Qualitative λ-models as Type Assignment Systems

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Dedicated to J. W. De Bakker in honour of his 25 years of work in Semantics

1. Introduction

Qualitative Domains were introduced by Girard [Gir, 1986] [Gir, 1988], as an alternate means to Scott Domains for providing a semantics for λ-calculus. The main difference between these two notions of domain lies in the way the function space is constructed. The qualitative function space is, in fact, made up only of stable functions, which are a proper subset of the set of all continuous functions. Moreover it is partially ordered with respect to a relation which induces a finer notion of approximation than the one induced by the pointwise ordering. The notion of stable function was first introduced by Berry [Ber, 1978], in the context of sequential functions.

We have studied an alternate description of qualitative λ-models, which allows for a natural definition of a formal system for reasoning about the interpretations of λ-terms. More precisely, we associate, to every qualitative λ-model D, a type assignment system for λ-terms T(D). The set of types of T(D) is isomorphic to the set of atoms of D, and moreover given a λ-term M it is possible to derive a type σ for M if and only if the element of D corresponding to σ belongs to the interpretation of M. A similar connection between Scott's D_∞ -λ-models and a suitable class of type assignment systems for λ-calculus was studied in [Bar, 1983], [Cop, 1984], [Hon, 1984].

In this paper we describe as a type assignment system the standard qualitative λ-model. This particular example can be easily generalized to arbitrary qualitative λ-models.

Describing a qualitative λ-model as a type assignment system gives us the possibility of using standard techniques for studying the fine structure of the model itself. In the particular case studied in this paper, for instance, a normalization property of the type derivation system immediately implies an Approximation Theorem (i.e., the interpretation of a term is the union of the interpretations of its syntactical approximants). Moreover the syntax of the type assignment system that we discuss in this paper provides a deep insight into the structure of qualitative domains, and in our opinion it illuminates the connection between qualitative domains and the coherent semantics for linear logic.

In Section 2 of this paper we define a type assignment system, and we show that it induces a λ-model S , where the interpretation of a term is the set of types derivable for it. In Section 3 we prove an Approximation Theorem for S , using a normalization property of the type assignment system. Finally, in Section 4 we define the standard qualitative λ-model D , with an inverse limit construction, and show that S and D are isomorphic.

Throughout the paper we assume the reader familiar with the basic notions and notations of λ-calculus as given in [Bar, 1984]. The definitions of qualitative domains and stable functions are recalled in Section 4.

2. The construction of the model S.

Now we introduce a formal system for assigning types to λ-terms and we show that it induces a λ-model.

Let $V=\{\phi_i \mid i\in\omega\}$ be an infinite set of variables. Starting from V, we define two languages, L and L'. Terms of the language L, ranged over by α, are defined as follows:

$$\alpha ::= \phi_1 \mid \phi_2 \mid \ldots \mid [\alpha_1,\ldots,\alpha_n]\rightarrow\alpha \mid [\,]\rightarrow\alpha \qquad (n\geq 1).$$

Terms of L', ranged over by ρ, are defined as follows:

$$\rho := \alpha \mid [\alpha_1,\ldots,\alpha_n] \mid [\,] \qquad (n\geq 1).$$

The intended meaning of $[\alpha_1,\ldots,\alpha_n]$ is the set whose elements are $\alpha_1,\ldots,\alpha_n$, and the intended meaning of $[\,]$ is the empty set. Accordingly, in the sequel, we will take terms of L and L' equivalent up to set-theoretic equality (i.e., reshuffling and repetitions of α_i's in subterms of the form $[\alpha_1,\ldots,\alpha_n]$). Set-theoretic equality is denoted with $=$.

In order to define the set of types, we introduce five predicates whose denotational meaning will be made precise in section 4. The five predicates are:

var $\subset$ L
type $\subset$ L
seq $\subset$ L'
comp $\subset$ L' $\times$ L'
nonc $\subset$ L' $\times$ L'.

