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Low Order Spectral Models of the Atmospheric Circulation*

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ABSTRACT

The spectral model consists of a system of coupled nonlinear ordinary differential equations. Low order barotropic models with 3 and 6 components have several stable solutions representing stream patterns with either a weak or a strong zonal component. With bifurcation theory these solutions are analyzed.
It is also shown that a 10 component model contains a strange attractor exhibiting alternately a weak and a strong zonal circulation pattern.
A comparable behaviour is found in 3 and 6 component models perturbed by noise.

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1. INTRODUCTION

In this contribution low order spectral solutions of the barotropic potential vorticity equation are studied. A spectral model consists of a system of coupled nonlinear ordinary differential equations for the time dependent coefficients of the spectral expansion. Truncation of this expansion yields a finite dimensional approximation of the atmospheric flow described by the vorticity equation.

First we consider a 3-dimensional model having two stable equilibria and one unstable equilibrium at the separatrice. These equilibria can be seen as preferent states of the atmosphere. The irregular alternation of these preferent states, as observed in circulation patterns, is not reflected by the simple 3-dimensional model. One way to compensate the effects of the severe truncation in the spectral model is to add stochastic forcing to the system. In section 3 an analysis of this stochastic problem is presented. Special attention is given to the expected time of residence near a preferent state. Moreover, a discrete state Markov process is formulated; it describes the stochastic alternation of preferent states. In section 4 higher dimensional spectral models are discussed. A bifurcation analysis shows that the equilibria of the 3-dimensional deterministic system are unstable in the higher dimensional model and that for changing parameter values periodic solutions may branch off. Important is the occurrence of chaotic solutions (strange attractors) that visit, in an irregular way, different regular limit solutions, which are situated in different parts of state space (regimes).

2. DERIVATION OF THE SPECTRAL MODEL

For a large scale barotropic flow over a slowly varying topography in a midlatitude beta plane we assume the following: let H be the characteristic height, k^{-1} the horizontal length sale en σ^{-1} the time scale. The topography has a characteristic amplitude h_0. The meridional scale of the flow is assumed to be much smaller than the radius of the earth r_0. The potential vorticity equation for this circulation model reads in nondimensional form

$$\frac{\partial}{\partial t} \nabla^2\Psi + J(\Psi,\nabla^2\Psi) + \gamma J(\Psi,h) + \overline{\beta}\frac{\partial\Psi}{\partial x} + \overline{C}\nabla^2(\Psi-\Psi^*) = 0,$$

where $\Psi(x,y)$ is the stream function h the position of the earth's surface and Ψ^* a forcing stream function. Furthermore,

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}, \quad \gamma = \frac{f_0 h_0}{\sigma H}, \quad \overline{\beta} = \frac{\beta_0}{\sigma k} \text{ and } \overline{C} = \frac{f_0\delta_E}{2\sigma H},$$

where

$$f_0 = 2\Omega\sin\phi_0, \quad \beta_0 = \frac{2\Omega\cos\phi_0}{r_0}$$

with ϕ_0 the central latitude and Ω the angular speed of rotation of the earth. Finally, δ_E is the depth of the Ekman layer near the surface. We investigate the existence of travelling wave solutions in a rectangular channel with length L and width $B = \frac{1}{2}bL$. The nondimensional length and width are 2π and πb. The boundary conditions are

$$\Psi(x,y,t) = \Psi(x+2\pi,y,t),$$

$$\frac{\partial\Psi}{\partial x} = 0 \text{ and } \frac{\partial}{\partial t}\int_0^{2\pi}\frac{\partial\Psi}{\partial y}\,dx = 0 \qquad \text{at} \qquad y = 0 \text{ and } y = \pi b.$$

Let ϕ_i, $i = 1,2,\ldots$ be an orthonormal set of eigenfunctions of the Laplace operator for the domain of the channel:

$$\phi_1 = \sqrt{2}\cos(y/b), \qquad \phi_2 = 2\cos x\sin(y/b),$$
$$\phi_3 = 2\sin x\sin(y/b), \quad \phi_4 = \sqrt{2}(\cos(2y/b), \ldots$$

Moreover, the functions Ψ^* and h are assumed to be of the form

$$\Psi^* = b(x_1^*\phi_1 + x_4^*\phi_4), \qquad h = \tfrac{1}{2}\phi_2.$$

Substitution of the expansion

$$\Psi(x,y,t) = b\sum_{n=1}^{\infty} x_n(t)\,\phi_n(x,y) \tag{1}$$

yields an infinite system of differential equations for $x_n(t)$, $n = 1,2,\ldots$.

