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MATHEMATICS

A GENERALIZATION OF THE METHOD OF *m* RANKINGS

BY

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1. *Introduction*

1.1. The method of m rankings due to M. FRIEDMAN [3] [1]) is treated by M. G. KENDALL in his book about rankcorrelation methods [6], chapters 6 and 7. KENDALL considers m "observers" $P_1, \ldots, P_m$. Every observer ranks n "objects" $O_1, \ldots, O_n$ and the results are written down in the following scheme.

(1.1.1)

	O_1	O_2	.	.	.	O_n
P_1	r_{11}	r_{12}	.	.	.	r_{1n}
P_2	r_{21}	r_{22}	.	.	.	r_{2n}
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
P_m	r_{m1}	r_{m2}	.	.	.	r_{mn}
	s_1	s_2	.	.	.	s_n

where the letters $r_{\mu\nu}$ [2]) denote the ranks and $s_1, \ldots, s_n$ their column totals.

In this paper for the ranking procedure the terminology used by M. G. KENDALL in [1] (e.g. the terms: "rank", "ranking", "tie", etc.) is applied.

1.2. The method of m rankings enables us to investigate whether the "observers" agree in their opinion about the ranks. For that reason one tests the hypothesis H_0 [3]), which in the case of absence of ties states, that the rankings are chosen at random from the collection of all permutations of the numbers $1, \ldots, n$ and that they are independent [4]).

[1]) Numbers between brackets of the type [ ] refer to the list of references.
[2]) If no restrictions are mentioned in this paper μ is supposed to run through the values $1, 2, \ldots, m$ and ν and ν' through the values $1, 2, \ldots, n$.
[3]) When there are ties, a slightly different hypothesis is tested (see 2.7).
[4]) We use the term "independent" for "mutually completely independent" according to J. NEYMAN.

The statistic used is:

$$\mathbf{S} \stackrel{\text{def}}{=} \sum_{\nu=1}^{n} \{\mathbf{s}_\nu - \tfrac{1}{2}\, m(n+1)\}^2. \quad (1.2.1)$$

[5] [6]

M. FRIEDMAN and M. G. KENDALL have computed the probability distribution of $\mathbf{S}$ for the case that H_0 is true and m and n are small. For large values of m one can use the statistic (introduced by M. FRIEDMAN):

$$\chi_r^2 \stackrel{\text{def}}{=} \frac{12\,\mathbf{S}}{mn(n+1)} \quad (1.2.2)$$

which, if H_0 is true, is distributed asymptotically as χ^2 with $n - 1$ degrees of freedom. If ties occur, a correction is applied (see 3.4). Also an asymptotic z-test, due to M. G. KENDALL exists, to be used if m or n is large (see [5] chapter 6).

For the generalization of the method of m rankings in this paper we only consider a statistic analogous to FRIEDMAN's χ_r^2.

1.3. The critical region of FRIEDMAN's test consists of all values of $\mathbf{S}$, which are not smaller than S_0, where S_0 is the greatest value of S, for which

$$\mathcal{P}\{\mathbf{S} \geqq S \,|\, H_0\} \leqq \alpha$$

and α is a given number ($0 < \alpha < 1$), the level of significance. If there is strong concordance between the observers $\mathbf{S}$ will assume a large value and H_0 will be rejected. Thus the test is a simple method to investigate "concordance" in rows of numbers (observations) of equal length. It is not necessary that the letters O_ν in scheme (1.1.1) refer to "objects" and the letters P_μ to "observers". For example $P_1, \ldots, P_m$ may be measurements of different quantities executed on different moments $O_1, \ldots, O_n$. In that case one supposes, that the measurements are observations of random variables $\mathbf{x}_{\mu\nu}$, one observation $x_{\mu\nu}$ of each $\mathbf{x}_{\mu\nu}$ being available. For each μ the observations $x_{\mu 1}, \ldots, x_{\mu n}$ are ranked according to increasing values. Then H_0 is valid if e.g. the sets $x_{\mu 1}, \ldots, x_{\mu n}$ are independent random samples taken either from the same or from different distributions. The test is often applied when one expects concordance caused by a common trend within each of the random vectors $(\mathbf{x}_{\mu 1}, \ldots, \mathbf{x}_{\mu n})$.

1.4. In practice it often occurs that the number of observations of $\mathbf{x}_{\mu\nu}$ is not one, but either another positive integer or zero. In that case we cannot apply FRIEDMAN's method of m rankings. J. DURBIN has given a generalization, which can be used in m-rankings-schemes in which observations are lacking, but these schemes are of a very special type and must be planned before the experiment. See [2] and 3.5 below.

