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The North Sea Problem I

General considerations concerning the hydrodynamical problem of the motion of the North Sea

by

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1. Introduction

On February 1st 1953 the South-Western part of the Netherlands was stricken by a flood disaster unsurpassed in the memory of this country. In order to design measures for preventing similar disasters in the future the government appointed a committee, consisting of prominent engineers with Ir A.G. Maris as Chairman. This so-called Delta-committee took several scientific institutions as advisers. The Mathematical Centre was asked to analyse the available statistical data on high tides in order to predict, as far as possible, the frequencies of extremely high tides. During the investigations this task was extended by an econometric study of the protection of the low areas against floods and also by a hydrodynamic study of the influence of a storm on the level of the sea.

D. van Dantzig was charged with the research on all these subjects. He carried this out in cooperation with a number of scientific workers of the statistical department and the applied mathematics department of the Mathematical centre. The present series of papers under the common title "The North Sea problem" contains a number of results obtained at the latter department with reference to the hydrodynamical problem.

Although Van Dantzig himself wrote relatively little on this subject and in his publications restricted himself mainly to reviewing the work of others, he knew to inspire his co-workers who profited time and again from his constructive mind and critical remarks. Already at the occasion of the International Congress at Amsterdam of 1954 Van Dantzig (1) delivered an address which made a strong impression on the audience and in which some aspects of the statistical and hydrodynamical North Sea problems were treated. Two years later (2) in a speech before the "Koninklijke Nederlandse Akademie van Wetenschappen" he reviewed the work

1) Deceased July 22nd 1959.

carried out at the applied mathematics department on the hydrodynamical problem. Further in 1958 (4) in an address on the occasion of a meeting of the GAMM at Saarbrücken he gave a survey of some recent results of the hydrodynamical problem. In the same year (3) he published a paper in the Proceedings of the Kon.Ned. Ak.v.Wet. containing his solution of a boundary value problem [1].

Already during the life of Van Dantzig it was planned to publish the research carried out at the Mathematical Centre in connection with the hydrodynamical problem in the form of a series of papers. The present paper is the first of a set of probably six papers written by the second author in memory of Van Dantzig. In this series a broad survey of the hydrodynamical North Sea problem is given. In it both old material contained in reports of the Mathematical Centre and new results obtained after Van Dantzig's death will be discussed. Part of the material of the present paper is borrowed from Van Dantzig (1) (4) and Lauwerier (1). For that reason Van Dantzig may be considered as a posthumous co-author although the second author is responsible for the contents.

An adequate mathematical treatment of the hydrodynamical problem is only possible at the cost of a number of simplifications the majority of which hardly affect the final result. In the first place the hydrodynamical equations are simplified. The vertical component of the velocity of the stream is neglected, the equations are linearized, and the coefficient of Coriolis is assumed constant. For the North Sea basin these simplifications are quite acceptable. In the second place the form of the basin is replaced by a simple mathematical model. This implies that the influence of irregularities of the coast and of variations of the depth are neglected. The North Sea can be conveniently represented by a rectangle which is bounded on three sides by coasts and which borders on an infinitely deep ocean on the remaining side. This means that the influence of e.g. the Channel is neglected. The grand total of all simplifications cannot be neglected if it is required to know the exact elevation at a given time at a given spot. But by ignoring local circumstances they permit us to obtain a clear picture of the overall motion and elevation of the surface.

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1) A second solution of this boundary value problem has been given in Lauwerier (3).

The rectangular model of the North Sea mentioned above suffers from two disadvantages. In the first place the leak of the Channel is neglected. In the second place the assumption of a uniform depth is rather drastic. In reality the depth increases gradually in the direction of the ocean. However, in a subsequent paper the stationary state of a sea with an exponentially increasing depth under a non-uniform windfield will be discussed. The mathematical difficulties would be considerably reduced if the rotation of the earth were negligible. However, it has been found time and again that the rotation is an essential feature of the problem which cannot be left out of account.

With the simplifications discussed above the problem can be described by an elliptic partial differential equation with oblique boundary conditions. This type of problem has not yet been solved entirely. A number of partial results which are obtained at the Mathematical Centre will be reported in this and the following papers.

