INTERSECTION TYPES FOR COMBINATORY LOGIC

Mariangiola Dezani-Ciancaglini,
Dip. Informatica, Corso Svizzera 185,
Torino, Italy

Roger Hindley,
Maths. Div., University College,
Swansea SA2 8PP, U.K.

Dedicated to J.W.de Bakker in honour of his 25 years of work in semantics.

ABSTRACT.

Two different translations of the usual formulation of intersection types for
\(\lambda\)-calculus into combinatory logic are proposed; in the first one the rule \((\leq)\) is
unchanged, while in the second one the rule \((\leq)\) is replaced by three new rules and
five axiom-schemes, which seem to be simpler than rule \((\leq)\) itself.

INTRODUCTION.

Intersection types were introduced as a generalization of the type discipline of
Church and Curry, mainly with the aim of describing the functional behaviour of all
solvable \(\lambda\)-terms. The usual \(\to\)-based type-language for \(\lambda\)-calculus was extended by
adding a constant \(\omega\) as a universal type and a new connective \(\land\) for the intersection
of two types. With suitable axioms and rules to assign types to \(\lambda\)-terms, this gave a
system in which (i) the set of types given to a \(\lambda\)-term does not change under
\(\beta\)-conversion, and (ii) the sets of normalizing and solvable \(\lambda\)-terms can be
characterized very neatly by the types of their members. (CDV1981] gives an
introduction and motivation of \(\land\) and \(\omega\), and BCDI1983] gives a summary of all the
most basic syntactic properties of the system.)

Moreover, in the new type-language we can build \(\lambda\)-models (filter models) in
which the interpretation of a \(\lambda\)-term coincides with the set of all types that can be
assigned to it. Filter models turn out to be a very rich class containing in particular
each inverse-limit space, and have been widely used to study properties of
\(D_\infty\)-\(\lambda\)-models; see BCD[1983], CDHL[1983] and CDZ[1987].

More recently, intersection types have been introduced in the programming
language Forsythe, which is a descendent of Algol 60, to simplify the structure of
types; see R[1988].

Systems of combinators are designed to perform the same tasks as systems of
\(\lambda\)-calculus, but without using bound variables. Curry's type discipline turns out to
be significantly simpler in combinatory logic than in λ-calculus. (For an introduction see HS[1986] Chapter 14.)

We propose here two different formulations of intersection types for combinatory logic. They are both essentially just translations of the λ-calculus system presented in BCD[1983], and have all the properties one would expect. However, there is at least one extra complication in combinatory logic. In the case of λ-calculus, the type-assignment rule (≤) is well known to be replaceable by the simpler rule (η) (§1 below). But in combinatory logic some more care must be taken in choosing a rule to replace rule (≤), and we do not know whether the second system we present below is the simplest possible (see §4).

For background λ-calculus, combinatory logic and type-theory, HS[1986] will be used as a basic reference.

1. INTERSECTION TYPES FOR λ-CALCULUS.

We introduce the intersection type-assignment system following BCD[1983], H[1982] and H[1988].

1.1 DEFINITION. (i) The set T of intersection types is inductively defined by:

\( \phi_0, \phi_1, ..., \phi \in T \) (type-variables)

\( \omega \in T \) (type-constant)

\( \sigma, \tau \in T \implies (\sigma \rightarrow \tau) \in T, \ (\sigma \land \tau) \in T. \)

(ii) A (type-assignment) statement is of the form M:σ with σ ∈ T and M a λ-term, called its subject. A basis B is a set of statements with only distinct variables as subjects. If x does not occur in B, then "B, x:σ" denotes B∪(x:σ).

On intersection types we define a pre-order relation which formalizes the subset relation and will be used in a type-assignment rule.

