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The Hilbert Problem for Generalized Functions

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1. Introduction

We consider analytic functions $g(w)$ in the complex w-plane with $w=u+iv$. If $g(w)$ is holomorphic in the upper half plane $v>0$, this will be indicated by writing $g^+(w)$. Similarly $g^-(w)$ denotes an analytic function which is holomorphic in the lower half plane $v<0$. Thus g^+ and g^- are sectionally holomorphic functions. The limits of $g^+(w)$ and $g^-(w)$ on the real axis are indicated by $g^+(u)$ and $g^-(u)$. Thus we have

$$(1.1)\qquad \begin{aligned} g^+(u) &= \lim_{v\to+0} g^+(u+iv)\,, \\ g^-(u) &= \lim_{v\to-0} g^-(u+iv)\,. \end{aligned}$$

We shall consider the following problem. Let $g(u)$ be some function of the real variable u; then functions $g^+(w)$ and $g^-(w)$ are sought such that

$$(1.2)\qquad g^+(u)+g^-(u)=g(u)\,.$$

In many problems of mathematical physics (*cf.* MUSKHELISHVILI [*1*]) a slightly more difficult problem occurs, *viz*

$$(1.3)\qquad g^+(u)+k(u)\,g^-(u)=g(u)\,,$$

where $k(u)$ also is a given function on the real axis. The latter problem is usually called the Hilbert problem. An extensive treatment of the latter problem is given by MUSKHELISHVILI [*2*] and NOBLE [*3*]. It is shown that by factorization of $k(u)$, *i.e.* by writing $k(u)$ as

$$(1.4)\qquad k(u)=k^+(u)\,k^-(u)\,,$$

the problem (1.3) may be reduced to the simpler problem (1.2). In fact, using (1.4), we may write the relation (1.3) as

$$(1.5)\qquad \frac{g^+(u)}{k^+(u)}+k^-(u)\,g^-(u)=\frac{g(u)}{k^+(u)}\,,$$

which indeed is of the form (1.2). The factorization of $k(u)$ is also a problem of the form (1.2) since, at least formally, the relation (1.4) may be written as

$$(1.6)\qquad \ln k^+(u)+\ln k^-(u)=\ln k(u)\,.$$

A rigorous treatment is given in MUSKHELISHVILI [*2*], but with the rather drastic restriction to classes of Hölder-continuous functions. In this paper we shall consider the simpler problem (1.2), henceforth referred to as the Hilbert problem, and, more particularly, we shall consider what happens if $g(u)$ is a generalized function.

In order to restrict the class of solutions of (1.2) it is usually required that $g^+(w)$ and $g^-(w)$ be of finite degree at infinity. Then two particular solutions of (1.2) differ by a polynomial only. Provided $g(u)$ satisfies certain conditions as regards integrability and behavior at infinity, the solution of (1.2) may be obtained in the following way:

$$(1.7)\qquad \begin{aligned} g^+(w) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)}{t-w}\,dt + P(w), &\quad \operatorname{Im} w > 0,\\ g^-(w) &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)}{t-w}\,dt - P(w), &\quad \operatorname{Im} w < 0, \end{aligned}$$

where $P(w)$ is an arbitrary polynomial.

A second way of solving (1.2) is as follows. Let $g(u)$ be the FT (Fourier transform) of $f(x)$

$$(1.8)\qquad g(u) = \int_{-\infty}^{\infty} e^{ixu} f(x)\,dx,$$

then (1.2) is solved by

$$(1.9)\qquad g^+(w) = \int_{0}^{\infty} e^{ixw} f(x)\,dx, \qquad g^-(w) = \int_{-\infty}^{0} e^{ixw} f(x)\,dx.$$

Of course both methods are entirely equivalent for ordinary functions. In fact, substitution of

$$(1.10)\qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux} g(u)\,du$$

in (1.9) and interchanging of the order of integrations immediately leads to (1.7). Here we may quote the classical result of TITCHMARSH [4], who states that the Hilbert problem can be solved in either way if $g(u) \in L^p(-\infty, \infty)$ with $p>1$. In that case the limits (1.1) should be taken in the sense of limits in the mean or limits almost everywhere.