These predicates are mutually defined by the rules given in the following definition.

Definition 1.

1) $\dfrac{\textbf{var}\ \phi}{\textbf{type}\ \phi}$ 2) $\dfrac{\textbf{type}\ \alpha \quad \textbf{seq}\ \rho}{\textbf{type}\ \rho\rightarrow\alpha}$ 3) $\dfrac{\textbf{var}\ \phi}{\textbf{comp}\ \phi\ ,\ \rho}$

4) $_{n\geq 0}\ \dfrac{(\textbf{comp}\ \alpha_i,\alpha_j)_{1\leq i,j\leq n}}{\textbf{seq}\ [\alpha_1,\ldots,\alpha_n]}$ 5) $_{n,m\geq 0}\ \dfrac{\textbf{seq}\ [\alpha_1,\ldots,\alpha_n,\alpha'_1,\ldots,\alpha'_m]}{\textbf{comp}\ [\alpha_1,\ldots,\alpha_n],[\alpha'_1,\ldots,\alpha'_m]}$

6) $\dfrac{}{\textbf{comp}\ \rho,\rho}$ 7) $\dfrac{\textbf{comp}\ \rho,\rho'}{\textbf{comp}\ \rho',\rho}$ 8) $\dfrac{\textbf{nonc}\ \rho,\rho'}{\textbf{nonc}\ \rho',\rho}$

9) $\dfrac{\textbf{comp}\ \rho,\rho' \quad \rho\neq\rho'}{\textbf{nonc}\ \rho\rightarrow\alpha\ ,\ \rho'\rightarrow\alpha}$ 10) $\dfrac{\textbf{comp}\ \rho,\rho' \quad \textbf{nonc}\ \alpha,\alpha'}{\textbf{nonc}\ \rho\rightarrow\alpha,\rho'\rightarrow\alpha'}$ 11) $\dfrac{\textbf{comp}\ \rho,\rho' \quad \textbf{comp}\ \alpha,\alpha' \quad \alpha\neq\alpha'}{\textbf{comp}\ \rho\rightarrow\alpha\ ,\ \rho'\rightarrow\alpha'}$

12) $\dfrac{\textbf{nonc}\ \rho,\rho'}{\textbf{comp}\ \rho\rightarrow\alpha\ ,\ \rho'\rightarrow\alpha}$.

Two remarks, which will be made precise in Section 4, are in order: Rule 9 is distintive of qualitative domains, while Rule 4 is distinctive of binary qualitative domains.

If **type** α holds, then we will say that α is a type. The set of types will be ranged over by σ,τ.

If **seq** ρ holds, then we will say that ρ is a sequence. The set of sequences will be ranged over by γ,δ.

We are now ready to introduce the formal system for assigning types to λ-terms.

Definition 2. i) A **basis** B is a set of pairs $x{:}\sigma$, where σ is a type. dom(B) will denote the

set $\{x \mid x:\sigma \in B\}$ and $B_{|E}$ will denote the restriction of B to the set of term variables E.
ii) Types are assigned to terms according to the following three rule schemes, where $B \vdash M:\sigma$ denotes that M has type σ under the assumptions recorded in B:

$$(var)\ \frac{}{\{x:\sigma\} \vdash x:\sigma}$$

$$(\rightarrow E)_{n\geq 0}\ \frac{B \vdash M:[\sigma_1,\ldots,\sigma_n]\rightarrow\sigma \qquad (B_i \vdash N:\sigma_i)_{1\leq i\leq n}}{B \cup (\cup_{1\leq i\leq n} B_i) \vdash MN:\sigma}$$

$$(\rightarrow I)_{n\geq 0}\ \frac{B \cup \{x:\sigma_1,\ldots,x:\sigma_n\} \vdash M:\sigma \qquad \mathbf{seq}\ [\sigma_1,\ldots,\sigma_n] \qquad x \notin dom(B)}{B \vdash \lambda x.M: [\sigma_1,\ldots,\sigma_n]\rightarrow\sigma.}$$

Notice the multiplicative behaviour (in Girard's terminology) of the basis in the rule $(\rightarrow E)$.

