International Series of
Numerical Mathematics, Vol. 73

STABILITY RESULTS FOR DISCRETE VOLTERRA EQUATIONS: NUMERICAL EXPERIMENTS

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In this paper we formulate a local stability criterion for linear multistep discretizations of first- and second-kind Volterra integral equations with finitely decomposable kernel. In a large number of numerical experiments this criterion is tested. We did not find examples which behaved unstable while the stability criterion predicted stability. However, we found several examples which behaved stable while the stability criterion predicted instability. A possible explanation may be the fact that the stability criterion is independent of the decomposition of the kernel, that is, it holds for the most ill-conditioned decomposition and consequently it may be rather pessimistic.

1. Introduction

We consider Volterra equations of the form

$$\theta y(t) = g_0(t) + \int_0^t k(t,s,y(s))ds, \quad t \in I := [0,T] \tag{1.1}$$

where θ is either 0 (*first-kind* equations) or 1 (*second-kind* equations).

It is well known that applying direct quadrature methods to the *first-kind equation* may give unsatisfactory results (cf.LINZ [5,p.67]). An often applied remedy (cf. [1,p.898] and also [5]) consists of differentiating equation (1.1) to obtain the (implicit) second-kind equation (assuming that g' and k_t exist)

$$0 = k(t,t,y(t)) + g_0'(t) + \int_0^t k_t(t,s,y(s))ds. \tag{1.2}$$

If the derivatives occurring in (1.2) cannot be evaluated analytically, g_0' and k_t may be replaced by a difference approximation [5].

When we apply direct quadrature methods to the *second-kind equation* ($\theta=1$ in (1.1)), we again may obtain poor results, particularly when $\partial k/\partial y$ is large. As in the case of first-kind equations, let us differentiate the equation to obtain the integro-differential equation

$$\theta y'(t) = k(t,t,y(t)) + F_t(t,t), \quad \theta = 1 \tag{1.3a}$$

where we have introduced the so-called *lag term*

$$F(t,s) := g_0(t) + \int_0^s k(t,x,y(x))dx. \tag{1.3b}$$

Again, the derivative F_t may be approximated by finite differences.

Let $\tilde{F}_n(t)$ denote the numerical lag term approximating $F(t,t_n)$:

$$\tilde{F}_n(t) := g_0(t) + h \sum_{\ell=0}^{\tilde{n}} w_{n,\ell} k(t,t_\ell,y_\ell), \; t_\ell = \ell h, \; \tilde{n} = \begin{cases} \tilde{\kappa}-1, n<\tilde{\kappa} \\ n, n\geq\tilde{\kappa} \end{cases}, \tag{1.4}$$

where $\tilde{\kappa}$ is sufficiently large to obtain the required order of accuracy. Let $\{a_i,b_i\}_{i=0}^{\kappa}$ define a linear multistep method $\{\rho,\sigma\}$ for ODEs and let $\tau(\zeta)$ define a κ-step difference formula, i.e.

$$g_0'(t_n) \approx h^{-1}\tau(E)g_0(t_n), \quad \tau(\zeta) = \sum_{i=0}^{\kappa} d_i \, \zeta^{\kappa-i-q}, \; q \geq 0, \tag{1.5}$$

where E is the forward shift operator; here, q is an integer which we should choose 0 (*forward* differences) if the kernel is only defined for $s \leq t$, and which may be chosen such that $\tau(\zeta)$ defines a *symmetric* difference formula if the kernel is defined for all (t,s). In this paper we will use *forward* *differences*. Approximating $F_t(t,t)$ in (1.3) by the κ-step difference formula and applying the linear multistep method, we obtain the formula

$$\theta\rho(E)y_n = h\sigma(E)k(t_n,t_n,y_n) + \sigma(E)\tau(E)\tilde{F}_n(t_n), \quad n \geq \kappa, \tag{1.6}$$

where $\tau(E)$ only affects the index n of the argument t_n in $\tilde{F}_n(t_n)$.

