STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

TW 67

Solutions of the Equation of Helmholtz in an Angle III. The Case of a Halfplane

H.A. Lauwerier

1961

KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
Reprinted from Proceedings, Series A, 64, No. 1 and Indag. Math., 23, No. 1, 1961

MATHEMATICS

SOLUTIONS OF THE EQUATION OF HELMHOLTZ IN AN ANGLE III [1]).

THE CASE OF A HALFPLANE

BY

H. A. LAUWERIER

(Communicated by Prof. A. VAN WIJNGAARDEN at the meeting of October 29, 1960)

1. *Introduction*

This paper is the third of a set of papers in which the problem of Green of the equation of Helmholtz in an angle with fairly general boundary conditions is discussed. References to the previous papers will be denoted by I etc. followed by the formula number.

In this paper the special case of a halfplane will be discussed. Although a number of results can be obtained in an almost trivial way by specialization of the general results obtained in the preceding papers a direct and independent treatment seems to be justified.

For a halfplane the problem of Green may be formulated in Cartesian coordinates (cf. I introduction) in the following way

$$(1.1)\quad y>0 \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-1\right) G(x, y, x_0, y_0) = -\delta(x-x_0)\,\delta(y-y_0),$$

$$(1.2)\qquad y=0,\ x>0 \qquad \cos\gamma_1 \frac{\partial G}{\partial y} - \sin\gamma_1 \frac{\partial G}{\partial x} = 0,$$

$$(1.3)\qquad y=0,\ x<0 \qquad \cos\gamma_2 \frac{\partial G}{\partial y} - \sin\gamma_2 \frac{\partial G}{\partial x} = 0.$$

It will be assumed that for $j=1$ and $j=2$

$$(1.4)\qquad -\tfrac{1}{2}\pi < \operatorname{Re}\gamma_j \leqq \tfrac{1}{2}\pi.$$

It has been proved in I section 5 that the corresponding homogeneous problem – the F-problem – has a solution which vanishes at infinity and is continuous for $y \geqq 0$ only in the case $\operatorname{Re}\gamma_1 > \operatorname{Re}\gamma_2$. If on the other hand $\operatorname{Re}\gamma_1 \leqq \operatorname{Re}\gamma_2$ there is a solution which is continuous for $y \geqq 0$ except at the origin where it has a singularity of the kind $r^{-1+\varepsilon}$ where $r=\sqrt{x^2+y^2}$ and $\varepsilon>0$ for $\operatorname{Re}\gamma_1 < \operatorname{Re}\gamma_2$ and a singularity of the kind $\ln r$ for $\operatorname{Re}\gamma_1 = \operatorname{Re}\gamma_2$. According to (I 5.11) and (I 5.10) this "best possible" solution takes in polar coordinates (r, φ) the form

$$(1.5)\qquad F(x, y) = \int_{-\infty}^{\infty} \cos(r \operatorname{sh} u)\, \phi(u+i\varphi)\, du,$$

[1]) Report TW 67 of the Mathematical Centre.

where

$$(1.6)\qquad \phi(w) \stackrel{\text{def}}{=} \frac{e(w, \gamma_2)}{e(w-\pi\,\mathrm{i}, \gamma_1)},$$

with $e(z, \gamma)$ defined by (I 4.5) with $\theta = \pi$, or explicitly for $|\mathrm{Im} z| < \frac{3}{2}\pi - |\mathrm{Re}\,\gamma|$

$$(1.7)\qquad e(z, \gamma) \stackrel{\text{def}}{=} \exp\left[\tfrac{1}{2}\int_{\infty}^{\infty} \frac{1-\cos tz}{t}\, \frac{\mathrm{sh}\gamma t}{\mathrm{sh}\pi t\ \mathrm{sh}\frac{1}{2}\pi t}\, dt\right].$$

A number of properties of this auxiliary function have been collected in I section 4 and a few more will be given in Appendix A at the end of this paper.

The behaviour of the solution (1.5) at the origin is as follows (cf. I 5.13)

$$(1.8)\qquad \begin{cases} \mathrm{Re}\,\gamma_1 > \mathrm{Re}\,\gamma_2 & F \text{ continuous at } r=0,\\ \mathrm{Re}\,\gamma_1 = \mathrm{Re}\,\gamma_2 & F = C\ln r + \mathrm{O}(1),\\ \mathrm{Re}\,\gamma_1 < \mathrm{Re}\,\gamma_2 & F = C r^{(\gamma_1-\gamma_2)/\pi} + \mathrm{O}(1),\end{cases}$$

where C is some constant.

According to (I 5.23) the solution of the F-problem can also be written in the form

$$(1.9)\qquad F(x, y) = \tfrac{1}{2}\int_{-\infty+\frac{1}{2}\pi\mathrm{i}}^{\infty+\frac{1}{2}\pi\mathrm{i}} \exp\{-r\mathrm{ch}(w-\mathrm{i}\varphi)\} H(w)\, dw,$$

where

$$(1.10)\qquad H(w) \stackrel{\text{def}}{=} \tfrac{1}{2}\{\phi(w-\tfrac{1}{2}\pi\mathrm{i}) + \phi(w+\tfrac{1}{2}\pi\mathrm{i})\}.$$

By using the relations (I 5.21) and (6.4) of the Appendix the solution (1.9) can be brought in the following form, however, with a different multiplicative constant

$$(1.11)\qquad F(x, y) = \tfrac{1}{2}\int_{-\infty}^{\infty} \exp -(\mathrm{i}x\mathrm{sh}w + y\mathrm{ch}w)\, \frac{e(w+\frac{1}{2}\pi\mathrm{i},\ \gamma_2-\frac{1}{2}\pi)}{e(w-\frac{1}{2}\pi\mathrm{i},\ \gamma_1+\frac{1}{2}\pi)}\, d\,\mathrm{sh}w.$$

An independent derivation of this result will be given in section 2.

The solution of the G-problem has been obtained in II sections 2 and 4. Again making the specialization $\theta = \pi$ we arrive at the following results.

If $\mathrm{Re}\,\gamma_1 \leqq \mathrm{Re}\,\gamma_2$ there is a single function of Green which is regular in $y \geqq 0$ with the exception of the logarithmic pole at (x_0, y_0). In polar coordinates the solution can be written as (cf. II 2.26)

$$(1.12)\qquad \begin{cases} G(x, y, x_0, y_0) = \dfrac{1}{2\pi^2}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(r\mathrm{sh}u)\sin(r_0\,\mathrm{sh}u_0)\, \frac{\phi(u+\mathrm{i}\varphi)}{\phi(u_0+\mathrm{i}\varphi_0)}\,\cdot\\ \qquad\qquad \cdot\, \dfrac{\mathrm{sh}(u_0+\mathrm{i}\varphi_0)}{\mathrm{ch}(u_0+\mathrm{i}\varphi_0)-\mathrm{ch}(u+\mathrm{i}\varphi)}\, du\, du_0.\end{cases}$$

If $\mathrm{Re}\,\gamma_1 > \mathrm{Re}\,\gamma_2$ there is a function of Green which is regular in $y \geqq 0$ with the exception of the logarithmic pole at (x_0, y_0). The Green's function can be made unique by requiring it to vanish at the origin. The latter solution can be written as (cf. II 2.32)

$$(1.13)\quad \begin{cases} G(x, y, x_0, y_0) = \dfrac{1}{2\pi^2} \displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin(r\mathrm{sh}u) \cos(r_0\mathrm{sh}u_0) \dfrac{\phi(u+\mathrm{i}\varphi)}{\phi(u_0+\mathrm{i}\varphi_0)} \cdot \\ \qquad \cdot \dfrac{\mathrm{sh}(u+\mathrm{i}\varphi)}{\mathrm{ch}(u_0+\mathrm{i}\varphi_0)-\mathrm{ch}(u+\mathrm{i}\varphi)} du\, du_0. \end{cases}$$

On the other hand, by using the expression (II 4.28) a solution of the G-problem may also be obtained in the form

$$(1.14)\quad \begin{cases} G(x, y, x_0, y_0) = \dfrac{1}{8\pi^2\mathrm{i}} \displaystyle\int_{-\infty+\frac{1}{2}\pi\mathrm{i}}^{\infty+\frac{1}{2}\pi\mathrm{i}} \exp\{-r\mathrm{ch}(w-\mathrm{i}\varphi)\} H(w) dw \\ \qquad \displaystyle\int_L \exp\{r_0\mathrm{ch}(w_0-\mathrm{i}\varphi_0)\} H^{-1}(w_0) \dfrac{\mathrm{sh}w_0}{\mathrm{ch}w_0-\mathrm{ch}w} dw_0, \end{cases}$$

where L is the path of I fig. 3.