Definition 3. Two basis B and B' are **coherent** iff $\{x:\sigma, x:\sigma'\} \subset B \cup B' \Rightarrow$ **comp** σ,σ'.

Some structural properties of this type assignment system are given in Lemma 4.

Lemma 4.i) Let $B \vdash M:\sigma$.Then $\{x:\tau_1,\ldots,x:\tau_m\} \subset B$ implies that there are at least m occurrences of x in M.
ii) Let $B \vdash \lambda x.M:\sigma$. Then $\sigma = \rho\rightarrow\tau$ for some ρ such that **seq** ρ.
iii) Let $B \vdash M:\sigma$, $B' \vdash M:\tau$ and let B and B' be coherent; then **comp** σ,τ.
iv) Let $B \vdash M:\sigma$, $B' \vdash M:\tau$, let B and B' be coherent and let $\sigma=\tau$; then $B=B'$.

Proof.i) and ii) are immediate.
iii) and iv) will be proved simultaneously by induction on M.
$M \equiv x$. Obvious.
$M \equiv PQ$. $B \vdash M:\sigma$ and $B' \vdash M:\tau$ implies that there are two derivations of the following shape:

$$(\rightarrow E)\ \frac{B_0 \vdash P:[\sigma_1,\ldots,\sigma_n]\rightarrow\sigma \quad (B_i \vdash Q:\sigma_i)_{1\leq i\leq n}}{B = B_0 \cup (\cup_{1\leq i\leq n} B_i) \vdash PQ:\sigma} \qquad (\rightarrow E)\ \frac{B'_0 \vdash P:[\tau_1,\ldots,\tau_m]\rightarrow\tau \quad (B'_i \vdash Q:\tau_i)_{1\leq i\leq m}}{B' = B'_0 \cup (\cup_{1\leq i\leq m} B'_i) \vdash PQ:\tau}$$

Since two subsets of two coherent basis are in turn coherent , by induction **comp** $[\sigma_1,\ldots,\sigma_n]\rightarrow\sigma$, $[\tau_1,\ldots,\tau_m]\rightarrow\tau$ and **comp** $(\sigma_i,\tau_j)_{1\leq i\leq n, 1\leq j\leq m}$, which implies **comp** σ,τ. Moreover, if $\sigma=\tau$ then $[\sigma_1,\ldots,\sigma_n] = [\tau_1,\ldots,\tau_n]$ and iv) follows directly by induction.
$M \equiv \lambda x.P$. If $B \vdash M:\sigma$ and $B' \vdash M:\tau$ then, by ii) we must have $\sigma = [\sigma_1,\ldots,\sigma_n]\rightarrow\sigma'$ and $\tau = [\tau_1,\ldots,\tau_m]\rightarrow\tau'$, with $m,n\geq 0$. This implies that there are two derivations of the following

shape:

$$(\to I)\ \frac{B\cup\{x:\sigma_1,\dots,x:\sigma_n\}\vdash P:\sigma' \quad \mathbf{seq}\ [\sigma_1,\dots,\sigma_n] \quad x\notin dom(B)}{B\vdash \lambda x.P: [\sigma_1,\dots,\sigma_n]\to\sigma'}$$

and

$$(\to I)\ \frac{B'\cup\{x:\tau_1,\dots,x:\tau_m\}\vdash P:\tau' \quad \mathbf{seq}\ [\tau_1,\dots,\tau_m] \quad x\notin dom(B')}{B'\vdash \lambda x.P: [\tau_1,\dots,\tau_m]\to\tau'.}$$

If **comp** $[\sigma_1,\dots,\sigma_n]$, $[\tau_1,\dots,\tau_m]$, then by induction **comp** σ',τ'. If $\sigma'\neq\tau'$ then, by definition of **comp**, this implies **comp** σ,τ, else, if $\sigma'=\tau'$,by induction on iv), $[\sigma_1,\dots,\sigma_n]=[\tau_1,\dots,\tau_m]$, and so $\sigma=\tau$. If **nonc** $[\sigma_1,\dots,\sigma_n]$, $[\tau_1,\dots,\tau_m]$, iii) follows by definition of **comp**. iv) is immediate by induction.

