ENUMERATION AND VISIBILITY PROBLEMS IN INTEGER LATTICES[‡]

(Extended Abstract)

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Abstract

We study enumeration and visibility problems in the d-dimensional integer lattice L_n^d of d-tuples of integers $\leq n$. In the first part of the paper we give several useful enumeration principles and use them to study the asymptotic behavior of the number of straight lines traversing a certain fixed number of lattice vertices of L_n^d, the line incidence problem and the edge visibility region. In the second part of the paper we consider an art gallery problem for point obstacles. More specifically we study the camera placement problem for the infinite lattice L^d. A lattice point is visible from a camera C (positioned at a vertex of L^d) if the line segment joining A and C crosses no other lattice vertex. For any given number $s \leq 3^d$ of cameras we determine the position they must occupy in the lattice L^d in order to maximize their visibility.

1 Introduction

The present paper is concerned with several enumeration and visibility problems in multidimensional integer lattices. Before providing an outline of the main results of the paper we remind the reader that $\zeta(z)$ denotes the Riemann zeta function, $\sum_{n\geq 1} n^{-z}$, $|z| > 1$, while L^d (respectively, L_n^d) is the complete lattice of d-tuples of non-negative integers (respectively, $\leq n$), where $d \geq 2$.

In the first part of the paper we are dealing with several enumeration problems which arise in the analysis of algorithms of combinatorial and computational geometry. These include: (1) the asymptotic number of different straight lines traversing at least k vertices of d-dimensional lattices, simplexes, etc., (2) the expected length and standard deviation of maximal (or other kinds of) segments of d-dimensional lattices, simplexes, etc., (3) the maximum number of incidences $I(m, n)$ between m points and n lines in the plane [ST83] [Ede87, chapter 6], [CEG*88, page 13], and (4) the complexity of computing the region of the plane illuminated by a line segment in the presence of other line segments (edge visibility region) [ORo87, pages 219-223]. We show how to compute asymptotically optimal bounds for problems (1), (2) and exact constants of known lower bounds for problems (3) and (4).

Underlying several themes of our present study we will encounter in the sequel several applications of generalizations of an old theorem, from 1849, of G. Lejeune Dirichlet. The theorem states that the probability that two integers chosen at random are relatively prime is $1/\zeta(2)$ [Knu81, page 324], [HW79, page 269]. This result can also be stated as follows: if Δ is a bounded plane region with area $area(\Delta)$ and $G(\Delta)$ is the set of lattice points of Δ whose coordinates are relatively prime integers then

$$|G(\Delta)| \sim \frac{\text{area}(\Delta)}{\zeta(2)}$$

as Δ grows by homothety to the full plane (see [HW79, page 409]). It turns out that our analysis of the above mentioned problems requires the asymptotic evaluation of multidimensional versions of sums of the form $\sum_{P\in G(\Delta)} f(P)$ in terms of $\int_\Delta f$, where f is a real function (monotone or Lipschitzian). Intuitively one can think of the function $f(P)$ as a weight "quantifying" the visibility of the point P from the origin while the sum $\sum_{P\in G(\Delta)} f(P)$ "quantifies" the "total" visibility from the origin. After proving the required extension we proceed with the precise evaluation of the above mentioned quantities.

In the second part of the paper we consider visibility questions on multidimensional integer lattices. Two points x and y of the d-dimensional lattice L^d are mutually visible (or can see one another) if there is no lattice point on the line segment joining them. If S is a set of lattice points we denote by $V_n(S)$ (respectively, $U_n(S)$) the set of lattice points which are visible from every (respectively, some) point of S. There have been several interesting results in the literature concerning visibility problems.

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F. Herzog and B. M. Stewart [HS71] consider the problem of realizability of patterns of visible and nonvisible lattice points. A pattern P_d in the d-dimensional lattice is defined to be an assignment of circles and crosses to the lattice points. They study the question of realizability of patterns, i.e. given a pattern P_d does there exist a point u in the lattice such that a point $u + x$ is visible (respectively, nonvisible) whenever x is a point of P_d marked with a circle (respectively, cross). In fact they show that a pattern P_d is realizable if and only if for any prime p the set $\{(x_1 \bmod p, \ldots, x_d \bmod p) : (x_1, \ldots, x_d) \in C\}$ is not a complete set of representatives modulo p on d-tuples of integers, where C is the set of circles of P_d.

H. Rumsey [Rum66] studies the density of the set $V(S) = \bigcup_n V_n(S)$, for S an arbitrary subset of L^d. Call two points x, y of the lattice p-equivalent if and only if $x \equiv y \bmod p$. Let $[x]_p$ be the equivalence class of a point x and let S/p be the set of equivalence classes $[x]_p$ with $x \in S$. Then Rumsey shows (generalizing the above mentioned theorem of Dirichlet) that for any finite set S of lattice points the density of the set $V(S)$ is given by the infinite product

$$\prod_{p \in \mathcal{P}} \left(1 - \frac{|S/p|}{p^d}\right),$$

where $\mathcal{P}$ is the set of prime numbers. In fact Rumsey gives a characterisation of the sets S for which the above formula is true. It should also be mentioned that the above formula for the density of $V(S)$ was previously obtained by Rearick [Rea60] for $|S| = 2$ and when the points of S are pairwise visible.

H. I. Abbott [Abb74] considers the problem of determining the minimum number $f(n)$ of cameras which are necessary in order to see all the points of the 2-dimensional lattice L_n^2, i.e. $f(n)$ = minimum s such that for some set S of s lattice points $V_n(S) = L_n^2$. He shows that

$$\frac{\ln n}{2 \ln \ln n} < f(n) < 4 \ln n.$$

The lower bound result follows easily by applying the Chinese remainder theorem. For the upper bound Abbott constructs recursively a sequence $x_1, x_2, \ldots, x_k$ such that for each i, x_{i+1} is a point x in the lattice L_n^2 for which the set-theoretic difference

$$V_n(x) - (V_n(x_1) \cup \cdots V_n(x_i))$$

is of maximal size and shows that $k = O(\ln n)$ iterations of this procedure suffice in order to cover all the vertices of the lattice. His method however gives no indication on how to locate "quickly" these points on the lattice. Nevertheless, he also shows using work of Erdös [Erd62] that there exists a constant $\alpha > 0$ such that for n sufficiently large every point of the lattice L_n^2 is visible from the set $\{(1,0)\} \cup \{(0,j) : j = 0, 1, \ldots, k\}$, where $k = O(\ln^\alpha n)$. However, this last configuration is far from optimal, as we will show later. It is straightforward to see that his methods extend easily in order to yield similar results for the d-dimensional lattice L_n^d.

