n met cmp M ≤ n-1

(d) M is a subset of the plane.

5.2. Example. The open ball, with an equator of rational points added, has cmp 1, and it can be compactified by a 1-dimensional set. Such a compactification is not easy to find, however.

5.3. Remark. Each space M_n (3.7) can clearly be compactified by a set of dimension n. Hence cmp $M_n \leq n$. As a matter of fact, M_n cannot be compactified by a set of dimension less than n. Thus if cmp $M_n \neq n$, conjecture 2.1 is false.

VI. SUM AND DECOMPOSITION THEOREMS.

Here the analogy with dimension theory is poor. The conclusion we might draw from the results of this section is that almost any reasonable conjecture in this area is false.

6.1. Example. Adding a single point can raise or lower the cmp of a space by 1. (Consider 3.7, and locally compact, non-compact spaces.)

6.2. Proposition. Adding a point cannot raise the cmp of a space by more than one. The only case adding a point can lower cmp is that which occurs when a locally compact space is compactified by adding a point.

6.3. Example. Adding a compact set can increase the cmp of a space by any amount. (3.7.)

6.4. Example. A, B closed in $M = A \cup B$, cmp A = cmp B = 0, while cmp M = 1.

6.5. Example. A, B closed in $M = A \cup B$, cmp A = cmp B = 1, cmp $A \cap B$ =0, and still cmp M = 2.

6.6. Theorem. If A and B are closed in $M = A \cup B$, then cmp M $\leq$ cmp A + + cmp B + 1.

6.7. Corollary. If $A = \bigcup_{i=1}^{n} A_i$, where for each i, A_i is closed in A, and cmp A_i = 0, then cmp A $\leq$ n. (See also 4.6.)

6.8. Example. It is not true that every space of cmp n can be decomposed into n+1 closed subsets having cmp 0.

6.9. Example. Not every space of cmp n can be written as a union of a space of cmp $\leq$ 0 and a space of dim $\leq$ n, even if these latter spaces are not required to be closed in their union.

VII. SEPARATION AND Cmp.

It appears to be difficult to prove separation properties for cmp, even in separable metric spaces. However, it seems likely that such properties will be closely bound up with the problem of compactifying a space by a set of small dimension.

We shall proceed as follows. We define a "separation cmp", called Cmp, and investigate its properties independently. Whether or not cmp M = Cmp M for all M, we can conjecture that 2.1 is true if cmp is replaced by Cmp.

7.1. Definition. We define, by induction, the expression Cmp M $\leq$ n. Cmp M = 0 if and only if cmp M = 0. For n $\geq$ 1, Cmp M $\leq$ n if and only if every closed set has arbitrarily small neighbourhoods whose boundaries have Cmp $\leq$ n-1.

7.2. Proposition. If M can be compactified by a k-dimensional set, then Cmp M $\leq$ k.

7.3. Problem. Is it true that for each M, Cmp M = cmp M? (Note that this equality is a consequence of 2.1.)

7.4. Proposition. If M is a subset of the plane, then Cmp M = cmp M. (5.1(d).)

7.5. Remark. It seems to be worthwile to look for a subspace M of E^3 such that Cmp M =2, while cmp M = 1.

7.6. Remark. Theorem 4.6 explains our starting point in defining Cmp.

7.7. Exercise. We must investigate the extent to which our known properties of cmp (for example, those in III) carry over to Cmp. In most cases, this appears to be only a routine check.

The numbering of these results follows the organization of the previously issued notes.

5.4. Proposition. If C_1, C_2 are two countable, dense in themselves sets on the surface of the open sphere O, then $M_1 = O \cup C_1$ and $M_2 = O \cup C_2$ are homeomorphic.

5.5. Example. Let O be the open ball, D an open "disc" on its surface, and B a countable dense set on the boundary of D.
Let $M = O \cup D \cup B$. Then M has cmp 1, and M can be compactified by a one-dimensional set.

5.6. Example. (Essentially due to M. E. Rudin.) Let O be the open ball, I an arc on the surface of O, and C a countable, dense in itself set on O which "spirals down" to I. Then $M = O \cup I \cup C$ has cmp 1, though $O \cup I$ has cmp 2. Using 5.4 a one-dimensional compactification of M can be constructed.

There is some hope that an answer to the main conjecture may be found by "dualizing" certain dimension theoretic results. For example, the decomposition theorem of dimension theory can be replaced by the following structure problem.

6.10. Structure Problem. A space has cmp $\leq n$ $(n \geq 1)$ if it is the intersection of at most n + 1 spaces of cmp 0, whose union is compact.

Observe that the decomposition theorem, together with a positive answer to the main conjecture gives a positive solution to this structure problem.

6.11. Definition. Suppose $A \subset B \subset C$. B is said to be closed in C modulo A if $B \setminus A$ is closed in $C \setminus A$.

The counterpart of the sum theorem of dimension might be expressed in the following interesting problem.

6.12. Problem: If 1^o) $M = \bigcap_i M_i$ 2^o) $\bigcup_i M_i = \overline{M}$ (compact), 3^o) each M_i is closed in $\overline{M}$ modulo M, and 4^o) for each i, cmp $M_i \leq n$, Then cmp $M \leq n$.

So far, problem 6.12. is known to have an affirmative solution only in the special case of two sets $M_1 \cap M_2 = M$ where cmp M_1 = cmp $M_2 = 0$. Thus we have here an unsolved conjecture concerning cmp 0.

4.7. Example. If $\dim M = n$ and $\text{cmp } M = 0$, there does not always exist a compactification $\overline{M}$ of M such that $\dim \overline{M} = n$ and $\dim (\overline{M} \setminus M) = 0$.

Remark. Objection has been raised concerning the appropriateness of the terms n - compact, and cmp. It would perhaps be more accurate to use a term such as "deficiency" or "defect" of compactness, and write $\text{def } M$ in place of $\text{cmp } M$.