□

Now we are able to define the model $\mathcal{S}$.

Definition 5. $\mathcal{S} \equiv (S, \circ, [\![\ ]\!])$, where:
$S \equiv <\{A\subset L \mid \forall\alpha_1,\alpha_2 \in A.\ \mathbf{comp}\ \alpha_1,\alpha_2\}, \subset>$.
$s_1 \circ s_2 = \{\alpha \mid [\alpha_1,\dots,\alpha_n]\to \alpha \in s_1,\ \{\alpha_1,\dots,\alpha_n\} \subset s_2\}$ $\quad (s_1,s_2 \subset S)$.
Given an environment ξ:Var→S, where Var is the set of term variables, ξ induces a basis $B_\xi = \{x:\alpha \mid \alpha\in \xi(x)\}$; then $[\![M]\!]_\xi = \{\sigma \mid \exists B.\ B\vdash M:\sigma \text{ and } B\subset B_\xi\}$.

Theorem 6. $\mathcal{S}$ is a λ-model.
Proof. First of all, it is necessary to prove that $\forall M\in\Lambda.\ \forall\xi.[\![M]\!]_\xi \in S$. This is an immediate consequence of Lemma 4.iii).
Then $\mathcal{S}$ can be proved to be a λ-model by showing that it satisfies the six conditions defining a λ-model given by Hindley and Longo in [Hin, 1980], i.e.:
1) $[\![x]\!]_\xi = \xi(x)$;
2) $[\![MN]\!]_\xi = [\![M]\!]_\xi \circ [\![N]\!]_\xi$;
3) $[\![\lambda x.M]\!]_\xi \circ s = [\![M]\!]_{\xi[s/x]}$;
4) $(\forall x\in FV(M).\ [\![x]\!]_\xi = [\![x]\!]_{\xi'}) \Rightarrow [\![M]\!]_\xi = [\![M]\!]_{\xi'}$;
5) $[\![\lambda x.M]\!]_\xi = [\![\lambda x.M[y/x]]\!]_\xi$, if $y \notin FV(M)$;
6) $(\forall s\in S.\ [\![M]\!]_{\xi[s/x]} = [\![N]\!]_{\xi[s/x]}) \Rightarrow [\![\lambda x.M]\!]_\xi = [\![\lambda x.N]\!]_\xi$.
This will be proved by induction on terms.
1) $[\![x]\!]_\xi = \{\sigma \mid \exists B.\ B\vdash x:\sigma \text{ and } B\subset B_\xi\} = \{\sigma \mid \sigma\in\xi(x)\}$;
2) $[\![MN]\!]_\xi = \{\sigma \mid \exists B.\ B\vdash MN:\sigma \text{ and } B\subset B_\xi\} =$
$\{\sigma \mid \exists B,B_1,\dots,B_n\subset B_\xi.\ B\vdash M:[\sigma_1,\dots,\sigma_n]\to\sigma \text{ and } (B_i\vdash N:\sigma_i)_{i\le n}\}$ = (by induction)
$\{\sigma \mid [\sigma_1,\dots,\sigma_n]\to\sigma\in [\![M]\!]_\xi \text{ and } (\sigma_i\in [\![N]\!]_\xi)_{i\le n}\} = [\![M]\!]_\xi \circ [\![N]\!]_\xi$ (by def. of ∘);
3) $[\![\lambda x.M]\!]_\xi \circ s = \{\sigma \mid \exists B.\ B\subset B_\xi \text{ and } B\vdash \lambda x.M: [\sigma_1,\dots,\sigma_n]\to\sigma \text{ and } (\sigma_i\in s)_{i\le n}\} =$
$\{\sigma \mid \exists B\subset B_\xi\ B\cup\{x:\sigma_1,\dots,x:\sigma_n\}\vdash M:\sigma \text{ and } (\sigma_i\in s)_{i\le n}\} =$
$\{\sigma \mid \exists B'\subset B_{\xi[s/x]}.B'\vdash M:\sigma\}$;
4) $(\forall x\in FV(M).\ [\![x]\!]_\xi = [\![x]\!]_{\xi'}) \Rightarrow (\forall x\in FV(M).\{\sigma \mid \sigma\in\xi(x)\} = \{\sigma \mid \sigma\in\xi'(x)\}) \Rightarrow$