The method {(1.4),(1.6)} will be called an *indirect linear multistep* (ILM) method [3]. Let $\tilde{p}$ be the order of the lag term approximation, let the starting values be sufficiently accurate, and let k and g_0 be sufficiently smooth. Then it can be proved that the ILM method is of order $\min\{\tilde{p},\kappa\}$ if $\theta = 0$ with $\sigma(\zeta)$ a Schur polynomial, and of order $\min\{p,\kappa,\tilde{p}\}$ if $\theta = 1$ with $\{\rho,\sigma\}$ being of order p.

It is the purpose of this paper to test the stability of the ILM method. In the particular case where $g_0(t)$ = constant and k is the *linear convolution* kernel

$$k(t,s,y) = [\xi + \eta(t-s)]y \tag{1.7}$$

the equation (1.3) reduces to the stability test equation investigated by BRUNNER and LAMBERT [2] and MATTHIJS [7]:

$$y'(t) = \xi y(t) + \eta \int_0^t y(s)ds; \tag{1.3'}$$

for $\theta = 1$ the ILM method then falls into the class of linear multistep methods studied by these authors so that their stability results apply. It was shown by MATTHIJS that for $\{\tilde{\rho},\tilde{\sigma}\}$-reducible lag term approximations, the application of a linear multistep method $\{\rho,\sigma\}$ to (1.3') is stable if the characteristic polynomial

$$\zeta^{\kappa-\tilde{\kappa}}[\rho(\zeta)\tilde{\rho}(\zeta) - h\xi\sigma(\zeta)\tilde{\rho}(\zeta) - h^2\eta\sigma(\zeta)\tilde{\sigma}(\zeta)] \tag{1.8}$$

is a Schur polynomial.

In Section 2 we show that an analogous characteristic equation is obtained in the case of *finitely decomposable* kernels, by associating a system of ODEs to (1.6) and by using standard arguments common in ODE theory. In Section 3, a more refined stability criterion is formulated; this result characterizes the *local stability behaviour* of the ILM method. Finally, in Section 4, a large number of experiments are presented in order to test the practical value of local stability criteria.

2. Finitely decomposable kernels

If the kernel $k(t,s,y)$ is finitely decomposable, it can be written in the form

$$k(t,s,y) = \sum_{\mu=1}^{m} g_\mu(t)f_\mu(s,y) =: <\vec{g}(t),\vec{f}(s,y)>, \tag{2.1}$$

where $\vec{g}$ and $\vec{f}$ are vectors with components g_μ and f_μ, $\mu = 1(1)m$, and where we have introduced the inner product $<,>$ in order to simplify the subsequent formulas.

Furthermore, we will assume that the lag term formula is $(\tilde{\rho},\tilde{\sigma})$-*reducible*, that is the quadrature rule used is assumed to correspond to a linear multistep formula $\{\tilde{\rho},\tilde{\sigma}\}$ for ODEs. The weights of such rules satisfy the relations [7,9]

$$\sum_{i=0}^{\tilde{\kappa}} \tilde{a}_i\, w_{n-i,j} = \begin{cases} 0 & \text{if } j = 0,1,..,n-\tilde{\kappa}-1 \\ \tilde{b}_{n-j} & \text{if } j = n-\tilde{\kappa},..,n \end{cases}, \quad n \geq \tilde{\kappa}. \tag{2.2}$$

where $\{\tilde{a}_i,\tilde{b}_i\}$ define $\{\tilde{\rho},\tilde{\sigma}\}$ and where $w_{n,j} = 0$ for $j > \max\{n,\tilde{\kappa}-1\}$.

Theorem 2.1. Let k be finitely decomposable and let the lag term formula be $(\tilde{\rho},\tilde{\sigma})$-reducible with $\tilde{\rho}(1) = 0$ and $\tilde{\kappa} = \kappa$. Then the ILM method is algebraically equivalent with the recurrence relations

$$\tilde{\rho}(E)\vec{u}_n = h\tilde{\sigma}(E)\vec{f}(t_n,y_n), \quad n \geq 0, \tag{2.3a}$$

$$\theta\rho(E)y_n = h\sigma(E)k(t_n,t_n,y_n) + \sigma(E)\tau(E)g_0(t_n) \tag{2.3b}$$

$$+ \sigma(E) < \tau(E)\vec{g}(t_n),\vec{u}_n>, \quad n \geq \kappa,$$

where the starting values $\vec{u}_j$, $j = 0,\ldots,\kappa-1$ satisfy the starting condition

$$g_0(t) + \langle \vec{g}(t), \vec{u}_j \rangle = \tilde{F}_j(t), \quad j = 0,\ldots,\kappa-1. \tag{2.4}$$