1.2 DEFINITION. The ≤ relation on intersection types is inductively defined by:

\( \tau \leq \tau, \quad \tau \leq \tau \land \tau, \)

\( \tau \leq \omega, \quad \sigma \land \tau \leq \sigma, \quad \sigma \land \tau \leq \tau, \)

\( \omega \leq \omega \rightarrow \omega, \quad (\sigma \rightarrow \rho) \land (\sigma \rightarrow \tau) \leq \sigma \rightarrow (\rho \land \tau), \)

\( \sigma \leq \rho \leq \tau \implies \sigma \leq \tau, \)

\( \sigma \leq \sigma', \ \tau \leq \tau' \implies \sigma \land \tau \leq \sigma' \land \tau', \)

\( \sigma \leq \sigma', \ \tau \leq \tau' \implies \sigma \rightarrow \tau \leq \sigma' \rightarrow \tau'. \)
1.3 DEFINITION. (i) $\text{T}_{\lambda}(\wedge, \omega, \leq)$ is the type assignment system defined by the following natural-deduction rules and axioms.

**Axioms** ($\omega$): $\ M : \omega$ (one axiom for each $\lambda$-term $M$).

**Rules:**

\[
\begin{align*}
\frac{[x:\sigma]}{\lambda x. M : \sigma \rightarrow \tau} & \quad (\rightarrow I) \\
\vdots & \\
\vdots & \quad (\rightarrow E) \\
\frac{M : \sigma \rightarrow \tau}{MN : \tau} & \\
\frac{M : \sigma \wedge \tau}{\frac{M : \sigma}{M : \tau}} \quad & \frac{M : \sigma \wedge \tau}{\frac{M : \sigma}{M : \tau}} \quad & \frac{M : \sigma \wedge \tau}{\frac{M : \sigma}{M : \tau}} \\
\frac{M : \sigma \quad \sigma \leq \tau}{M : \tau} & \quad (\leq)
\end{align*}
\]

(*) if $x$ is not free in assumptions above $M : \tau$, other than $x : \sigma$.

(ii) We write $B \vdash_{\lambda} M : \sigma$ if $M : \sigma$ is derivable from the basis $B$ in this system.

The main syntactic property of this type system is the following theorem of invariance under $\beta$-equality and $\eta$-reduction. (For a proof see CDV[1981] Lemma 1 and Theorem 1, or H[1982] §5.)

1.5 THEOREM. (i) $\text{T}_{\lambda}(\wedge, \omega, \leq)$ is invariant under $\beta$-equality; that is, if $M =_{\beta} N$ and $B \vdash_{\lambda} M : \sigma$, then $B \vdash_{\lambda} N : \sigma$.

(ii) $\text{T}_{\lambda}(\wedge, \omega, \leq)$ is invariant under $\eta$-reduction; that is, if $z \notin \text{FV}(M)$ and $z$ does not occur in $B$, and $B, z : \sigma \vdash_{\lambda} Mz : \tau$, then $B \vdash_{\lambda} M : (\sigma \rightarrow \tau)$.

The invariance under $\eta$-reduction allows a replacement of rule $(\leq)$ which preserves type assignment, as follows.
1.6 DEFINITION. (i) Let TA_λ(\lambda, \omega, \eta) be the type-assignent system obtained from TA_λ(\lambda, \omega, \leq) by replacing rule (\leq) by

\[
\frac{(\lambda x. Mx):\sigma}{M:\sigma}. \quad \text{(if } x \text{ is not free in } M)\]

(ii) Let B \vdash_{\lambda, \eta} M: \sigma denote derivability in the resulting system.

1.7 THEOREM. TA_\lambda(\lambda, \omega, \leq) and TA_\lambda(\lambda, \omega, \eta) are equivalent; that is,

B \vdash_\lambda M: \sigma \iff B \vdash_{\lambda, \eta} M: \sigma.

This equivalence can be proved directly fairly easily, or by using BCD[1983] (in particular Lemma 4.2, Remark 2.10, and the remark just before 4.3).

2. CORRESPONDENCE BETWEEN \lambda AND CL.

The reader is assumed to know at least the basic definitions of combinatory logic (see Chapter 2 of HS[1986]). The atomic combinators are assumed here to be S, K, I.

2.1 DEFINITION (Abstraction in Combinatory Logic).

(i) A functional (fml) term is any of S, SX, SY, KX, KX, I (for any X, Y).

(ii) We present four alternative definitions for \lambda x.X. (The second one has been discussed in HS[1986] §§9.34-35, and the other three are common in the literature. Note that the definition of \lambda \beta uses \lambda \eta.)