An independent derivation of (1.14) and similar results will be given in section 3.

In section 4 we consider the problem (1.1) with the boundary condition (1.2) and with $G=0$ for $y=0$, $x<0$. This is equivalent to taking $\gamma_2=-\frac{1}{2}\pi$ and requiring that $G=0$ at the origin. However, an independent treatment is more appropriate.

In section 5 we consider the important subcase that G vanishes at the negative X-axis and that its *normal* derivative vanishes at the positive X-axis. This case which is of interest in diffraction theory has been treated in some detail. In a certain sense the solution of this problem may be considered almost trivial since the free solutions of the F-problem can be written down at once (cf. 5.4) whereas the Green's function can be constructed from the free solutions after the example of (I 3.6). In this way the solution (5.5) is obtained. By following the treatment of the more general case of section 4, however, the solution is obtained in a different form (5.15) and (5.16). For the auxiliary function θ which is used here and which is discussed in Appendix B several expressions may be derived. One of these is equivalent to the well-known integral of Macdonald which he obtained in connection with Sommerfeld's problem.

2. *The F-problem*

Following (II 4.5) a solution of the F-problem which vanishes at infinity may be represented by

$$(2.1)\qquad F(x, y) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp -(\mathrm{i}x\mathrm{sh}w + y\mathrm{ch}w)\, f(w) dw.$$

The boundary conditions (1.2) and (1.3) give

$$(2.2)\qquad \int_{-\infty}^{\infty} e^{-\mathrm{i}x\mathrm{sh}w} f_1(w) d\,\mathrm{sh}w = 0 \qquad \text{for } x>0,$$

$$(2.3)\qquad \int_{-\infty}^{\infty} e^{-\mathrm{i}x\mathrm{sh}w} f_2(w) d\,\mathrm{sh}w = 0 \qquad \text{for } x<0,$$

where for $j=1$ and $j=2$

$$f_j(w)\mathrm{ch}w \overset{\mathrm{def}}{=} \mathrm{ch}(w-\mathrm{i}\gamma_j)\, f(w). \tag{2.4}$$

A sufficient condition for (2.2) is that $f_1(w)$ when considered in the z-plane with $z=\mathrm{sh}w$ is holomorphic in the lower halfplane $\mathrm{Im}\, z<0$ and of order $\mathrm{O}(z^{-1})$ at infinity. Similarly $f_2(w)$ should be holomorphic in the upper halfplane $\mathrm{Im}\ z>0$.

It follows that $f_1(w)$ when considered as a function of w is holomorhic in the lower strip $-\pi<\mathrm{Im}\, w<0$ and that it is symmetric with respect to $-\frac{1}{2}\pi\mathrm{i}$. Likewise $f_2(w)$ is holomorphic in the upper strip $0<\mathrm{Im}\, w<\pi$ and symmetric with respect to $\frac{1}{2}\pi\mathrm{i}$.

By elimination of $f(w)$ from the relations (2.4) a homogeneous Hilbert problem is obtained on the real axis viz.

$$\frac{f_2(w)}{\mathrm{ch}\,(w-\mathrm{i}\gamma_2)} = \frac{f_1(w)}{\mathrm{ch}\,(w-\mathrm{i}\gamma_1)}\,. \tag{2.5}$$

This problem has the formal solution

$$f_j(w) = \exp - \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \ln \frac{\mathrm{ch}(t-\mathrm{i}\gamma_1)}{\mathrm{ch}(t-\mathrm{i}\gamma_2)}\, \frac{d\mathrm{sh}t}{\mathrm{sh}t-\mathrm{sh}w}, \tag{2.6}$$

where $-\pi<\mathrm{Im}\, w<0$ for $j=1$ and $0<\mathrm{Im}\, w<\pi$ for $j=2$. In order to remove the apparent divergency of the right-hand side of (2.6) we put

$$Q(t) \overset{\mathrm{def}}{=} \frac{\mathrm{ch}(t-\mathrm{i}\gamma_1)}{\mathrm{ch}(t-\mathrm{i}\gamma_2)} \exp\,\{\mathrm{i}(\gamma_1-\gamma_2)\ \mathrm{sgn}\ t\}. \tag{2.7}$$

We note that for $t \to \pm\infty$ we have $Q(t)=1+\mathrm{O}(\mathrm{sh}^{-2}t)$. Then we may construct the convergent solution

$$f_j(w) = \{\mathrm{sh}w\}^{(\gamma_2-\gamma_1)/\pi} \exp\left\{-\mathrm{i}\gamma_j - \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \ln Q(t)\, \frac{\mathrm{ch}t\, dt}{\mathrm{sh}t-\mathrm{sh}w}\right\}, \tag{2.8}$$

where $\{\mathrm{sh}w\}^{(\gamma_2-\gamma_1)/\pi}$ has its cut along the negative real axis.

A simple calculation shows that for $w \to 0$ and either $\mathrm{Im}\, w>0$ or $\mathrm{Im}\, w<0$

$$\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \ln Q(t)\, \frac{\mathrm{ch}t\, dt}{\mathrm{sh}t-\mathrm{sh}w} = \frac{\gamma_2-\gamma_1}{\pi} \ln \mathrm{sh}w + \mathrm{O}(1), \tag{2.9}$$

so that for $w \to 0$ the function $f_j(w)$ have a finite generally non-vanishing limit.

Logarithmic differentiation of (2.8) for e.g. $j=2$ gives

$$\frac{d}{dw} \ln f_2(w) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \{\mathrm{th}(t-\mathrm{i}\gamma_2) - \mathrm{th}(t-\mathrm{i}\gamma_1)\}\, \frac{\mathrm{ch}w\, dt}{\mathrm{sh}t-\mathrm{sh}w}. \tag{2.10}$$

By using the following two well-known Fourier transforms

$$\frac{\mathrm{ch}w}{\mathrm{sh}t-\mathrm{sh}w} = \int_{-\infty}^{\infty} \sin\,\{u(\tfrac{1}{2}\pi\mathrm{i}-w)\}\, \frac{\exp\,\{u(-\frac{1}{2}\pi+\mathrm{i}t)\}}{\mathrm{sh}\pi u}\, du, \tag{2.11}$$

where $0 < \text{Im}\, w < \pi$, and

$$(2.12)\qquad \int_{-\infty}^{\infty} e^{itu}\, \text{th}(t - i\gamma)\, dt = \frac{\pi i e^{-\gamma u}}{\text{sh}\frac{1}{2}\pi u},$$

where u is real and $-\frac{1}{2}\pi < \text{Re}\, \gamma < \frac{1}{2}\pi$,
the right-hand side of (2.10) can be reduced to

$$(2.13)\qquad \tfrac{1}{2} \int_{-\infty}^{\infty} \sin\, \{u(\tfrac{1}{2}\pi i - w)\} \frac{\exp\{-(\frac{1}{2}\pi + \gamma_2)u\} - \exp\{-(\frac{1}{2}\pi + \gamma_1)u\}}{\text{sh}\, \pi u\ \text{sh}\frac{1}{2}\pi u}\, du.$$

Using the definition (1.7) the latter expression can be written as

$$(2.14)\qquad \frac{d}{dw} \ln \frac{e(\frac{1}{2}\pi i - w, \frac{1}{2}\pi + \gamma_2)}{e(\frac{1}{2}\pi i - w, \frac{1}{2}\pi + \gamma_1)}.$$