$(B_{\xi|FV(M)} = B_{\xi'|FV(M)} \Rightarrow$ (by Lemma 4.i) $\{\sigma \mid \exists B.\ B \vdash M:\sigma$ and $B \subset B_\xi\} =$
$\{\sigma \mid \exists B.\ B \vdash M:\sigma$ and $B \subset B_{\xi'}\} \Rightarrow [\![M]\!]_\xi = [\![M]\!]_{\xi'}$;

5) immediate from the definition of $[\![\ ]\!]$;

6) $[\![\lambda x.M]\!]_\xi =$ (by Lemma 4.ii)$\{[\sigma_1,\dots,\sigma_n]\to\sigma \mid \exists B \subset B_\xi.\ B \vdash \lambda x.M: [\sigma_1,\dots,\sigma_n]\to\sigma\} =$
$\{[\sigma_1,\dots,\sigma_n]\to\sigma \mid \exists B \subset B_\xi.\ B \cup \{x:\sigma_1,\dots,x:\sigma_n\} \vdash M:\sigma$ and **seq** $[\sigma_1,\dots,\sigma_n]$ and $x \notin dom(B)\} =$
$\{[\sigma_1,\dots,\sigma_n]\to\sigma \mid \exists B' \subset B_{\xi[[\sigma_1,\dots,\sigma_n]/x]}.\ B' \vdash M:\sigma\} =$ (since $\forall s \in S.\ [\![M]\!]_{\xi[s/x]} = [\![N]\!]_{\xi[s/x]}$) $\{[\sigma_1,\dots,\sigma_n]\to\sigma \mid \exists B \subset B_{\xi[[\sigma_1,\dots,\sigma_n]/x]}.\ B' \vdash N:\sigma\} = [\![\lambda x.N]\!]_\xi$. □

3. The Approximation Theorem.

Every derivation D of $B \vdash M:\sigma$ in the above type assignment system is normalizable. Here normalizable means that D can be transformed into a derivation D' of $B \vdash M':\sigma$ where no application of the rule $(\to I)$ in D' is immediately followed by an application of the rule $(\to E)$ and M' is a β-reduct of M. Using this fact we will show that the interpretation of a term in ***S*** is the collection of the interpretations of its syntactical approximants. This will be called the Approximation Theorem for the model ***S***.

Definition 7. i) Let D be the deduction: $B \vdash M:\sigma$. A **cut** in D is an application of the rule $(\to I)$ immediately followed by an application of the rule $(\to E)$;
ii) The **degree** of a cut is the number of type symbols occurring in the premises of the application of the rule $(\to E)$ determining the cut.
iii) The **degree** of a deduction D, G(D), is the pair $<d,n>$ where n is the number of cuts in D and d is the maximum degree of all cuts in D.
iv) A deduction D is **normal** iff $G(D) = <0,0>$.