2. The mathematical problem

The linearized equations of motion are [1)2)]

$$(2.1)\quad \begin{cases} \dfrac{\partial u_z}{\partial t} - \Omega v_z = -g\dfrac{\partial \varphi}{\partial x} - \dfrac{1}{\rho}\dfrac{\partial p_a}{\partial x} - \dfrac{1}{\rho}\dfrac{\partial U_z}{\partial z} \\ \dfrac{\partial v_z}{\partial t} + \Omega u_z = -g\dfrac{\partial \varphi}{\partial y} - \dfrac{1}{\rho}\dfrac{\partial p_a}{\partial y} - \dfrac{1}{\rho}\dfrac{\partial V_z}{\partial z} . \end{cases}$$

Integration of these equations with respect to z gives

$$(2.2)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Omega v + gh\dfrac{\partial \varphi}{\partial x} = -\dfrac{h}{\rho}\dfrac{\partial p_a}{\partial x} + \dfrac{1}{\rho}(U_s - U_b) \\ \dfrac{\partial v}{\partial t} + \Omega u + gh\dfrac{\partial \varphi}{\partial y} = -\dfrac{h}{\rho}\dfrac{\partial p_a}{\partial y} + \dfrac{1}{\rho}(V_s - V_b) . \end{cases}$$

The surface of the sea is subjected to a tractive force which has the same direction as the velocity of the wind at sealevel and which has an absolute value determined by the following semi-empirical law [3)]

$$(2.3)\quad \sqrt{U_s^2 + V_s^2} = k\,\rho_a\, v_s^2 ,$$

where k is a dimensionless constant for which usually the value k=0.0025 is taken.

The fundamental assumption is made that the bottom friction is proportional to the total stream [4)]

$$(2.4)\quad \frac{1}{\rho} U_b = \lambda u , \quad \frac{1}{\rho} V_b = \lambda v ,$$

where λ is uniform and constant.

If next we put

$$(2.5)\quad \begin{aligned} U &= \frac{1}{\rho} U_s - \frac{h}{\rho}\frac{\partial p_a}{\partial x} \\ V &= \frac{1}{\rho} V_s - \frac{h}{\rho}\frac{\partial p_a}{\partial y} \end{aligned}$$

the equations of motion (2.2) can be written in the form [5)]

$$(2.6)\quad \begin{cases} (\dfrac{\partial}{\partial t} + \lambda)u - \Omega v + gh\dfrac{\partial \varphi}{\partial x} = U \\ (\dfrac{\partial}{\partial t} + \lambda)v + \Omega u + gh\dfrac{\partial \varphi}{\partial y} = V . \end{cases}$$

To this we add the equation of continuity [6)]

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1) Cf. list of symbols at the end of this paper.
2) Cf. J. Proudman (2), 44.
3) Cf. J. Proudman (1), 135.
4) Cf. J. Proudman (3) and W.F. Schalkwijk (1).
5) Cf. J.C. Schönfeld (1)
6) Cf. J. Proudman (1), ch.2.

(2.7) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0.$$

The system (2.6) and (2.7) is the starting-point for a great number of investigations. We note that the quantities λ, Ω and gh are assumed to be uniform and constant. By a proper choice of the units it can be attained that gh=1.

The sea will be represented by a domain D, its boundary by C, the coastal part of C by C_1, the oceanic part by C_2. The boundary conditions express the fact that the normal component of the total stream vanishes along C_1 and that the elevation is continuous along C_2. Since the invariable level of the ocean can be taken as the zero level it can be assumed that $\zeta=0$ along C_2.

We imagine that at a certain moment t=0 the sea is at rest. Then the system (2.6) and (2.7) will be subjected to a Laplace transformation according to

(2.8) $$\bar{\zeta}(x,y,p) = \int_0^\infty e^{-pt}\, \zeta(x,y,t)dt,$$

and similarly for u,v and U,V.
The system (2.6) and (2.7) is transformed into

(2.9) $$\begin{cases} (p+\lambda)\bar{u} - \Omega\,\bar{v} + \frac{\partial \bar{\zeta}}{\partial x} = \bar{U} \\ (p+\lambda)\bar{v} + \Omega\,\bar{u} + \frac{\partial \bar{\zeta}}{\partial y} = \bar{V} \end{cases}$$

and
(2.10) $$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + p\bar{\zeta} = 0.$$

The same system is obtained if a solution of (2.6) and (2.7) is sought in which the dependent variables contain the time in the form of an exponential factor exp pt only. This is of importance e.g. in the case of free and forced oscillatory motions.