\lambda \eta:

(a) \lambda \eta x.Y \equiv KY if x \notin FV(Y),
(b) \lambda \eta x.x \equiv I,
(c) \lambda \eta x.Ux \equiv U if x \notin FV(U),
(f) \lambda \eta x.UV \equiv S(\lambda \eta x.U)(\lambda \eta x.V) if (a)-(c) do not apply.

\lambda \beta:

(a), (b) as above,
(c) \lambda \beta x.Ux \equiv U if x \notin FV(U) and U is fml,
(f) \lambda \beta x.UV \equiv S(\lambda \eta x.U)(\lambda \eta x.V) if (a)-(c) do not apply.

\lambda \text{abf}:

(a), (b) as above, and (f) used when (a) and (b) do not apply.
\[ \lambda^{\text{fab}} : (\lambda x. \lambda x. \lambda x. U V) \equiv S(\lambda x. \lambda x. U)(\lambda x. \lambda x. V), \]

(a) \( \lambda^{\text{fab}} x. y \equiv K y \) if \( y \) is an atom distinct from \( x \),
(b) \( \lambda^{\text{fab}} x. x \equiv I \).

2.2 DEFINITION (H-transformations). Each abstraction determines an H-mapping from \( \lambda \)-calculus to combinatory logic: \( (\lambda x. M)_H \equiv \lambda^x. (M_H) \). (Details are in HS[1986] Chapter 9.) We call these mappings \( H_\beta, H_\eta, H_{abf}, H_{fab} \). Let \( X_\lambda \) denote the \( \lambda \)-term associated in the standard way with the CL-term \( X \), and let \( =_{\text{cb}} \) denote combinatory \( \beta \)-equality (i.e. \( X =_{\text{cb}} Y \iff X_\lambda =_\beta Y_\lambda \)).

2.3 LEMMA. (i) For all CL-terms \( X \):
   
   \begin{align*}
   X_\lambda H_\eta & \equiv X, \text{ in particular } S_\lambda H_\eta \equiv S; \\
   X_\lambda H_\beta & \equiv X, \text{ in particular } S_\lambda H_\beta \equiv S; \\
   X_\lambda H_{abf} & =_{cb} X \text{ and } S_\lambda H_{abf} \neq S; \\
   X_\lambda H_{fab} & =_{cb} X \text{ and } S_\lambda H_{fab} \neq S.
   \end{align*}

   (ii) For all \( \lambda \)-terms \( M \) and for \( H_\beta \) or \( H_{abf} \) or \( H_{fab} : M_{H_\lambda} =_\beta M \).

   The proof for \( H_{abf} \) is in HS[1986] §§9.20-28, and the others are similar; see HS[1986]§9.35 for hints on the proof for \( H_\beta \).

3. INTERSECTION TYPES FOR CL-TERMS.

   We introduce now an assignment of intersection types to CL-terms which can be viewed as a translation of \( TA_\lambda (\land, \omega, \leq) \) into combinatory logic. Its relation to \( TA_\lambda (\land, \omega, \leq) \) will be precisely stated in Theorem 3.3.

   In this section, type-assignment statements have form \( X : \sigma \) where \( X \) is a CL-term. Bases are sets \( \{x_1 : \sigma_1, x_2 : \sigma_2, \ldots\} \) with \( x_1, x_2, \ldots \) distinct, as usual.

3.1 DEFINITION. (i) \( TA_{\text{CL}}(\land, \omega, \leq) \) is the system whose rules are \( (\rightarrow E), (\land I), (\land E), (\leq) \), and whose axiom-schemes are \( (\omega) \) and

   \begin{align*}
   (\rightarrow I) & : \sigma \rightarrow \sigma, \\
   (\rightarrow K) & : \sigma \rightarrow \tau \rightarrow \sigma, \\
   (\rightarrow S) & : \sigma \rightarrow (\tau \rightarrow \rho) \rightarrow (\sigma \rightarrow \tau) \rightarrow \sigma \rightarrow \rho.
   \end{align*}

   (ii) We write \( B \vdash_{\text{CL}} X : \sigma \) if \( X : \sigma \) is derivable from the basis \( B \) in this system.
3.2 **LEMMA.** (i) $B, x: \sigma \vdash_{CL} x: \tau \Rightarrow \sigma \leq \tau$.

