A CHARACTERIZATION OF THE STATE SPACES OF ELEMENTARY NET SYSTEMS

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ABSTRACT

The state space of an elementary net system can be represented as an edge-labeled directed graph. Here we give an exposition of a characterization (from [ER4]) of the class of edge-labeled directed graphs which correspond to the state space representations of elementary net systems.

INTRODUCTION

The theory of *Petri nets* has originated in the famous paper by C.A. Petri ([P]) and since then this theory has provided a spectrum of concepts and tools facilitating the description and the analysis of concurrent systems (see, e.g., [BRR1], [BRR2], and [Re]). At present the theory of Petri nets is a well-accepted theory of concurrency.

One of the important aspects of this theory is the way in which basic aspects of concurrent systems, such as concurrency, non-determinism and confusion, are identified both conceptually and mathematically. This is best seen in the fundamental system model of the theory called *elementary net systems* (see, e.g., [T] and [Ro]).

The notion of the state space of a concurrent system is one of the basic concepts of the theory of concurrent systems. Within the theory of elementary net systems it is formalized through the notion of the case graph. Understanding the notion of a case graph is one of the important aims of the theory of Petri nets. It increases our understanding of the behaviour of an elementary net system and it forms an important link to the theory of transition systems.

Recently a characterization of state spaces of elementary net systems was obtained in [ER3] and [ER4]. It is based on the theory of *partial 2-structures* which is related to the theory of *2-structures* (see, e.g., [ER1] and [ER2]).

The aim of this paper is to give an exposition of the characterization result directed to those interested in the theory of concurrent systems and in particular in the theory of Petri nets. In this exposition we will ignore many (interesting!) aspects of the theory of partial 2-structures; rather we will focus on these notions that lead to the characterization result. The paper is somewhat informal - it does not contain proofs which can be found in [ER3] and [ER4]. The reader who is intrigued by this paper is advised to study [ER3] and [ER4] to see how central problems concerning the theory of concurrent systems fit well into the framework of partial 2-structures.

PRELIMIARIES

We assume the reader to be familiar with basic notions of the theory of Petri nets, and in particular with the basic theory of elementary net systems (see, e.g., [Ro] and [T]). We assume that elementary net systems (EN systems) we consider satisfy the following conditions: the underlying net of an EN system is pure and simple, and for each event e of an EN system, ${}^\bullet e \neq \emptyset$ and $e^\bullet \neq \emptyset$.

We also assume that the reader is familiar with the rudiments of graph theory and in particular with the notion of an edge-labeled graph.

An *edge-labeled graph* g will be specified in the form $g = (V,E,\Delta)$, where V is the set of *nodes*, Δ is the alphabet of *labels*, and $E \subseteq V \times \Delta \times V$ is the set of *labeled edges*. An *initialized edge-labeled graph* is a system $h = (g,v_{in})$ where g is an edge-labeled graph and v_{in} is a distinguished node of g called the *initial node of* h.

We assume that all graphs we deal with are finite.

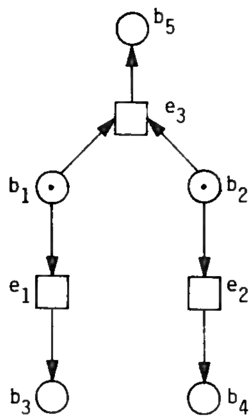
Finally, for an ordered pair of sets (A,B), the *ordered symmetric difference of* (A,B) is the pair (A-B, B-A); it is denoted by *osd*(A,B).

