The Decent Philosophers: An exercise in operational semantics of concurrent systems

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25 years of Jaco's extraordinarily successful work in semantics causes a particular occasion to arrange for him a big dinner with friends and colleagues. Edsger W. Dijkstra almost 20 years ago suggested how to organize such a dinner: The table is round, the dish is spaghetti, participants (called *philosophers*) share forks with their respective neighbours, using both their right and left fork when taking a meal. Each philosopher may take several meals during the dinner.

A lot of different courses of the dinner are possible in principle. As an example, a greedy philosopher might have several meals before his left neighbour had one, or a philosopher might starve to death due to conspiring neighbours.

The participants at Jaco's dinner of course all will be very decent: Each philosopher takes his left and his right fork alternating with his corresponding neighbour. In the following we study dinners of decent philosophers. It will turn out that this appears trivial only at a very first glance.

How to describe dinners of decent philosophers?

This question will be tackled depending on the number of participating philosophers.

In case two philosphers A and B, say, are the only participants, the decent dinner in the usual style is

$$\ldots a\; b\; a\; b\; a \ldots \tag{1}$$

Each occurrence of "a" and of "b" in (1) denotes a particular meal of A and of B, respectively. We are not interested in how the dinner begins or ends; so (1) describes the dinner in progress, without explicit bound to the left as well as to the right. (1) obviously indicates that the meals of the philosophers are taken sequentially.

In case three decent philosophers A, B and C participate in the dinner we likewise get the sequence

$$\ldots a\; b\; c\; a\; b\; c\; a \ldots \tag{2}$$

as well as the sequence

$$\ldots a\; c\; b\; a\; c\; b\; a \ldots, \tag{3}$$

the meaning of which is obvious from the above explanation of (1).

Four decent philosophers are enough to cause concurrency to some degree. Assuming the right neighbour of A, B, C and D to be B, C, D and A, respectively, we may represent their dinner as

$$\cdots \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} \cdots \tag{4}$$

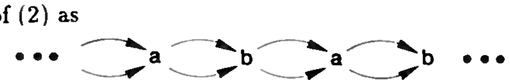
with $\binom{x}{y}$ denoting the philosophers X and Y to take a meal at the same time. Formally, (4) describes a sequence of sets of meals, a *step sequence*, whereas in (1), (2) and (3) we had sequences of single meals, i.e. *event sequences.*

Before continuing we should discuss which properties of the decent philosphers' dinner are represented in (1) – (4): The single meals of the involved philosphers are ordered exactly such that neighbouring philosophers' meals alternate. No further assumptions or restrictions are represented. Particularly no means are available to observe order of far apart philosophers meals. All order on the meals occurring through the entire dinner is imposed by the alternate ordering of neighbour's dinners.

Now we turn to the case of five decent philosophers. One more or less quickly realizes that no event sequence and no step squence of meals would property describe their dinner: Each such sequence guaranteeing alternation among neighbours, inevitably causes unmotivated, additional order among far apart philosophers! So the quest is:

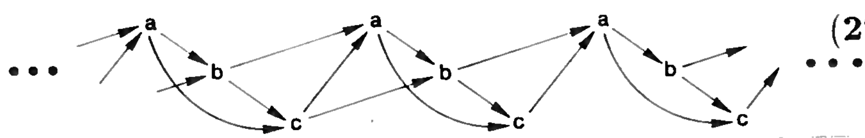
How to represent the dinner of five decent philosophers?

We tackle this problem by a fresh look at the representation of decent philosophers' dinner, taking now into account the forks, too. As a graphical convention, let $x \longrightarrow y$ denote a meal of the philosopher X, followed by the joint fork being passed over to a neighbour philosopher Y of X, followed by a meal of Y. This leads to a new representation of (2) as