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[5]) According to the hypothesis tested, the ranks are random variables. The random character of a variable is denoted by printing its symbol in bold type. Values assumed by a random variable are often denoted by the same symbol, printed in italics.

[6]) The symbol $\stackrel{\text{def}}{=}$ denotes an equality, defining the left hand member.

1.5. In this paper we shall consider a much wider generalization, where the number of observations of $\mathbf{x}_{\mu\nu}$ may be any arbitrary non-negative integer $k_{\mu\nu}$. To achieve this, we rank for each μ, the observations corresponding to "observer" P_μ. The ranks of observations of $\mathbf{x}_{\mu\nu}$ are said to belong to cell (μ, ν). The present method can also be used if some cells are empty, because some experiments have failed.

As we have mn parameters $k_{\mu\nu}$ our test is more complicated than FRIEDMAN's test. The high number of parameters also forces us to restrict ourselves to an asymptotic test.

1.6. Summary of the paper's contents:
In 2 we describe the computation of our statistic (**2.1–2.6**), we state sufficient conditions for this statistic to have asymptotically a χ^2-distribution (**2.7–2.9**), and add some remarks concerning the application of the test (**2.10–2.13**). In **3** we discuss some special cases of our test and **4** is a mathematical appendix containing the proofs of theorems, on which our results are based.

2. *Description of the test*

2.1. We have seen in **1.5**, that in the μ^{th} ranking of our scheme we have $k_\mu \overset{\text{def}}{=} \sum_\nu k_{\mu\nu}$ ranks. In each ranking "ties" (see **1.1**) are allowed. The number of ranks in a tie is called the size of that tie. In particular a rank that is not equal to any other rank in the same ranking is considered as a tie of size 1 [7]). We denote by $t_{\mu\gamma}$ the number of ties of size γ [8]) and by g_μ the size of the greatest tie in the μ^{th} ranking.

2.2. For each μ, we derive from the ranks of the μ^{th} ranking, the "reduced" ranks by substracting $\frac{1}{2}(k_\mu + 1)$, the arithmetical mean of the ranks of the μ^{th} ranking. The sum of the reduced ranks in cell (μ, ν) is denoted by $\tilde{u}_{\mu\nu}$. If $k_{\mu\nu} = 0$ we put $\tilde{u}_{\mu\nu} = 0$. We use the symbol $\tilde{u}_{\mu\nu}$ because, if $k_{\mu\nu} \geqq 1$, $\tilde{u}_{\mu\nu}$ is the reduced value of WILCOXON's statistic, usually denoted by U (see [8] and [13]), of the sample of observations of cell (μ, ν) against the sample of all other observations of P_μ taken together. Instead of scheme (1.1.1) we now get:

(2.2.1)

	O_1	O_2	.	.	.	O_n
P_1	$\tilde{u}_{11}$	$\tilde{u}_{12}$	.	.	.	$\tilde{u}_{1n}$
P_2	$\tilde{u}_{21}$	$\tilde{u}_{22}$	.	.	.	$\tilde{u}_{2n}$
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
P_m	$\tilde{u}_{m1}$	$\tilde{u}_{m2}$	.	.	.	$\tilde{u}_{mn}$
	$\tilde{u}_1$	$\tilde{u}_2$	.	.	.	$\tilde{u}_n$

where the quantities $\tilde{u}_\nu = \sum_\mu \tilde{u}_{\mu\nu}$ will be called the column-totals.

[7]) But if all ranks are unequal we usually say that there are "no ties" in the scheme.

[8]) In this paper γ is supposed to run through the values $1, 2, \ldots, g_\mu$.

2.3. We compute the quantities:

$$(2.3.1)\qquad \begin{cases} \sigma_{\nu\nu'} \overset{\text{def}}{=} -\sum\limits_{\mu} k_{\mu\nu}\, k_{\mu\nu'}\, K_\mu & (\text{if } \nu \neq \nu'), \\ \sigma_{\nu\nu} \overset{\text{def}}{=} \sum\limits_{\mu} k_{\mu\nu}\,(k_\mu - k_{\mu\nu})\, K_\mu\,, \end{cases}$$

where

$$(2.3.2)\qquad K_\mu \overset{\text{def}}{=} \frac{k_\mu^3 - \sum\limits_{\gamma} \gamma^3\, t_{\mu\gamma}}{12 k_\mu (k_\mu - 1)}.$$

The matrix $(\sigma_{\nu\nu'})$ is denoted by V; for abbreviation we often use σ_ν instead of $\sqrt{\sigma_{\nu\nu}}$.