Hence, apart from some multiplicative constant we have

$$(2.15)\qquad f_2(w) = \frac{e(\frac{1}{2}\pi i - w, \frac{1}{2}\pi + \gamma_2)}{e(\frac{1}{2}\pi i - w, \frac{1}{2}\pi + \gamma_1)},$$

and likewise

$$(2.16)\qquad f_1(w) = \frac{e(\frac{1}{2}\pi i + w, -\frac{1}{2}\pi + \gamma_2)}{e(\frac{1}{2}\pi i + w, -\frac{1}{2}\pi + \gamma_1)}.$$

In the function defined by either side of the equality (2.5) is denoted by $K(w)$ it follows by using the functional relation (6.9) of Appendix A that apart from some multiplicative constant

$$(2.17)\qquad K(w) \stackrel{\text{def}}{=} \frac{e(\frac{1}{2}\pi i + w, -\frac{1}{2}\pi + \gamma_2)}{e(\frac{1}{2}\pi i - w, \frac{1}{2}\pi + \gamma_1)}.$$

The general solution of the Hilbert problem (2.5) can now be written down as

$$(2.18)\qquad f_j(w) = \text{ch}\,(w - i\gamma_j)\, K(w)\, \text{sh}^m w,$$

where $m = 0, 1, 2, \dots$.
Accordingly the F-problem has the general solution

$$(2.19)\qquad F_m(x, y) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp -(ix\text{sh}w + y\text{ch}w)\, K(w)\text{sh}^m w\, d\text{sh}w.$$

This solution is regular with the possible exception of the origin (0, 0). It follows from (2.8) and (2.15) that for $\text{Re}\, w \to \pm\infty$

$$(2.20)\qquad K(w) = O\{(\text{ch}w)^{(\gamma_2 - \gamma_1 - \pi)/\pi}\}.$$

Hence for $m \geqq 1$ we have at the origin

$$(2.21)\qquad F_m(x, y) = O\{r^{-m + (\gamma_1 - \gamma_2)/\pi}\}.$$

This result also holds for $m = 0$ and $\text{Re}\, \gamma_1 < \text{Re}\, \gamma_2$. However, if $\text{Re}\, \gamma_1 > \text{Re}\, \gamma_2$, $F_0(x, y)$ has a finite limit at the origin and finally if $\gamma_1 = \gamma_2$ it is of logarithmic order at the origin.

A more precise result can be obtained by using the lemma of Appendix A. If e.g. $m=0$ and γ_1, γ_2 are real with $\gamma_1<\gamma_2$ we have from (2.17), (6.10) and (6.11) for $\operatorname{Re} w \to +\infty$

$$K(w)=\frac{\exp\{(\gamma_2-\gamma_1-\pi)\,w/\pi+\frac{1}{2}(\gamma_1+\gamma_2)\mathrm{i}\}}{\mu(\frac{1}{2}\pi-\gamma_2)\,\mu(\frac{1}{2}\pi+\gamma_1)}\{1+\mathrm{O}(\mathrm{e}^{-w})\}, \tag{2.22}$$

so that by using (6.17) and (6.18) with $\alpha=\frac{1}{2}(\gamma_1+\gamma_2)$ and $\beta=(\gamma_2-\gamma_1)/\pi$

$$F_0(x,y)=\frac{\Gamma((\gamma_2-\gamma_1)/\pi)}{2\mu(\frac{1}{2}\pi-\gamma_2)\,\mu(\frac{1}{2}\pi+\gamma_1)}(\tfrac{1}{2}r)^{-(\gamma_2-\gamma_1)/\pi}\cdot \tag{2.23}$$
$$\cdot\cos\{\gamma_1(1-(\varphi/\pi))+\gamma_2(\varphi/\pi)\}+\text{constant}+\mathrm{O}(r^{1-(\gamma_2-\gamma_1)/\pi}).$$

A limit operation shows that for $\gamma_1=\gamma_2=\gamma$

$$F_0(x,y)=-(1+\cos\gamma)\,(\cos\gamma\ln r+\varphi\sin\gamma)+\text{constant}+\mathrm{O}(r\ln r). \tag{2.24}$$

We note that for $\gamma_1=\gamma_2$ in view of (6.9) the function $K(w)$ reduces to

$$K(w)=\frac{1+\cos\gamma}{\operatorname{ch}(w-\mathrm{i}\gamma)}, \tag{2.25}$$

so that for $\gamma_1=\gamma_2$

$$F_0(x,y)=\tfrac{1}{2}(1+\cos\gamma)\int_{-\infty}^{\infty}\exp-(\mathrm{i}x\operatorname{sh}w+y\operatorname{ch}w)\frac{\operatorname{ch}w\,dw}{\operatorname{ch}(w-\mathrm{i}\gamma)}. \tag{2.26}$$

Using the notation of (II 5.1) this may be written as

$$F_0(x,y)=\frac{1+\cos\gamma}{2\cos\gamma}\{R_0(x,y)+K_0(\sqrt{x^2+y^2})\}, \tag{2.27}$$

so that $F_0(x,y)$ can be interpreted as the result of a logarithmic pole of strength $\cos\gamma(1+\cos\gamma)$ and a tail of normal dipoles starting at (0, 0) and making the angle $-\frac{1}{2}\pi+\gamma$ with the positive X-axis.

3. *The G-problem*

Following (II 4.5) a solution of the G-problem may be represented by

$$2\pi G(x,y,x_0,y_0)=K_0(\sqrt{(x-x_0)^2+(y-y_0)^2})+ \tag{3.1}$$
$$+\tfrac{1}{2}\int_{-\infty}^{\infty}\exp-(\mathrm{i}x\operatorname{sh}w+y\operatorname{ch}w)\,g(w)dw.$$

The singular part on the right-hand side may be represented by (cf. II 3.2)

$$K_0(\sqrt{(x-x_0)^2+(y-y_0)^2})=\tfrac{1}{2}\int_{-\infty}^{\infty}\exp-\{\mathrm{i}(x-x_0)\operatorname{sh}w+|y-y_0|\operatorname{ch}w\}dw. \tag{3.2}$$

The boundary conditions give in a similar way as in the previous section for $j=1$ and $j=2$

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}x\operatorname{sh}w}\,g_j(w)d\operatorname{sh}w=0, \tag{3.3}$$

where now

$$(3.4) \qquad g_j(w)\mathrm{ch}w \overset{\mathrm{def}}{=} \mathrm{ch}(w-\mathrm{i}\gamma_j)g(w) - \mathrm{ch}(w+\mathrm{i}\gamma_j)\exp(\mathrm{i}x_0\mathrm{sh}w - y_0\mathrm{ch}w).$$

Elimination of $g(w)$ from the two relations (3.4) gives at the real axis

$$(3.5) \qquad \frac{g_2(w)\mathrm{ch}w}{\mathrm{ch}(w-\mathrm{i}\gamma_2)} - \frac{g_1(w)\mathrm{ch}w}{\mathrm{ch}(w-\mathrm{i}\gamma_1)} = \left\{\frac{\mathrm{ch}(w+\mathrm{i}\gamma_1)}{\mathrm{ch}(w-\mathrm{i}\gamma_1)} - \frac{\mathrm{ch}(w+\mathrm{i}\gamma_2)}{\mathrm{ch}(w-\mathrm{i}\gamma_2)}\right\}\cdot \cdot\exp(\mathrm{i}x_0\mathrm{sh}w - y_0\mathrm{ch}w).$$