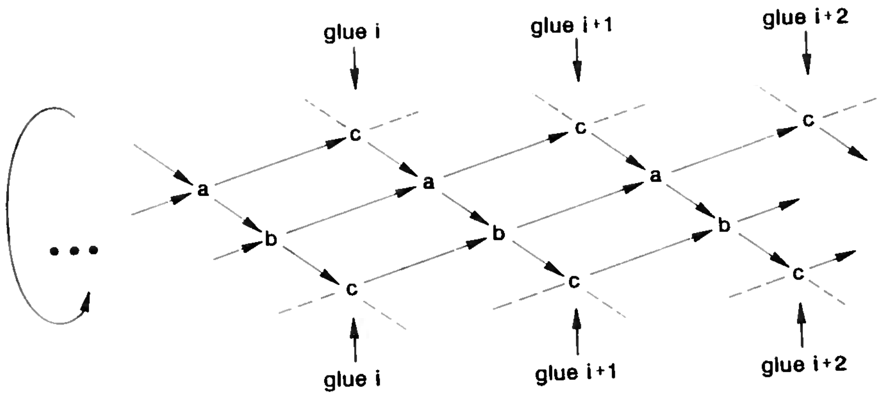
(1′)

because the right as well as the left neighbour coincide in the two philosphers system. Consequently, for (2) we obtain (2′):



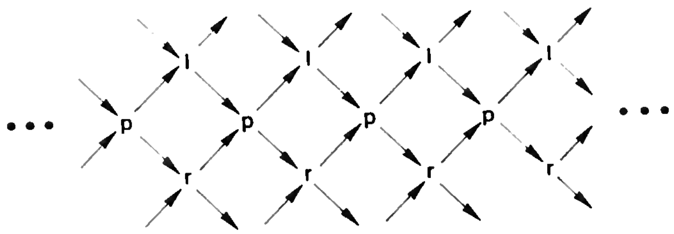
(2′)

A three dimensional representation of (2′) shows its highly symmetrical structure: In the following figure (2″), unordered "c" are assumed to be identical, i.e. (2″) may be conceived an *inscribed cylinder* by glueing the unordered "c"–occurrences and also the straight and corresponding dotted arrows, as indicated.



(2″)

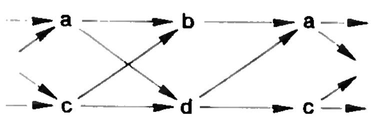
(2″) points at a general representation scheme of the decent philosophers' dinner: The meals of each philosopher P are represented by "p" occurrences along a virtual horizontal line, accompanied by $P's$ left an right neighbours meals "ℓ" and "r", according to the following scheme (5):



(5)

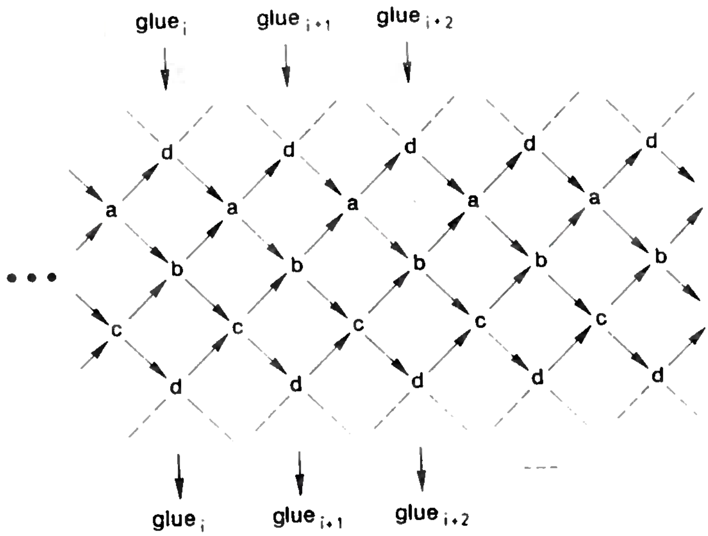
As an exercise we might ask for representations (3′) and (3″) corresponding to (3), just as (2′) and (2″) correspond to (2). (3″) in fact can be gained from (2″) by a different glueing policy: Glue the upper c in the "glueing cut" i with the lower c in cut $i+1$. (This becomes even more obvious by a totally symmetrical representation as in (5)).