In the present paper we are concerned with a slightly different problem; the camera placement problem in multidimensional lattices. We are given s cameras $C_1, \ldots, C_s$ which are supposed to be located on the nodes of the d-dimensional lattice L_n^d. We are interested in determining a set $S = \{A_1, \ldots, A_s\}$ of positions (lattice points) for these cameras in such a way that if camera C_i is positioned at location A_i, for $i = 1, \ldots, s$, then the number of lattice points visible by at least one of the cameras is maximized, i.e. under what conditions on the set S of possible camera locations is the quantity $|U_n(S)|$ maximized?

It is easy to see (using the above mentioned theorem of Dirichlet) that in the case of a single camera and any location A, $|V_n(A)| = |U_n(A)|$ is asymptotically equal to $\frac{n^d}{\zeta(d)}$. Moreover, it can be shown that the set of lattice points visible from a fixed location A contains arbitrarily large cubic gaps [Apo76, theorem 5.29], [Rad64], [HS71], i.e. for any integer $k > 0$ there exists a lattice point $P = (p_1, \ldots, p_d)$ such that none of the points in the cube $\{P + x : 1 \leq x_i \leq k\}$ is visible from A. This immediately raises the question of where to locate an additional camera in order to maximize visibility. If $s = 2$ then it is still not hard to show using the principle of inclusion/exclusion that the optimal visibility for two cameras is achieved when the two cameras are pairwise visible.

The second part of the paper begins with an extension of Rumsey's work on the density of visibility sets which is suitable to our analysis of the camera placement problem. We study the general case of this problem both for finite (using sieve methods which enable us to count the number of points of a set not belonging to certain prescribed subsets) and infinite (using probabilistic methods) lattices. We give a necessary condition for an arbitrary set S of s cameras to be in optimal configuration, namely that for every prime p with $s \leq p^d$ the cameras are pairwise p-visible. This implies that for any $s \leq 2^d$, the number of points visible from s cameras is maximized exactly when the camera positions are pairwise visible. Thus although the above cited theorem of Abbott implies that for n large enough (actually, $2^d \leq \ln n / 2 \ln \ln n$) it is impossible to see all the points of L_n^d with only 2^d cameras, the optimal configuration of 2^d cameras is achieved exactly when the cameras are pairwise visible. For example, as an immediate consequence of our results, straightforward calculations (using values of the Riemann zeta function) show that with four cameras in "pairwise visible" (which is also the optimal) configuration one can see (asymptotically in n) about 99,86 percent of the points of L_n^2. In addition we show that the optimal configuration for $s \leq 3^d$ cameras is obtained exactly when the cameras are p-visible for all $p > 2$ and each equivalence class $x \in S/2$ has either $\lfloor |x|/2^d \rfloor$ or $\lceil |x|/2^d \rceil$ elements. Detailed proofs of all these results will be given in the full version of the paper [KP90].

2 Enumeration Problems

The enumeration problems considered in this section will turn out to be consequences of enumeration principles regarding the number of lattice points inside a convex compact set. Subsection 2.1 includes our general enumeration theorems while subsection 2.2 gives the following applications: (1) enumerating the number of different lines each traversing k vertices of the d-dimensional lattice L_n^d, (2) computing the expected length and standard deviation of maximal (or other kinds of) segments of d-dimensional lattices, simplexes, etc., (3) enumerating the maximum number of incidences between m points and n lines in the plane, and (4) analysing the edge visibility region.

2.1 General Results

In this subsection we abbreviate by L the complete lattice of d-tuples of non-negative integers ($d \geq 2$). Let Δ be a convex compact subset of $\mathbb{R}^d$ and let f be a real function on Δ. Let $G(\Delta)$ be the set of lattice points $x = (x_1, \ldots, x_d)$ in Δ such that $\gcd(x_1, \ldots, x_d) = 1$. We would like to find an estimate on the sum $\sum_{P \in G(\Delta)} f(P)$. We prove the following two theorems which can be useful in many lattice enumeration problems.

Theorem 2.1 *Let Δ be a convex compact subset of $\mathbb{R}^d$. Let f be a real positive continuous function on Δ which is monotone in all its arguments. Then we have that*

$$\left| \sum_{P \in G(\Delta)} f(P) - \frac{1}{\zeta(d)} \cdot \int_\Delta f \right| =$$

$$O\left(\max f \cdot \begin{cases} \delta \log \delta & \text{if } d = 2 \\ \delta^{d-1} & \text{otherwise} \end{cases} \right)$$

where δ is the diameter of Δ.

Theorem 2.2 *Let Δ be a convex compact subset of $\mathbb{R}^d$. Let f be a real positive function on Δ which satisfies the Lipschitz condition*

$$A = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty.$$

Then we have that

$$\left| \sum_{P \in G(\Delta)} f(P) - \frac{1}{\zeta(d)} \cdot \int_\Delta f \right| =$$

$$O\left((\delta \cdot A + \max f) \cdot \begin{cases} \delta \log \delta & \text{if } d = 2 \\ \delta^{d-1} & \text{otherwise} \end{cases} \right)$$

where δ is the diameter of Δ.

We first prove a lemma.

Lemma 2.1 *Under the assumptions of theorem 2.1 we have that*

$$\left| \int_\Delta f - \sum_{P \in \Delta_1} f(P) \right| = O(\delta^{d-1} \cdot \max f)$$

where $\Delta_1 = \Delta \cap L$.

Proof. (Outline) First it will be necessary to extend f on $\mathbb{R}^d$. We may assume, without loss of generality, that f is non-decreasing. Extend f on $\mathbb{R}^d$ by setting $f(x) := \inf\{f(y) : y \in \Delta, x \leq y\}$ with the convention $\inf \emptyset = \sup_\Delta f$. It is not hard to prove that the extension is still positive, non decreasing, upper semicontinuous and that $\sup_{\mathbb{R}^d} f = \sup_\Delta f$.