By elimination of $\bar{u}$ and $\bar{v}$ from (2.9) and (2.10) the following non-homogeneous equation of Helmholtz is obtained

(2.11) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \kappa^2\right)\bar{\zeta} = \bar{F},$$

$$\bar{F} \stackrel{\text{def}}{=} \left(\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y}\right) + \operatorname{tg}\gamma\left(\frac{\partial \bar{V}}{\partial x} - \frac{\partial \bar{U}}{\partial y}\right),$$

(2.13) $$\kappa^2 \stackrel{\text{def}}{=} p(p+\lambda)+\Omega^2 p(p+\lambda)^{-1}, \quad \operatorname{Re}\kappa \geq 0,$$

and
(2.14) $$\operatorname{tg}\gamma \stackrel{\text{def}}{=} \Omega(p+\lambda)^{-1}, \quad -\tfrac{1}{2}\pi < \operatorname{Re}\gamma \leq \tfrac{1}{2}\pi .$$

The combinations U_x+V_y and V_x-V_y are often called the divergence and the rotation of the windfield respectively.

The boundary conditions for $\bar{\varphi}$ become

$$\bar{\varphi} = 0 \quad \text{along } C_2, \tag{2.15}$$

$$\frac{\partial \bar{\varphi}}{\partial n} + \text{tg}\,\gamma \frac{\partial \bar{\varphi}}{\partial s} = \bar{f} \quad \text{along } C_1, \tag{2.16}$$

with

$$\bar{f} \overset{\text{def}}{=} \bar{W}_n + \text{tg}\,\gamma\, \bar{W}_s, \tag{2.17}$$

where W_n and W_s are respectively the normal component and the tangential component of the wind-stress vector (U,V). We note that the normal n is directed outwardly.

If $\bar{\varphi}$ is known the components of the total stream can be derived from (2.9) as follows

$$\left\{\begin{aligned} \frac{\kappa^2}{p}\,\bar{u} &= -\left(\frac{\partial \bar{\varphi}}{\partial x} + \text{tg}\,\gamma \frac{\partial \bar{\varphi}}{\partial y}\right) + (\bar{U}+\text{tg}\,\gamma\,\bar{V}) \\ \frac{\kappa^2}{p}\,\bar{v} &= -\left(\frac{\partial \bar{\varphi}}{\partial y} - \text{tg}\,\gamma \frac{\partial \bar{\varphi}}{\partial x}\right) + (\bar{V}-\text{tg}\,\gamma\,\bar{U}). \end{aligned}\right. \tag{2.18}$$

From (2.11) and (2.18) it follows that $\bar{u}$ and $\bar{v}$ satisfy the following non-homogeneous equations of Helmholtz

$$(\Delta-\kappa^2)\bar{u} = -\frac{1}{p+\lambda}\frac{\partial}{\partial y}\left(\frac{\partial \bar{V}}{\partial x} - \frac{\partial \bar{U}}{\partial y}\right) - p(\bar{U}+\text{tg}\,\gamma\,\bar{V}) \tag{2.19}$$

and

$$(\Delta-\kappa^2)\bar{v} = \frac{1}{p+\lambda}\frac{\partial}{\partial x}\left(\frac{\partial \bar{V}}{\partial x} - \frac{\partial \bar{U}}{\partial y}\right) - p(\bar{V}-\text{tg}\,\gamma\,\bar{U}). \tag{2.20}$$

3. Green's theorem

The solution of the problem (2.11) with the boundary conditions (2.15) and (2.16) can be facilitated by making use of Green's theorem.

For simplicity we shall assume that the domain D is simply connected and that its boundary is piecewise smooth. The points (x,y), (x_o,y_o) etc. will be denoted shortly as P, P_o etc. The distance between P and P_o will be indicated by $\rho(P_1,P_2)$. We now consider two functions φ and ψ with continuous second derivatives in D with the exception of at most a single point. We assume that φ has a logarithmic singularity in A of the kind $-(2\pi)^{-1}\ln\rho(P,A)$ and that ψ has a similar singularity in B. Then we have

$$\iint_D \left(\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \kappa^2\varphi\,\psi\right)dxdy = -\iint_D \varphi(\Delta-\kappa^2)\psi\, dxdy + \tag{3.1}$$
$$- \int_C \varphi\frac{\partial\psi}{\partial n}\, ds + \varphi(B).$$