(ii) Let $\lambda^*$ be any of $\lambda I, \lambda^B, \lambda^b, \lambda^f$. Then

$$B, x: \sigma \vdash_{CL} Y: \tau \Rightarrow B \vdash_{CL} (\lambda^* x. Y): \sigma \rightarrow \tau.$$ 

**Proof.** (i) By an easy induction on deductions.

(ii) Induction on the deduction of $Y: \tau$. We will prove the result for all four $\lambda^*$’s at once, and will use the induction hypothesis for $\lambda I$ in proving the induction step for $\lambda^B$.

**Case 1:** $Y: \tau$ is $x: \sigma$. \(\vdash Y \equiv x, \vdash \lambda^* x. Y \equiv I\). But $I: \sigma \rightarrow \sigma$ is an axiom.

**Case 2:** $Y: \tau$ is either in $B$, or is an $S$, $K$ or $I$ axiom, or an $\omega$-axiom with $Y$ an atom $\not\equiv x$. \(\vdash Y\) is an atom and $x \not\in \text{FV}(Y)$, so $\lambda^* x. Y \equiv KY$. Hence, by the axiom $K: \tau \rightarrow \sigma \rightarrow \tau$ and rule $(-E)$, $B \vdash_{CL} KY: \sigma \rightarrow \tau$.

**Case 3:** $Y: \tau$ is an $\omega$-axiom. \(\vdash \tau \equiv \omega\). Now $(\lambda^* x. Y): \omega$ is an $\omega$-axiom. And, since $\sigma \leq \omega$, we have $\omega \leq \omega \rightarrow \omega \leq \sigma \rightarrow \omega$. Hence $(\lambda^* x. Y): \sigma \rightarrow \omega$ by rule $(\leq)$.

**Case 4:** The last step in the deduction of $Y: \tau$ is $(\leq)$ or $(\land E)$:

$$
\begin{array}{c}
B \\
\vdots \\
\vdots \\
Y: \rho \\
\hline
Y: \tau
\end{array}
\quad
\begin{array}{c}
(\rho \leq \tau)
\end{array}

\text{Then } (\sigma \rightarrow \rho) \leq (\sigma \rightarrow \tau), \text{ so we use the induction hypothesis and rule } (\leq).

**Case 5:** Rule $(\land I)$:

$$
\begin{array}{c}
Y: \tau_1 \\
\vdots \\
Y: \tau_2 \\
\hline
Y: (\tau_1 \land \tau_2)
\end{array}
\quad
\begin{array}{c}
(\tau_1 \equiv \tau_1 \land \tau_2)
\end{array}
$$

By induction hypothesis, $B \vdash_{CL} (\lambda^* x. Y): \sigma \rightarrow \tau_i$ for $i = 1, 2$. But $(\sigma \rightarrow \tau_1) \land (\sigma \rightarrow \tau_2) \leq \sigma \rightarrow (\tau_1 \land \tau_2)$, so rules $(\land I)$ and $(\leq)$ give the result.

**Case 6:** Rule $(-E)$: Say $Y \equiv UV$, and we have:

$$
\begin{array}{c}
B \\
\vdots \\
B \\
\vdots \\
U: \rho \rightarrow \tau \\
\hline
UV: \tau
\end{array}
\quad
\begin{array}{c}
V: \rho
\end{array}
$$
Subcase 6a: \( x \not\in \text{FV}(UV) \) and \( \lambda^* x . (UV) \equiv K(UV) \). Since \( x \not\in \text{FV}(UV) \), \( x \) cannot occur in the given deduction. Hence \( B \vdash_{CL} UV : \tau \). So by the axiom
\[
K : \tau \rightarrow \sigma \rightarrow \tau \quad \text{and rule } (\rightarrow E), \quad B \vdash_{CL} K(UV) : \sigma \rightarrow \tau.
\]

Subcase 6c: \( V \equiv x, \ x \not\in \text{FV}(U) \), and \( \lambda^* x . (UV) \equiv U \). Since \( B, x : \sigma \vdash x : \rho \), we have \( \sigma \leq \rho \) by (i). \( \therefore (\rho \rightarrow \tau) \leq (\sigma \rightarrow \tau) \). But \( B \vdash_{CL} U : (\rho \rightarrow \tau) \) since \( x \not\in \text{FV}(U) \); hence by \((\leq)\), \( B \vdash_{CL} U : (\sigma \rightarrow \tau) \).