The proof given here is a generalization of the proof of the main result in [Nos48]. Let S be the square with corners the 2^d points $(x_1, \ldots, x_d)$ where $x_i = 1, 0$. For each lattice point P let S_P^+ be the square $P + S$ and S_P^- the square $P - S$. Put $\overline{\Delta} = \{P : d(P, \partial\Delta) \leq \sqrt{d}\}$ and let $\Delta^+ = \Delta \cup \overline{\Delta}, \Delta^- = \Delta - \overline{\Delta}$. By hypothesis, f is non-decreasing and positive; hence we have

$$\left| \int_\Delta f - \sum_{P \in \Delta_1} f(P) \right| \leq \int_{\Delta^+} f - \int_{\Delta^-} f \leq \int_{\overline{\Delta}} f.$$

Moreover we have that

$$\int_{\overline{\Delta}} f \leq \mathrm{area}(\overline{\Delta}) \cdot \max_{\overline{\Delta}} f \leq \mathrm{area}(\overline{\Delta}) \cdot \max_{\Delta} f.$$

Next we prove that $area(\overline{\Delta}) = O(\delta^{d-1})$. Indeed since Δ is convex the area of $\overline{\Delta}$ is less than 2 times the area of $\overline{\Delta} \setminus \Delta$. According to the Steiner-Minkowski formula [BZ88, page 141] , the area of this last set can be expressed in the form

$$area(\overline{\Delta} \setminus \Delta) = \sum_{i=1}^{d} \ell_i(\Delta) \cdot d^{\frac{i}{2}}. \tag{1}$$

where the functions $\ell_i(.)$ are bounded over the set of convex subsets of the unit ball and satisfy the identities $\ell_i(k\Delta) = k^{d-i} \ell_i(\Delta)$. Hence we can write (assuming, without loss of generality, that $0 \in \Delta$.)

$$area(\overline{\Delta} \setminus \Delta) = \sum_{i=1}^{d} \delta^{d-i} \cdot \ell_i(\frac{1}{\delta} \cdot \Delta) \cdot d^{\frac{i}{2}} = O(\delta^{d-1}),$$

which completes the proof of our lemma. □

Proof of theorem 2.1. (Outline) Let Δ_k the set of lattice points of Δ whose coordinates are divisible by k. By using the observations

$$G(\Delta) = \Delta_1 \setminus \bigcup_p \Delta_p$$

and

$$\gcd(k,k') = 1 \Longrightarrow \Delta_k \cap \Delta_{k'} = \Delta_{k\cdot k'},$$

where p ranges over primes, and a standard sieve argument (see, for example, [Nar83]) we can show that

$$\sum_{P\in G(\Delta)} f(P) = \sum_{k\geq 1} \mu(k) \cdot \sum_{P\in\Delta_k} f(P)$$

where μ is the Möbius function. Now we use the previous lemma in order to estimate the sum $\sum_{P\in\Delta_k} f(P)$ above. Let $h_k(P) = k \cdot P$. Then using the fact that $\Delta_k = k(\frac{1}{k}\Delta \cap L)$, for $k \geq 1$, we obtain

$$\sum_{P\in\Delta_k} f(P) = \sum_{P\in\frac{1}{k}\Delta\cap L} f\circ h_k(P)$$

Hence it follows from the previous lemma that

$$\left| \sum_{P\in\frac{1}{k}\Delta\cap L} f\circ h_k(P) - \int_{\frac{1}{k}\Delta} f\circ h_k \right| =$$

$$O\left((\frac{\delta}{k})^{d-1} \cdot \max_{\frac{1}{k}\Delta} f\circ h_k \right)$$

Trivial calculations show that

$$\max_{\frac{1}{k}\Delta} f\circ h_k = \max_{\Delta} f,$$

$$\int_{\frac{1}{k}\Delta} f\circ h_k = \frac{1}{k^d}\cdot \int_{\Delta} f$$

It follows easily by summing over k that

$$\left| \sum_{P\in G(\Delta)} f(P) - \sum_{k\leq\delta} \frac{\mu(k)}{k^d} \cdot \int_{\Delta} f \right| =$$

$$O\left(\sum_{k\leq\delta} (\frac{\delta}{k})^{d-1} \cdot \max_{\Delta} f \right).$$

The right-hand side is readily simplified to

$$O\left(\max_{\Delta} f \cdot \begin{cases} \delta\log\delta & \text{if } d=2 \\ \delta^{d-1} & \text{otherwise} \end{cases} \right)$$

Using the well-known identities

$$\sum_{k\geq 1} \frac{\mu(k)}{k^d} = \frac{1}{\zeta(d)} \text{ and } \left| \sum_{k>\delta} \frac{\mu(k)}{k^d} \right| = O\left(\frac{1}{\delta^{d-1}}\right),$$

e.g. see [Knu81, exercise 10, section 4.5.2], and $area(\Delta) = O(\delta^d)$ the proof of the theorem can be completed without difficulty. □

The proof of theorem 2.2 is similar. We will make use of theorems 2.1 and 2.2 for functions of polynomial type on a convex domain. However it is worth mentioning that our results extend to (not necessarily convex) rectifiable domains in the plane $\mathbb{R}^2$. In that case the error term that appears in theorems 2.1 and 2.2 is expressed as a function of the area and the length of the domain instead of its diameter [KP90].

2.2 Applications

The enumeration principles proved in the previous section can be applied to many problems in combinatorial and computational geometry.

2.2.1 Computing the Number of Lines

As a first application of theorem 2.1 we enumerate the number of different lines traversing at least $k+1$ lattice points of the d-dimensional cube, the d-dimensional simplex of size n or a product of simplexes of lower dimension. We formalize this as follows. Let $\mathcal{J}$ be a partition of $\{1,\ldots,d\}$ and let n be a function of $\mathcal{J}$ into $\mathbb{Z}$. Set

$$\mathcal{D}(n) = \left\{ x : 0 \leq \sum_{i\in I} x_i < n_I,\ \forall I \in \mathcal{J} \right\},$$

where $x = (x_1,\ldots,x_d)$ runs over d-tuples of integers. Clearly, with a suitable choice of the function n both the d-dimensional cubes and simplexes can be defined as above. Let $\delta(n,k)$ be the number of different lines of positive slope each traversing at least $k+1$ lattice points of the domain $\mathcal{D}(n)$. The following theorem gives an asymptotic evaluation of $\delta(n,k)$.