This is a non-homogeneous Hilbert problem which can be solved at once by using the factorization in the previous section. If $K(w)$ is defined by (2.17) and $f_j(w)$ by (2.16) the relation (3.5) can be written as

$$(3.6) \qquad \frac{g_2(w)}{f_2(w)} - \frac{g_1(w)}{f_1(w)} = h(w),$$

where

$$(3.7) \qquad h(w) \overset{\mathrm{def}}{=} \left\{\frac{\mathrm{ch}(w+\mathrm{i}\gamma_1)}{\mathrm{ch}(w-\mathrm{i}\gamma_1)} - \frac{\mathrm{ch}(w+\mathrm{i}\gamma_2)}{\mathrm{ch}(w-\mathrm{i}\gamma_2)}\right\} \frac{\exp(\mathrm{i}x_0\mathrm{sh}w - y_0\mathrm{ch}w)}{K(w)\mathrm{ch}w}.$$

The functions $g_2(w)$, $f_2(w)$ and $g_2(w)/f_2(w)$ are holomorphic in the upper strip $0<\mathrm{Im}\, w<\pi$ and symmetric with respect to $\frac{1}{2}\pi\mathrm{i}$. The functions $g_1(w)$, $f_1(w)$ and $g_1(w)/f_1(w)$ are holomorphic in the lower strip $-\pi<\mathrm{Im}\, w<0$ and symmetric with respect to $-\frac{1}{2}\pi\mathrm{i}$. Then the problem (3.6) has the obvious particular solution

$$(3.8) \qquad g_j(w) = \frac{f_j(w)}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} h(w_0)\frac{\mathrm{ch}w_0 dw_0}{\mathrm{sh}w_0 - \mathrm{sh}w},$$

where $-\pi<\mathrm{Im}\, w<0$ for $j=1$ and $0<\mathrm{Im}\, w<\pi$ for $j=2$.
Of course we need a single particular solution only since the homogeneous problem has been solved already in the previous section.

Now it follows from (3.4) and (3.8) that the G-problem has a particular solution with

$$(3.9) \qquad g(w) = \frac{\mathrm{ch}(w+\mathrm{i}\gamma_j)}{\mathrm{ch}(w-\mathrm{i}\gamma_j)}\exp(\mathrm{i}x_0\mathrm{sh}w - y_0\mathrm{ch}w) + \frac{K(w)\mathrm{ch}w}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} h(w_0)\frac{d\mathrm{sh}w_0}{\mathrm{sh}w_0 - \mathrm{sh}w},$$

where either $j=1$ or $j=2$ with an appropriate meaning of the integral on the right-hand side. By using the well-known Plemelj formula we have the alternate expression

$$(3.10) \qquad \left\{ \begin{aligned} g(w) = \tfrac{1}{2}\left\{\frac{\mathrm{ch}(w+\mathrm{i}\gamma_1)}{\mathrm{ch}(w-\mathrm{i}\gamma_1)} + \frac{\mathrm{ch}(w+\mathrm{i}\gamma_2)}{\mathrm{ch}(w-\mathrm{i}\gamma_2)}\right\}\exp(\mathrm{i}x_0\mathrm{sh}w - y_0\mathrm{ch}w) + \\ + \frac{K(w)\mathrm{ch}w}{2\pi\mathrm{i}} \fint_{-\infty}^{\infty} h(w_0)\frac{\mathrm{ch}w_0 dw_0}{\mathrm{sh}w_0 - \mathrm{sh}w}, \end{aligned} \right.$$

where the Cauchy integral takes its principal value.

4. *A special case*

In this section the following important case will be considered. To find the Green's function which satisfies in the upper halfplane $y > 0$ the Helmholtz equation

(4.1) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right) G(x, y, x_0, y_0) = -\delta(x - x_0)\, \delta(y - y_0),$$

and the boundary conditions

(4.2) $$y = 0,\ x > 0 \qquad \cos\gamma \frac{\partial G}{\partial y} - \sin\gamma \frac{\partial G}{\partial x} = 0,$$

(4.3) $$y = 0,\ x < 0 \qquad G = 0.$$

In order to simplify the discussion the non-essential assumption will be made that γ is real and that it is restricted to the interval

(4.4) $$-\tfrac{1}{2}\pi < \gamma < \tfrac{1}{2}\pi.$$

The solution of the corresponding F-problem will be determined first. As in section 2 we may put

(4.5) $$F(x, y) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp -\{ix \operatorname{sh} w + y \operatorname{ch} w\} f(w) dw.$$

The boundary conditions give relations of the type (2.2) and (2.3) with

(4.6) $$f_1(w) \operatorname{ch} w \overset{\text{def}}{=} C \operatorname{ch}(w - i\gamma) f(w),$$

and

(4.7) $$f_2(w) \operatorname{ch} w \overset{\text{def}}{=} f(w),$$

where C is some constant and where $f_1(w)$ and $f_2(w)$ have the same properties as in section 2.

It follows now immediately from the functional relation (6.9) of Appendix A that we may take

(4.8) $$\begin{cases} f_1(w) = e(w + \tfrac{1}{2}\pi i,\ \tfrac{1}{2}\pi - \gamma) \\ f_2(w) = e(w - \tfrac{1}{2}\pi i,\ -\tfrac{1}{2}\pi - \gamma). \end{cases}$$

Hence the following explicit solution is obtained

(4.9) $$F(x, y) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp -\{ix \operatorname{sh} w + y \operatorname{ch} w\}\, e(w - \tfrac{1}{2}\pi i,\ -\tfrac{1}{2}\pi - \gamma) \operatorname{ch} w\, dw.$$

The behaviour of $F(x, y)$ at the origin follows easily by applying the lemma of Appendix A.

It follows from (6.10) that for $w \to \infty$

(4.10) $$e(w - \tfrac{1}{2}\pi i, -\tfrac{1}{2}\pi - \gamma) \operatorname{ch} w = \frac{\exp\{(\tfrac{1}{2} - (\gamma/\pi))\, w + \tfrac{1}{2}(\tfrac{1}{2}\pi + \gamma)\, i\}}{2\mu(\tfrac{1}{2}\pi + \gamma)} \{1 + O(e^{-w})\},$$

so that

(4.11) $$F(x, y) = \frac{\Gamma(\tfrac{1}{2} - (\gamma/\pi))}{2\mu(\tfrac{1}{2}\pi + \gamma)} (\tfrac{1}{2} r)^{-\frac{1}{2} + (\gamma/\pi)} \cos\{\gamma + (\tfrac{1}{2} - (\gamma/\pi))\varphi\} + O(r^{\frac{1}{2} + (\gamma/\pi)}).$$

The solution of the G-problem can be represented again by the expression (3.1). Then the boundary conditions lead to the conditions (3.3) where now

$$(4.12)\quad \begin{cases} g_1(w)\mathrm{ch}w \overset{\mathrm{def}}{=} \mathrm{ch}(w-\mathrm{i}\gamma)g(w)-\mathrm{ch}(w+\mathrm{i}\gamma)\exp(\mathrm{i}x_0\mathrm{sh}w-y_0\mathrm{ch}w). \\ g_2(w)\mathrm{ch}w \overset{\mathrm{def}}{=} \qquad g(w) \qquad + \qquad \exp(\mathrm{i}x_0\mathrm{sh}w-y_0\mathrm{ch}w). \end{cases}$$

Elimination of $g(w)$ gives for real w the following non-homogeneous Hilbert problem

$$(4.13)\qquad \mathrm{ch}(w-\mathrm{i}\gamma)g_2(w)-g_1(w)=2\cos\gamma\exp(\mathrm{i}x_0\mathrm{sh}w-y_0\mathrm{ch}w).$$