In view of the symmetry of the left-hand side we obtain the equality

$$\psi(A) - \varphi(B) = \iint_D \left\{ \psi(\Delta-\kappa^2)\varphi - \varphi(\Delta-\kappa^2)\psi\right\} dxdy + \tag{3.2}$$
$$+ \int_C \left\{ \psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right\} ds.$$

After some elementary reductions it follows from (3.2) that

$$\psi(A) - \varphi(B) = \iint_D \left\{ \psi(\Delta-\kappa^2)\varphi - \varphi(\Delta-\kappa^2)\psi\right\} dxdy +$$
$$+ \int_{C_1} \psi\left\{ \frac{\partial\varphi}{\partial n} - \mathrm{tg}\,\gamma\frac{\partial\varphi}{\partial s}\right\} ds + \tag{3.3}$$
$$- \int_{C_1} \varphi\left\{ \frac{\partial\psi}{\partial n} + \mathrm{tg}\,\gamma\frac{\partial\psi}{\partial s}\right\} ds + \mathrm{tg}\,\gamma \int_{C_1} \frac{\partial}{\partial s}(\varphi\,\psi)ds +$$
$$+ \int_{C_2} \left\{ \psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right\} ds.$$

From (3.3) a number of interesting conclusions can be drawn which will be listed below.

a If $G(P,P_o,\mathrm{tg}\,\gamma)$ satisfies the Helmholtz equation

$$(\Delta-\kappa^2)G = 0 \tag{3.4}$$

the ocean condition

$$G = 0 \quad \text{along } C_2, \tag{3.5}$$

the adjoint coast condition

(3.6) $$\frac{\partial G}{\partial n} - \operatorname{tg}\gamma \frac{\partial G}{\partial s} = 0 \qquad \text{along } C_1,$$

then

(3.7) $$\bar{\zeta}(P_o) = -\iint_D G(P,P_o)\bar{F}(P)dxdy - \int_{C_1} G(P,P_o)\bar{f}(P)ds.$$

Proof.

Put $\varphi \equiv G$, $A \equiv P_o$; $\psi \equiv \bar{\zeta}$, B absent.

b For the function of Green determined by (3.4), (3.5) and (3.6) the following oblique symmetry relation holds

(3.8) $$G(P_1,P_2,\operatorname{tg}\gamma) = G(P_2,P_1,-\operatorname{tg}\gamma).$$

Proof.

Put $\varphi \equiv G(P,P_2,\operatorname{tg}\gamma)$, $A \equiv P_2$;

$\psi \equiv G(P,P_1,-\operatorname{tg}\gamma)$, $B \equiv P_1$.

c If there is a function G which satisfies the Helmholtz equation (3.4) with a singularity at P_o and the ocean condition but not necessarily the coast condition we obtain easily from (3.3)

(3.9) $$\bar{\zeta}(P_o) = Z(P_o) + \int_{C_1} \bar{\zeta}(P)\left(\frac{\partial}{\partial n} - \operatorname{tg}\gamma\frac{\partial}{\partial s}\right)G(P,P_o)ds,$$

where $Z(P_o)$ represents the right-hand side of (3.7) which is a known function of P_o.

The equation (3.9) may be considered as a singular integral equation along the coast C_1.

d If there is a function G which satisfies the Laplace equation

(3.10) $$\Delta G = 0$$

with a singularity at P_o and which satisfies both the ocean condition (3.5) and the coast condition (3.6), we obtain in a similar way

(3.11) $$\bar{\zeta}(P_o) = Z(P_o) - \kappa^2 \iint_D \bar{\zeta}(P)G(P,P_o)dxdy.$$

This represents a two-dimensional Fredholm equation. A solution of (3.11) in the form of a Neumann series would lead to an expansion of $\bar{\zeta}$ in powers of κ^2.

It has been found possible to obtain the function of Green in an explicit form for a few simple regions which are bounded by at most two straight lines. The relatively simple case of a half-plane with a coast condition will be discussed in the following section. The difficult case of an angle has been treated by Van Dantzig (4) and Lauwerier (2), (3).