Theorem 2.3 *Let $\mathcal{J}$ be a partition of $\{1,\ldots,d\}$ and let n be a function of $\mathcal{J}$ into $\mathbb{Z}$. The number $\delta(n,k)$ of different straight lines of positive slope each traversing at least $k+1$ different lattice points of the domain $\mathcal{D}(n)$ is given by the formula*

$$\frac{1}{\zeta(d)} \cdot \prod_{I\in\mathcal{J}} \frac{n_I^{2\cdot|I|}}{(2\cdot|I|)!} \cdot \left\{ \frac{1}{k^d} - \frac{1}{(k+1)^d} \right\}$$

$$+O\left(\begin{cases} \frac{|n|^3}{k^2}\log\frac{|n|}{k} & \textit{if } d=2 \\ \frac{|n|^{2d-1}}{k^d} & \textit{otherwise} \end{cases} \right)$$

where $|n| = sup_{\mathcal{J}} n$.

Proof. (Outline) Let $p = (p_1,\ldots,p_d)$ be a given slope such that $\gcd(p) = 1$ and let $S(p,k)$ be the set of lines each traversing at least $k+1$ different lattice points of $\mathcal{D}(n)$. It is then clear that

$$\delta(n,k) = \sum_{\gcd(p)=1} g_k(p,n), \qquad (2)$$

where $g_k(p,n)$ is the cardinal of the set $S(p,k)$. Therefore we expect that the theorem will follow from the above identity and theorem 2.1 or 2.2. Indeed we can show that $g_k(p,n)$ is a polynomial expression of the coordinates $p_1,\ldots,p_d$ of p and that $\max_{p\in\mathbb{R}^d} g_k(p,n) = O(\frac{|n|^d}{k})$ and $\max_{p\in\mathbb{R}^d} \frac{\partial g_k(p,n)}{\partial p_i} = O(|n|^{d-1})$. □

Motivated from equation (2) we can prove a stronger "weighted" version of the previous theorem by enumerating the lines with some weight. This we have used in our study of the expected length and standard deviation of maximal segments in the lattice L_n^d. More precisely we have the following theorem.

Theorem 2.4 *Let h be a real positive homogeneous function of degree* $a \geq 1$ *which is* C^1 *on* $(I\!R_+^d)^*$ *and let* $S(p,k)$ *be the set of lines of positive slope* $p = (p_1, \ldots, p_d)$ *each traversing at least* $k+1$ *different lattice points of the d-dimensional grid of size n. The number*

$$\delta(h,n,k) = \sum_{\gcd(p)=1} h(p) \cdot |S(p,k)|$$

is given by the formula

$$\frac{n^{a+2d}}{\zeta(d)} \cdot \left(\frac{1}{k^{a+d}} - \frac{1}{(k+1)^{a+d}}\right) \cdot \omega(h)$$

$$+O\left(\begin{cases} \frac{n^{a+3}}{k^{a+2}} \log \frac{n}{k} & \textit{if } d=2 \\ \frac{n^{a+2d-1}}{k^{a+d}} & \textit{otherwise} \end{cases}\right)$$

where

$$\omega(h) = \int_{[0,1]^d} \prod_i (1-x_i) h(x_1, \cdots, x_d) dx_1 \ldots dx_d.$$

Proof. **(Outline)** Elementary calculus shows that the function $h(p) \cdot |S(p,k)|$ is $O(\frac{n^{a+d-1}}{k^a})$-Lipschitz on the d dimensional grid. Then the result follows by application of theorem 2.2. □

For example the expected length and standard deviation of maximal segments in the two dimensional lattice of size n are respectively $(0.695\cdots) \cdot n$ and $(0.185\cdots) \cdot n$ (see [KP90] for more details).

2.2.2 Analysis of the Incidence Problem

We conclude this section by an application of our theorem 2.2 to the computation of constants occurring in lower bounds of two combinatorial problems arising in Computational Geometry. The first problem is *the incidence problem* in arrangement of lines as defined in [ST83],[Ede87, chapter 6] or [CEG*88].

In [ST83] it is shown that the maximum number of incidences, $I(m,n)$, between m points and n lines in the plane is $\Theta(m^{2/3}n^{2/3} + m + n)$, moreover we can read in [CEG*88, page 13] that

$$I(m,n) \leq 3\sqrt[3]{6}m^{2/3}n^{2/3} + 25n + 2m.$$

Here we prove the following result.

Theorem 2.5 *If*

$$m = o(n^2) \textit{ and } n \leq \frac{3}{16\zeta(2)}m^2(1+o(1))$$

then for all $\epsilon > 0$ *we have for m and n sufficiently large* $I(m,n) \geq \left\{\sqrt[3]{12/\pi^2} - \epsilon\right\} m^{2/3}n^{2/3}$.

Proof. **(Outline)** The lower bound example of [Ede87, chapter 6] is based on arranging the points in a square grid and choosing the lines close to highly populated rows of points. We follow this example and apply the theorem 2.1 to make precise computations. Let ℓ be a line of the grid of size $p\,(\sim \sqrt{m})$. We denote by $contr(\ell)$ the number of points of the grid that lie on ℓ. Let L be the set of lines of slope (positive and negative) $\leq \alpha p$ of the grid and n_α the number of such lines. The real α is to be later determinated so that $n \sim n_\alpha$. We put $contr(L) = \sum_{\ell \in L} contr(\ell)$. Using theorem 2.1 we get

$$n_\alpha = \frac{2}{\zeta(2)}p^4 \cdot f_1(\alpha) + O(\alpha^2 p^3 \log \alpha p)$$

and

$$contr(L) = \frac{1}{\zeta(2)}p^4 \cdot f_2(\alpha) + O(\alpha p^3 \log \alpha p)$$

where $f_1(\alpha)$ and $f_2(\alpha)$ are polynomial expressions in α.