Subcase 6f: \( \lambda^* x . (UV) \equiv S(\lambda^* x . U)(\lambda^* x . V) \) (where \( \lambda^* \) is \( \lambda \) or \( \lambda^{ab} \) or \( \lambda^{ab} \)). By induction hypothesis for \( \lambda^* \), we have \( B \vdash_{CL} (\lambda^* x . U) : \sigma \rightarrow \rho \rightarrow \tau \), \( B \vdash_{CL} (\lambda^* x . V) : \sigma \rightarrow \rho \). Hence the result, by an \( S \)-axiom and \((\rightarrow E)\). \( \square \)

3.3 THEOREM. (i) \( B \vdash_{CL} X : \tau \iff B \vdash_{\lambda} X_{\lambda} : \tau \).

(ii) \( B \vdash_{\lambda} M : \tau \Rightarrow B \vdash_{CL} M_{H : \tau} \) for \( H_\eta, H_\beta, H_{ab}, H_{fab} \).

(iii) For \( H_\beta, H_{ab}, H_{fab} \), we also have the converse of (ii).

Proof. We prove all parts together. (i) "\( \Rightarrow \)" is trivial.

(ii): Induction on \( \vdash_{\lambda} \). The only difficult case is rule \((\rightarrow I)\), which comes by Lemma 3.2.

(iii): Let \( H \) be any of \( H_\beta, H_{ab}, H_{fab} \) and let \( B \vdash_{CL} M_{H : \tau} \). \( \therefore \) by (i) "\( \Rightarrow \)",
\[
B \vdash_{\lambda} M_{H_{\lambda} : \tau}.
\]
But \( M_{H_{\lambda}} =_\beta M \) by Lemma 2.3(ii). \( \therefore \) by Theorem 1.5(i), \( B \vdash_{\lambda} M : \tau \).

(i) "\( \Leftarrow \)": Let \( B \vdash_{\lambda} X_{\lambda} : \tau \). Then \( B \vdash_{CL} X_{\lambda H_{\beta}} : \tau \) by (ii). \( \therefore \) \( B \vdash_{CL} X : \tau \) because \( X_{\lambda H_{\beta}} = X \) by Lemma 2.3(i). \( \square \)

Note that Theorem 3.3(iii) does not hold for \( H_\eta \). A counter-example is \( M \equiv \lambda x y . x y \); we have \( M_{H_\eta} = \downarrow \) which has type \( \phi \rightarrow \phi \) in the CL-system (\( \phi \) being a type-variable), but it can be shown that \( M \) does not have this type in the \( \lambda \)-system.

The following theorem shows that \( TA_{CL, \lambda, \omega, \downarrow} \) is invariant under \( \beta \)-equality and \( \eta \)-reduction.

3.4 THEOREM. (i) If \( B \vdash_{CL} X : \tau \) and \( Y =_c X \), then \( B \vdash_{CL} Y : \tau \).

(ii) If \( B, x : \sigma \vdash_{CL} Yz : \tau \) and \( z \not\in \text{FV}(Y) \) and \( z \) is not in \( B \), then \( B \vdash_{CL} Y : (\sigma \rightarrow \tau) \).
Proof. (i): By 3.3(i), (iii) and 1.5(i).

(ii) Induction on the deduction of \( Y z : \tau \), as follows.