Under the hypothesis H_0 (to be defined in detail in **2.7**) we have

$$(2.3.3)\qquad \sigma_{\nu\nu'} = \mathcal{E}\, \tilde{\boldsymbol{u}}_\nu\, \tilde{\boldsymbol{u}}_{\nu'}$$

and so V is the matrix of the variances and covariances of the column-totals. (Proof see **4.1**, Theorem I.)

2.4. In the following text we shall omit all rows in which all ranks are equal or all $k_{\mu\nu}$ are zero except for one value of ν only. As for these rows all quantities $\tilde{u}_{\mu\nu} \equiv 0$, they do not contribute to the values of the quantities $\sigma_{\nu\nu'}$. They are called "superfluous rows".

2.5. It is possible that in our scheme so many numbers $k_{\mu\nu}$ are zero, that we have two or more complementary sets of objects, so that in every ranking only the objects of *one* of these sets occur. These sets are called non-compared sets of objects; the number of non-compared sets of objects will be denoted by s. Formally the number s of non-compared sets of objects is the greatest number of mutually exclusive subsets $I_t (t = 1, \dots, s)$ into which the set of numbers $I = \{1, \dots, n\}$ can be divided so that $k_{\mu\nu}\, k_{\mu\nu'} = 0$

$$\text{for all } \mu,\ \nu,\ \nu',\ t,\ t' \text{ with } \mu = 1, \dots, m;\ \nu \in I_t;$$
$$\nu' \in I_{t'};\ t = 1, \dots, s;\ t' = 1, \dots, s;\ t' \neq t.$$

If a submatrix $(\sigma_{\nu\nu'})$ of V with $\nu \in J \subset I$ and $\nu' \in I - J$ is called a "submatrix with complementary sides", then if $s > 1$, there is at least one "submatrix with complementary sides" in V, all elements of which are zero.

In **4.2** we shall prove Theorem II:

If and only if there are s non-compared sets of objects the rank of the matrix V is $n - s$.

2.6. If $s = 1$ [9]) the statistic of our test is defined in the following way. Consider the matrix obtained from:

$$(2.6.1)\qquad V_u \overset{\text{def}}{=} \begin{pmatrix} \sigma_{11} & \cdot & \cdot & \cdot & \sigma_{1n} & \tilde{\boldsymbol{u}}_1 \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \sigma_{n1} & \cdot & \cdot & \cdot & \sigma_{nn} & \tilde{\boldsymbol{u}}_n \\ \tilde{\boldsymbol{u}}_1 & \cdot & \cdot & \cdot & \tilde{\boldsymbol{u}}_n & 0 \end{pmatrix}$$

[9]) The case $s \geqq 2$ will be considered in **2.11**.

by omitting an arbitrary row and an arbitrary column, except for the last row and the last column, and compute its determinant Δ_u; consider also the matrix obtained from:

$$(2.6.2) \qquad V \stackrel{\text{def}}{=} \begin{pmatrix} \sigma_{11} & \cdot & \cdot & \cdot & \sigma_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{n1} & \cdot & \cdot & \cdot & \sigma_{nn} \end{pmatrix}$$

by omitting an arbitrary row and an arbitrary column, and compute its determinant Δ. Then our statistic is:

$$(2.6.3) \qquad \chi_r^2 \stackrel{\text{def}}{=} \frac{|\Delta_u|}{|\Delta|}.$$

The statistic χ_r^2 is independent of the choice of the rows and columns omitted in V_u and V, as in both matrices, each row (column) except for the last one in V_u is a linear combination of the other rows (columns). (Cf. (4.2.4) below.)

2.7. Before we can treat the asymptotic distribution of χ_r^2, we first have to describe the hypothesis H_0 on which it is based.

The result of an experiment is, according to our test, brought into a scheme of m rankings or sets of ranks, each of which is divided into subsets, called "cells", corresponding to the objects.

We consider the collection of all possible results of experiments where the numbers $k_{\mu\nu}$ (see **2.1**) as well as the ranks occurring in the rankings, are the same as those found in the experiment actually performed. Hypothesis H_0 postulates that we have:

1 For each ranking all possible manners of dividing the given set of ranks into cells of the prescribed sizes have the same probability.

2 The different rankings are independent.

If the ranks are based upon observations of random variables $\boldsymbol{x}_{\mu\nu}$ the hypothesis H_0 will be valid if all $\boldsymbol{x}_{\mu\nu}$ are independent and the variables $\boldsymbol{x}_{\mu\nu}$ with the same suffix μ have the same distribution functions.