Combining the two previous equations and the fact that $contr(L)$ is a lower bound for $I(p^2, n_\alpha)$ we can show that

$$\liminf_{m,n\to\infty} \frac{I(m,n)}{n^{2/3}m^{2/3}} \geq (\zeta(2)/2)^{-1/3}. \square$$

2.2.3 Analysis of the Edge Visibility Region

The second problem we want to analyse is the *Edge visibility region* as defined in [ORo87, pages 219-223]. The problem is to compute the region of the plane illuminated by a line segment in the presence of other line segments. Suri and O'Rourke [SO86] establish an $\Omega(n^4)$ worst-case lower bound for constructing this region where n is the number of segments. Their analysis is based on the evaluation of the number $N(n)$ of distinct intersections lying in the half-plane $y > 2$ between lines passing through points $(1,i)$ and $(2,j)$ for $0 \leq i,j < n$. In [ORo87,SO86] it is shown that a lower bound for this number is the sum

$$\sum_{a \leq b \leq n, \gcd(a,b)=1} \min(b, n-b) \cdot (n-b),$$

which is then evaluated as an $\Omega(n^4)$ using the result of Dirichlet. We show now how to compute an equivalent of $N(n)$.

Lemma 2.2 *The number* $2 \cdot N(n)$ *is exactly the number of different lines of positive slope of the 2-dimensional grid of size n.*

Proof. **(Outline)** Use the duality which maps the line passing through the points $(1,x)$ and $(2,y)$ on the point (x,y). It is not hard to show that by duality concurrent lines are transformed into points on a line. □

Using the above lemma and theorem 2.3 we get

$$N(n) \sim \frac{1}{\zeta(2)}\frac{3}{32} \cdot n^4.$$

3 Visibility Problems

In this section we concentrate on the study of an art gallery question.

3.1 Camera Placement Problem

An interesting (and in general still open) art gallery problem was posed by Moser [Mos85] in 1966: given a set P of points in the plane how many guards located at points of P are needed to see the unguarded points of P? The special case of this problem where the points of P are located on the vertices of the integer lattice L_n^2 has been studied by Abbott [Abb74]. In this section we will be concerned with a related but different art gallery question for point obstacles: the camera placement problem on integer lattices. Namely, where on the infinite lattice L^d does one position a set of s cameras in order to maximize their visibility? A naive search over all possible n^d lattice positions of L_n^d is impractical since it would require about

$$\binom{n^d}{s} = O(n^{ds})$$

searches in order to check and verify all possible configurations for the s cameras.

Before proceeding any further it will be necessary to define more rigorously what we mean by optimal configuration of a set of cameras. Our analysis will be based on a theorem of Rumsey [Rum66] regarding the ratio of the set $V_n(S)$ of points of the lattice L_n^d which are visible from all the points of S simultaneously, namely

$$\lim_{n\to\infty} \frac{|V_n(S)|}{n^d} = \prod_{p\in\mathcal{P}} \left(1 - \frac{|S/p|}{p^d}\right). \tag{3}$$

The above quantity is denoted by $d_{\mathcal{P}}(S)$. It follows easily from the principle of inclusion/exclusion that the limit of the ratio of the set $U_n(S)$ of points of the lattice L_n^d which are visible from at least one point of S is given by the formula

$$\lim_{n\to\infty} \frac{|U_n(S)|}{n^d} = \sum_{E\subseteq S, E\neq\emptyset} (-1)^{|E|-1} d_{\mathcal{P}}(E). \tag{4}$$

We call the above quantity the density of the configuration S and denote it by $u(S)$. A configuration S consisting of s points is called optimal if for any other s-point configuration S' the density of S exceeds the density of S'.

Now we can determine what is the optimal configuration for a single point. Equation (3) shows that the density of the set of lattice points visible from a single camera is always $1/\zeta(2)$ regardless of the position of the camera. For two points it is not difficult to see that by combining equations (3), (4) we can conclude that the visibility is maximized exactly when the cameras are pairwise visible. For $s > 2$ equation (4) becomes rather unmanageable. To proceed any further it will be necessary to make a thorough analysis of the relative position and distribution of the points of the given configuration.

3.1.1 Admissible Systems

In the sequel we give several basic definitions and establish notation that will be essential in our subsequent study. Let $\mathcal{P} = \{2, 3, 5\cdots\}$ be the set of prime numbers, p ranges over the set of primes and $\mathcal{Q}$ over subsets of $\mathcal{P}$. Two lattice points A and B are p-visible if p is not a divisor of $\gcd(A-B)$; A and B are $\mathcal{Q}$-visible if for all $p \in \mathcal{Q}$, p is not a divisor of $\gcd(A-B)$. In particular two points A, B which are $\mathcal{P}$-visible are visible in the geometric sense, i.e. the line segment joining A and B avoids all the lattice points but A, B.

For S a set of lattice points we use the following notations

- $V_{\mathcal{Q}}(S)$ the set of points which are $\mathcal{Q}$-visible from each point of S
- $d_{\mathcal{Q}}(S)$ the density (if it exists!) of the corresponding set $V_{\mathcal{Q}}(S)$.

Now the above mentioned result of Rumsey can be stated as follows.

Theorem 3.1 ([Rum66]) *If S is a finite set of points then the set $V_{\mathcal{P}}(S)$ has a density given by*

$$d_{\mathcal{P}}(S) = \prod_{p\in\mathcal{P}} \left(1 - \frac{|S/p|}{p^d}\right). \square$$

We see then that $d_{\mathcal{P}}(S)$ depends only on the $\gcd(A-B)$, where A and B run over elements of the set S. Clearly, theorem 3.1 gives the density of the set of points X such that

$$X \not\equiv A \bmod p,\ \forall p \in \mathcal{P},\ \forall A \in S.$$

It is a particular case of the following problem.