1. ABSTRACT STATE SPACES OF ELEMENTARY NET SYSTEMS

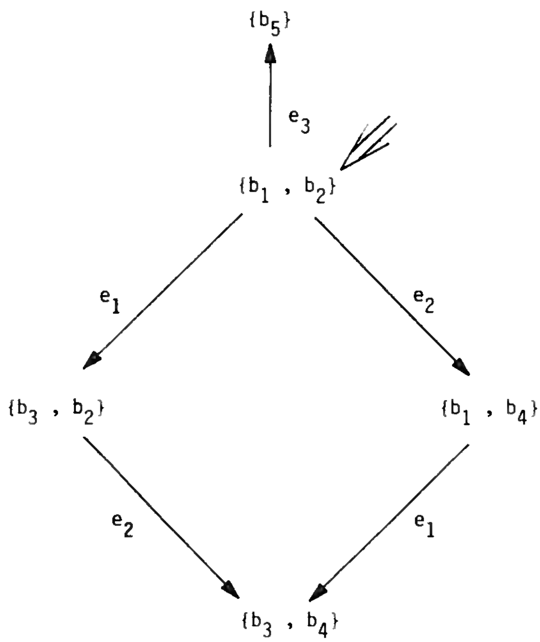
The notion of the *case graph* of an EN system plays an important role in the theory of EN systems - it formalizes the notion of the *state space* of an EN system. It is well known (see, e.g., [RT]) that rather than to consider the case graph of an EN system one can consider the *sequential case graph* of an EN system - the case graph can be uniquely recovered from it using the so called "diamond rule".

For an EN system N its sequential case graph will be denoted by SCG(N).

Example 1.1. Consider the following EN system N_0 :



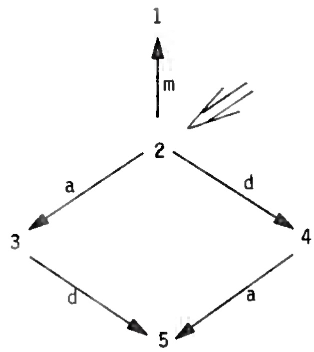
Then SCG(N_0) is as follows:



■

We note that, for an EN system N, SCG(N) is an initialized edge-labeled graph. If we now take an initialized edge-labeled graph h which is isomorphic with SCG(N), then h is an *abstract sequential case graph of N*. Note that nodes of h do not have to be sets and edge-labels may be arbitrary letters; in this way h is an abstract representation of SCG(N).

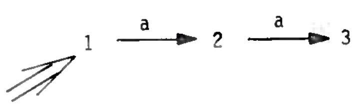
Example 1.2. The following initialized edge-labeled graph is an abstract sequential case graph of the EN system N_0 from Example 1.1:



■

It is easily seen that *not* each initialized edge-labeled graph is an abstract sequential case graph of an EN system.

Example 1.3. The following initialized edge-labeled graph is not an abstract sequential case graph of an EN system:



The reason is that otherwise there must exist an EN system N, an event e of N (an "interpretation" of the letter a), and a case C_2 of N (an "interpretation" of the node 2) such that $e^\bullet \subseteq C_2$ and $^\bullet e \cap C_2 = \emptyset$; a contradiction. ■

The problem discussed in this paper is the problem of characterizing those initialized edge-labeled graphs which are abstract sequential case graphs of EN systems. Once this is achieved we have a characterization of state spaces of EN systems.

2. INITIALIZED LABELED PARTIAL 2-STRUCTURES AND THEIR REGIONS

Initialized labeled partial 2-structures introduced in [ER3] provide a framework for the mathematical theory of state spaces of EN systems. They are defined as follows.

Definition 2.1. An *initialized labeled partial 2-structure*, $i\ell p2s$ for short, is an initialized edge-labeled graph $h = (g, v_{in})$ with $g = (V,E,\Delta)$ such that:

(i) if $(u,a,v) \in E$, then $u \neq v$, and

(ii) if $(u,a,v) \in E$ and $(u,b,v) \in E$, then $a = b$. ∎

Thus an $i\ell p2s$ is an initialized edge-labeled graph without loops and such that between any two nodes there is at most one (labeled) edge. We will use also the notation $h = (V,E,\Delta;v_{in})$ for an $i\ell p2s$ as above.

Clearly, for each EN system N, an abstract sequential case graph of N is an $i\ell p2s$. On the other hand, for each EN system N, SCG(N) is an $i\ell p2s$ of a special sort: (1) its nodes are sets, and (2) each edge is labeled by the ordered symmetric difference of sets its connect. The (2) above is based on an additional assumption that we will make in representing SCG(N): each event e of N is identified with its characteristic pair $({}^{\bullet}e, e^{\bullet})$. This is a natural assumption which we can safely make because we have assumed that we deal with simple sets only.

These special sorts of initialized labeled partial 2-structures will be called initialized labeled partial *set* 2-structures.