3. THE 3-DIMENSIONAL MODEL WITH STOCHASTIC FORCING

Taking $x_4^* = 0$ and $x_n(t) = 0$ for $n = 4,5,\ldots$, we obtain by substitution of (1) in the vorticity equation a system of differential equations for the remaining coefficients

$$\frac{dx_1}{dt} = bx_3 - C(x_1 - x_1^*), \tag{2a}$$

$$\frac{dx_2}{dt} = -ab(x_1 - \tfrac{1}{2}\beta)x_3 - Cx_2, \tag{2b}$$

$$\frac{dx_3}{dt} = ab(x_1 - \tfrac{1}{2}\beta)x_2 - \tfrac{1}{2}ax_1 - Cx_3 \tag{2c}$$

with

$$a = \frac{2b}{1+b^2}, \qquad \beta = \frac{3\pi}{4\sqrt{2}}\,\overline{\beta} = 2.55, \quad C = \frac{3\pi}{4\sqrt{2}}\,\overline{C} = .2$$

or

$$\frac{dx_i}{dt} = f_i(x), \qquad i = 1,2,3. \tag{3}$$

The stationary points $\overline{x}$ satisfy the equation $f(\overline{x}) = 0$. Depending on the parameter values either one or three real valued roots are found. In fig. 1 the first component of the equilibrium $\overline{x}$ is given as a function of x_1^*. Fig. 2 gives the three circulation patterns that correspond with the three equilibria for $x_1^* = 10$. The two stable equilibria with attraction domains Ω_i are denoted by $\overline{x}^{(i)}$, $i = 1,3$ $(x_1^{(1)} > x_3^{(1)})$ and the unstable one at the separatrice Γ by $\overline{x}^{(2)}$.

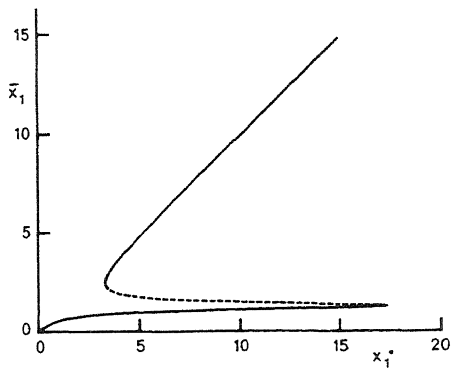


Fig. 1 Equilibrium solution $\overline{x}_1$ as a function of x_1^* for b = 1

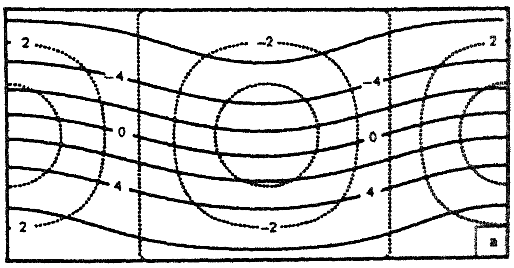
Next we consider the system (3) with each term forced by white noise of intensity ε:

$$dx_i = f_i(x)dt + \varepsilon dW_i(t), \quad = 1,2,3, \tag{4}$$

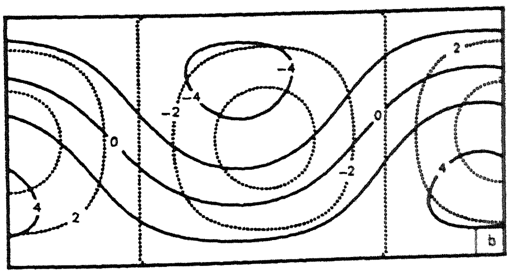
where $W_i(t)$, $i = 1,2,3$ are independent Wiener processes. This stochastic input compensates the absence of higher order spectral terms. The stochastic dynamical system (4) can be approximated by a diffusion process. Let $p(x,t)$ be the probability density distribution that the system is in state x at time t. Then $p(x,t)$ satisfies the so-called Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \tfrac{1}{2}\varepsilon^2 \Delta p - \nabla .\ (pf(x)) \quad \text{or} \quad \frac{\partial p}{\partial t} = M_\varepsilon p. \tag{5}$$

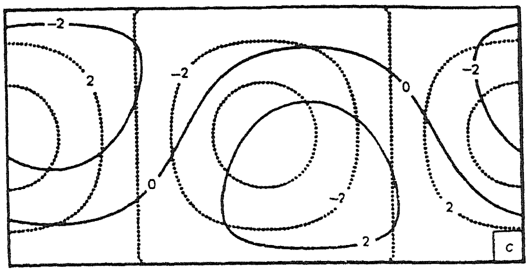
Let at time $t = 0$ the system be in x. Then $T(x)$ is defined as the first passage time of arriving at the separatrice $\Gamma = \partial\Omega_i$ of the deterministic system. Its expected value $T(x)$ satisfies Dynkin's equation