**Axioms:** \( Y z : \tau \) cannot be an \( S, K, I \)-axiom. The only possibility is an \( \omega \)-axiom, with \( \tau \equiv \omega \). But \( \omega \leq \omega \to \omega \leq \sigma \to \omega \) (since \( \sigma \leq \omega \)), so we have

\[
\begin{align*}
(\omega)\text{-ax} \\
Y : \omega \\
\hline
(\omega \leq \sigma \to \omega) \\
Y : (\sigma \to \omega).
\end{align*}
\]

**Rule \((\to E)\):** Say we have, for some \( \rho \),

\[
\begin{align*}
Y : \rho \to \tau \\
z : \rho \\
\hline
Y z : \tau.
\end{align*}
\]

But \( z : \rho \) is deduced from \( B \), \( z : \sigma \) and \( z \) does not occur in \( B \). Hence \( \sigma \leq \rho \) by 3.2(i).

\[ \therefore (\rho \to \tau) \leq (\sigma \to \tau), \text{ so by } Y : (\rho \to \tau) \text{ and rule } (\leq), \ B \vdash_{CL} Y : (\sigma \to \tau). \]

**Rule \((\leq) \text{ or } (\land E)\):** Say we have

\[
\begin{align*}
Y z : \rho \\
\hline
Y z : \tau.
\end{align*}
\]

By induction hypothesis, \( B \vdash_{CL} Y : (\sigma \to \rho) \). Hence, by \((\leq)\), \( B \vdash_{CL} Y : (\sigma \to \tau) \).

**Rule \((\land)\):** Say \( \tau \equiv (\tau_1 \land \tau_2) \) and we have

\[
\begin{align*}
Y z : \tau_1 \\
Y z : \tau_2 \\
\hline
Y z : (\tau_1 \land \tau_2).
\end{align*}
\]

By induction hypothesis, \( B \vdash_{CL} Y : (\sigma \to \tau_i) \), \( i = 1, 2 \). \( \therefore \) by \((\land)\) and \((\leq)\), since \((\sigma \to \tau_1) \land (\sigma \to \tau_2) \leq \sigma \to (\tau_1 \land \tau_2)\), we have \( B \vdash_{CL} Y : \sigma \to (\tau_1 \land \tau_2) \). \( \square \)

3.5 NOTE. Following H[1982], let us define the set NTS of Normal Types to be the set of all types \( \sigma \) such that: either \( \sigma \equiv \omega \) or \( \sigma \equiv \sigma_1 \land \ldots \land \sigma_n \) with some bracketing and with each \( \sigma_i \) having the form \( \sigma_{i,1} \to \ldots \to \sigma_{i,m(i)} \to \phi_i \). Normal types corresponded closely to the types in CDV[1981], which were slightly more restricted than those in BCD[1983] and later papers, including this one. In H[1982] it was proved that the restriction was trivial, in the sense that every deduction \( B \vdash_{\lambda} M : \tau \) could be paralleled by a deduction \( B^* \vdash_{\lambda} M : \tau^* \) containing only normal types, where the map \( * : T \to \text{NTS} \) applied to a type gave its "normal form". But in CL the
restriction seems not to be so trivial. For example, in CL there is a problem with the axiom \( I:\sigma \land \tau \rightarrow (\sigma \land \tau) \). The type in this is not normal, and the nearest normal type to it is \( ((\sigma \land \tau) \rightarrow \tau) \land (\sigma \land \tau) \rightarrow \tau \). So if types were restricted to being normal, quite a complicated form of the axiom scheme for \( I \) would be needed to give a reasonable equivalence to the \( \lambda \)-system. Similarly for \( S \) and \( K \).

4. REPLACING RULE \( (\leq) \).

In this section we propose an alternative formulation of intersection type-assignment to CL-terms in which rule \( (\leq) \) has been replaced by something simpler. Let \( B \equiv S(KS)K \) and \( B' \equiv SB(KI) \).

4.1 DEFINITION. (i) \( TA_{CL}(\land,\omega,\eta) \) is the system for CL-terms whose axiom-schemes are \((\omega), (\rightarrow I), (\rightarrow K), (\rightarrow S)\) and

\[
\begin{align*}
(I_1) & \quad I : \sigma \rightarrow \omega \\
(I_2) & \quad I : \omega \rightarrow (\omega \rightarrow \omega) \\
(I_3) & \quad I : (\sigma_1 \land \sigma_2) \rightarrow \sigma_i \quad (i = 1,2) \\
(I_4) & \quad I : ((\sigma \rightarrow \tau) \land (\sigma \rightarrow \rho)) \rightarrow (\sigma \rightarrow (\tau \land \rho))
\end{align*}
\]

and whose rules are \((\rightarrow E), (\land I), (\land E)\) and

\[
\begin{align*}
(l_5) & \quad \frac{X : \sigma}{X : \sigma} \quad \text{(I\!X)} \\
(\eta_1) & \quad \frac{l : \sigma}{l : \sigma} \quad \text{(I\!\!l)} \\
(\eta_2) & \quad \frac{B'l : \sigma}{l : \sigma} \quad \text{(B\!l)}
\end{align*}
\]