<u>Proof</u>. From the $(\tilde{\rho},\tilde{\sigma})$-reducibility of the lag term formula it follows that

$$\tilde{\rho}(E)\tilde{F}_n(t) = h\tilde{\sigma}(E)k(t,t_n,y_n), \quad n \geq 0. \tag{2.5}$$

Furthermore, it follows from (2.1) and (1.4) that

$$\tilde{F}_n(t) = g_0(t) + \langle \vec{g}(t), \vec{u}_n \rangle, \qquad n \geq 0, \tag{2.6}$$

where

$$\vec{u}_n := h \sum_{\ell=0}^{\tilde{n}} w_{n,\ell} \vec{f}(t_\ell, y_\ell).$$

From (2.5) and (2.6) it follows that

$$\langle \vec{g}(t), \tilde{\rho}(E)\vec{u}_n - h\tilde{\sigma}(E)\vec{f}(t_n,y_n) \rangle = 0, \qquad n \geq 0,$$

from which (2.3a) is derived.

Relation (2.3b) is obtained on substitution of (2.6) into the ILM formula (1.6). Finally, the starting conditions follow from (2.6). □

<u>2.1 Relation with ODEs</u>

For $\theta = 1$, the recurrence relation (2.3) is recognized as a linear multistep discretization of the system of ODEs

$$\begin{cases} \vec{u}'(t) = \vec{f}(t,y) \\ y'(t) = k(t,t,y(t)) + \frac{1}{h}\tau(E)g_0(t) + \frac{1}{h}\langle \tau(E)\vec{g}(t), \vec{u}(t) \rangle \end{cases} \tag{2.7}$$

using different linear multistep methods $\{\tilde{\rho},\tilde{\sigma}\}$ and $\{\rho,\sigma\}$ with integration step h.

In SÖDERLIND [8], such linear multistep methods were called *linear multistep compound* (LMC) methods. The (linear) stability of LMC methods with respect to the test equation

$$\vec{x}'(t) = J\vec{x}, \text{ J constant matrix, } \vec{x} = [\vec{u},y]^T \tag{2.8}$$

is characterized by the roots of the characteristic equation

$$\det[P(\zeta) - \Sigma(\zeta)hJ] = 0, \tag{2.9}$$

where

$$P(\zeta) := \begin{pmatrix} \tilde{\rho}(\zeta)I_m & 0 \\ 0 & \rho(\zeta) \end{pmatrix}, \Sigma(\zeta) := \begin{pmatrix} \tilde{\sigma}(\zeta)I_m & 0 \\ 0 & \sigma(\zeta) \end{pmatrix},$$

with I_m denoting the m×m unit matrix. If (2.9) is a Schur polynomial then the LMC solution converges to 0 as $t_n \to \infty$. The system (2.7) suggests choosing for J the Jacobian matrix

$$J = \begin{pmatrix} 0 & \frac{\partial \vec{f}}{\partial y}(\bar{t},\bar{y}) \\ h^{-1}(\tau(E)\vec{g}(\bar{t}))^T & \frac{\partial k}{\partial y}(\bar{t},\bar{t},\bar{y}) \end{pmatrix}$$

at some point $(\bar{t},\bar{y})$. The eigenvalues of J are given by m-1 zero-eigenvalues and two eigenvalues satisfying the equation

$$\lambda^2 - \frac{\partial k}{\partial y}(\bar{t},\bar{t},\bar{y})\lambda - h^{-1}\tau(E)\frac{\partial k}{\partial y}(\bar{t},\bar{t},\bar{y}) = 0,$$

where $\tau(E)$ only affects the first argument of $\partial k/\partial y$. It is now easily verified that (2.9) reduces to the equation

$$\rho(\zeta)\tilde{\rho}(\zeta) - h\frac{\partial k}{\partial y}(\bar{t},\bar{t},\bar{y})\sigma(\zeta)\tilde{\rho}(\zeta) - h\tau(E)\frac{\partial k}{\partial y}(\bar{t},\bar{t},\bar{y})\sigma(\zeta)\tilde{\sigma}(\zeta) = 0. \tag{2.10}$$

Notice the resemblance with the characteristic polynomial (1.8).