2.8. Our knowledge concerning the asymptotic distribution of χ_r^2 is based upon the following theorems:

Theorem III. *If hypothesis H_0 is valid and:*

III_1 *the number of columns, n, is bounded,*

III_2 *the number of rows, m, tends to infinity,*

III_3 *for $\nu = 1, \ldots, n$,*

$$\lim_{m\to\infty} \sum_{\mu} \sigma_\nu^{-3}\, \mathcal{E}\,|\tilde{\boldsymbol{u}}_{\mu\nu}|^3 = 0,$$

III_4 *the rank of the matrix*

$$R \stackrel{\text{def}}{=} \begin{pmatrix} \varrho_{11} & \varrho_{12} & \cdot & \cdot & \cdot & \varrho_{1n} \\ \varrho_{21} & \varrho_{22} & \cdot & \cdot & \cdot & \varrho_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varrho_{n1} & \varrho_{n2} & \cdot & \cdot & \cdot & \varrho_{nn} \end{pmatrix},$$

where $\varrho_{\nu\nu'} = \lim_{m\to\infty} (\sigma_\nu \sigma_{\nu'})^{-1} \sigma_{\nu\nu'}$ *equals* $n-1$,

then the distribution of any $n-1$ *column-totals is asymptotically equivalen, with the* $(n-1)$*-dimensional normal distribution with the same covariance matrix as the exact distribution of these column-totals.* (This is a submatrix of V, see 2.6.)

From this theorem it can easily be deduced that the statistic χ^2_r, defined by (2.6.3) has under the same conditions asymptotically a χ^2-distribtuion with $n-1$ degrees of freedom.

This theorem is a consequence of the Central-limit-theorem for random vectors (see for instance [12] p. 318, where this theorem is proved for the two-dimensional case).

Theorem IV. *If for any row (row-suffix* μ_0*), the corresponding part of hypothesis* H_0 *is valid and:*

IV_1 *the number of objects* n *is bounded,*

IV_2 *there is a set* L *of* l *column-suffixes* $(l \geqq 2)$ *so that if* $\nu \in L$:

$$\lim_{k_{\mu_0}\to\infty} k_{\mu_0\nu}/k_{\mu_0} > 0$$

IV_3 $\lim_{k_{\mu_0}\to\infty} g_{\mu_0}/k_{\mu_0} < 1$, *where* g_{μ_0} *is the size of the greatest tie in the* $\mu_0{}^{\text{th}}$ *ranking*

then the simultaneous distribution of $l-1$ *quantities* $\tilde{\mathbf{u}}_{\mu_0\nu}$, *all with* $\nu \in L$ *is asymptotically equivalent with the* $(l-1)$*-dimensional normal distribution with the same covariance matrix as the exact distribution of these* $\tilde{\mathbf{u}}_{\mu_0\nu}$.

Theorem IV is due to W. H. Kruskal [7]. Similar theorems have been treated by T. J. Terpstra [11] and P. J. Rijkoort [9]. It follows that:

$$\boldsymbol{H} \overset{\text{def}}{=} \frac{1}{k_\mu K_\mu} \sum_{\nu\in L} \frac{\tilde{\boldsymbol{u}}_\nu^2}{k_{\mu\nu}} \tag{2.8.1}$$

has asymptotically a χ^2-distribution with $n-1$ degrees of freedom.

2.9. Theorem V.

For a scheme, all numbers k_μ of which are bounded:

(2.9.1) $\quad k_\mu \leqq M$

III_3 and III_4 can be replaced by the following more convenient conditions:

(2.9.2) $\quad \lim_{m\to\infty} m^{-1} \sum_\mu k_\mu > 0$, sufficient for III_3 and:

(2.9.3) $\quad$ the matrix:

$$C \overset{\text{def}}{=} \begin{pmatrix} \varkappa_{11} & \cdot & \cdot & \cdot & \varkappa_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \varkappa_{n1} & \cdot & \cdot & \cdot & \varkappa_{nn} \end{pmatrix}$$

where $\varkappa_{\nu\nu'} = \lim_{m\to\infty} m^{-1} \sum_\mu \varkappa_{\mu\nu} \varkappa_{\mu\nu'}$ is not a matrix of the type $\begin{pmatrix} PO' \\ OQ \end{pmatrix}$

in which P and Q are square matrices and O and O' consist of zeros only. (Proof see 4.3.)

2.10. It follows from the theorems considered in **2.8** that the statistic χ_r^2 defined by (2.6.3) asymptotically has a χ^2-distribution with $n-1$ degrees of freedom, if the scheme consists of a set of rows, obeying $\mathrm{III}_1, \ldots, \mathrm{III}_4$ and a set obeying $\mathrm{IV}_1, \ldots, \mathrm{IV}_3$.