4. Solution of the problem of Green for a half-plane

The function of Green G_o in the full plane satisfying

(4.1) $$(\Delta - \kappa^2)G(x,y,x_o,y_o)=0$$

and which for Re $\kappa > 0$ vanishes exponentially at infinity according to

(4.2) $$G = O(\exp - \varepsilon\sqrt{x^2+y^2}),$$

where ε is an arbitrarily small positive quantity is given by

(4.3) $$G_o(x,y,x_o,y_o) \overset{\text{def}}{=} (2\pi)^{-1} K_o(\kappa\rho),$$

where

(4.4) $$\rho = \sqrt{(x-x_o)^2 + (y-y_o)^2}.$$

We note the integral representations

(4.5) $$G_o = (4\pi)^{-1} \int_{-\infty}^{\infty} \exp-(\kappa\rho \operatorname{ch}t)dt$$

and

(4.6) $$G_o = (4\pi)^{-1} \int_{-\infty}^{\infty} \exp-\kappa\{|y-y_o|\operatorname{ch}t + i(x-x_o)\operatorname{sh}t\}\, dt.$$

From (4.3) the solution of the following problem of Green can be derived. To find a function of Green G_1 satisfying (4.7) in the half-plane $y > 0$ and the boundary condition

(4.7) $$G = 0 \qquad \text{for} \quad y = 0.$$

By means of the well-known reflection principle it follows that

(4.8) $$G_1(x,y,x_o,y_o) \overset{\text{def}}{=} (2\pi)^{-1}K_o(\kappa\rho) - (2\pi)^{-1}K_o(\kappa\rho^*),$$

where

(4.9) $$\rho^* = \sqrt{(x-x_o)^2 + (y+y_o)^2}.$$

It can easily be shown that

(4.10) $$(\cos^2\gamma \frac{\partial^2}{\partial y^2} - \sin^2\gamma \frac{\partial^2}{\partial x^2})G_1 = 0 \qquad \text{for} \quad y=0,$$

for the left-hand side of (4.10) can be written in the form

$$\left\{\cos^2\gamma(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - \frac{\partial^2}{\partial x^2}\right\} G =$$

$$= (\kappa^2\cos^2\gamma - \frac{\partial^2}{\partial x^2})G = 0$$

in view of (4.7).
Therefore the function G_2 defined by

(4.11) $$G_2(x,y,x_0,y_0) = (\cos\gamma \frac{\partial}{\partial y} + \sin\gamma \frac{\partial}{\partial x})G_1(x,y,x_0,y_0)$$

satisfies the boundary condition

(4.12) $$(\cos\gamma \frac{\partial}{\partial y} - \sin\gamma \frac{\partial}{\partial x})G_2 = 0 \quad \text{for} \quad y = 0.$$

The operator $\cos\gamma \frac{\partial}{\partial y} - \sin\gamma \frac{\partial}{\partial x}$ represents for real γ a directional derivative, the direction of which makes the angle $\gamma - \frac{1}{2}\pi$ with the positive X-axis.
The solution (4.11) can be interpreted as being generated by a dipole at (x_0,y_0) in the direction $-\gamma - \frac{1}{2}\pi$ and a dipole at the reflected source $(x_0,-y_0)$ in the direction $-\gamma + \frac{1}{2}\pi$ (see figure 1).