The equation (2.10) is independent of the decomposition of the kernel. For instance, if k(t,s,y) is of the convolution type $K^*(t-s)y$, then the kernel enters into (2.10) only by the values of $K^*(0)$ and $h^{-1}\tau(E)K^*(0) \approx K^*_t(0)$. Hence, when a stability criterion is based on (2.10), we use only a very limited amount of information on the kernel. In the following section we will derive a stability criterion that takes into account more information on the kernel. Moreover, the first-kind case ($\theta=0$) is included at the same time.

3. A local stability criterion

Let $k(t,s,y)$ be of the linear form $K(t,s)y$ with $K(t,s)$ of separable form: $K(t,s) = \langle\vec{g}(t),\vec{f}(s)\rangle$. Then we can write the recurrence relation (2.3) in the form

$$\sum_{i=0}^{\kappa^*} B_i(n)\vec{v}_{n-i} = \vec{w}_n, \quad \kappa^* = \max\{\kappa,\tilde{\kappa}\} \tag{3.1}$$

where

$$\vec{v}_n := [y_n,\vec{u}_n]^T, \quad \vec{w}_n := [\sigma(E)\tau(E)g_0(t_{n-\kappa}),\vec{0}]^T,$$

$$B_i(n) := \begin{pmatrix} \theta a_i - b_i hK(t_{n-i},t_{n-i}) & -b_i\tau(E)\vec{g}^T(t_{n-i}) \\ -\tilde{b}_i h\vec{f}(t_{n-i}) & \tilde{a}_i I_m \end{pmatrix},$$

with the convention that $a_i = b_i = 0$ for $i > \kappa$ and $\tilde{a}_i = \tilde{b}_i = 0$ for $i > \tilde{\kappa}$.

In analogy to the linear stability analysis used in ODEs we will call the recurrence relation (3.1) *locally stable at* $t_{\bar{n}}$ if the recurrence relation

$$\sum_{i=0}^{\kappa^*} B_i(\bar{n})\vec{v}_{n-i} = \vec{0}, \quad \bar{n} \text{ fixed} \tag{3.2}$$

is stable, that is if its solutions converge. This leads to the condition

$$\det\left[\sum_{i=0}^{\kappa^*} B_i(\bar{n})\zeta^{n-i}\right] \text{ is a Schur polynomial.} \tag{3.3}$$

Analogous to the stability analysis in [4] the following theorem can be proved:

Theorem 3.1. The recurrence relation (3.1) is locally stable at $t_{\bar{n}}$ if the polynomial

$$\theta\rho(\zeta)\tilde{\rho}(\zeta) - h\tilde{\rho}(\zeta)\sum_{i=0}^{\kappa} b_i K(t_{\bar{n}-i}, t_{\bar{n}-i})\zeta^{\kappa-i}$$
$$- h^2 \sum_{i=0}^{\kappa}\sum_{j=0}^{\tilde{\kappa}} b_i \tilde{b}_j \left(\frac{1}{h}\sum_{\ell=0}^{\kappa} d_\ell K(t_{\bar{n}+\kappa-i-\ell}, t_{n-j})\right)\zeta^{\kappa+\tilde{\kappa}-i-j} \quad (3.4)$$

is a Schur polynomial. □

In the actual application of this theorem one may consider the approximation

$$\frac{1}{h}\sum_{\ell=0}^{\kappa} d_\ell K(t_{\bar{n}+\kappa-i-\ell}, t_{\bar{n}-j}) \approx K_t(t_{\bar{n}-i}, t_{\bar{n}-j}) \quad (3.5)$$

which slightly simplifies the polynomial (3.4).

In the particular case of convolution kernels where $K(t,s) = K^*(t-s)$, the polynomial (3.4) reduces to

$$\theta\rho(\zeta)\tilde{\rho}(\zeta) - hK^*(0)\tilde{\rho}(\zeta)\sigma(\zeta) - h^2 \sum_{i=0}^{\kappa}\sum_{j=0}^{\tilde{\kappa}} b_i \tilde{b}_j K_t^*((j-i)h)\zeta^{\kappa+\tilde{\kappa}-i-j} \quad (3.6)$$

where we have used (3.5). Notice that (3.6) does not depend on n.