The following example,

$$(1.11)\qquad \frac{1}{1+u^2} = \frac{i}{2}\,\frac{1}{u+i} - \frac{i}{2}\,\frac{1}{u-i},$$

can easily be treated by means of either (1.7) or (1.9) and is covered by TITCHMARSH's theorem. However, difficulties arise in the following case:

$$(1.12)\qquad g(u) = \operatorname{sgn} u, \quad g^+(w) = \frac{1}{2} - \frac{1}{\pi i}\ln w, \quad g^-(w) = \frac{1}{2} + \frac{1}{\pi i}\ln w$$

with the usual interpretation of $\ln w$ at the negative real axis. Even simpler cases which cannot be derived by the means given above are

$$(1.13)\qquad g(u) = \ln|u|, \qquad g^+(w) = \tfrac{1}{2}\ln w, \qquad g^-(w) = \tfrac{1}{2}\ln w,$$

and

$$(1.14)\qquad g(u) = e^{iu}, \qquad g^+(w) = e^{iw}, \qquad g^-(w) = 0.$$

In this paper it will be shown that by using the theory of generalized functions or distributions the latter results may be obtained in a fully legitimate way from (1.9). In order to prepare our way some fundamental notions of the theory of distributions are briefly discussed in the following section in which we have chosen the approach as given by GEL'FAND & SHILOV [5] in their most commendable book. In the fourth section the notions of upper and lower regular functions are introduced. The main results, which extend those of TITCHMARSH, are given in the form of two theorems. In Section 5 we discuss the so-called bisection of a distribution, *i.e.* its separation into a part with a positive support and a part with a negative support. By using the results of Sections 3 and 4 the generalized Hilbert problem can easily be solved, and this is in fact done in Section 5. The technique is illustrated by a few typical examples. In the last section some remarks are made on related topics such as Hilbert transform and Wiener-Hopf factorization.

2. Theory of distributions in a nutshell

In this section some topics of the theory of distributions will be summarized. We shall mainly follow the introduction to this theory as given by GEL'FAND & SHILOV [5] in their book, which is highly recommended. It is sufficient to consider distributions $f(x)$ for real x only.

A distribution or generalized function $f(x)$ is defined as a linear continuous functional (f, φ) on a class of (real) testing functions K in which a suitable topology is defined. Thus (f, φ) satisfies the following conditions:

1. $(f, \alpha_1\varphi_1+\alpha_2\varphi_2)=\alpha_1(f, \varphi_1)+\alpha_2(f, \varphi_2)$ (linearity);
2. $\varphi_n\to\varphi$ implies $(f, \varphi_n)\to(f, \varphi)$ (continuity).

The class K is the collection of all infinitely differentiable functions vanishing outside some interval. Convergence in K to zero $\varphi_n\to 0$ means that all φ_n vanish outside one and the same interval and that $\max_x |\varphi_n^{(j)}(x)|\to 0$ for $n\to\infty$ and for each fixed j. If $f(x)$ is a locally integrable function, we simply have

$$(f, \varphi)\overset{\text{def}}{=}\int \overline{f(x)}\,\varphi(x)\,dx. \tag{2.1}$$

Distributions that are not of this form are called singular. A typical example is DIRAC's delta-function $\delta(x)$, distributionally defined as

$$(\delta, \varphi)=\varphi(0). \tag{2.2}$$

The distributions form the linear space K', the conjugate of K. Convergence in K' is defined as follows:

$$f_n\to f \text{ means } (f_n, \varphi)\to(f, \varphi) \text{ for all } \varphi\in K.$$

A distribution f has a derivative f' which is defined by

$$(f', \varphi)=-(f, \varphi'). \tag{2.3}$$

The delta-function $\delta(x)$ is the derivative of the unit-step function $\vartheta(x)$

$$\vartheta(x)=\begin{cases}0 & \text{for } x<0,\\ 1 & \text{for } x\geqq 0.\end{cases} \tag{2.4}$$

On the other hand, f is called a primitive of f'. A distribution has a primitive which is unique apart from an additive constant. By using fundamental theorems of RIESZ and HAHN-BANACH it can be shown that every singular distribution can be obtained from a locally integrable function by a finite number, say m, of differentiations. The least number for which this is true is called the order of the distribution. The example $\delta = \vartheta'$ shows that $\delta(x)$ is a singular distribution of order one.