Applying the theorems III and IV for finite schemes in practice we will translate "bounded" by "small", "infinite" by "large" and "asymptotically" by "approximately". As for the limit theorems used, we have no estimate of the difference between the exact and the limit distributions, we cannot be more precise in our formulation. In special cases, however, where the exact distribution could be calculated, the χ^2-approximation appeared to underestimate the level of significance.

2.11. If the number s of non compared sets of objects (see **2.5**) is greater than 1, the rankings in which the objects of one of the non compared sets occur constitute a scheme, and for each thus obtained scheme we can define a statistic of the type χ_r^2 according to **2.6.** Under the conditions mentioned in **2.7–2.8**, these statistics will asymptotically have χ^2-distributions with numbers of degrees of freedom that are one less than the numbers of elements of the non compared sets. Furthermore they are independently distributed under H_0; hence their sum will then asymptotically have a χ^2-distribution with $n-s$ degrees of freedom.

2.12. The statistic defined by (2.6.3) can be written as a positive definite quadratic form in $n-1$ column-totals, for instance $\tilde{\boldsymbol{u}}_1, \ldots, \tilde{\boldsymbol{u}}_{n-1}$. So it may by linear transformation be transformed into a quadratic form of the type $\sum_i a_i \boldsymbol{v}_i^2$ where $a_i > 0$ $(i = 1, \ldots, n-1)$ and $\sum_i \boldsymbol{v}_{ij}^2 = \sum_i \tilde{\boldsymbol{u}}_i^2$. Consequently χ_r^2 will be large if there is a strong variation in the numbers $\tilde{\boldsymbol{u}}_1, \ldots, \tilde{\boldsymbol{u}}_{u-1}$ ($\sum_i \tilde{\boldsymbol{u}}_i^2$ is large). We expect such a strong variation if the observers are concordant.

2.13. If we compute KRUSKAL's H (see **2.8**) for every row of our scheme, the sum of these statistics will, under the appropriate conditions (theorem IV), asymptotically have a χ^2-distribution, with a number of degrees of freedom equal to the sum of the degrees of freedom of the individual terms. In this way we obtain another test for rankings fulfilling the conditions of theorem IV. It is, however, not a test against concordance but against inhomogeneity in each of the rows separately, or in terms of the random variables $\boldsymbol{x}_{\mu\nu}$, a test of H_0 against alternatives involving that the differences of many pairs of these variables with the same suffix μ have a median different from zero.

3. *Special cases*

3.1. In this paragraph we consider some special cases for which the scope of the computation of χ_r^2 can be reduced. We shall also see that many non-parametric tests can be considered as special cases of ours.

3.2. We shall prove in **4.4** (theorem VI) that the statistic χ_r^2 is a linear compositum of the squares $\tilde{u}_1^2, \ldots, \tilde{u}_n^2$, if and only if there are positive numbers c_ν such that

(3.2.1) $$\sigma_{\nu\nu'} = - c_\nu c_{\nu'} \qquad (\nu' \neq \nu)$$

(whence $\sigma_{\nu\nu} = cc_\nu$, where $c = \sum_\nu c_\nu$).

In that case we have:

(3.2.2) $$\chi_r^2 = \sum_\nu \tilde{u}_\nu^2 / cc_\nu .$$

For $n = 3$, condition (3.2.1) is always satisfied. (Put $c_1 = \sqrt{-\sigma_{12}\sigma_{31}/\sigma_{23}}$, etc., the statistic then becomes:

$$\chi_r^2 = -\frac{\sigma_{23}\tilde{u}_1^2 + \sigma_{31}\tilde{u}_2^2 + \sigma_{12}\tilde{u}_3^2}{\sigma_{12}\sigma_{23} + \sigma_{31}\sigma_{12} + \sigma_{23}\sigma_{31}} .)$$

For $n > 3$, condition (3.2.1) can only be realised by designing the experiment appropriately. For instance, if

(3.2.3) $$k_{\mu\nu} = a_\mu b_\nu ,$$

then

$\chi_r^2 = \dfrac{\sum_\nu \tilde{u}_\nu^2 / b_\nu}{b \sum_\mu a_\mu^2 K_\mu}$, where $b = \sum_\nu b_\nu$ and K_μ is defined by (2.3.2). (Put $c_\nu = b_\nu \sqrt{\sum_\mu a_\mu^2 K_\mu}$.)

If $m = 1$, condition (3.2.3) is fulfilled if we put (omitting the suffix $\mu = 1$): $a = 1$, $b_\nu = k_\nu$. We then have:

(3.2.4) $$\chi_r^2 = \frac{1}{kK} \sum_\nu \frac{\tilde{u}_\nu^2}{k_\nu} .$$

This is a special case ($l = n$) of the statistic defined by (2.8.1).