Definition 2.2. An *initialized labeled partial set 2-structure*, $i\ell ps2s$ for short, is an $i\ell p2s$ $h = (V,E,\Delta;v_{in})$ such that:

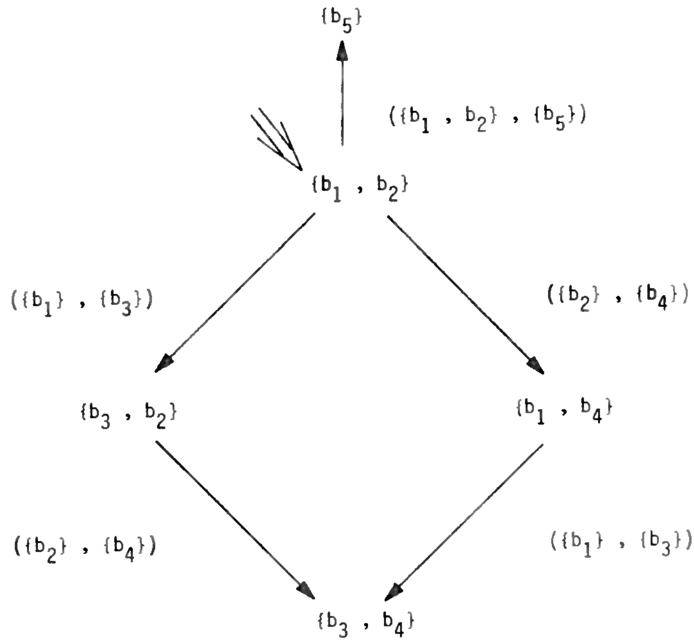
(1) each $x \in V$ is a finite set,

(2) all elements of Δ are of the form (A,B) where A,B are finite sets such $A \cap B = \emptyset$ and $A \cup B \neq \emptyset$, and

(3) for each $(X,A,Y) \in E$, $A = osd(X,Y)$. ∎

In order to avoid often repeating the long acronyms $i\ell p2s$ and $i\ell ps2s$, in the sequel of this paper we will use instead the acronyms p2s and ps2s, respectively.

Example 2.1. According to our convention of identifying an event with its characteristic pair, for the EN system N_0 from Example 1.1, SCG(N_0) is as follows:



Hence it is a ps2s. ■

The notion of an isomorphism between initialized labeled partial 2-structures is as follows.

Definition 2.3. Let $h_1 = (V_1 , E_1 , \Delta_1 ; v_{in}^1)$ and $h_2 = (V_2 , E_2 , \Delta_2 ; v_{in}^2)$ be initialized labeled partial 2-structures and let φ be a mapping from V_1 into V_2 .

(1) φ is a *morphism from* h_1 *into* h_2 , iff $\varphi(v_{in}^1) = v_{in}^2$, and for all (x,a,y), $(u,a,v) \in E_1$ such that $\varphi(x) \neq \varphi(y)$ and $\varphi(u) \neq \varphi(v)$, there exists $A \in \Delta_2$ such that $(\varphi(x), A, \varphi(y))$, $(\varphi(u), A, \varphi(v)) \in E_2$.

(2) φ is an *isomorphism from* h_1 *onto* h_2 , iff φ is a bijective morphism and φ^{-1} is a morphism. ■

We say that h_1 , h_2 are *isomorphic* iff there exists an isomorphism from h_1 onto h_2 .

Hence given an EN system *N*, an initialized edge-labeled graph h is an abstract sequential case graph of *N* iff h is isomorphic with SCG(*N*).

The basic technical notion of the theory of (initialized) labeled partial 2-structures is the notion of a region of a p2s. It is defined as follows.

Definition 2.4. Let $h = (V,E,\Delta;v_{in})$ be a p2s and let $Z \subseteq V$. Z is a *region of* g iff for all (x,a,y), $(u,b,v) \in E$, $a = b$ implies that:

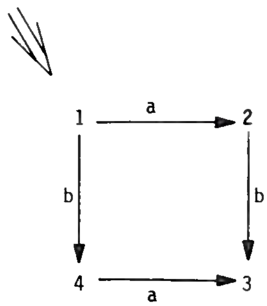
(1) if $x \in Z$ and $y \notin Z$, then $u \in Z$ and $v \notin Z$, and

(2) if $x \notin Z$ and $y \in Z$, then $u \notin Z$ and $v \in Z$. ■

For a subset Z of V and an edge $e = (x,a,y)$ we say that e is *crossing* Z iff either $x \in Z$ and $y \notin Z$ or $x \notin Z$ and $y \in Z$; if the former holds then e is *leaving* Z, and if the latter holds then e is *entering* Z. Hence Z is a region of h iff either all edges of h with the same label are not crossing Z or all edges of h with the same label are crossing Z in the same way, meaning that they are either all leaving Z or all entering Z.