We shall now give a few important examples of regular and singular distributions. In general there are two ways of constructing singular distributions; either by means of repeated differentiation of a locally integrable function, or by analytic continuation with respect to some complex parameter.

(*a*) The distribution x_+^λ is defined as follows. For Re $\lambda > -1$ it is identified with the function

(2.5) $$x_+^\lambda = x^\lambda \text{ for } x > 0, \qquad x_+^\lambda = 0 \text{ for } x < 0.$$

For other values of the complex parameter λ it is defined by analytic continuation of the functional (x_+^λ, φ). In this way a distribution is obtained for all complex values of λ with the exception of $\lambda = -1, -2, -3, \ldots$. This distribution satisfies

(2.6) $$\frac{d}{dx} x_+^\lambda = \lambda x_+^{\lambda-1}, \qquad \lambda \neq 0, -1, -2, \ldots.$$

In a similar way x_-^λ is defined by starting from

(2.7) $$x_-^\lambda = 0 \text{ for } x > 0, \qquad x_-^\lambda = |x|^\lambda \text{ for } x < 0.$$

(*b*) The distributions $|x|^\lambda$ and $|x|^\lambda \operatorname{sgn} x$ are defined by

(2.8) $$|x|^\lambda = x_+^\lambda + x_-^\lambda, \qquad |x|^\lambda \operatorname{sgn} x = x_+^\lambda - x_-^\lambda.$$

(*c*) The distribution x^{-m} ($m = 1, 2, 3, \ldots$) can be derived from the distributions defined above. However, an independent definition is obtained from the differentiation rule

(2.9) $$x^{-1} = \frac{d}{dx} \ln|x|,$$

and

(2.10) $$x^{-m} = \frac{-1}{m-1} \frac{d}{dx} x^{-m+1}, \qquad m = 0, 1, 2, \ldots.$$

(*d*) The distributions $(x \pm i0)^\lambda$ are defined as

(2.11) $$(x \pm i0)^\lambda = \lim_{\lambda \to \pm 0} (x + iy)^\lambda.$$

We have for $\lambda \neq -1, -2, -3, \ldots$

(2.12) $$(x + i0)^\lambda = x_+^\lambda + e^{i\lambda\pi} x_-^\lambda, \qquad (x - i0)^\lambda = x_+^\lambda + e^{-i\lambda\pi} x_-^\lambda,$$

and next for $m = 1, 2, 3, \ldots$

(2.13) $$(x \pm i0)^{-m} = x^{-m} \pm (-1)^m \frac{\pi i}{(m-1)!} \delta^{(m-1)}(x).$$

By Fourier transformation of the testing functions $\varphi \in K$ a space Z of testing functions $\psi(u)$ is obtained according to

$$\psi(u) = \int_{-\infty}^{\infty} e^{ixu} \varphi(x)\, dx. \tag{2.14}$$

The testing functions of Z are entire functions in the complex w-plane with $w = u + iv$. They have the characteristic property that for $k = 0, 1, 2, \ldots$

$$|w^k \psi(w)| < A_k\, e^{-a|v|} \tag{2.15}$$

where the constants A_k and the coefficient a depend only on the testing function ψ.

The Fourier transform of the distribution $f \in K'$ is defined as a functional g on Z such that

$$(g, \psi) = 2\pi (f, \varphi). \tag{2.16}$$

For ordinary functions this is nothing else than the well known Parseval equality. For distributions this is taken as a definition. The distributions $g(u)$ form the space Z', the conjugate of Z. Thus by (2.16) the $(1-1)$ correspondence between K and Z is carried over to K' and Z'. If $f(x)$ is a locally integrable function, the Fourier transform $g(u)$ can also be obtained as the distributional limit of ordinary functions

$$g(u) = \lim_{n\to\infty} \int_{-n}^{n} e^{ixu} f(x)\, dx. \tag{2.17}$$

We conclude this section with a condensed table of some important Fourier transforms.