We consider the pairs ordered in lexicographic order (i.e., $<d,n> \sqsubseteq <d',n'>$ iff $(d \sqsubseteq d')$ or ($d=d'$ and $n \sqsubseteq n'$)).

Lemma 8. $D: B \vdash M:\sigma$ and $G(D) > 0$ implies that there exists D' such that $D' \vdash M':\sigma$, where M β-reduces to M' and $G(D') < G(D)$.
Proof. We have to distinguish two cases, according to the number of premises of the $(\to E)$ rule which determines the cut.
1) At least one of the cuts with the maximum degree in D is of the following shape:

$$\dfrac{\dfrac{B \vdash P:\tau \quad x \notin dom(B)}{B \vdash \lambda x.P:[\ ]\to\tau}(\to I)}{B \vdash (\lambda x.P)Q:\tau.}(\to E)$$

This implies that D: $B \cup B' \vdash C[(\lambda x.P)Q]:\sigma$, for a suitable context C[ ].
Then, if x occurs in P at all, x occurs in subterms of P $S_i[x]$ $(i \geq 0)$, which occur in subderivations of D of the shape:

$$D_i: \quad \dfrac{B_i' \vdash R:[\ ]\to\sigma_i}{B_i' \vdash RS_i[x]:\sigma_i.}(\to E)$$

Then D' is obtained from D by performing the following three operations:

i) replacing every D_i with:

$$\frac{D_i' \qquad B_i'\vdash R:[\,]\to\sigma_i}{B_i'\vdash RS_i\,[Q\backslash x]:\sigma_i}\,(\to E)$$

ii) replacing $(\lambda x.P)Q$ and every descendent of it with $P[Q\backslash x]$
iii) deleting the cut.
Thus we have D': $B\cup B'\vdash C[P[Q\backslash x]]:\sigma$ and $G(D') < G(D)$.
2) All the cuts with the maximum degree in D are of the following shape:

$$\frac{\dfrac{B\cup\{x:\sigma_1,\ldots,x:\sigma_n\}\vdash P:\sigma \quad \mathbf{seq}\ [\sigma_1,\ldots,\sigma_n] \quad x\notin dom(B)}{B\vdash\lambda x.P:[\sigma_1,\ldots,\sigma_n]\to\sigma}(\to I) \qquad (B_i\vdash Q:\sigma_i)_{i\le n}}{B\cup(\cup_{i\le n}B_i)\vdash(\lambda x.P)Q:\sigma.}(\to E)$$

Pick one of these. This implies that D: $B\cup B'\vdash C[(\lambda x.P)Q]:\sigma$, for a suitable context C[ ]. Now, there are $m\ge n$ occurrences of x in P, by Lemma 4.i). Exactly n of these occurrences occur in subderivations of D_i $(1\le i\le n)$, consisting of an application of the (var) rule:

$$D_i: \quad \frac{}{x:\sigma_i\vdash x:\sigma_i}\,(var).$$

The remaining m-n occurrences of x are in subterms of P for which no type has been derived in D. Then D' is obtained from D by performing the following four operations:
i) replacing every D_i with D_i' $:B_i\vdash Q:\sigma_i$
ii) handling the remaining m-n occurrences of x for which no type has been derived in D as in 1)
iii) replacing $(\lambda x.P)Q$ and every descendent of it with $P[Q\backslash x]$
iv) deleting the cut.
Thus we have D': $B\cup B'\vdash C[P[Q\backslash x]]:\sigma$ and $G(D') < G(D)$.

□

The following theorem is an easy consequence of the lemma we have just proved.

Theorem 9. If $D:B\vdash M:\sigma$ then there exists a normal derivation D' and a term M' such that M β-reduces to M'and $D':B\vdash M':\sigma$.

We will now recall the notion of approximate normal form first introduced in [Wad,1978] in order to discuss the interpretation of non-terminating λ-terms.