(ii) We write \( B \vdash X : \sigma \) if \( X : \sigma \) is derivable from the basis \( B \) in this system.

We shall prove that \( TA_{CL}(\land,\omega,\leq) \) and \( TA_{CL}(\land,\omega,\eta) \) are equivalent.

4.2 LEMMA. If \( \sigma \leq \sigma' \), then \( I : \sigma \rightarrow \sigma' \).

Proof. Induction on the proof of \( \sigma \leq \sigma' \). We consider only the non-trivial cases. Axiom \( \sigma \leq \sigma \land \sigma \).
Transitivity: Suppose $1: \sigma \rightarrow \tau$ and $1: \tau \rightarrow \rho$. Deduce $1: \sigma \rightarrow \rho$ thus:

$$
B : (\tau \rightarrow \rho) \rightarrow (\sigma \rightarrow \tau) \rightarrow \sigma \rightarrow \rho \\
B \vdash \sigma \rightarrow \rho \quad (\rightarrow E)
$$

$$
B \vdash (\sigma \rightarrow \tau) \rightarrow \sigma \rightarrow \rho \\
\vdash (\sigma \rightarrow \tau) \rightarrow \sigma \rightarrow \rho \quad (\eta_1) \quad (\eta_1)' \quad (\eta_1)'' \quad (\eta_1)'''
$$

$$
B \vdash \sigma \rightarrow \tau \quad (\rightarrow E) \\
\vdash \sigma \rightarrow \tau \quad (\rightarrow E)
$$

$$
B \vdash (\sigma \rightarrow \tau) \rightarrow \sigma \rightarrow \rho \\
\vdash (\sigma \rightarrow \tau) \rightarrow \sigma \rightarrow \rho \quad (\eta_1) \quad (\eta_1)'' \quad (\eta_1)'''
$$

$$
\vdash \sigma \rightarrow \rho \\
\vdash \sigma \rightarrow \rho
$$

Replacement in $\eta$. Assume $1: \sigma \rightarrow \sigma'$ and $1: \tau \rightarrow \tau'$. Deduce $1: (\sigma \land \tau) \rightarrow (\sigma' \land \tau')$ thus:

$$
B : (\sigma \rightarrow \sigma') \rightarrow ((\sigma \land \tau) \rightarrow (\sigma \land \tau)) \rightarrow \sigma \rightarrow \sigma' \\
B \vdash (\sigma \rightarrow \sigma') \rightarrow (\sigma \land \tau) \rightarrow \sigma \rightarrow \sigma' \quad (\rightarrow E) \\
\vdash (\sigma \rightarrow \sigma') \rightarrow (\sigma \land \tau) \rightarrow \sigma \rightarrow \sigma' \quad (\eta_1) \\
\vdash (\sigma \rightarrow \sigma') \rightarrow (\sigma \land \tau) \rightarrow \sigma \rightarrow \sigma' \quad (\eta_1)'' \quad (\eta_1)'''
$$

$$
B \vdash (\sigma \rightarrow \sigma') \rightarrow (\sigma \land \tau) \rightarrow \sigma \rightarrow \sigma' \\
\vdash (\sigma \rightarrow \sigma') \rightarrow (\sigma \land \tau) \rightarrow \sigma \rightarrow \sigma' \quad (\eta_1) \\
\vdash (\sigma \rightarrow \sigma') \rightarrow (\sigma \land \tau) \rightarrow \sigma \rightarrow \sigma' \quad (\eta_1)'' \quad (\eta_1)'''
$$

$$
B \vdash (\sigma \land \tau) \rightarrow \sigma \rightarrow \rho \\
\vdash (\sigma \land \tau) \rightarrow \sigma \rightarrow \rho \quad (\eta_1) \quad (\eta_1)'' \quad (\eta_1)'''
$$

$$
\vdash (\sigma \rightarrow \rho) \quad (\eta_1) \quad (\eta_1)'' \quad (\eta_1)'''
$$