Turning to four philosophers, a revised representation of (4) is



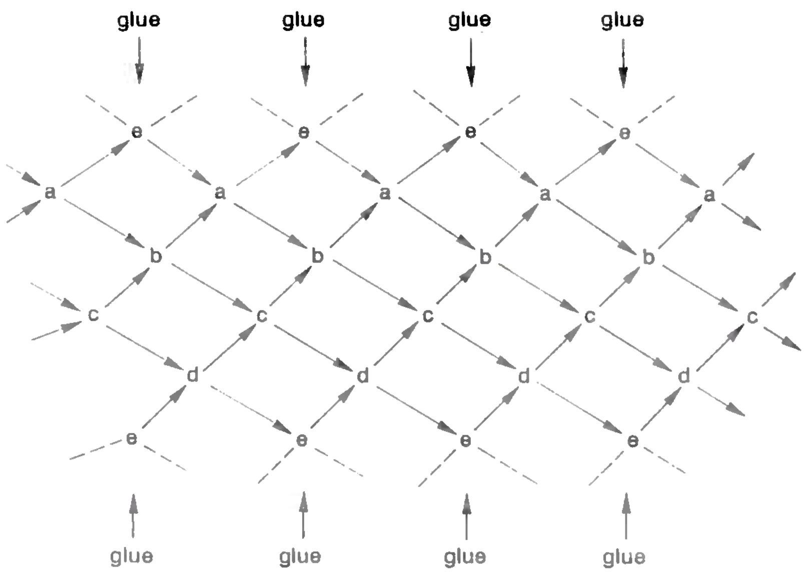
(4′)

which according to the scheme (5) has the following three-dimensional symmetrical representation (4″):



$(4'')$

Based on the above considerations, particularly on (5), we have now the means to represent a dinner of five decent philosophers A, B, C, D, E:

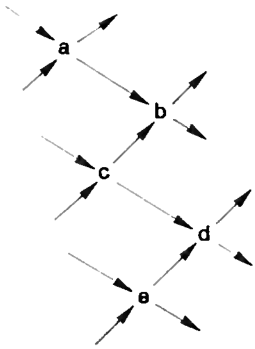


(6)

What is now the essential structural difference between (4″) and (6)? Why has (4″) got a representation as a step sequence, (4), whereas (6) has not? Before discussing this topic we have to drop a word on what the figures (1′) ... (6) in this chapter formally denote. In fact they represent *labelled, partially ordered sets.* Their elements are visible only indirectly by their labels, the partial order is represented as usual in Hasse diagrams, with $x \longrightarrow y$ denoting an element labelled x, to be a direct predecessor of an element labelled y.

Now we observe that the partial order of (4″) (which was identical to that of (4′), has a *transitive complement*: This relation, representing the involved "unorder" (or "concurrency"), decomposes into two–elementary equivalence classes with elements labelled either "a" and "c" or "b" and "d", respectively.

(6) on the contrary has sub–structures such as shown in (7)



(7)

where "e" is unordered with "a" and "a" is unordered with "d", but "e" ordered with "d".

It generally holds that a dinner of decent philosophers can be represented as an event– or step–sequence only in case the concurrency involved (i.e. the complement of the meals' order) happens to be transitive.

What about the general case?

The consequences of the above considerations are far reaching: Operational semantics of non–sequential systems in general cannot be formulated in the framework of transition systems!

In order to overcome this problem in our concrete example, Jaco might avoid the dinner to have five participants. But any bigger number of participants will cause the same problem: (7) is a sub–structure of any dinner with five or more decent philosophers. So, one should instead base operational semantics of concurrent system onto the some what more general ground of partially ordered sets.

Some more or less interesting questions remain: Is there always a unique dinner for any number n of philosophers? Certainly not for $n = 3$, as we have seen above. But intuitively this case appears an exception, caused by the need for "too tight synchronization". How to formulate this formally? How to prove uniqeness of all dinners for $n \neq 3$?

More generally: What denotational semantics corresponds to partially ordered labelled sets? What should it abstract form? What reflexive domains are adequate? How get compositionality?

Jaco will surely find an adequate solution during another 25 years at CWI!