By making use of the functions $f_1(w)$ and $f_2(w)$ from (4.6) and (4.7) this Hilbert problem can be reduced to the elementary form

$$(4.14)\qquad \frac{g_2(w)}{Cf_2(w)}-\frac{g_1(w)}{f_1(w)}=\frac{2\cos\gamma}{f_1(w)}\exp(\mathrm{i}x_0\mathrm{sh}w-y_0\mathrm{ch}w),$$

the solution of which is of the form (3.8). Eventually we arrive at the result

$$(4.15)\qquad 2\pi G(x,y,x_0,y_0)=K_0(\sqrt{(x-x_0)^2+(y-y_0)^2})+ \\ +\tfrac{1}{2}\int_{-\infty}^{\infty}\exp-(\mathrm{i}x\mathrm{sh}w+y\mathrm{ch}w)g(w)dw$$

with

$$(4.16)\qquad g(w)=\tfrac{1}{2}\left\{\frac{\mathrm{ch}(w+\mathrm{i}\gamma)}{\mathrm{ch}(w-\mathrm{i}\gamma)}-1\right\}\exp(\mathrm{i}x_0\mathrm{sh}w-y_0\mathrm{ch}w)+ \\ +\frac{2\cos\gamma}{1+\cos\gamma}\,\frac{\mathrm{ch}w}{e(w-\frac{1}{2}\pi\mathrm{i},\frac{1}{2}\pi+\gamma)}\,\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}x_0\mathrm{sh}w_0-y_0\mathrm{ch}w_0)}{e(w_0+\frac{1}{2}\pi\mathrm{i},\frac{1}{2}\pi-\gamma)}\cdot\frac{\mathrm{ch}w_0dw_0}{\mathrm{sh}w_0-\mathrm{sh}w}.$$

The behaviour of G at the origin can be derived from the lemma of Appendix A. However, it can easily be predicted, at least formally, by substitution of

$$r^\lambda\sin\mu(\pi-\varphi)+\text{higher-order terms}$$

in (4.1) and (4.2). It is easily seen that

$$(4.17)\qquad \lambda=\mu=\tfrac{1}{2}+\frac{\gamma}{\pi},$$

so that for $r\to 0$

$$(4.18)\qquad G=a\,r^{\frac{1}{2}+\gamma/\pi}\cos\left\{\gamma-\left(\tfrac{1}{2}+\frac{\gamma}{\pi}\right)\varphi\right\}+\text{higher-order terms},$$

The constant a can be determined by using the lemma of the Appendix which involves the asymptotic behaviour of $g(w)$ of (4.16).

Since we have for $w\to+\infty$

$$(4.19)\quad g(w)=\frac{2\mathrm{i}\cos\gamma}{1+\cos\gamma}\,\frac{\exp-\{(\frac{1}{2}+(\gamma/\pi))(w-\frac{1}{2}\pi\mathrm{i})\}}{\pi\mu(\frac{1}{2}\pi+\gamma)}F(x_0,y_0,-\gamma)\{1+O(e^{-w})\},$$

where $F(x_0,y_0,-\gamma)$ is defined by (4.9) with γ replaced by $-\gamma$, it follows after some elementary reductions that

$$(4.20)\qquad a=\frac{2^{\frac{1}{2}-\gamma/\pi}\mu(\frac{1}{2}\pi-\gamma)}{\pi\Gamma(3/2+\gamma/\pi)}F(x_0,y_0,-\gamma)$$

or combining (4.18) and (4.20)

$$(4.21)\quad \begin{cases} G(x,y,x_0,y_0) = \\ = \dfrac{2\mu(\frac{1}{2}\pi-\gamma)}{\pi\Gamma(^3/_2+\gamma/\pi)}\,(\tfrac{1}{2}r)^{\frac{1}{2}+(\gamma/\pi)} \cos\{\gamma-(\tfrac{1}{2}+(\gamma/\pi))\varphi\}\,F(x_0,y_0,-\gamma) + \\ \qquad + O(r^{^3/_2+(\gamma/\pi)}). \end{cases}$$

5. *The subcase* $\gamma=0$

The problem of the previous section with $\gamma=0$ deserves special attention in view of its importance in connection with diffraction theory. The problem is to find a Green's function in the halfplane $y>0$ satisfying

$$(5.1)\qquad (\Delta-1)\,G(x,y,x_0,y_0) = -\delta(x-x_0)\,\delta(y-y_0),$$

$$(5.2)\qquad y=0 \qquad x>0 \qquad \frac{\partial G}{\partial y}=0,$$

$$(5.3)\qquad y=0 \qquad x<0 \qquad G=0.$$

The free solutions of the corresponding F-problem are in polar coordinates apparently

$$(5.4)\qquad K_{n+\frac{1}{2}}(r)\cos(n+\tfrac{1}{2})\varphi, \qquad I_{n+\frac{1}{2}}(r)\cos(n+\tfrac{1}{2})\varphi$$

for $n=0, 1, \ldots$.

The Green's function can be written down at once by using the trick of (I 3.6) viz.

$$(5.5)\quad \begin{cases} G=\dfrac{2}{\pi}\displaystyle\sum_{n=0}^{\infty} I_{n+\frac{1}{2}}(r)K_{n+\frac{1}{2}}(r_0)\cos(n+\tfrac{1}{2})\varphi\cos(n+\tfrac{1}{2})\varphi_0, & r<r_0, \\ G=\dfrac{2}{\pi}\displaystyle\sum_{n=0}^{\infty} K_{n+\frac{1}{2}}(r)I_{n+\frac{1}{2}}(r_0)\cos(n+\tfrac{1}{2})\varphi\cos(n+\tfrac{1}{2})\varphi_0, & r>r_0. \end{cases}$$

It may be left to the reader to verify that the solution of the F-problem (4.9) for this case reduces to $2K_{\frac{1}{2}}(r)\cos\frac{1}{2}\varphi$.

Alternative expressions for the Green's function may be obtained by specialization of (4.15) and (4.16). However, we prefer the following somewhat more direct approach.

Quoting the expression (3.1) we may put in polar coordinates

$$(5.6)\qquad 2\pi G(r,\varphi,r_0,\varphi_0) = K_0(\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}) + \\ + \tfrac{1}{2}\int_{-\infty}^{\infty} \exp\{-ir\,\mathrm{sh}(w-i\varphi\}g(w)dw.$$

The relations (4.12) are here

$$(5.7)\qquad \begin{cases} g_1(w) = g(w) - \exp\{ir_0\mathrm{sh}(w+i\varphi_0)\} \\ \mathrm{ch}w\; g_2(w) = g(w) + \exp\{ir_0\mathrm{sh}(w+i\varphi_0)\}. \end{cases}$$

Hence the Hilbert problem for this particular case is

$$(5.8)\qquad \mathrm{ch}w\; g_2(w) - g_1(w) = 2\exp\{ir_0\mathrm{sh}(w+i\varphi_0)\}.$$

In view of the factorization

$$\operatorname{ch} w = 2 \operatorname{ch}\tfrac{1}{2}(w - \tfrac{1}{2}\pi i)\operatorname{ch}\tfrac{1}{2}(w + \tfrac{1}{2}\pi i)$$

this problem is easily solvable. We shall, however, not follow the traditional approach but we note that the solution of (5.8) may be written down at once if one uses the auxiliary function

$$\psi(r, z) \stackrel{\text{def}}{=} e^{r\operatorname{ch} z} \operatorname{erfc}(\operatorname{ch}\tfrac{1}{2}z \sqrt{2r}). \tag{5.9}$$

This function has the translation property

$$\psi(r, z + 2\pi i) + \psi(r, z) = 2\, e^{r\operatorname{ch} z}. \tag{5.10}$$

Hence a simple inspection shows that (5.8) is solved by

$$g_1(w) = -\psi(r_0, w + \tfrac{1}{2}\pi i + \varphi_0 i) - \psi(r_0, w + \tfrac{1}{2}\pi i - \varphi_0 i) \tag{5.11}$$

and

$$\operatorname{ch} w\, g_1(w) = \psi(r_0, w - \tfrac{3}{2}\pi i + \varphi_0 i) - \psi(r_0, w + \tfrac{1}{2}\pi i - \varphi_0 i). \tag{5.12}$$

The link between these two methods is given by the following formula which is proved in Appendix B (cf. 7.4)

$$\psi(r, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-r\operatorname{ch} u} \frac{1}{\operatorname{ch}\frac{1}{2}(u - z)}\, du. \tag{5.13}$$

where $-\pi < \operatorname{Im} z < \pi$.