Replacement in $\rightarrow$. Assume $1: \sigma \rightarrow \sigma'$ and $1: \tau \rightarrow \tau'$. Deduce $1: (\sigma' \rightarrow \tau) \rightarrow (\sigma \rightarrow \tau')$ as follows. In this deduction, let $\xi \equiv (\sigma \rightarrow \tau)$, $\eta \equiv (\sigma \rightarrow \tau')$, and $\zeta \equiv (\sigma' \rightarrow \tau)$.

$$
B : ((\tau \rightarrow \tau') \rightarrow \eta) \rightarrow (\zeta \rightarrow \eta) \rightarrow \xi \rightarrow \eta \\
B \vdash (\tau \rightarrow \tau') \rightarrow \eta \rightarrow (\zeta \rightarrow \eta) \rightarrow \xi \rightarrow \eta \quad (\eta_1) \\
\vdash (\tau \rightarrow \tau') \rightarrow \eta \rightarrow (\zeta \rightarrow \eta) \rightarrow \xi \rightarrow \eta \quad (\eta_1)'' \quad (\eta_1)'''
$$

$$
B \vdash (\zeta \rightarrow \eta) \rightarrow (\tau \rightarrow \tau') \rightarrow \eta \rightarrow (\zeta \rightarrow \eta) \rightarrow \xi \rightarrow \eta \quad (\eta_2) \\
\vdash (\zeta \rightarrow \eta) \rightarrow (\tau \rightarrow \tau') \rightarrow \eta \rightarrow (\zeta \rightarrow \eta) \rightarrow \xi \rightarrow \eta \quad (\eta_2)'' \quad (\eta_2)'''
$$

$$
\vdash (\xi \rightarrow \eta) \\
\vdash (\xi \rightarrow \eta)
$$

$\Box$
4.3 **THEOREM.** \( B \vdash_{\text{CL}} X: \sigma \iff B \vdash_{\text{CL} \eta} X: \sigma \).

**Proof.** "\( \Rightarrow \)". The only thing to show is that \( (\leq) \) is an admissible rule in \( \text{TA}_{\text{CL} \delta} (\land, \omega, \eta) \); that is, to show that if \( B \vdash_{\text{CL} \eta} X: \sigma \) and \( \sigma \leq \tau \), then \( B \vdash_{\text{CL} \eta} X: \tau \).

By Lemma 4.2, \( \vdash_{\text{CL} \eta} \Gamma \sigma \sigma \rightarrow \tau \). Then we can deduce

\[
\begin{align*}
1: \sigma \rightarrow \tau & \quad X: \sigma \\
\hline
1X: \tau & \quad (\rightarrow \text{E}) \\
\hline
X: \tau . & \quad (1) \\
\hline
\end{align*}
\]

"\( \Leftarrow \)". Immediate from 3.4(ii). \( \square \)

4.4 **NOTE.** Rule \( (\leq) \) can also be replaced by a strengthened \( \Gamma \)-axiom-scheme saying \( 1: \sigma \rightarrow \tau (\sigma \leq \tau) \), and an \( \Gamma \)-rule:

\[
\begin{align*}
1X: \sigma & \\
\hline
X: \sigma .
\end{align*}
\]

Using this axiom-scheme and rule, we get \( X: \sigma \vdash X: \tau \) when \( \sigma \leq \tau \), as follows:

\[
\begin{align*}
1: \sigma \rightarrow \tau & \quad X: \sigma \\
\hline
1X: \tau & \quad (\rightarrow \text{E}) \\
\hline
X: \tau .
\end{align*}
\]

Conversely, the axiom and \( \Gamma \)-rule are easily proved admissible in \( \text{TA}_{\text{CL} \delta} (\land, \omega, \leq) \).
REFERENCES.