3.3. The statistic χ_r^2 defined by (2.6.3) is symmetrical in all $\tilde{u}_\nu$ if and only if all covariances $\sigma_{\nu\nu'}$ are equal. (Proof see **4.5**, theorem VII.)

We then have $\sigma_\nu^2 = \sigma^2$ and (by (3.2.1) and (3.2.2)):

(3.3.1) $$\chi_r^2 = \frac{(n-1)\mathbf{S}}{n\sigma^2} , \quad \text{where } \mathbf{S} \overset{\text{def}}{=} \sum_\nu \tilde{u}_\nu^2 .$$

In each scheme with $n = 2$, the statistic χ_r^2 is symmetrical. Because of $\tilde{u}_1 + \tilde{u}_2 = 0$ we then have $\tilde{u}_1^2 = \tilde{u}_2^2 = \tilde{u}^2$, and $\chi_r^2 = \tilde{u}^2/\sigma^2$ has aymptotically a χ^2-distribution with one degree of freedom, i.e. $\tilde{u}/\sigma$ is asymptotically normal, under the appropritate conditions. Special cases are the sign test [1], where $n = 2$ and $k_\mu = 2$ for all μ and WILCOXON's test [13], [7], where $n = 2$ and $m = 1$. Applying our theorem III to the sign test, we see that $2\tilde{u}m^{-\frac{1}{2}}$ is asymptotically normal for $m \to \infty$, if ties (superfluous rows) are omitted. J. HEMELRIJK has proved that the sign test is asymptotically more powerful if ties are omitted than if we divide them equally among the positive and negative observations. (See [5].) This leads us to conjecture that the power of our generalized test of m rankings would be decreased if superfluous rows had not been omitted.

3.4. The χ^2-statistic for the ordinary method of m rankings with correction for ties is an example of a symmetric $\boldsymbol{\chi}_r^2$ with $k_{\mu\nu} = 1$. We then have:

$$\boldsymbol{\chi}_r^2 = \frac{12\,\mathbf{S}}{mn(n+1)-T}, \tag{3.4.1}$$

where

$$T = \frac{1}{n-1} \sum_{\mu} \sum_{\gamma} (\gamma^3-\gamma)\, t_{\mu\gamma}. \tag{3.4.2}$$

If there are no ties (i.e. $g_\mu = 1$), $T = 0$ and $\boldsymbol{\chi}_r^2$ is equal to the statistic defined by (1.2.2).

3.5. The χ^2-statistic for the DURBIN-scheme without ties (see [2]) is also a symmetric $\boldsymbol{\chi}_r^2$, with $k_\mu = k$, all $k_{\mu\nu} = 0$ or 1 and $\sum_\mu k_{\mu\nu} k_{\mu\nu'} = \lambda$ for $\nu' \neq \nu$. We then have:

$$\sigma_{\nu\nu'} = \tfrac{1}{12}\,\lambda(k+1)$$

and

$$\boldsymbol{\chi}_r^2 = 12\,\mathbf{S}/n\lambda\,(k+1).$$

4. *Mathematical appendix*

4.1. Theorem I (See 2.3).

If H_0 (2.7) is valid we have in the notation of 2.1–2.3:

$$\left\{\begin{aligned} \mathcal{E}\,\tilde{\mathbf{u}}_\nu^2 &= \sum_\mu k_{\mu\nu}(k_\mu-k_{\mu\nu})\,K_\mu, \\ \mathcal{E}\,\tilde{\mathbf{u}}_\nu \tilde{\mathbf{u}}_{\nu'} &= -\sum_\mu k_{\mu\nu} k_{\mu\nu'}\,K_\mu, \end{aligned}\right. \tag{4.1.1}$$

where $\nu' \neq \nu$ and

$$K_\mu \overset{\text{def}}{=} \frac{k_\mu^3 - \sum_\gamma \gamma^3 t_{\mu\gamma}}{12\,k_\mu(k_\mu-1)}, \quad \text{cf. (2.3.2).} \tag{4.1.2}$$

Proof: M. G. KENDALL's expression for the variance of his rank-correlation statistic $\mathbf{S}$ has been adapted by J. HEMELRIJK [4] to the variance of WILCOXON's $\boldsymbol{U}$, his formula can easily be reduced to:

$$\mathcal{E}\,\tilde{\boldsymbol{U}}^2 = \frac{1}{12}\,\frac{mn\{(m+n)^3-\sum_\gamma \gamma^3 t_\gamma\}}{(m+n)(m+n-1)}, \tag{4.1.3}$$

where m and n are the numbers of elements of the samples considered, t_γ is the number of ties of size γ and H_0 is the hypothesis that all manners of arranging the $m+n$ ranks in the two samples have the same probability (Cf. 2.7).