Problem 3.1 *Given a finite set S of lattice points and for every point A of S a square-free natural number g_A, what is the density of the set of points X such that*

$$X \equiv A \bmod p \Longleftrightarrow p \mid g_A\ ? \tag{5}$$

Theorem 3.2 *The system (5) has a solution if and only if the following two conditions are satisfied for any prime p,*

- *coherence condition:* $p \mid g_A \Longrightarrow (p \mid g_B \Longleftrightarrow p \mid \gcd(A-B))$
- *maximality condition:* $|\{A \in S : p \nmid g_A\}/p| < p^d$

Moreover this set of solutions has a density given by

$$\frac{1}{(\mathrm{lcm}\{g_A : A \in S\})^d} \cdot \prod_{p \in \mathcal{Q}} \left(1 - \frac{|S/p|}{p^d}\right),$$

where $\mathcal{Q}$ is the set of primes relatively prime to the lcm *of the g_A's.*

Proof. (Outline) If the system has a solution then the coherence and maximality conditions are easily verified. Let Ω be the set of solutions of equation (5) and let G be the set of points X satisfying the congruences $X \equiv A \bmod g_A$, where $A \in S$. Clearly we have $\Omega \subseteq V_{\mathcal{Q}}(S) \cap G$. Now use the coherence and maximality conditions to show that in fact equality holds $\Omega = V_{\mathcal{Q}}(S) \cap G$. Finally use the work of Rumsey on the density of periodic and visibility sets to obtain the result concerning the density of the above mentioned set. This proves the desired result. □

In our subsequent study we will be mainly concerned with the following extension of the previous problem concerning the realizability of families $g_{i,j}$ of integers by lattice points A_i.

Problem 3.2 *Solve in $A_i, 1 \leq i \leq s$ the system*

$$A_i \equiv A_j \bmod p \Longleftrightarrow p | g_{i,j},$$

where the $g_{i,j}$ are given and satisfy $g_{i,j} = g_{j,i}$ and $g_{i,i} = 0$, for $1 \leq i, j \leq s$.

Theorem 3.3 *The problem 3.2 has a solution if and only if the following two conditions are satisfied for any prime p,*

- *coherence condition:*
  $p \mid g_{i,j}, g_{j,k} \Longrightarrow p \mid g_{i,k}$
- *maximality condition:*
  $|\{1, \ldots, s\}/p| \leq p^d$,

where $\{1, \ldots, s\}/p$ is the quotient space of $\{1, \ldots, s\}$ by the relation $i \sim j$ iff $p \mid g_{i,j}$.

Proof. See [KP90]. □

Now we have developed the necessary machinery to proceed with our study of the optimal placement of a set of cameras. In the sequel we will study the following problem.

Problem 3.3 *Given s, maximize $u(S)$, under the condition $|S| = s$.*

Let S be a configuration of points of the lattice L_n^d. We know that the set of points which are visible from at least one point of S has a density given by

$$u(S) \quad = \sum_{E \subset S, |E| \geq 1} (-1)^{|E|-1} d_{\mathcal{P}}(E).$$

Moreover we know that $u(S)$ depends only on the prime factors of $\gcd(A_i - A_j)$, for $A_i, A_j \in S$. This leads us to defining $g_{i,j}$ as the product of the prime factors of the $\gcd(A_i - A_j)$'s and let g be the family of the $g_{i,j}$'s. Moreover we define $u(g) := u(S)$ where

$$u(S) = \sum_{\substack{E \subseteq \{1, \cdots, s\} \\ |E| \geq 1}} (-1)^{|E|-1} \prod_{p \in \mathcal{P}} \left(1 - \frac{|E/g(p)|}{p^d}\right)$$

and $E/g(p)$ is the quotient space of E by the relation $i \sim j$ if and only if $p \mid g_{i,j}$.

The previous considerations have made it clear how, given a family $g = (g_{i,j})_{1 \leq i < j \leq s}$ of square free integers which satisfies the **coherence** and **maximality conditions** 3.3, to construct a set S of s points such that $u(S) = u(g)$. Let us call **admissible system** (of size s) such a family of $g_{i,j}$'s. In the rest of this section we will concentrate on the solution of the following problem.

Problem 3.4 *Maximise $u(g)$ over the set of admissible systems g of a given size s.*

3.1.2 Optimal Placement of Cameras

In the sequel we will use of the following notation:

- $u_g(\mathcal{Q}, S_1)$ is the density of the set of points which are $\mathcal{Q}$-visible from at least one point of S_1, for the system g.
- $u_g(\mathcal{Q}, \overline{T_1})$ is the density of the set of points which are not $\mathcal{Q}$-visible from each point of T_1, for the system g.
- $u_g(\mathcal{Q}, S_1 \text{ and/or } S_2 \cdots \text{and/or } \overline{T_1} \text{ and/or } \overline{T_2} \cdots)$ is the density of the set of points which are $\mathcal{Q}$-visible from at least one point of S_1 and/or $S_2 \cdots$ and/or not $\mathcal{Q}$-visible from each point of T_1 and/or $T_2 \cdots$, for the system g.

where S_i and T_i are subsets of $\{1, \ldots, s\}$. In particular we have $u(g) = u_g(\mathcal{P}, \{1, \ldots, s\})$. Our first lemma also provides an algorithm for relocating the given set of cameras in order to improve their visibility.

Lemma 3.1 *If g and h are two admissible systems of size s then we have*

$$(\forall 1 \leq i, j \leq s, g_{i,j} \mid h_{i,j}) \Longrightarrow u(g) \geq u(h),$$

with equality if and only if $\forall i, j \; g_{i,j} = h_{i,j}$.

Proof. (Outline) Put $S = \{1, \ldots, s\}$. In the sequel we use the notation

$$d_{\mathcal{Q}}(E, g) = \prod_{p \in \mathcal{Q}} \left(1 - \frac{|E/g(p)|}{p^d}\right).$$

The main idea of the proof is to construct a sequence

$$h^{(0)} := h, \ldots, h^{(i)}, \ldots h^{(k)} = g$$

of admissible families each of size s. The family $h^{(i+1)}$ is obtained from the family $h^{(i)}$ by dividing an equivalence class in $h^{(i)}$ by an appropriate prime number (as indicated in the sequel). Since the resulting sequence of admissible families satisfies $u(h^{(i)}) < u(h^{(i+1)})$ the proof of the theorem will be complete.

In the sequel we indicate how to resolve the induction step. This amounts to treating the special case where for some prime $p_0 \in \mathcal{P}$ and some index i_0 we have that $g_{i,j} = h_{i,j} \, \forall \, i, j$, except that

$$g_{i_0,j} = \frac{h_{i_0,j}}{p} \, \forall j \in S' := \{j : p_0 \mid h_{i_0,j}\}.$$

Let Ω be the domain $\{E \subseteq S : i_0 \in E, S' \cap E \neq \emptyset\}$ and let $S'' = S \setminus (S' \cup \{i_0\})$. Straightforward arguments on the number of equivalence classes of the sets concerned show that

- $\forall E \subseteq S, \forall p \neq p_0, |E/g(p)| = |E/h(p)|$,
- $\forall E \in \Omega, |E/g(p_0)| = |E/h(p_0)| + 1$,
- $\forall E \in \overline{\Omega}, |E/g(p_0)| = |E/h(p_0)|$.