$f(x)$	$g(u)$
$f'(x)$	$-i\,u\,g(u)$
$i\,x\,f(x)$	$g'(u)$
1	$2\pi\,\delta(u)$
$\delta(x)$	1
$\frac{1}{2}\operatorname{sgn} x$	i/u
$(i\pi x)^{-1}$	$\operatorname{sgn} u$
$\vartheta(x)$	$i/u + \pi\,\delta(u)$
$(i\pi x)^{-1} + \delta(x)$	$2\vartheta(u)$
$x_+^{\lambda-1}$	$\Gamma(\lambda)\,(u+i0)^{-\lambda} \exp \frac{1}{2}\lambda\pi i$
$(x-i0)^{-\lambda} \exp -\frac{1}{2}\lambda\pi i$	$2\pi\, u_+^{\lambda-1}$
$(\ln x_+)'$	$\frac{1}{2}\pi i + \Gamma'(1) - \ln(u+i0)$
$(\ln x_-)'$	$\frac{1}{2}\pi i - \Gamma'(1) + \ln(u-i0)$
$-\frac{1}{2}(\operatorname{sgn} x \ln\lvert x\rvert)' + \Gamma'(1)\,\delta(x)$	$\ln\lvert u\rvert$
$i \ln x_+$	$\dfrac{\ln(u+i0)}{u+i0} - \dfrac{\frac{1}{2}\pi i + \Gamma'(1)}{u+i0}$
$-i \ln x_-$	$\dfrac{\ln(u-i0)}{u-i0} - \dfrac{-\frac{1}{2}\pi i + \Gamma'(1)}{u-i0}$
$\ln\lvert x\rvert$	$-\pi(\operatorname{sgn} u \ln\lvert u\rvert)' + 2\pi\,\Gamma'(1)$

Table of generalized Fourier transforms

3. Upper and lower regular functions

An analytic function $g(w)$ of the complex variable $w=u+iv$ is said to be an upper regular function if it is regular in the half plane $v>0$ and if there exists a constant p such that in any half plane $v\geqq\delta>0$ the following inequality is satisfied:

$$|g(w)|<C_\delta(1+|w|)^p, \tag{3.1}$$

where the constant C_δ may depend on δ. A lower regular function is defined in a similar way with respect to the lower half plane $v<0$. The classes of upper and lower regular functions will be denoted respectively by G^+ and G^-.

Let $g(u+iv)\in G^+$. This function may be interpreted as a distribution with the parameter v with respect to the testing space Z. It will be shown that by taking the distributional limit $v\to+0$ a distribution is obtained which is the Fourier transform of a distribution from K' which vanishes for $x<0$.

Before proving this theorem we first give an example. Let $g(w)=iw^{-1}\in G^+$. Then for $v\to+0$ we obtain the distribution $iu^{-1}+\pi\,\delta(u)$ which is the FT of the ordinary function $\vartheta(x)\in K'$.

The proof of the statement made above is as follows. We first assume that $p<-1$. Consider the integral $\int\exp(-iwx)\,g(w)\,dw$ taken around the rectangle with corners at $\pm a+iv_1$, $\pm a+iv_2$ where $0<v_1<v_2$. For $a\to\infty$ the contributions of the vertical sides tend to zero, which shows that the absolutely convergent integral

$$e^{vx}\int_{-\infty}^{\infty}e^{-iux}g(u+iv)\,du \tag{3.2}$$

does not depend on v and represents a function of x, say $2\pi f(x)$. From the inequality (3.1) it follows that $f(x)=O(e^{vx})$ so that for $x<0$ we must have $f(x)\equiv0$. The relation (3.2) means that $g(u+iv)$ is the Fourier transform in the ordinary sense of the function $e^{-vx}f(x)$. The function $f(x)$ may not have a Fourier transform in the ordinary sense but it does have one in the distributional sense. The latter distribution, belonging to Z', will be called $g(u)$. We shall now show that $g(u+iv)\to g(u)$. In fact using the definition (2.16), we have for any testing function $\psi(u)\in Z$

$$\big(g(u+iv),\psi(u)\big)=2\pi\big(e^{-vx}f(x),\varphi(x)\big)\to2\pi\big(f(x),\varphi(x)\big)=$$
$$=\big(g(u),\psi(u)\big).$$

If now $p\geqq-1$, there is a positive integer n such that the theorem holds for $w^{-n}g(w)$. Hence this function has a distributional limit $h(u)$. Then it is easily seen that $g(w)$ tends to the distributional limit $g(u)=u^n h(u)$. Since $h(u)=\mathrm{FT}\,k(x)$ implies $(-iu)^n h(u)=\mathrm{FT}\,k^{(n)}(x)$, it follows that the limit $g(u)$ is the FT of a distribution which vanishes for $x<0$ and which is the n^{th} derivative of an ordinary function.

