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NOTE

ON A PROBLEM IN THE COLLECTIVE BEHAVIOR OF AUTOMATA*

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Varshavsky defines the function $L(n)$ as the maximum finite length of a configuration which can be grown from one activated automaton in a linear cell space of identical finite state automata having n internal states. It is shown that L increases faster than any computable function, even if the flow of information in the linear cell space is restricted to one direction.

1. Introduction

Varshavsky [4] investigated the function $L : \mathbf{N} \to \mathbf{N}$ informally defined by: $L(n)$ is the maximum length of a configuration which can be grown from one activated automaton in a linear cell space of identical finite state automata having n internal states. Each automaton in the linear cell space receives input from both neighbors. In [4] it is shown that $L(3) = 7$, $L(4) \geqslant 45$, and a very fastly increasing computable function is derived which is a lower bound on L. Here we observe that, even for a restricted version of Varshavsky's problem where each automaton recieves input from its left neighbor only, there is no computable function which is an upper bound on L, that is, L increases faster than any computable function.

Define a *one directional linear cell space* (1 LCS) as a 4 tuple $C = \langle W_C, \delta_C, w_C, ¢\rangle$, where W_C is a finite nonempty *alphabet* and $¢$ is a distinguished letter in W_C called the *passive letter*; δ_C is a total mapping from $W_C \times W_C$ into W_C such that $\delta_C(¢, ¢) = ¢$ and $\delta_C(a, b) \neq ¢$ for all $(a, b) \in W_C \times (W_C - \{¢\})$; $w_C \in (W_C - \{¢\})(W_C - \{¢\})^*$ is called the *initial configuration.*

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We imagine C as operating on an infinite string $¢^\infty w_C^{(t)} ¢^\infty$ over W_C, all the constituent letters of which are ¢'s except for a finite substring $w_C^{(t)}$ over $W_C - \{¢\}$ called the *configuration* at time t. C produces an infinite sequence of configurations $w_C^{(0)}, w_C^{(1)}, \ldots$ as follows. The string at time $t = 0$ is $¢^\infty w_C^{(0)} ¢^\infty$, where $w_C^{(0)} = w_C$. If $w_C^{(k)} = a_1 a_2 \ldots a_n$ is the configuration at time $t = k$, then $w_C^{(k+1)}$ is the configuration at time $t = k + 1$, where $w_C^{(k+1)}$ is defined by

$$¢^\infty w_C^{(k+1)} ¢^\infty = ¢^\infty \delta_C(¢, a_1)\, \delta_C(a_1, a_2) \ldots \delta_C(a_{n-1}, a_n)\, \delta_C(a_n, ¢)\, ¢^\infty .$$

Because of the restrictions on δ_C, length $(w_C^{(k+1)}) \geqslant$ length $(w_C^{(k)})$.

Let m be a function from $\{C: C \text{ is a 1 LCS}\}$ into $\mathbf{N}$ defined as follows:

$$m(C) = \begin{cases} 0 & \text{if for each } i \in \mathbf{N} \text{ there is a } t \in \mathbf{N} \text{ such that length } (w_C^{(t)}) \geqslant i, \\ \sup\{\text{length } (w_C^{(t)}): t \geqslant 0\} & \text{otherwise}. \end{cases}$$

Then Varshavsky's function L is given by

$$L(n) = \sup\{m(C): C = \langle W_C, \delta_C, w_C, ¢\rangle \text{ with } \#W_C = n \text{ and length } (w_C) = 1\}.$$

Finally, we need the notion of a Tag system. A *Tag system* T is a 4 tuple $T = \langle W_T, \delta_T, w_T, \beta\rangle$, where W_T is a finite nonempty *alphabet*; δ_T is a total mapping from W_T into W_T^*; $w_T \in W_T W_T^*$ is the *initial string* and β is a natural number called the *deletion number*. The operation of a Tag system is inductively defined as follows. The string produced at time $t = 0$ is $w_T^{(0)} = w_T$. If $w_T^{(k)} = a_1 a_2 \ldots a_n$ is the string produced at time $t = k$, then $w_T^{(k+1)} = a_{\beta+1} a_{\beta+2} \ldots a_n \delta_T(a_1)$ is the string produced at time $t = k + 1$.

2. Varshavsky's function is noncomputable

Lemma 1 (Minsky [2]). *Let k be a natural number. It is undecidable whether or not an arbitrary Tag system with deletion number 2 will ever produce a string of length less than or equal to k. In particular this is undecidable for $k = 0$.*

We shall now proceed to show that if there is a computable function f such that $L(n) \leqslant f(n)$ for all n, then this contradicts Lemma 1.

Lemma 2. *Let T be any Tag system with deletion number 2. There is an algorithm which, given T, produces a 1 LCS C with $w_C = w_T$ such that there is a time t_0 such that $w_C^{(t)} = w_C^{(t_0)}$ for all $t \geqslant t_0$ iff there is a time t_0' such that $w_T^{(t_0')} = \lambda$, i.e., the string of length* 0.

The proof of Lemma 2 can be easily derived from a slight modification in the construction in the appendix of [1].

Now it is easy to see that if $T = \langle W_T, \delta_T, w_T, 2\rangle$ is a Tag system, then $T' = \langle W_T \cup \{s\}, \delta_T \cup \{\delta_{T'}(s) = w_T\}, s, 2\rangle$, $s \notin W_T$, is a Tag system such that $w_{T'}^{(t+1)} = w_T^{(t)}$ for all $t > 0$. Therefore Lemma 1 also holds if we restrict our attention to Tag systems with deletion number 2 and an initial string of one letter and disregard the length of the initial string with respect to k.

Theorem. *There is no computable function f such that $L(n) \leqslant f(n)$ for all n. I.e., $L(n)$ grows faster than any computable function.*

Proof. Suppose there were such a function f. Then we can decide, for each 1LCS C, whether or not there exists a t_0 such that $w_C^{(t)} = w_C^{(t_0)}$ for all $t \geqslant t_0$ (remember that length $(w_C^{(t+1)}) \geqslant$ length $(w_C^{(t)})$ for all t). By Lemma 2 and the subsequent discussion this contradicts Lemma 1, i.e., it would imply the decidability of the halting problem for Tag systems which is known to be undecidable.

We might point out that Varshavsky's original problem can be shown to be equivalent to the halting problem for Turing machines by encoding the finite control and the scanned symbol in each cell of the linear cell space. Actually, Varshavsky's functions are variations on Rado's Busy Beaver function which was shown to be noncomputable in [3].

References

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