(a) the equilibrium $\bar{x}_1$



(b) the equilibrium $\bar{x}_2$



(c) the equilibrium $\bar{x}_3$

Fig. 2 Dimensional stream function patterns for the equilibrium states of the 3-dimensional spectral model. Dashed lines represent contours of the orography.

$$L_\varepsilon T = -1 \quad \text{in} \quad \Omega_i, \tag{6a}$$
$$T = 0 \quad \text{at} \quad \Gamma, \tag{6b}$$

where L_ε is the formal adjoint of M_ε:

$$L_\varepsilon \equiv \tfrac{1}{2}\varepsilon^2\Delta + f(x) . \nabla .$$

The elliptic singular perturbation problem (6) has an asymptotic solution of the form

$$T(x) \approx C_i \; e^{K_i/\varepsilon^2} \quad \text{in } \Omega_i \text{ outside a neighborhood of } \Gamma, \tag{7a}$$

$$T(x) \approx C_i \; e^{K_i/\varepsilon^2} \sqrt{\tfrac{2}{\pi}} \int_0^{s(x)} e^{-\frac{1}{2}t^2} \, dt \text{ in } \Omega_i \text{ near } \Gamma \tag{7b}$$

with

$$s(x) = \frac{2}{\varepsilon} \left\{ \int_0^{r(x)} \frac{\partial f}{\partial \nu} \, r dr \right\}^{\frac{1}{2}},$$

where ν is the normal at Γ and $r(x)$ the distance to Γ. The constants C_i and K_i are determined as follows. Let $p(x)$ satisfy the stationary Fokker-Planck equation in Ω_i and be of the form

$$p(x) = w(x)e^{-Q(x)/\varepsilon^2} \tag{8}$$

with

$$Q(\overline{x}^{(i)}) = 0 \quad \text{and} \quad Q(x) > 0 \quad \text{for} \quad x \neq \overline{x}^{(i)}.$$

The functions $Q(x)$ and $w(x)$ are determined by the ray method. Substitution of (7) and (8) in the formula for the divergence theorem gives

$$\int_{\Omega_i} \{pL_\varepsilon T - TM_\varepsilon p\}dV = \int_\Gamma [\tfrac{1}{2}\varepsilon^2\{p\frac{\partial T}{\partial \nu} - T\frac{\partial p}{\partial \nu}\} + pTf(x).\nu]ds.$$

For $\varepsilon \to 0$ this equation must hold asymptotically which yields the values of K_i and C_i. We only give

$$K_i = \lim_{x \to \overline{x}_2} Q(x), \qquad K_1 = .23 \quad \text{and} \quad K_2 = .52.$$

It is concluded that most of the time the system is in an ε-neighborhood of one of the two stable equilibria. The attraction dom of these equilibria is most likely left through the separatrice Γ in an ε-neighborhood of the unstable equilibrium $\overline{x}_2$. The expected residence time in domain Ω_i is

$$T_i \approx C_i\ e^{K_i/\varepsilon^2}.$$

Near the unstable equilibrium the system remains a time of order

$$T_2 \approx \frac{1}{\lambda}\ln(\frac{1}{\varepsilon}),$$

where λ is the largest positive eigenvalue of the deterministic system linearized at $\overline{x}_2$. Estimates of ε for atmospheric models are given by Egger and Shilling (1983). They found $\varepsilon^2 \approx .2$.

A discrete Markov process is formulated as follows. Let Q_{ij} denote the transition probability per unit of time from state i to j $(i,j = 1,2,3)$ and let $p_i(t)$ denote the probability of being in state i at time t. Then $p_i(t)$ satisfy

$$\frac{dp_1}{dt} = -(Q_{12}+Q_{21})p_1 - Q_{21}p_3 + Q_{21},$$
$$\frac{dp_3}{dt} = -Q_{23}p_1 - (Q_{32}+Q_{23})p_3 + Q_{23},$$
$$r_2 = 1 - p_1 - p_3,$$

where

$$Q_{23} = Q_{21} = \frac{1}{2T_2}, \qquad Q_{12} = \frac{1}{T_1} \quad \text{and} \quad Q_{32} = \frac{1}{T_3} .$$

In fig. 3 the probability functions $p_i(t)$ are given for a process that starts in state 1 with probability 1.