Again assuming $p<-1$, the inverse Fourier transform $f(x)$ of $g(u)$ is bounded at $+\infty$, since

$$f(x)=O(\exp\varepsilon x) \tag{3.3}$$

for all $\varepsilon>0$.

If on the other hand we start from a function $f(x)$ which vanishes for $x<0$ and which is bounded as (3.3), then the FT of $e^{-vx}f(x)$ yields an upper regular-function $g(w)$ for which the theorem holds.

Similar arguments apply in the case of a lower regular function $g(w)$ and a function $f(x)$ which vanishes for $x>0$.

The results derived above may be stated in the form of the following theorems which we give in the version for upper regular functions only.

Theorem 3.1. *An upper regular function $g(u+iv)$ has for $v\to+0$ a distributional limit $g(u)\in Z'$ which is the Fourier transform of a distribution $f(x)\in K'$ which vanishes for $x<0$. Moreover $f(x)$ has a continuous primitive of some order which is bounded at $+\infty$ as $O(\exp\varepsilon x)$ for all $\varepsilon>0$.*

Theorem 3.2. *Let $F(x)$ be a continuous function which vanishes for $x<0$ and which is bounded at $+\infty$ as $O(\exp\varepsilon x)$ for all $\varepsilon>0$. Then the Fourier transform of any distributional derivative of $F(x)$ may be obtained as the distributional limit of an upper regular function.*

4. Bisection of a distribution

In general there is no unique way of defining a separation of a given distribution $f(x)\in K'$ into a part with a positive support $(0,\infty)$ and a part with a negative support $(-\infty,0)$. Examples where difficulties occur are $\delta(x)$ and x^{-1}. However, if $f(x)$ is a locally integrable function, we have the trivial splitting

$$(4.1)\qquad f_+(x)=\begin{cases}f(x) & \text{for } x>0,\\ 0 & \text{for } x<0,\end{cases}\qquad f_-(x)=\begin{cases}0 & \text{for } x>0.\\ f(x) & \text{for } x<0.\end{cases}$$

This definition may be carried over without difficulty to a distribution which is regular at $x=0$, *i.e.* which is equivalent to a locally integrable function in some neighborhood of $x=0$.

We consider next a distribution $f(x)$ that is singular at $x=0$. However, this singularity is of finite order, and there exists a primitive $F(x)$ of $f(x)$ such that $F^{(m)}(x)=f(x)$ and $F(x)$ is regular at $x=0$. Let m be the least number for which this is true, then we define

$$(4.2)\qquad f_+(x)=\left(\frac{d}{dx}\right)^m F_+(x),\qquad f_-(x)=\left(\frac{d}{dx}\right)^m F_-(x).$$

Since $F(x)$ is not unique, this definition is likewise not unique. However since two m^{th} primitives of $f(x)$ differ by at most a polynomial of degree $m-1$, the bisection of $f(x)$ is determined up to a polynomial in the first $m-1$ derivatives of $\delta(x)$. Thus if f_+ and f_- form a particular bisection, the general bisection is given by

$$(4.3)\qquad \begin{aligned}&f_+(x)+a_0\,\delta(x)+a_1\,\delta'(x)+\cdots+a_{m-1}\,\delta^{(m-1)}(x),\\ &f_-(x)-a_0\,\delta(x)-a_1\,\delta'(x)-\cdots-a_{m-1}\,\delta^{(m-1)}(x),\end{aligned}$$

where $a_0, a_1, \ldots, a_{m-1}$ are arbitrary constants. We note that although the bisection of $f(x)$ is not unique the distributions $x^m f_+(x)$ and $x^m f_-(x)$ are uniquely determined.

In Section 2 it is shown by (2.8) that $|x|^\lambda$ and $|x|^\lambda\operatorname{sgn}x$ can be bisected in some standard way provided $\lambda\neq-1,-2,\ldots$. In the exceptional cases we may apply (4.3). Thus the general bisection of x^{-1} is determined by

$$x_+^{-1} = \frac{d}{dx} \ln x_+ + a\,\delta(x), \tag{4.4}$$

where a is an arbitrary constant. The singular distribution $(\ln x_+)'$ can easily be shown to be determined by the functional

$$\int_0^1 \frac{\varphi(x) - \varphi(0)}{x}\, dx + \int_1^\infty \frac{\varphi(x)}{x}\, dx. \tag{4.5}$$

We note that the product $x\,x_+^{-1}$ always gives $\vartheta(x)$.