We observe that the particular decomposition (2.1) of the kernel does not occur in (3.4). Thus, formally we can apply (3.4) to non-decomposable kernels as well, provided that $K(t,s)$ is also defined for $t < s$.

If $O(h^3)$ terms in (3.6) are neglected, the characteristic polynomial reduces to

$$\theta\rho(\zeta)\tilde{\rho}(\zeta) - hK^*(0)\tilde{\rho}(\zeta)\sigma(\zeta) - h^2 K_t^*(0)\tilde{\sigma}(\zeta)\sigma(\zeta). \quad (3.6')$$

For $\theta = 1$ this polynomial is equivalent to (1.8); for $\theta = 0$ we obtain a polynomial of the form $\tilde{\rho}(\zeta)+h(K_t^*/K^*)(0)\tilde{\sigma}(\zeta)$, indicating that first-kind equations require that $-hK_t^*/K^*$ should lie in the stability region of the LM method $\{\tilde{\rho},\tilde{\sigma}\}$ (we recall that first-kind equations also require that σ is a Schur polynomial, otherwise we have no convergence).

Finally, it should be remarked that the considerations above refer to the stability of the sequence of vectors $\{\vec{v}_n\}$, whereas in actual computation we are only concerned with stability of the first components $\{y_n\}$ of $\{\vec{v}_n\}$. Consequently, these considerations might be conservative in practice.

4. Numerical experiments

In order to test the local stability result of the preceding Section we have integrated a large number of Volterra equations of convolution type. In each experiment we have computed: (i) the number of correct significant digits obtained at the end point T, i.e. the value of

$$sd := -\log\left|\frac{y_N - y(T)}{y(T)}\right|, \quad N := T/h,$$

unless otherwise stated (ii) the value of $\zeta_{max} = \max_j |\zeta_j|$, where ζ_j are the zeros of the polynomial (3.4). ζ_{max} serves as a predictor of stability or instability.

In the tables of results we use the notation AM_p-BD_q indicating that the lag term is based on a p-th order Adams-Moulton formula and the ILM formula is based on a q-th order Backward Differentiation formula.

In all experiments the starting values were derived from the exact solution.

From our experiments we draw the following conclusions

(i) The solutions of all second-kind equations behaved stably if $\zeta_{max} \leq 1$.

(ii) The solutions of all first-kind equations behaved stably if $\zeta_{max} \leq 1$ and if $|K^*(0)|$ is not small.

(iii) $\zeta_{max} > 1$ does not necessarily imply instability this may be explained by observing that $\zeta_{max} > 1$ indicates an unstable behaviour of $\{\vec{v}_n\}$, and not necessarily of $\{y_n\}$.

(iv) The ILM method yields poor results for first-kind equations with $|K^*(0)|$ small.