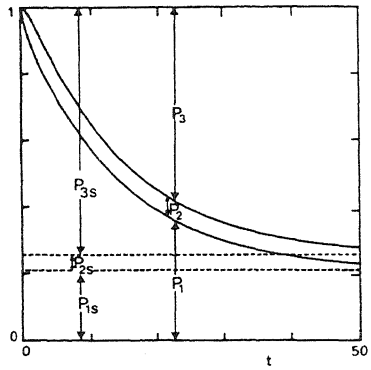
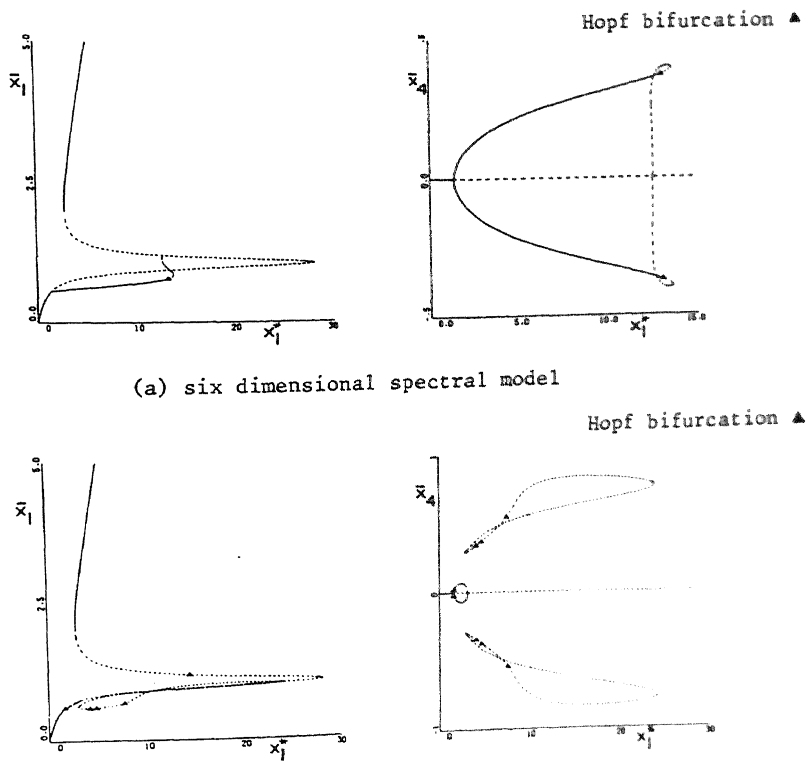


Fig. 3 Evolution of the probability distribution of the Markov process starting in state 1. The dotted lines represent the stationary distribution.

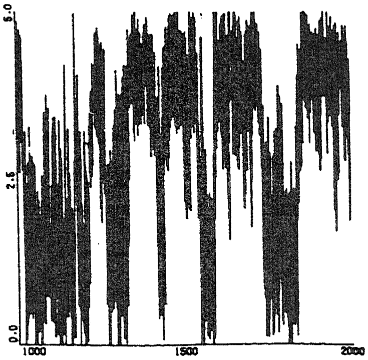
4. HIGHER DIMENSIONAL SPECTRAL MODELS

In higher dimensional spectral models the system will exhibit irregular dynamics from itself. No stochastic forcing is needed to obtain vacillation between states with a zonal flow of different intensities. In this section we summarize some results of De Swart (1987). The purpose is to formulate a spectral model with the lowest dimension that still has chaotic behavior with two clearly different scales of motion (planetary/synoptic) and that has a zonal component (x_1) that varies over a sufficiently large realistic range. In fig. 4 for a six and a ten dimensional model the bifurcation diagrams connected to the equilibria of the three dimensional model are given. One of the stable equilibria under goes a pitchfork bifurcation with the stable branches turning unstable through a Hopf bifurcation. However, the other equilibrium always remains stable. This is due to the symmetry in the forcing stream function. Therefore we take $x_4^* \neq 0$. It is verified that only in the ten dimensional model all equilibria get unstable for some x_4^*. Using physical arguments it is understood that 10 dimensions must be the minimum as only then energy exchange between wave triades is possible. In fig. 5a the x_1 component of a solution is given. Its largest Lyapunov exponent has a positive value, which indicates the presence of a strange attractor. Examining the course of a trajectory projected in the x_2,x_3-plane, we observe that this strange attractor remains from time to time close to three different periodic orbits. The behavior strongly resembles the discrete state Markov process of the preceding section, see fig. 5b. The deterministic chaotic model can be used to study the predictability of atmospheric flow from a theoretical point of view.

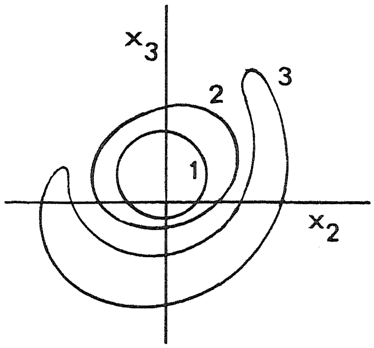


(b) ten dimensional spectral model

Fig. 4 Bifurcation diagrams for higher dimensional spectral models. Solid (dotted) lines represent stable (unstable) stationary solutions.



(a) the x_1-component



(b) sketch of unstable periodic solutions

Fig. 5 A chaotic solution of the 10-dimensional model

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