5. The generalized Hilbert problem

In this section we shall discuss the following generalization of the Hilbert problem: to split a given distribution $g(u) \in Z'$ into two parts $g^+(u)$ and $g^-(u)$ which are the limits of upper and lower regular functions $g^+(w)$ and $g^-(w)$, respectively.

We know already that the solution of this problem is not unique and that to any particular solution an arbitrary polynomial may be added. In this way all solutions are obtained. If the problem has a solution, we know from Theorem (3.1) that $g(u)$ may be considered as the derivative of some order of a continuous function which is bounded at infinity as $O(\exp \varepsilon |x|)$ for all $\varepsilon > 0$.

On the other hand Theorem (3.2) shows that the generalized Hilbert transform can be solved for any distribution $g(u) \in Z'$ which can be derived from a continuous primitive which is bounded at infinity in the way indicated above. The more or less standard method of bisection described in the previous section leads to a certain class of lowest-order solutions. In fact the Hilbert problem may be solved as follows. First the inverse FT of $g(u)$ is determined. This is a distribution $f(x) \in K'$. Bisection of this gives the two parts (4.3) with arbitrary constants. Then the FT of these parts yield a class of solutions with m arbitrary constants

$$\begin{aligned} g^+(w) &= \int e^{ixw} f_+(x)\, dx + c_0 + c_1 w + \cdots + c_{m-1} w^{m-1}, \\ g^-(w) &= \int e^{ixw} f_-(x)\, dx - c_0 - c_1 w - \cdots - c_{m-1} w^{m-1}. \end{aligned} \tag{5.1}$$

To these solutions higher polynomials may also be added, but then the order at infinity of $g^+(w)$ and $g^-(w)$ will necessarily be increased.

In practice it is often easier to solve the Hilbert problem first for some derivative of $g(u)$. Then the solution for $g(u)$ may be determined by integration.

In order to illustrate the technique a few examples will be given.

(a) $$g(u) = \exp i u.$$

This is the FT of $\delta(x-1)$. Bisection gives $f_+(x) = \delta(x-1)$ and $f_-(x) = 0$. This leads at once to the result (1.14).

(b) $$g(u) = \delta(u).$$

This is the FT of $(2\pi)^{-1}$. Bisection gives $f_+(x) = (2\pi)^{-1}\vartheta(x)$ and $f_-(x) = (2\pi)^{-1}$ $\vartheta(-x)$. Hence we obtain

$$g_+(w) = \frac{-1}{2\pi i}\,\frac{1}{w}, \qquad g_-(w) = \frac{1}{2\pi i}\,\frac{1}{w}.$$

The corresponding Hilbert split is

(5.2) $$\frac{1}{u-\mathrm{i}0}-\frac{1}{u+\mathrm{i}0}=2\pi\,\mathrm{i}\,\delta(u).$$

(c) $$g(u)=\operatorname{sgn} u.$$

Since $g'(u)=2\delta(u)$, this problem may be reduced to the previous example. In fact, by integration we find

$$g_+(w)=\frac{-1}{\pi\,\mathrm{i}}\ln w+c_1,\qquad g_-(w)=\frac{1}{\pi\,\mathrm{i}}\ln w+c_2,$$

where c_1 and c_2 are constants. By letting $w\to+1$ we see that $c_1+c_2=1$. Hence a possible solution is obtained by taking $c_1=c_2=\frac{1}{2}$. In this way the result (1.12) is obtained. The corresponding Hilbert split may be written as

(5.3) $$\ln(u+\mathrm{i}0)-\ln(u-\mathrm{i}0)=2\pi\,\mathrm{i}\,\vartheta(-u).$$

A direct treatment of this problem is slightly more complicated. The inverse FT of sgn u is $f(x)=(\mathrm{i}\,\pi x)^{-1}$. Bisection gives $f_+(x)=(\mathrm{i}\,\pi)^{-1}(\ln x_+)'$. Hence $g_+(w)$ becomes

$$-\frac{w}{\pi}\int_0^\infty e^{\mathrm{i}xw}\ln x\,dx=\frac{1}{\pi\,\mathrm{i}}\int_0^\infty e^{-t}\ln\frac{\mathrm{i}t}{w}\,dt,$$

so that

$$g_+(w)=\frac{1}{\pi\,\mathrm{i}}\left(\Gamma'(1)+\frac{1}{2}\pi\,\mathrm{i}-\ln w\right),$$

which is an equivalent result.