The Green's function now follows from (5.6), (5.7) and either (5.11) or (5.12). Taking e.g. (5.11) we obtain

$$\begin{aligned} 2\pi G(r, \varphi, r_0, \varphi_0) = {} & K_0(\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)}) + \\ & + K_0(\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\varphi + \varphi_0)}) - \tfrac{1}{2} \int_{-\infty}^{\infty} \exp - \{r\operatorname{ch}(w - i\varphi)\}\{\psi(r_0, w + i\varphi_0) + \\ & + \psi(r_0, w - i\varphi_0)\}\, dw, \end{aligned} \tag{5.14}$$

which is most useful for $0 \leqq \varphi < \tfrac{1}{2}\pi$.

If we define for all values of γ

$$\theta(r, r_0, \gamma) = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-r\operatorname{ch} v} \psi(r_0, v + i\gamma)\, dv \tag{5.15}$$

the solution (5.14) and the alternative form obtained from (5.12) can be written as

$$\begin{aligned} 2\pi G(r, \varphi, r_0, \varphi_0) = {} & K_0(\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)}) - \theta(r, r_0, \varphi - \varphi_0) + \\ & + K_0(\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\varphi + \varphi_0)}) - \theta(r, r_0, \varphi + \varphi_0), \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} 2\pi G(r, \varphi, r_0, \varphi_0) = {} & K_0(\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)}) - \theta(r, r_0, \varphi - \varphi_0) + \\ & - K_0(\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\varphi + \varphi_0)}) + \theta(r, r_0, \varphi + \varphi_0 - 2\pi). \end{aligned} \tag{5.17}$$

In fact the translation property (5.10) leads by substitution in (5.15) to

$$\theta(r, r_0, \gamma) + \theta(r, r_0, \gamma + 2\pi) = 2K_0(\sqrt{r^2 + r_0^2 - 2rr_0 \cos \gamma}). \tag{5.18}$$

It is possible to express $\theta(r, r_0, \gamma)$ in a variety of ways. We note that substitution of (5.13) in (5.15) leads to a double integral which is symmetric in r and r_0. To the latter expression one is also led by solving (5.8) in the ordinary way and straightforward substitution in (5.7) and (5.6). In this way the discussion of the ψ-function may be avoided. In Appendix B two simple expressions of θ in the form of a single integral are derived. We note in particular (7.6) and (7.10), the latter due originally to Macdonald.

6. *Appendix A*

The following two integral expressions are frequently needed

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\operatorname{ch} zt}{\operatorname{ch}\frac{1}{2}\pi t}\,dt = \sec z \qquad \text{for } |\operatorname{Re} z| < \tfrac{1}{2}\pi, \tag{6.1}$$

and

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\operatorname{sh} zt}{\operatorname{sh}\frac{1}{2}\pi t}\,dt = \operatorname{tg} z \qquad \text{for } |\operatorname{Re} z| < \tfrac{1}{2}\pi. \tag{6.2}$$

These formulae may be derived from Erdélyi *et al.* Tables I (1.9.1) and (2.9.2) but they can also be proved independently by means of a simple application of the calculus of residues.

We shall now consider the function $e(z, \gamma)$ as defined by (I 4.5) for arbitrary θ. Before making the specialization $\theta = \pi$ a few general results will be proved. With the following definition quoted above

$$\ln e(z,\gamma) \overset{\text{def}}{=} \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos tz}{t}\,\frac{\operatorname{sh}\gamma t}{\operatorname{sh}\theta t\,\operatorname{sh}\frac{1}{2}\pi t}\,dt, \tag{6.3}$$

we have the functional relation

$$\frac{e(z, \gamma+\frac{1}{2}\pi)}{e(z+\frac{1}{2}\pi \mathrm{i}, \gamma)} = c_0 \operatorname{ch}\tfrac{1}{2}\nu(z-\mathrm{i}\gamma), \tag{6.4}$$

where c_0 is a constant.

The proof runs as follows. From (6.3) it follows that

$$\frac{d}{dz}\ln\frac{e(z, \gamma+\frac{1}{2}\pi)}{e(z+\frac{1}{2}\pi \mathrm{i}, \gamma)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin tz\,\operatorname{sh}(\gamma+\frac{1}{2}\pi)t - \sin t(z+\frac{1}{2}\pi\mathrm{i})\operatorname{sh}\gamma t}{\operatorname{sh}\theta t\,\operatorname{sh}\frac{1}{2}\pi t}\,dt =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin t(z-\mathrm{i}\gamma)}{\operatorname{sh}\theta t}\,dt = \tfrac{1}{2}\nu \operatorname{th}\tfrac{1}{2}\nu(z-\mathrm{i}\gamma),$$

where at the final stage (6.2) has been used. Integration of the result leads at once to (6.4).

By changing the sign of z we obtain from (6.4) the equivalent functional relation

$$\frac{e(z, \gamma+\frac{1}{2}\pi)}{e(z-\frac{1}{2}\pi \mathrm{i}, \gamma)} = c_0 \operatorname{ch}\tfrac{1}{2}\nu(z+\mathrm{i}\gamma). \tag{6.5}$$

Hence the functional relation (I 4.13) now appears to be an immediate consequence of (6.4) and (6.5).

By applying (6.4) twice the following functional relation can be derived

$$\frac{e(z-\frac{1}{2}\pi\mathrm{i},\gamma+\frac{1}{2}\pi)}{e(z+\frac{1}{2}\pi\mathrm{i},\gamma-\frac{1}{2}\pi)}=c_1\{\mathrm{ch}\nu(z-\mathrm{i}\gamma)+\cos\tfrac{1}{2}\nu\pi\}, \tag{6.6}$$

where the constant c_1 is given by $c_1=\frac{1}{2}c_0(\gamma)c_0(\gamma-\frac{1}{2}\pi)$.

Next we shall prove the special case

$$e(\pi\mathrm{i},\gamma)=\frac{\cos\frac{1}{2}\nu(\frac{1}{2}\pi+\gamma)}{\cos\frac{1}{2}\nu(\frac{1}{2}\pi-\gamma)}. \tag{6.7}$$

By logarithmic differentiation of $e(\pi\mathrm{i},\gamma)$ with respect to γ we obtain by using (6.3)

$$\frac{d}{d\gamma}\ln e(\pi\mathrm{i},\gamma)=-\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{(\mathrm{ch}\pi t-1)\,\mathrm{ch}\gamma t}{\mathrm{sh}\theta t\,\mathrm{sh}\frac{1}{2}\pi t}\,dt=-\int_{-\infty}^{\infty}\frac{\mathrm{sh}\frac{1}{2}\pi t\,\mathrm{ch}\gamma t}{\mathrm{sh}\theta t}\,dt.$$

Application of (6.2) gives

$$\frac{d}{d\gamma}\ln e(\pi\mathrm{i},\gamma)=-\tfrac{1}{2}\nu\{\mathrm{tg}\,\tfrac{1}{2}\nu(\tfrac{1}{2}\pi+\gamma)+\mathrm{tg}\tfrac{1}{2}\nu(\tfrac{1}{2}\pi-\gamma)\}.$$

Integration of this result and noting that $e(\pi\mathrm{i},0)=1$ gives the required expression.