Definition 10. i) The set $\mathcal{A}$ of the **approximate normal forms** is defined inductivily as:
- a term variable belongs to $\mathcal{A}$;
- the constant Ω belongs to $\mathcal{A}$;
- if $A_1,\ldots,A_n$ belong to $\mathcal{A}$, then $\lambda x_1,\ldots,x_m.zA_1\ldots A_n$ belongs to $\mathcal{A}$, for any term variables $x_1,\ldots,x_m,z$.

ii) If $M\in\Lambda$, the set of the **approximants** of M is:
$\mathcal{A}(M)=\{A\in\mathcal{A}\mid\exists M'.$ M β-reduces to M' and A and M' match up to subterms of M' corresponding to occurrences of Ω in A$\}$.

Approximation Theorem. $B\vdash M:\sigma$ iff $\exists A\in\mathcal{A}(M)$. $B\vdash A:\sigma$.
(i.e., $[\![M]\!]_\xi=\{[\![A]\!]_\xi\mid A\in\mathcal{A}(M)\}$).

Proof. ($\Rightarrow$) $B\vdash M:\sigma$ implies (by Lemma 8) $\exists D:B\vdash M':\sigma$ and M β-reduces to M' and D is normal. Let A be the approximant of M obtained from M'by replacing with Ω every subterm of M' to which no type has been assigned by D. Clearly $B\vdash A:\sigma$. If D has assigned a type to every

subterm of M', then M' is in normal form and A=M'.
($\Leftarrow$) Let A be an approximant of M such that $\exists D:B\vdash A:\sigma$. Then M reduces to M', where M' is obtained from A by replacing the occurrences of Ω with suitable subterms, say $N_1,\dots,N_p$. Every occurrence of Ω in A must occur in a subderivation of the shape:

$$(\to E)\frac{B'\vdash A':[\,]\to\tau}{B'\vdash A'\Omega:\tau.}$$

So a derivation $D':B\vdash M':\sigma$ can be obtained from D simply by replacing in D the occurrences of Ω with $N_1,\dots,N_p$ respectively. Since $[\![\ ]\!]$ satisfies β-equality, by Theorem 6, we have that $B\vdash M':\sigma$ implies $B\vdash M:\sigma$.

□

The Approximation Theorem is a powerful tool for investigating the theory induced by a model. In this case it implies immediately, for instance, that the theory of the model $\mathcal{S}$ is sensible, and that $[\![Y]\!]$ is Tarski's least fixed point operator.
Moreover, using the Approximation Theorem and following the argument in [Ron, 1982], one can characterize the theory of $\mathcal{S}$ as: $\forall\xi.\ [\![M]\!]_\xi \sqsubseteq [\![N]\!]_\xi$ if and only if
$\forall C[\,]$.(if C[M] reduces to a head-normal-form with no initial abstractions then the same holds for C[N]).
This is the same theory as the one induced by the filter model [Bar,1983]. Equationally it is the same as the theory of Scott's P_ω.
We will end this section pointing out the following interesting fact:

Multiplicity Theorem. Let $\lambda x.A$ be an approximate normal form. There are exactly n occurrences of the variable x in A if and only if n is the maximum integer such that there is a derivation $D:B\vdash\lambda x.A:[\sigma_1,\dots,\sigma_n]\to\sigma$.

Proof. The theorem follows immediately noticing that, given an approximate normal form A it is always possible to build a derivation D where every subterm of A has a type, but Ω's .

□

4. Isomorphism Theorem.

In this section we give a semantics to the type assignment system that we introduced in Section 2. More precisely we will show that the model $\mathcal{S}$ is isomorphic to a particular binary qualitative domain $\mathcal{D}$ which is a λ-model.
First let us recall some definitions about qualitative λ- models [Gir, 1986].