(d) $$g(u)=\ln|u|.$$

We first consider the derivative u^{-1}. This is the FT of $-\frac{1}{2}\mathrm{i}\operatorname{sgn} x$, bisection of which gives $f_+(x)=-\frac{1}{2}\mathrm{i}\,\vartheta(x)$ and $f_-(x)=\frac{1}{2}\mathrm{i}\,\vartheta(-x)$. The FT is $(2w)^{-1}$ in both cases. Integration at once gives the result (1.13). The corresponding Hilbert split is

(5.4) $$\ln(u+\mathrm{i}0)+\ln(u-\mathrm{i}0)=2\ln|u|.$$

6. Miscellaneous

In this section are briefly discussed a few related topics which may be considered for further research.

The Hilbert transform of an ordinary function $g(u)$ is usually defined by

(6.1) $$h(u)=\mathrm{HT}\{g(u)\}=\frac{1}{\pi\,\mathrm{i}}\int_0^\infty\frac{g(t+u)-g(t-u)}{t}\,dt.$$

If $g(u)$ is the FT of the ordinary function $f(x)$, we also have

(6.2) $$h(u)=\mathrm{FT}\{\mathrm{i}\,f(x)\operatorname{sgn} x\}.$$

The latter relation may be taken as the definition of the Hilbert transform of a distribution $g(u)\in Z'$. According to the notation of Section 4 we may write

(6.3) $$h(u)\overset{\mathrm{def}}{=}\mathrm{i}\,\mathrm{FT}\{f_+(x)-f_-(x)\}.$$

The latter definition is unique to within a polynomial the degree of which is determined by the order of the distribution $f(x)$ at $x=0$. As a consequence of (6.3) we have the inversion formulae

$$h(u) = \mathrm{HT}\{g(u)\}, \qquad g(u) = -\mathrm{HT}\{h(u)\}. \tag{6.4}$$

Examples.

(a) $$\mathrm{HT}(u^{-1}) = \pi\,\delta(u),$$

(b) $$\mathrm{HT}(\operatorname{sgn} u) = -\frac{2}{\pi}\ln|u| + \text{constant}.$$

The discussion of the Hilbert problem (1.3) needs the factorization (1.4). This may be carried out by means of (1.6). In some cases it is possible to extend the meaning of the latter relation by admitting discontinuities in $k(u)$. This remark may be illustrated by considering the following example

$$g^+(u) + \operatorname{sgn} u\, g^-(u) = g(u). \tag{6.5}$$

The factorization (1.4) is formally solved by the Hilbert problem

$$\ln k^+(u) + \ln k^-(u) = \mathrm{i}\,\pi\,\vartheta(-u). \tag{6.6}$$

This has the solution

$$\ln k^+(u) = \tfrac{1}{2}\ln(u+\mathrm{i}\,0), \qquad \ln k^-(u) = -\tfrac{1}{2}\ln(u-\mathrm{i}\,0). \tag{6.7}$$

In fact we may take

$$k^+(w) = w^{\frac{1}{2}}, \qquad k^-(w) = w^{-\frac{1}{2}}, \tag{6.8}$$

so that (6.5) can be replaced by the simpler problem

$$\frac{g^+(u)}{(u+\mathrm{i}\,0)^{\frac{1}{2}}} + \frac{g^-(u)}{(u-\mathrm{i}\,0)^{\frac{1}{2}}} = \frac{g(u)}{(u+\mathrm{i}\,0)^{\frac{1}{2}}}. \tag{6.9}$$

The final remark we wish to make concerns the formal solution (1.7). It would be worthwhile to give a direct meaning, without using Fourier transforms, to the Cauchy integrals when $g(u)$ is a distribution of some class.

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