Substitution of $z=\frac{1}{2}\pi\mathrm{i}$ in (6.6) gives in view of (6.7) an explicit expression for the constant c_1. An elementary calculation shows that

$$1/c_1=1+\cos\nu\gamma$$

so that (6.6) may be replaced by

$$\frac{e(z-\frac{1}{2}\pi\mathrm{i},\gamma+\frac{1}{2}\pi)}{e(z+\frac{1}{2}\pi\mathrm{i},\gamma-\frac{1}{2}\pi)}=\frac{\mathrm{ch}\,\nu(z-\mathrm{i}\gamma)+\cos\frac{1}{2}\nu\pi}{1+\cos\nu\gamma}. \tag{6.8}$$

We shall now make the specialization $\theta=\pi$, $\nu=1$. Then this functional relation reduces to (cf. I 4.14)

$$\frac{e(z-\frac{1}{2}\pi\mathrm{i},\gamma+\frac{1}{2}\pi)}{e(z+\frac{1}{2}\pi\mathrm{i},\gamma-\frac{1}{2}\pi)}=\frac{\mathrm{ch}(z-\mathrm{i}\gamma)}{1+\cos\gamma}. \tag{6.9}$$

The asymptotic behaviour of $e(z,\gamma)$ with $\theta=\pi$ for $\mathrm{Re}\,z\to\pm\infty$ may be given by (cf. also I section 4)

$$\ln e(z,\gamma)=\frac{\gamma}{\pi}\ln(\mathrm{e}^{z}+\mathrm{e}^{-z})+\ln\mu(\gamma)+O(\mathrm{e}^{-|\mathrm{Re}\,z|}), \tag{6.10}$$

where the constant term $\ln\mu(\gamma)$ is given by

$$\mu(\gamma)=\exp-\frac{1}{\pi}\int_{0}^{\gamma}\frac{\frac{1}{2}\pi-u\sin u}{\cos u}\,du, \tag{6.11}$$

with

$$2(1+\sin\gamma)\mu(\gamma)\mu(\pi-\gamma)=1,\ \mu(0)=1,\ \mu(\tfrac{1}{2}\pi)=\mu(\pi)=\tfrac{1}{2}. \tag{6.12}$$

The proof of (6.10) is as follows. From (6.3) it is obtained that

$$\ln e(z,\gamma) = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos tz}{t\,\mathrm{sh}\,\tfrac{1}{2}\pi t}\left\{\frac{\gamma}{\pi} + \left(\frac{\mathrm{sh}\,\gamma t}{\mathrm{sh}\,\pi t} - \frac{\gamma}{\pi}\right)\right\} dt =$$

$$= \frac{\gamma}{\pi}\ln e(z,\pi) + m(\gamma) + \mathrm{O}(\exp - |\mathrm{Re}\, z|),$$

where

$$m(\gamma) = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{1}{t\,\mathrm{sh}\,\tfrac{1}{2}\pi t}\left(\frac{\mathrm{sh}\,\gamma t}{\mathrm{sh}\,\pi t} - \frac{\gamma}{\pi}\right) dt. \tag{6.13}$$

Since $e(z,\pi) = \mathrm{ch} z$ (cf. I 4.11) we have

$$\mu(\gamma) = 2^{-\gamma/\pi}\, \mathrm{e}^{m(\gamma)}. \tag{6.14}$$

It is obvious that $m(0) = m(\pi) = 0$. Differentiation of (6.13) gives (by aid of the calculus of residues)

$$m'(\gamma) = \tfrac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\mathrm{ch}\,\gamma t}{\mathrm{sh}\,\tfrac{1}{2}\pi t\,\mathrm{sh}\,\pi t} - \frac{1}{\pi t\,\mathrm{sh}\,\tfrac{1}{2}\pi t}\right) dt =$$

$$= \frac{1}{4}\int_{-\infty}^{\infty}\left(\frac{\mathrm{ch}\,\tfrac{1}{2}\pi t\,\mathrm{ch}\,\gamma t}{\mathrm{sh}^2\,\tfrac{1}{2}\pi t} - \frac{\mathrm{ch}\,\gamma t}{\mathrm{ch}\,\tfrac{1}{2}\pi t} - \frac{2}{\pi t\,\mathrm{sh}\,\tfrac{1}{2}\pi t}\right) dt =$$

$$= -\frac{1}{2\pi}\,⨍_{-\infty}^{\infty} \mathrm{ch}\,\gamma t\, d\,\frac{1}{\mathrm{sh}\,\tfrac{1}{2}\pi t} - \frac{1}{4}\int_{-\infty}^{\infty}\frac{\mathrm{ch}\,\gamma t}{\mathrm{ch}\,\tfrac{1}{2}\pi t}dt - \frac{1}{2\pi}\,⨍_{-\infty}^{\infty}\frac{1}{t\,\mathrm{sh}\,\tfrac{1}{2}\pi t}dt =$$

$$= \frac{\gamma}{\pi}\mathrm{tg}\,\gamma - \tfrac{1}{2}\sec\gamma + \frac{1}{\pi}\ln 2,$$

so that

$$m(\gamma) = \frac{\gamma}{\pi}\ln 2 - \frac{1}{\pi}\int_0^{\gamma}\frac{\tfrac{1}{2}\pi - u\sin u}{\cos u}\,du. \tag{6.15}$$

We note that

$$m'(\pi-\gamma) = m'(\gamma) + \frac{1-\sin\gamma}{\cos\gamma},$$

so that by integration

$$m(\pi-\gamma) + m(\gamma) + \ln\,(1+\sin\gamma) = 0. \tag{6.16}$$

By using (6.14) the relations (6.11) and (6.12) follow at once from (6.15) and (6.16).

Finally the following lemma will be proved

Lemma. Let $g(w)$ be an analytic function of w which is regular on the real axis, for which $g(w) = \overline{g(-w)}$ and which has the following asymptotic behaviour

$$g(w) = \exp\,(\mathrm{i}\alpha + \beta w)\{1 + \mathrm{O}(\mathrm{e}^{-w})\} \tag{6.17}$$

for $\mathrm{Re}\, w \to +\infty$, where α and β are real and $0<\beta<1$, then for $r \to 0$

$$(6.18)\qquad \tfrac{1}{2}\int_{-\infty}^{\infty} \exp -(\mathrm{i}x\,\mathrm{sh}w + y\,\mathrm{ch}w)\, g(w)dw = \\ = \Gamma(\beta)(\tfrac{1}{2}r)^{-\beta}\cos\{\alpha - \beta(\tfrac{1}{2}\pi - \varphi)\} + \text{constant} + O(r^{1-\beta}),$$

where $x = r\cos\varphi$, $y = r\sin\varphi$, with $0<\varphi<\pi$.

Proof. The left-hand side of (6.18) can be written as

$$\mathrm{Re}\int_0^{\infty} \exp\{-\mathrm{i}r\mathrm{sh}(w-\mathrm{i}\varphi)\}\, g(w)dw = \mathrm{Re}\, e^{\mathrm{i}\alpha}\int_0^{\infty}\exp\{-\mathrm{i}r\mathrm{sh}(w-\mathrm{i}\varphi)+\beta w\}dw + \\ + \text{constant} + O(r^{1-\beta}) = \mathrm{Re}\, e^{\mathrm{i}\alpha}\int_0^{\infty}\exp(-\tfrac{1}{2}\mathrm{i}r\, e^{-\mathrm{i}\varphi}u)u^{\beta-1}du + \text{id.} = \\ = \Gamma(\beta)\mathrm{Re}\, e^{\mathrm{i}\alpha}(\tfrac{1}{2}\mathrm{i}re^{-\mathrm{i}\varphi})^{-\beta} = \text{required expression.}$$

7. *Appendix B*

The main object of this appendix is the proof of some integral expressions which are connected with a well-known result derived by Macdonald in his discussion of Sommerfeld's problem. We shall start with a few useful auxiliary expressions.

For $a>0$ and $b>0$ we have

$$(7.1)\qquad K_0(\sqrt{a^2+b^2}) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-a\mathrm{ch}u + \mathrm{i}b\mathrm{sh}u}\, du,$$

$$(7.2)\qquad \int_{-\infty}^{\infty} K_0(\sqrt{a^2t^2+b^2})dt = \frac{\pi e^{-b}}{a},$$

$$(7.3)\qquad \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-a\mathrm{ch}u}\,\mathrm{ch}\tfrac{1}{2}u\, du = \sqrt{\frac{\pi}{2a}}\, e^{-a}.$$

The result (7.1) is well-known and has already been used quite often on the preceding pages. The result (7.2) can easily be derived from (7.1). The result (7.3) can be obtained in a simple way by taking $\mathrm{sh}\tfrac{1}{2}u$ as new variable of integration. The following expression, however, is somewhat more difficult to prove.