As $\tilde{\mathbf{u}}_{\mu\nu}$ is the reduced statistic of WILCOXON of the sample of observations of cell (μ, ν) against the sample of all other observations of P_μ taken together (see 2.2), we have:

$$\mathcal{E}\,\tilde{\mathbf{u}}_{\mu\nu}^2 = k_{\mu\nu}(k_\mu - k_{\mu\nu})\,K_\mu \tag{4.1.4}$$

and by

$$\mathcal{E}\,(\tilde{\mathbf{u}}_{\mu\nu}+\tilde{\mathbf{u}}_{\mu\nu'})^2 = \mathcal{E}\,\tilde{\mathbf{u}}_{\mu\nu}^2+\mathcal{E}\,\tilde{\mathbf{u}}_{\mu\nu'}^2+2\,\mathcal{E}\,\tilde{\mathbf{u}}_{\mu\nu}\tilde{\mathbf{u}}_{\mu\nu'},$$

$$\mathcal{E}\,\tilde{\mathbf{u}}_{\mu\nu}\tilde{\mathbf{u}}_{\mu\nu'} = -\,k_{\mu\nu}k_{\mu\nu'}K_\mu. \tag{4.1.5}$$

Now the formulae (4.1.1) are obvious, as the rankings are independent if H_0 is true.

4.2. Theorem II (See 2.5).

If and only if there are s non compared sets of objects the rank of matrix V is $n-s$.

Proof: O. Tausski has drawn attention to the following theorem: let (a_{ik}) be an $n \times n$ matrix with complex elements such that:

$$|a_{ii}| \geqq \sum_{k=1}^{n} |a_{ik}| \qquad (i = 1, \ldots, n)$$

with equality in at most $n-1$ cases. Assume further that the matrix cannot be transformed into a matrix of the form:

$$\begin{pmatrix} P & U \\ O & Q \end{pmatrix} \tag{4.2.1}$$

by the same permutation of the rows and columns, where P and Q are square matrices and O consists of zeros. It follows that $\det(a_{ik}) \neq 0$. (See [10], theorem III.)

In the proof of this theorem it is shown, that if the matrix is of the form (4.2.1), $\det(a_{ik}) = 0$.

Our matrix V has the following properties:

$$\sigma_{\nu\nu'} = \sigma_{\nu'\nu} \leqq 0 \qquad (\nu' \neq \nu), \tag{4.2.2}$$

$$\sigma_{\nu\nu} > 0, \tag{4.2.3}$$

if superfluous rows are omitted, see 2.4, and:

$$\sum_{\nu'} \sigma_{\nu\nu'} = \sum_{\nu'} \sigma_{\nu'\nu} = 0. \tag{4.2.4}$$

By (4.2.4) we have immediately:

$$\det V = 0. \tag{4.2.5}$$

Hence the rank of V is not greater than $n-1$.

The matrix $V_{\nu\nu}$, obtained from V by omitting the ν^{th} row and the ν^{th} column, fulfills the conditions of the theorem mentioned; hence the rank of V is only smaller than $n-1$, if each $V_{\nu\nu}$ is a (symmetric) matrix of the form (4.2.1). Then V will also be of this form. Conversely, if V is of the form (4.2.1) it is trivial that its rank is $n-2$.

We have seen in 2.5 that "V is of the form (4.2.1)" is aequivalent with: "the number of non compared sets of objects is greater than 1".

If $s > 1$, we can, by repeating our argument to the matrices P and Q, both having properties analogous to (4.2.2)–(4.2.4), prove that the rank of V is $n - s$.

4.3. Proof of Theorem V (See **2.9**).

If (2.9.1) is valid and superfluous rows (**2.4**) are omitted we have:

$$K_\mu \overset{\text{def}}{=} \frac{k_\mu^3 - \sum_\gamma \gamma^3 t_{\mu\gamma}}{12 k_\mu (k_\mu - 1)} \geq \frac{k_\mu^3 - (k_\mu - 1)^3 - 1}{12 k_\mu (k_\mu - 1)} = \frac{1}{4}\,.$$

Now by (4.1.4) and (4.1.5)

(4.3.1) $$\mathcal{E}\, \tilde{\boldsymbol{u}}_{\mu\nu} \tilde{\boldsymbol{u}}_{\mu\nu'} = 0 \text{ if and only if } k_{\mu\nu}\, k_{\mu\nu'} = 0.$$