Definition 11. i) A **qualitative domain** D is a set of sets such that:
- $\emptyset\in D$
- D is closed under directed unions
- if $a\in D$ and $b\subset a$, then $b\in D$;

ii) The union of the set of the atomic elements of D i.e. $\{ z \mid \{z\}\in D \}$, is denoted with $|D|$;
iii) Let D and D' be two qualitative domains. A function $F: D\to D'$ is **stable** iff:
- $a\subset b\in D \Rightarrow F(a)\subset F(b)$
- $F(\cup_{i>0} a_i)=\cup_{i>0} F(a_i)$, provided $a_i\subset a_j$ for $i\leq j$
- $a\cup b \in D \Rightarrow F(a\cap b) = F(a)\cap F(b)$;

iv) Let D and D' be two qualitative domain, and let $F: D\to D'$ be a stable function. The **trace** of F is:
$Tr(F) = \{(a,z) \mid a \text{ is a finite element of } D,\ z\in|D'|,\ z\in F(a) \text{ and } z\notin F(a') \text{ for all } a'\subset a\}$;
v) A qualitative domain D is a λ-model iff there are two stable functions H and K such that:

$H: D \to [D \to_s D]$, $K: [D \to_s D] \to D$ and $H \circ K = Id_{[D \to sD]}$
where $[D \to_s D]$ denotes the qualitative domain of the traces of the stable functions from D to D, partially ordered by inclusion.

Definition 12. Let D be the qualitative domain defined as the standard inverse limit solution of the following equation:

$$D = P(V) \times [\, D \to_s D\,]$$

where P(V) denotes the power set of the set V of variables, ordered by inclusion.
Up to isomorphisms, D is defined as:

$D = \lim_{n \geq 0} D_n$, where $D_0 = \{\emptyset\}$, $D_{n+1} = P(V) \times [D_n \to_s D_n]$.

Isomorphism Theorem. S and D are isomorphic.
Proof (sketch). One can easily verify that S is a qualitative domain; in particular $|S| = \{\alpha \mid$ **type** $\alpha\}$. An isomorphism I between the supports of S and D can be defined in the following way:
$I(\phi) = (\{\phi\}, \emptyset)$
$I([\sigma_1, \ldots, \sigma_n] \to \sigma) = (\emptyset, F)$, where $Tr(F) = \{(\{I(\sigma_1), \ldots, I(\sigma_n)\}, I(\sigma))\}$.
Then I can easily be extended to all points of S and D, since the definition of **comp** is such that
$\{\alpha\} \cup \{\alpha'\} \in S \iff$ **comp** $\alpha, \alpha' \iff \exists d \in D\ (\forall \phi \in \{\alpha\} \cup \{\alpha'\}. \{\phi\} \in \pi_1(\text{unfold}(d))$ and $\forall [\sigma_1, \ldots, \sigma_n] \to \sigma \in \{\alpha\} \cup \{\alpha'\}.\ (\{I(\sigma_1), \ldots, I(\sigma_n)\}, I(\sigma)) \in \pi_2(\text{unfold}(d)))$.

□

We can now show that the five predicates introduced in Section 2 are abstract syntactic counterparts of qualitative-domain-theoretic concepts. Let μ and μ' be elements of S and let I be the isomorphism between S and D. The following relations hold:

var (μ)	$\iff$	$I(\mu)$ is a non-functional atom of D
type (μ)	$\iff$	$I(\mu)$ is an atom of D
seq (μ)	$\iff$	$I(\mu)$ is an element of D
comp (μ, μ') and **type** (μ) and **type** (μ')	$\iff$	$\{I(\mu), I(\mu')\}$ is an element of D
comp (μ, μ') and **seq** (μ) and **seq** (μ')	$\iff$	$I(\mu) \cup I(\mu')$ is an element of D
nonc (μ, μ') and **type** (μ) and **type** (μ')	$\iff$	$\{I(\mu), I(\mu')\}$ is not an element of D
nonc (μ, μ') and **seq** (μ) and **seq** (μ')	$\iff$	$I(\mu) \cup I(\mu')\}$ is not an element of D.

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