Table I. $\theta = 1$: Results for second-kind equations obtained at T = 20

	Problem	h	AM_4-AM_5 sd	ζ_{max}	AM_4-BD_4 sd	ζ_{max}	BD_4-AM_5 sd	ζ_{max}	BD_4-BD_4 sd	ζ_{max}
1.	$g_0 = \frac{1}{2}t^2e^{-t}$	1/10	2.3	1.0	2.3	1.02	3.4	1.0	2.9	1.0
	$k = \frac{1}{2}(t-s)^2 \exp(-(t-s))y$	1/20	3.3	1.0	3.5	1.01	5.7	1.0	4.0	1.0
	$y = \frac{1}{3}[1-e^{-3/2\ t}(\cos(\frac{1}{2}\sqrt{3}t)+\sqrt{3}\sin(\frac{1}{2}\sqrt{3}t)]$									
2.	$g_0 = 2-2/(t+2)$	1/10	3.4	1.04	3.4	1.05	5.3	.98	5.4	1.05
	$k = -2/(t-s+2)^2.y$	1/20	4.5	1.02	4.5	1.02	6.3	.99	6.3	1.03
	$y = 1$									
3.	$g_0 = 1+t-\cos(t)$	1/10	3.8	1.00	3.8	1.01	4.9	0.99	4.9	1.00
	$k = -\cos(t-s)y$	1/20	5.0	1.00	5.0	1.00	6.5	1.00	6.5	1.00
	$y = t$									
4.	$g_0 = 2t+3$	1/10	3.0	.91	2.9	.90	2.8	.90	3.0	.91
	$k = (-2(t-s)-3)y$	1/20	2.8	.95	3.9	.95	2.7	.95	3.3	.95
	$y = 4e^{-2t}-e^{-t}$									
5.	$g_0 = 1 - \frac{1}{2}\sqrt{2}\ \mathrm{erf}(t)$	1/10	-.8	1.11	-.7	1.09	-.7	1.12	-.6	1.11
	$k = \exp(-(t-s)^2)y$	1/20	-1.0	1.05	-1.0	1.05	-.9	1.05	-.9	1.05
	$y = 1$									
6.	$g_0 = \frac{1}{2}\gamma(1-t^2)\ln(1+t)$	1/10	4.7	1.11	4.7	1.00	5.9	1.56	5.9	1.00
	$+ \frac{3}{4}\gamma t^2$	1/20	5.7	1.03	5.7	1.00	7.0	1.28	7.0	1.00
	$-(\frac{1}{2}\gamma+1)t+1$									
	$k = -\gamma\ \ln(1+t-s)y$	1/10	-48	2.95	-3.7	1.12	5.8	5.95	-1.3	1.12
	$y = 1-t$	1/20	-23	1.86	-17	1.16	-11	3.98	-19	1.18
	$\gamma = 10, 1000$									
7.	$g_0 = 2t+3-\sin(t)-3\cos(t)$	1/10	4.7	.90	4.7	.90	4.8	.90	4.8	.90
	$k = (-2(t-s)-3)y$	1/20	5.9	.95	5.9	.95	5.9	.95	6.0	.95
	$y = \sin(t)$									
8.	$g_0 = 1$	1/10	2.2	1.05	2.2	1.05	4.0	1.05	4.0	1.05
	$k = e^{-2(t-s)}y$	1/20	3.3	1.03	3.3	1.03	4.9	1.03	5.0	1.03
	$y = 2-e^{-t}$									
9.	$g_0 = 1$	1/10	3.1	.90	3.1	.90	2.9	.90	2.9	.90
	$k = -2e^{t-s}y$	1/20	4.3	.95	4.4	.95	4.1	.95	4.1	.95
	$y = -1+2e^{-t}$									

Table I. (continued) $\theta = 1$: Results obtained at $T = 20$

Problem	h	AM_4-AM_5 sd	ζ_{max}	AM_4-BD_4 sd	ζ_{max}	BD_4-AM_5 sd	ζ_{max}	BD_4-BD_4 sd	ζ_{max}
10. $g_0 = \cos(t)$	1/10	2.0	1.00	1.6	1.00	1.3	1.00	1.2	1.00
$k = -(t-s)\cos(t-s).y$	1/20	3.5	1.00	2.8	1.00	2.4	1.00	2.3	1.00
$y = \frac{2}{3}\cos(\sqrt{3}t)+1/3$									
11. $g_0 = t$	1/10	3.5	1.11	3.5	1.11	2.8	1.11	2.8	1.11
$k = \sin(t-s).y$	1/20	4.5	1.05	4.5	1.05	4.0	1.05	4.0	1.05
$y = t(1+t^2/6)$									
12. $g_0 = e^t - 2\sin(t)$	1/10	2.6	1.22	2.3	1.22	2.0	1.23	1.9	1.22
$k = 2\cos(t-s).y$	1/20	3.8	1.11	3.5	1.10	3.2	1.11	3.1	1.11
$y = e^t(1+t^2)$									
13. $g_0 = \sinh t$	1/10	3.6	1.00	3.7	.99	3.6	1.00	3.7	1.00
$k = -\cosh(t-s).y$	1/20	4.8	1.00	4.9	1.00	4.9	1.00	5.0	1.00
$y = 2\sinh(\sqrt{5}t/2)e^{-\frac{1}{2}t}/\sqrt{5}$									
14. $g_0 = 1+\frac{1}{2}\gamma(1-e^{-t^2})$	1/10	2.2	1.00	2.2	1.00	4.8	1.00	4.4	1.00
$k = -\gamma(t-s)\cdot$	1/20	3.7	1.00	3.7	1.00	5.1	1.00	5.1	1.00
$.\exp(-(t-s)^2).y$									
$y = 1$	1/10	-53	1.96	-8	1.14	-6	1.10	-6	1.12
γ = 10, 1000, 1900	1/20	-30	1.22	-22	1.16	-18	1.14	-25	1.18
3000, 7500, 12000									
14000	1/10	-64	2.22	-.6	1.05	2.8	.99	.5	1.03
	1/20	-73	1.57	-24	1.18	-20	1.15	-24	1.18
	1/10	-69	2.35	2.2	.98	-48	1.67	4.3	.95
	1/20	-98	1.82	-21	1.16	-16	1.12	-20	1.15
	1/10	-74	2.52	-49	1.80	-75	2.38	4.3	.79
	1/20	-132	2.21	-5	1.05	5.0	.96	-1	1.03
	1/10	-76	2.57	-57	1.98	-80	2.55	4.3	.72
	1/20	-143	2.35	3.6	.97	-100	1.79	5.1	.95
	1/10	-77	2.58	-59	2.03	-82	2.60	4.3	.70
	1/20	-145	2.39	-34	1.21	-114	1.95	5.1	.92