It follows that:

(4.3.2) $$\lim_{m\to\infty} \sigma_\nu^2/m = \lim_{m\to\infty} m^{-1} \sum_\mu \mathcal{E}\, \tilde{\boldsymbol{u}}_{\mu\nu}^2 > 0 \quad \text{(if (2.9.2) is valid).}$$

We also have, by

$$|\tilde{\boldsymbol{u}}_{\mu\nu}| \leqq \tfrac{1}{2} k_\mu k_{\mu\nu} \leqq \tfrac{1}{2} M^2 \quad \text{i.e.} \quad \mathcal{E}\, |\tilde{\boldsymbol{u}}_{\mu\nu}|^3 \leqq \frac{M^6}{8},$$

(4.3.3) $$\lim_{m\to\infty} m^{-1} \sum_\mu \mathcal{E}\, |\tilde{\boldsymbol{u}}_{\mu\nu}|^3 \leqq \tfrac{1}{8} M^6.$$

Now by (4.3.2) and (4.3.3) we see that the conditions (2.9.1) and (2.9.2) are sufficient for III_4.

By (4.3.1) and (4.3.2) we have, that if (2.9.2) is valid:

$$\varrho_{\nu\nu'} = \lim_{m\to\infty} \frac{m^{-1} \sum_\mu \mathcal{E}\, \tilde{\boldsymbol{u}}_{\mu\nu} \tilde{\boldsymbol{u}}_{\mu\nu'}}{(\sigma_\nu^2/m)^{1/2} (\sigma_{\nu'}^2/m)^{1/2}} = 0$$

if

$$\varkappa_{\nu\nu'} = \lim_{m\to\infty} \sum_\mu k_{\mu\nu}\, k_{\mu\nu'} = 0.$$

If follows easily that the conditions (2.9.1)–(2.9.3) are sufficient for III_4.

4.4. Theorem VI (See **3.2**).

If and only if there are positive numbers c_ν such that:

(3.2.1) $$\sigma_{\nu\nu'} = -\, c_\nu c_{\nu'} \qquad (\nu' \neq \nu)$$

then positive numbers $c_\nu^* = (c c_\nu)^{-1}$ with $c = \sum_\nu c_\nu$ exist such that:

(4.4.1) $$\chi_r^2 = \sum_\nu c_\nu \tilde{\boldsymbol{u}}_\nu^2\,.$$

Proof: If the matrix of $\varDelta_u$ is derived from V_u and the matrix of $\varDelta$ from V by omitting the n^{th} rows and columns we see that χ_r^2 (defined by (2.6.3)) is a quadratic form in $\tilde{\boldsymbol{u}}_1, \ldots, \tilde{\boldsymbol{u}}_{n-1}$, the matrix of which is the inverse of the matrix

$$V_{nn} = \begin{pmatrix} \sigma_{11} & \cdot & \cdot & \cdot & \sigma_{1,n-1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{n-1,1} & \cdot & \cdot & \cdot & \sigma_{n-1,n-1} \end{pmatrix}.$$

If (4.4.1) is valid we also have (by $\sum_\nu \tilde{u}_\nu = 0$):

$$\chi_r^2 = \sum_{\nu=1}^{n-1} (c_\nu^* + c_n^*)\, \tilde{u}_\nu^2 + 2\, c_n^* \sum_{\nu=2}^{n-1} \sum_{\nu'=1}^{\nu-1} \tilde{u}_\nu \tilde{u}_{\nu'}$$

also a quadratic form in $\tilde{u}_1, \ldots, \tilde{u}_{n-1}$. Its matrix must be the inverse of matrix V_{nn}. Using this relation theorem VI is easily proved.

4.5. Theorem VII (See 3.3).

If and only if all covariances $\sigma_{\nu\nu'}$ are equal, χ_r^2 is symmetrical in the column-totals $\tilde{u}_\nu$.

Proof: If all covariances are equal, the symmetry of χ_r^2 is trivial by theorem VI.

If χ_r^2 is symmetric in $\tilde{u}_1, \ldots, \tilde{u}_n$, it remains a symmetric quadratic form if we eliminate one of the variables $\tilde{u}_1, \ldots, \tilde{u}_n$ using $\sum_\nu \tilde{u}_n = 0$. The matrix of such a quadratic form is the inverse of a matrix $V_{\nu\nu}$ obtained from V (2.6.2) by omitting the ν^{th} row and column. In this $V_{\nu\nu}$ all diagonal and all non-diagonal elements must be equal. It follows that all covariances $\sigma_{\nu\nu'}$ $(\nu' \neq \nu)$ are equal.

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