We will use R_h to denote the set of all regions of h and, for an $x \in V$, $R_h(x)$ denotes the set of all regions of h containing x.

Example 2.2. Let h_0 be the following p2s:



Then

R_h = { {1,2,3,4}, {1,2}, {3,4}, {1,4}, {2,3}, $\varnothing$ },

$R_h(1)$ = { {1,2,3,4}, {1,2}, {1,4} },

$R_h(2)$ = { {1,2,3,4}, {1,2}, {2,3} },

$R_h(3)$ = { {1,2,3,4}, {3,4}, {2,3} }, and

$R_h(4)$ = { {1,2,3,4}, {3,4}, {1,4}}. ■

3. A CHARACTERIZATION OF THE ABSTRACT SEQUENTIAL CASE GRAPHS OF EN SYSTEMS

In this section we state a characterization result for abstract sequential case graphs of EN systems (proved in [ER3], [ER4]). It is based on the notion of a region, and more explicitely it is based on the following fundamental construction.

Definition 3.1. Let $h = (V,E,\Delta;v_{in})$ be a p2s.

(1) The *regional* h-*mapping*, denoted reg_h, is the mapping φ from V into 2^{R_h} defined by:

for every $x \in V$, $\varphi(x) = R_h(x)$.

(2) The *regional version of* h, denoted *regv*(h), is the ps2s $(V',E',\Delta';v'_{in})$ such that

(i) $V' = \{ R_h(x) : x \in V \}$,

(ii) for all $(X,Y) \in V' \times V'$, $(X, osd(X,Y), Y) \in E'$ iff there exists
$(x, osd(x,y), y) \in E$ such that $X = R_x$ and $Y = R_y$,

(iii) $\Delta' = \{ osd(X,Y) : (X, osd(X,Y), Y) \in E' \}$, and

(iv) $v'_{in} = \varphi(v_{in})$. ■

Example 3.1. For the p2s h_0 from Example 2.2, $regv(h_0)$ is as follows:

$$\begin{tikzcd}
\{ Z_1 , Z_2 , Z_4 \} \arrow[rr, "(\{Z_4\} , \{Z_5\})"] \arrow[d, "(\{Z_2\} , \{Z_3\})"'] & & \{ Z_1 , Z_2 , Z_5 \} \arrow[d, "(\{Z_2\} , \{Z_3\})"] \\
\{ Z_1 , Z_3 , Z_4 \} \arrow[rr, "(\{Z_4\} , \{Z_5\})"'] & & \{ Z_1 , Z_3 , Z_5 \}
\end{tikzcd}$$

■

To state our main result we need also the following notion.

Definition 3.3. A ps2s $h = (V,E,\Delta;v_{in})$ is *forward closed*, denoted $FC(h)$, iff, for all $(A,B) \in \Delta$ and all $X \in V$ such that $A \subseteq X$ and $B \cap X = \emptyset$, there exists $Y \in V$ such that $(X,(A,B),Y) \in E$. ■

Theorem 3.1. An initialized edge-labeled graph h is an abstract sequential case graph of an EN system iff

(1) h is a p2s,

(2) each node of h is reachable from the initial node of h,

(3) reg_h is an isomorphism, and

(4) $FC(regv(h))$. ■

Example 3.2.

For the p2s h_0 from Example 2.2, reg_{h_0} is an isomorphism of h_0 onto $regv(h_0)$ (see Example 3.1). Clearly each node of h_0 is reachable from the initial node of h_0 , and $FC(regv(h_0))$. Hence by Theorem 3.1, h_0 is an abstract sequential case graph of an EN system.

On the other hand for the p2s h_1 from Example 1.3, its regional version has one node only and consequently h_1 is not isomorphic with *regv*(h_1). Hence, indeed, h_1 is *not* an abstract sequential case graph of an EN system. ■

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