Using the above properties we obtain

$$u(g) - u(h) =$$

$$\sum_E (-1)^{|E|-1} \{d_{\mathcal{P}}(g, E) - d_{\mathcal{P}}(h, E)\} =$$

$$\sum_\Omega (-1)^{|E|-1} d_{\mathcal{P} \setminus p_0}(g, E) \cdot \{d_{p_0}(g, E) - d_{p_0}(h, E)\} =$$

$$\sum_\Omega (-1)^{|E|-1} d_{\mathcal{P} \setminus p_0}(g, E) \cdot \frac{1}{p_0^d} =$$

$$\frac{1}{p_0^d} \cdot u_g(\mathcal{P} \setminus p_0, i_0 \text{ and } S' \text{ and } \overline{S''}) > 0.$$

The difference $u(h) - u(g)$ is clearly positive because up to a constant positive factor it appears as the density of the set of points which are $\mathcal{P} \setminus p_0$-visible from i_0 and from at least one point of S' and not $\mathcal{P} \setminus p_0$-visible from each point of S''. This completes the proof of the induction step, and hence also the proof of the lemma. □

As a consequence of the lemma we obtain the following rather surprising fact: if S is an optimal configuration then the number $|S/p|$ of equivalence classes of S modulo p depends only on $|S|$ and the prime p and is otherwise independent of the chosen configuration. More formally we have the following theorem.

Theorem 3.4 *If S is an optimal configuration then*

$$\forall p \in \mathcal{P}, \; |S/p| = \min(|S|, p^d). \tag{6}$$

Proof. (Outline) First we prove the necessity of (6). $|S/p| \leq p^d$ is obvious since there can exist at most p^d different d-tuples modulo p. This implies that $|S/p| \leq \min(|S|, p^d)$. Let $s = |S|$. If $s \leq p^d$ then identity (6) follows easily from the previous lemma. So let us assume that $s > p^d$. We need to show that $|S/p| = p^d$. Assume on the contrary that $|S/p| < p^d$. Assume that the d-tuple $(t_1, \ldots, t_d)$ is a representative of a missing equivalence class and let A, B be two different lattice points of S such that $p \mid \gcd(A - B)$. Use the Chinese remainder theorem to replace A with a new point A' satisfying $A' \equiv (t_1, \ldots, t_d) \bmod p$ and for primes $q \neq p$, $A' \equiv A \bmod q$. Let $S' = (S - \{A\} \cup \{A'\}$. Using the previous lemma it is easy to show that $u(S') > u(S)$, contradicting the optimality of S. □

It is now possible to prove the optimality condition for $\leq 2^d$ cameras.

Theorem 3.5 *A configuration S of $\leq 2^d$ lattice points is optimal if and only if it consists of pairwise visible points.*

Proof. (Outline) Use theorem 3.4. □

For $s \leq 3^d$ cameras we have the following theorem.

Theorem 3.6 *A configuration S of $\leq 3^d$ points is optimal if and only if the following two conditions are satisfied*

$$\forall p \in \mathcal{P}, \; |S/p| = \min(|S|, p^d)$$

and

$$\forall x \in S/2, \; |x| = \left\lfloor \frac{|S|}{2^d} \right\rfloor \text{ or } |x| = \left\lceil \frac{|S|}{2^d} \right\rceil$$

Proof. (Outline) Let $|S| = s$. First we prove that the conditions are necessary. We have seen in theorem 3.4 that the first condition is necessary. So without loss of generality we may assume that the first condition is realized. In that case it is easily seen that the second condition is equivalent to

$$\forall x, y \in S/2 \; ||x| - |y|| \leq 1.$$

Let $d_1 = c_1 \cup \{i\}$ and c_2 be two equivalent classes of $S/2$ where i is a distinguished element of d_1. Assume on the contrary that $|d_1| > |c_2| + 1$. A contradiction will be obtained if we can show that the configuration obtained by removing i from d_1 and adding it to c_2 is a better one. Let S' be the configuration obtained from S by deleting i drom d_1 and adding it to c_2. That this can be done follows easily from the Chinese remainder theorem. Let ϕ be an injection from c_2 to c_1 and let $c_1' = c_1 \setminus \phi(c_2)$. Then we obtain easily that for all $E \subseteq S$ such that i is not an element of E,

- if $E \cap c_1 \neq \emptyset, E \cap c_2 = \emptyset$ then $|E \cup \{i\}/p|_{S'} = 1 + |E \cup \{i\}/p|_S$,

- if $E \cap c_1 \neq \emptyset, E \cap c_2 \neq \emptyset$ then $|E \cup \{i\}/p|_{S'} = |E \cup \{i\}/p|_S$,
- if $E \cap c_1 = \emptyset, E \cap c_2 = \emptyset$ then $|E \cup \{i\}/p|_{S'} = |E \cup \{i\}/p|_S$,
- if $E \cap c_1 = \emptyset, E \cap c_2 \neq \emptyset$ then $|E \cup \{i\}/p|_{S'} = -1 + |E \cup \{i\}/p|_S$,

where $|E \cup \{i\}/p|_S$ and $|E \cup \{i\}/p|_{S'}$ denote the number of equivalence classes of $E \cup \{i\}$ modulo p, in the configurations S, S', respectively. Using these properties we obtain easily that

$$u(S') - u(S) =$$

$$\sum_{\substack{E \subseteq S \\ E \cap c_1 = \emptyset \\ E \cap c_2 \neq \emptyset}} (-1)^{|E|} d_{\mathcal{P} \setminus 2}(E \cup \{i\}) \cdot \frac{1}{2^d}$$

$$- \sum_{\substack{E \subseteq S \\ E \cap c_2 = \emptyset \\ P \cap c_1 \neq \emptyset}} (-1)^{|E|} d_{\mathcal{P} \setminus 2}(E \cup \{i\}) \cdot \frac{1}{2^d} =$$

$$\frac{1}{2^d} \cdot \sum_{\substack{E \subseteq S \\ E \cap c_2, E \cap \phi(c_2) = \emptyset \\ E \cap c_1 \setminus \phi(c_2) \neq \emptyset}} (-1)^{|E|} d_{\mathcal{P} \setminus 2}(E \cup \{i\}) =$$

$$\frac{1}{2^d} \cdot u_{g(S)}(\mathcal{P} \setminus 2, i \text{ and } c_1' \text{ and } \overline{S \setminus c_2 \cup \phi(c_2)}) > 0,$$

which proves the necessity of the second condition.