For $a>0$ and $-\pi<\alpha<\pi$ we have

$$(7.4)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a\mathrm{ch}u}\frac{1}{\mathrm{ch}\tfrac{1}{2}(u-\mathrm{i}\alpha)}du = e^{a\cos\alpha}\,\mathrm{erfc}\sqrt{a(1+\cos\alpha)}.$$

If the left-hand side of (7.4) is denoted by $f(a,\alpha)$ then by the following partial differentiation the inconvenient factor in the denominator can be removed

$$\frac{\partial}{\partial a}\{e^{-a\cos\alpha}f(a,\alpha)\} = -\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-a(\mathrm{ch}u+\cos\alpha)}\,\mathrm{ch}\tfrac{1}{2}(u+\mathrm{i}\alpha)\, du.$$

Now taking advantage of the fact that only the even part of the integrand adds to the result and by using (7.3) we obtain

$$\frac{\partial}{\partial a}\{\mathrm{e}^{-a\cos\alpha}f(a,\alpha)\} = -\left(\frac{1+\cos\alpha}{\pi a}\right)^{\frac{1}{2}}\mathrm{e}^{-a(1+\cos\alpha)}.$$

Integrating this between a and ∞ we are immediately led to the required result (7.4).

Now we consider the following double integral

$$\theta(a,b,\gamma) \overset{\text{def}}{=} \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-a\,\mathrm{ch}u-b\,\mathrm{ch}v}\frac{1}{\mathrm{ch}\frac{1}{2}(u+v-\mathrm{i}\gamma)}\,du\,dv, \tag{7.5}$$

where $a>0$, $b>0$ and $-\pi<\gamma<\pi$.

Introducing new variables x and y by means of $u=x+y$ and $v=x-y$ we obtain

$$\theta(a,b,\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dx}{\mathrm{ch}(x-\frac{1}{2}\gamma)}\int_{-\infty}^{\infty}\exp-\{a\,\mathrm{ch}(x+y)+b\,\mathrm{ch}(x-y)\}dy =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} K_0(\sqrt{(a+b)^2\mathrm{ch}^2x-(a-b)^2\mathrm{sh}^2x})\,\frac{dx}{\mathrm{ch}(x-\frac{1}{2}\mathrm{i}\gamma)},$$

where use has been made of (7.1). Since $\theta(a,b,\gamma)$ is an even function of γ we may also write

$$\theta(a,b,\gamma) = \frac{1}{\pi}\int_{-\infty}^{\infty} K_0(\sqrt{(a+b)^2+4ab\,\mathrm{sh}^2x})\,\frac{\cos\frac{1}{2}\gamma\,\mathrm{ch}x}{\mathrm{sh}^2x+\cos^2\frac{1}{2}\gamma}\,dx.$$

If now the substitution $\mathrm{sh}x=t\cos\frac{1}{2}\gamma$ is made we find the following result

$$\theta(a,b,\gamma) = \frac{1}{\pi}\int_{-\infty}^{\infty} K_0(\sqrt{(a+b)^2(t^2+1)-c^2t^2})\,\frac{dt}{t^2+1}, \tag{7.6}$$

where $c^2=a^2+b^2-2ab\cos\gamma$.

It appears that in fact $\theta(a,b,\gamma)$ only depends on the two parameter groups $s=a+b$ and c. An alternative expression for θ can easily be obtained from (7.6) in the following way. If the right-hand side of (7.6) is differentiated with respect to s again the denominator disappears viz.

$$\frac{\partial\theta}{\partial s} = \frac{s}{\pi}\int_{-\infty}^{\infty}\frac{K_0'(\sqrt{s^2(t^2+1)-c^2t^2})}{\sqrt{s^2(t^2+1)-c^2t^2}}\,dt. \tag{7.7}$$

The right-hand side of this relation is elementary since it follows from (7.2) by differentiation with respect to b^2 that

$$\int_{-\infty}^{\infty}\frac{K_0'(\sqrt{a^2t^2+b^2})}{\sqrt{a^2t^2+b^2}}\,dt = -\frac{\pi\mathrm{e}^{-b}}{ab}. \tag{7.8}$$

Therefore (7.7) reduces to

$$\frac{\partial\theta}{\partial s} = -\frac{e^{-s}}{\sqrt{s^2-c^2}}$$

so that by integration it is finally obtained that

$$\theta(a,b,\gamma) = \int\limits_{a+b}^{\infty} \frac{e^{-s}}{\sqrt{s^2-c^2}}\,ds, \tag{7.9}$$

or in an equivalent form due originally to Macdonald

$$\theta(a,\,b,\,\gamma) = \int\limits_{w_0}^{\infty} e^{-c\,\mathrm{ch}w}\,dw, \tag{7.10}$$

where $\mathrm{ch}w_0 = \frac{a+b}{c}$, $w_0 > 0$.

This rather elegant result may also be derived in the following more direct way. Let a and b be sides of a triangle ABC and γ the angle at C. Let the remaining elements be α, β and c. Further let h be the altitude from C and p, q the projections of a and b upon AB.

Then we have

$$p = a\cos\beta, \qquad q = b\cos\alpha, \qquad h = a\sin\beta = b\sin\alpha.$$

Now the expression (7.5) will be reduced in the following way. Imaginary translation of u and v gives

$$\theta(a,b,\gamma) = \frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\mathrm{i}a\,\mathrm{sh}(u-\mathrm{i}\beta)-\mathrm{i}b\,\mathrm{sh}(v-\mathrm{i}\alpha)}\frac{1}{\mathrm{ch}\frac{1}{2}(u+v)}\,du\,dv,$$

which can be written as

$$\theta(p,q,h) = \frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-h(\mathrm{ch}u+\mathrm{ch}v)-\mathrm{i}(p\,\mathrm{sh}u+q\,\mathrm{sh}v)}\frac{1}{\mathrm{ch}\frac{1}{2}(u+v)}\,du\,dv.$$

By differentiation with respect to h the denominator is removed viz.

$$\frac{\partial\theta}{\partial h} = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-h(\mathrm{ch}u+\mathrm{ch}v)-\mathrm{i}(p\,\mathrm{sh}u+q\,\mathrm{sh}v)}\,\mathrm{ch}\tfrac{1}{2}(u-v)\,du\,dv.$$

If now u and v are transformed back to their original value we obtain

$$\frac{\partial\theta}{\partial h} = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-a\,\mathrm{ch}u-b\,\mathrm{ch}v}\,\mathrm{ch}\tfrac{1}{2}(u-v+\mathrm{i}(\alpha-\beta))\,du\,dv =$$

$$= -\frac{1}{2\pi}\cos\tfrac{1}{2}(\alpha-\beta)\int\limits_{-\infty}^{\infty} e^{-a\,\mathrm{ch}u}\,\mathrm{ch}\tfrac{1}{2}u\,du\int\limits_{-\infty}^{\infty} e^{-b\,\mathrm{ch}v}\,\mathrm{ch}\tfrac{1}{2}v\,dv,$$

so that by using (7.3).

$$\frac{\partial\theta}{\partial h} = -\frac{\cos\frac{1}{2}(\alpha-\beta)}{\sqrt{ab}}\,e^{-(a+b)},$$

which by simple trigonometry may be changed into

$$\frac{\partial\theta}{\partial h} = -\frac{\sin\alpha+\sin\beta}{\sqrt{(a+b)^2-c^2}}\,e^{-(a+b)}. \tag{7.11}$$

The introduction of $s=a+b$ as new independent parameter together with p and q suggests itself. Since

$$\frac{\partial s}{\partial h} = \sin\alpha+\sin\beta,$$

we obtain from (7.11)

$$\frac{\partial\theta}{\partial s} = -\frac{e^{-s}}{\sqrt{s^2-c^2}},$$

from which (7.10) immediately follows as shown above.

REFERENCES

MacDonald, H. M., Proc. Lond. Math. Soc. (2) 410 (1915).
Lauwerier, H. A., Solutions of the equation of Helmholtz in an angle. I and II.
I. Kon. Ned. Ak. v. Wet. Proc. **A 62**, No. 5, 475–488 (1959).
II. ibid. **A 63** No. 4, 355–372 (1960).