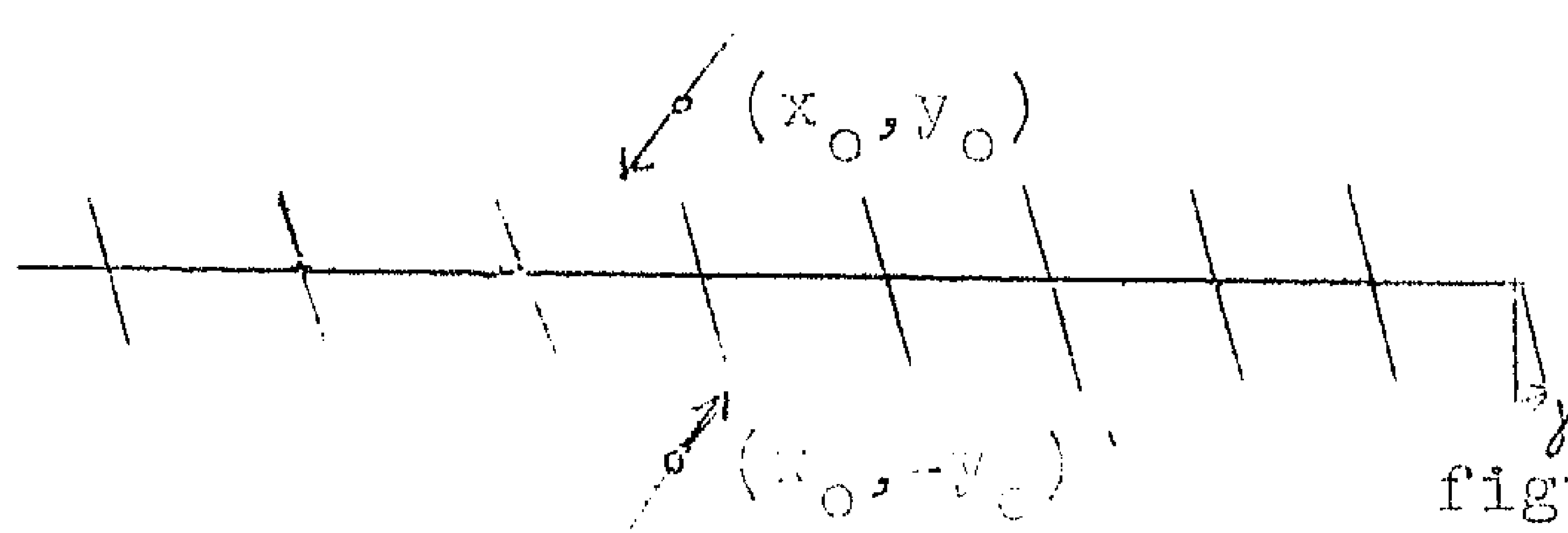

figure 1

From the solution G_2 the function of Green G_3 which satisfies (4.1), the boundary condition (4.12) and which has a logarithmic pole at (x_0,y_0), can easily be derived by integrating the dipole at (x_0,y_0) of G_2 over the half-line

$$x = x_0 + t \sin\gamma \ , \quad y = y_0 + t \cos\gamma \ , \quad 0 < t < \infty$$

assuming that $-\frac{1}{2}\pi < \gamma < \frac{1}{2}\pi$.
Then the halfline of dipoles, which are lying head to tail, reduces to a single pole at (x_0,y_0). The reflected dipole is integrated over the half-line

$$x = x_0 + t \sin\gamma \ , \quad y = -y_0 - t \cos\gamma \ , \quad 0 < t < \infty .$$

In that case, however, the dipoles do not annihilate each other and there remains a line of dipoles as shown in figure 2