Next we prove the sufficiency of the two conditions. For this it suffices to show that any two configurations S, S' of the same size both satisfying the two conditions have the same visibility. But it is clear that $S/2, S'/2$ have the same number of equivalence classes of each type $\lfloor |S|/2^d \rfloor$, $\lceil |S|/2^d \rceil$, respectively. This implies easily that there is a unique up to isomorphism configuration. And thus $u(S)$ is independent of the chosen configuration S. □

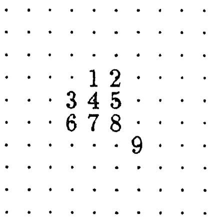

Figure 1: A nine-point optimal configuration

In the plane, optimal configurations for $s \leq 9$ points are depicted in figure 1. It is easy to show using the previous results that for each $s \leq 9$ the optimal s-point configuration consists of the points $1, \ldots, s$. Of course other optimal configurations are possible. If $d = 3$ then optimal configurations for $s \leq 27$ points are constructed similarly.

3.2 Probabilistic Method and Extensions

The main difficulty in studying the optimality of a given configuration S of s lattice points lies in part in the unwieldiness of the alternating sum formula for the density $u(S)$ of the lattice points visible from a camera in S. The main concept that proved helpful in our study of the camera placement problem was that of admissible systems. Intuitively, the coherence and maximality conditions of an admissible system for a configuration S capture the essential information concerning visibility questions of a point A from a point B, namely the prime divisors of $\gcd(A - B)$, for $A, B \in S$. This makes it possible to manipulate configurations by changing the locations of their points in order to eventually determine a configuration with better visibility. We then showed that in optimal configurations of size s, the cameras must be clustered in equivalence classes (for p prime) of specific size which depends only on the size s and the prime p. This enabled us to give the optimality characterizations of the previous section.

Still the key idea in overcoming the inherent complexity of optimizing $u(S)$ lies in the inductive formula for computing $u(S)$ which is proved by allowing the primes to 'play a game of chance' [Kac59, chapter 4]. We have the following theorem.

Theorem 3.7 *For any configuration S and any prime p the density $u(S)$ is given by the following formula*

$$\sum_{c \in S/p} \frac{u(\mathcal{P} \setminus p, S \setminus c)}{p^d} + \left(1 - \frac{|S/p|}{p^d}\right) \cdot u(\mathcal{P} \setminus p, S)$$

Details of the proof can be found in [KP89]. It is interesting to note that using the above formula we can obtain an elegant proof of theorem 3.4 as well as of the optimality for $s \leq 3^d$. Better yet this formula gives important insight about the combinatorial nature of our optimization problem (see [KP89] and [KP90] for more details).

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References

[Abb74] H. L. Abbott. Some results in combinatorial geometry. *Discrete Mathematics*, 9:199–204, 1974.

[Apo76] T. M. Apostol. *Introduction to Analytic Number Theory*. Springer Verlag, 1976.

[BZ88] Y. D. Burago and V. A. Zalgaller. *Geometric Inequalities.* Springer Verlag, 1988. Translated from the Russian.

[CEG*88] K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, and E. Wezl. *Combinatorial Complexity Bounds and Arrangements of Curves and Surfaces.* Technical Report R88/1470, University of Illinois at Urbana-Champaign, November 1988.

[Ede87] H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, 1987.

[Erd62] P. Erdös. On the integers relatively prime to n and on a number theoretic function considered by Jacobsthal. *Math. Scand.*, 10:163 - 170, 1962.

[HS71] F. Herzog and B. M. Stewart. Patterns of visible and nonvisible lattice points. *American Mathematical Monthly*, 78:487 - 496, 1971.

[HW79] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers.* Oxford Science Publications, 1979. 355 pages.

[Kac59] M. Kac. *Statistical Independence in Probability, Analysis and Number Theory.* Volume 12 of *The Carus Mathematical Monographs*, Mathematical Association of America, 1959.

[Knu81] D. Knuth. *The Art of Computer Programming: Seminumerical Algorithms. Computer Science and Information Processing*, Addison Wesley, second edition, 1981. 688 pages.

[KP89] E. Kranakis and M. Pocchiola. *Enumeration and Visibility Problems in integer lattices.* Technical Report 13, LIENS, December 1989.

[KP90] E. Kranakis and M. Pocchiola. Visibility in integer lattices. 1990. In preparation.

[Mos85] W. O. J. Moser. Problems on extremal properties of a finite set of points. In Goodman et al, editor, *New York Academy of Sciences*, pages 52 - 64, 1985.

[Nar83] W. Narkiewicz. *Number Theory.* World Scientific, second edition, 1983. 371 pages.

[Nos48] M. Nosarzewska. Evaluation de la différence entre l'aire d'une région plane convexe et le nombre des points aux coordonnées entières couverts par elle. *Colloq. Math.*, 1, 1948.

[ORo87] J. O'Rourke. *Art Gallery Theorems and Algorithms. International Series of Monographs on Computer Science*, Oxford University Press, 1987. 282 pages.

[Rad64] H. Rademacher. *Lectures on Elementary Number Theory.* Blaisdell, New York, 1964.

[Rea60] D. F. Rearick. Ph.D. Thesis, California Institute of Technology, 1960.

[Rum66] H. Rumsey Jr. Sets of visible points. *Duke Mathematical Journal*, 33:263–274, 1966.

[SO86] S. Suri and J. O'Rourke. Worst-case optimal algorithms for constructing visibility polygons with holes. In *Proceedings of the 2nd Annual ACM Symposium on Computational Geometry*, 1986.

[ST83] E. Szemerédi and W. Trotter Jr. Extremal problems in discrete geometry. *Combinatorica*, 3:381–392, 1983.