Table I. (continued) $\theta = 1$: Results obtained at T = 20

Problem	h	AM_4-AM_5 sd	ζ_{max}	AM_4-BD_4 sd	ζ_{max}	BD_4-AM_5 sd	ζ_{max}	BD_4-BD_4 sd	ζ_{max}
15. $g_0 = 1-2\varepsilon^{3/2}/3+t$	1/10	2.7	.99	2.7	.95	7.0	1.23	7.0	.96
$+2(t+\varepsilon)^{3/2}/3$	1/20	3.0	.99	3.0	.97	7.5	1.13	7.5	.97
$k = -(1+\sqrt{t-s+\varepsilon})y$									
$y = 1$	1/10	2.4	1.01	2.4	.95	7.0	1.29	7.0	.95
$\varepsilon = 10^{-2}, 10^{-6}$	1/20	2.5	1.00	2.5	.97	7.4	1.18	7.5	.98

Table II. $\theta = 0$: Results for first-kind equations obtained at T = 20

Problem		h	AM_4-BD_4 sd	ζ_{max}	BD_4-BD_4 sd	ζ_{max}
16. $g_0 = a\cos(t)-\sin(t)-ae^{at}$	a=1	1/10	4.2	1.01	3.9	1.00
$k = (a^2+1)\cos(t-s).y$		1/20	5.5	1.00	5.1	1.00
$y = e^{at}$						
	a=-1*	1/10	3.5	1.01	2.9	1.00
		1/20	4.6	1.00	4.0	1.00
17. $g_0 = -\sinh(at)$	a=1*	1/10	3.6	.90	3.8	.90
$k = a\exp(a(t-s)).y$		1/20	3.5	.95	2.5	.95
$y = e^{-at}$						
	a=-1	1/10	5.4	1.10	3.7	1.11
		1/20	7.2	1.05	4.8	1.05
18. $g_0 = 1-t-e^{-t}$		1/10	3.3	.90	3.5	.90
$k = (1+t-s)y$		1/20	3.4	.95	3.2	.95
$y = te^{-t}$						
19. $g_0 = -a(1-\cos(t))+\frac{1}{2}t\sin(t)$	a=.9	1/10	-21	1.05	-22	1.00
$k = [a-\cos(t-s)]y$		1/20	-20	1.01	-21	1.00
$y = \sin(t)$						
	a=1.1	1/10	4.2	.95	3.3	1.00
		1/20	5.5	.99	4.9	1.00

20.	g_0 = 1+at−cos(t)	a=−.1	1/10	−.2	.40	−.2	.63
	k = [a−sin(t−s)]y		1/20	−.2	.61	−.2	.61
	y = 1	a=−.01	1/10	−46	1.73	.5	.50
			1/20	−50	1.34	.5	.57

* In these cases, sd corresponds to absolute error

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