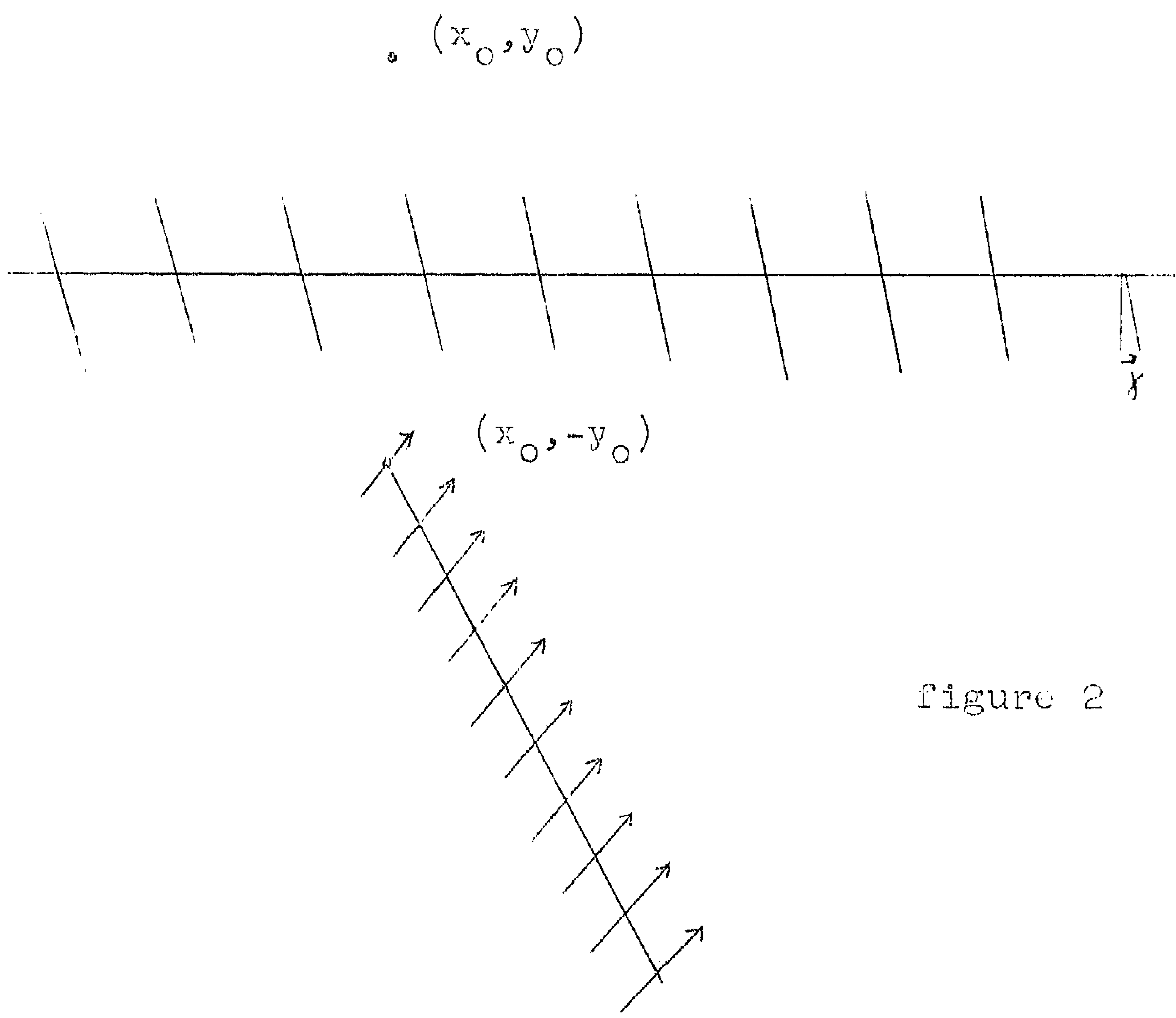

figure 2

This corresponds analytically with the formula x)

$$G_3(x,y,x_o,y_o) \overset{def}{=} G_o(x,y,x_o,y_o) +$$
$$+ \left(\cos\gamma \frac{\partial}{\partial y} + \sin\gamma \frac{\partial}{\partial x}\right) \int_0^\infty G_o(x,y,x_o+t\sin\gamma, -y_o-t\cos\gamma)dt.$$

By using (4.3) and (4.6) this may be written in the form

$$(4.14) \quad G_3(x,y,x_o,y_o) = \frac{1}{2\pi} K_o(\kappa\rho) - \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{ch(t+i\gamma)}{ch(t-i\gamma)} \exp-\kappa\{(y+y_o)cht+i(x-x_o)sht\}\, dt.$$

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x) These two expressions have been first given by H.A. Lauwerier (1). The above-given elegant derivation is due to G.W. Veltkamp.

List of symbols

x, y, z	Cartesian coordinates. The undisturbed surface of the water is given by $z=0$ and the bottom by $z=h$;
t	the time;
u_z, v_z	the components of the current at depth z;
u, v	the components of the total stream, $u = \int_0^h u_z dz$, $v = \int_0^h v_z dz$;
ζ	the elevation of the water-surface. The undisturbed level is given by $\zeta = 0$;
p_a	the atmospheric pressure;
U_z, V_z	the components of the force of friction by which the water above the depth z acts on the water below that depth;
U_s, V_s	the components of the friction of the wind on the water-surface;
U_b, V_b	the components of the friction of the water on the bottom;
Ω	the coefficient of the Coriolis force, $\Omega = 2\omega_a \sin\varphi$, where ω_a is the velocity of the rotation of the earth and φ the geographic latitude;
λ	a coefficient of friction;
g	the acceleration of the earth's gravity;
ρ	the density of the water, assumed uniform ($\rho = 1.027$ g/cm^3);
ρ_a	the density of the air ($\rho_a = 0.00125$ g/cm^3);
v_s	the velocity of the wind at sealevel;
c	$c = \sqrt{gh}$;
p	the variable of the Laplace transformation;
κ	defined by $\kappa^2 = p(p+\lambda)^{-1}\{(p+\lambda)^2 + \Omega^2\}$, Re $\kappa \geqq 0$;
tg γ	defined by tg $\gamma = \Omega(p+\lambda)^{-1}$, $-\frac{1}{2}\pi <$ Re $\gamma \leqq \frac{1}{2}\pi$.

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