On bounded rank positive semidefinite matrix completions of extreme partial correlation matrices

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Abstract

We study a new geometric graph parameter $\text{egd}(G)$, defined as the smallest integer $r \geq 1$ for which any partial symmetric matrix which is completable to a correlation matrix and whose entries are specified at the positions of the edges of $G$, can be completed to a matrix in the convex hull of correlation matrices of rank at most $r$. This graph parameter is motivated by its relevance to the bounded rank Grothendieck constant: $\text{egd}(G) \leq r$ if and only if the rank-$r$ Grothendieck constant of $G$ is equal to 1. The parameter $\text{egd}(G)$ is minor monotone. We identify several classes of forbidden minors for $\text{egd}(G) \leq r$ and give the full characterization for the case $r = 2$. We show an upper bound for $\text{egd}(G)$ in terms of a new tree-width-like parameter $\text{lag}(G)$, defined as the smallest $r$ for which $G$ is a minor of the strong product of a tree and $K_r$. We show that, for $G \neq K_{3,3}$ 2-connected on at least 6 nodes, $\text{egd}(G) \leq 2$ if and only if $\text{lag}(G) \leq 2$.

Keywords: matrix completion, semidefinite programming, correlation matrix, Gram representation, graph minor, tree-width, Grothendieck constant.

1 Introduction

In this paper we investigate a new graph invariant $\text{egd}(G)$, motivated by its relevance to bounded rank Grothendieck inequalities and to bounded

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rank semidefinite matrix completions. This new geometric graph parameter has also some close connections to some Colin de Verdière spectral graph parameters and to some topological tree-width-like graph parameters.

We start with some notation. Throughout, $\mathcal{S}_n$ will denote the set of $n \times n$ symmetric matrices, $\mathcal{S}_n^+$ the cone of positive semidefinite (psd) matrices and $\mathcal{S}_n^{++}$ the cone of positive definite matrices. A psd matrix with an all ones diagonal is called a correlation matrix. The set

$$\mathcal{E}_n = \{ X \in \mathcal{S}_n^+ : X_{ii} = 1 \ \forall i \in [n] \}$$

of all $n \times n$ correlation matrices is known as the elliptope. For an integer $r \geq 1$, define also the (in general non-convex) bounded rank elliptope

$$\mathcal{E}_{n,r} = \{ X \in \mathcal{E}_n : \text{rank } X \leq r \}.$$ 

Given a graph $G = (V = [n], E)$, $\pi_E$ denotes the projection from $\mathcal{S}_n$ onto the subspace $\mathbb{R}^E$ indexed by the edge set of $G$, and we define the projected elliptope:

$$\mathcal{E}(G) = \pi_E(\mathcal{E}_n).$$

The elements of $\mathcal{E}(G)$ can be seen as the partial symmetric matrices with entries specified at positions corresponding to edges of $G$ that can be completed to a correlation matrix.

For any integer $r \geq 1$, we have the following chain of inclusions:

$$\pi_E(\mathcal{E}_{n,r}) \subseteq \pi_E(\text{conv}(\mathcal{E}_{n,r})) \subseteq \pi_E(\mathcal{E}_n) = \mathcal{E}(G). \quad (1)$$

Hence a natural question is to determine what is the smallest value of $r \geq 1$ for which equality holds in the above chain of inclusions. Equality between the sets on the left and on the right side of (1) has been considered in [17], where the following graph parameter is introduced and studied.

**Definition 1.1** The Gram dimension of a graph $G = ([n], E)$, denoted by $\text{gd}(G)$, is defined as the smallest integer $r \geq 1$ such that

$$\mathcal{E}(G) = \pi_E(\mathcal{E}_{n,r}).$$

Here we investigate when equality holds at the right inclusion of (1), which leads to the following graph parameter.

**Definition 1.2** The extreme Gram dimension of a graph $G = ([n], E)$, denoted by $\text{egd}(G)$, is the smallest integer $r \geq 1$ for which

$$\mathcal{E}(G) = \pi_E(\text{conv}(\mathcal{E}_{n,r})).$$
Equivalently, using the Krein–Milman theorem, \( \text{egd}(G) \) is the smallest integer \( r \geq 1 \) for which
\[
\text{ext} \mathcal{E}(G) \subseteq \pi_E(\mathcal{E}_{n,r}),
\]
where \( \text{ext} \mathcal{E}(G) \) is the set of extreme points of \( \mathcal{E}(G) \). We denote by \( \mathcal{G}_r \) the class of graphs \( G \) with \( \text{egd}(G) \leq r \).

Alternatively, \( \text{gd}(G) \) and \( \text{egd}(G) \) can be defined using the following notion of Gram representation, which also clarifies the origin of the names for the graph parameters.

**Definition 1.3** Given a graph \( G = (V, E) \) and a vector \( x \in \mathbb{R}^E \), a Gram representation of \( x \) in \( \mathbb{R}^r \) is a set of unit vectors \( p_1, \ldots, p_n \in \mathbb{R}^r \) such that
\[
x_{ij} = p_i^T p_j \quad \forall \{i, j\} \in E.
\]
The Gram dimension of \( x \in \mathcal{E}(G) \), denoted by \( \text{gd}(G,x) \), is the smallest integer \( r \geq 1 \) for which \( x \) has such a Gram representation in \( \mathbb{R}^r \). Therefore, the (extreme) Gram dimension of \( G \) can be reformulated as
\[
\text{gd}(G) = \max_{x \in \mathcal{E}(G)} \text{gd}(G,x), \quad \text{egd}(G) = \max_{x \in \text{ext} \mathcal{E}(G)} \text{gd}(G,x).
\] (2)

It is shown in [17] that the graph parameter \( \text{gd}(G) \) is minor monotone, and the full list of forbidden minors is identified for graphs with \( \text{gd}(G) \leq r \) for the values \( r = 2, 3 \) and 4. Moreover it is shown there that there are tight connections between the Gram dimension and results about Euclidean graph realizations of Belk and Connelly [2, 3] and the parameter \( \nu^=(G) \) of van der Holst [13].

While the Gram dimension \( \text{gd}(G) \) permits to give an upper bound on the rank of optimal solutions to any semidefinite program with aggregated sparsity pattern \( G \) (see [17]), the extreme Gram dimension permits to upper bound the rank of optimal solutions to optimization programs over the elliptope.

Our new parameter is also related to the celebrated Grothendieck constant. Recall the inclusion \( \pi_E(\text{conv}(\mathcal{E}_{n,r})) \subseteq \mathcal{E}(G) \). Then the smallest constant \( \kappa \geq 1 \) for which
\[
\mathcal{E}(G) \subseteq \kappa \cdot \pi_E(\text{conv} \mathcal{E}_{n,r})
\]
is known as the *rank*-\( r \) *Grothendieck constant* of \( G \), denoted as \( \kappa(r,G) \). For \( r = 1 \), this constant has been introduced and studied by Grothendieck [11]
for bipartite graphs (although in a different language), and for general graphs by Alon et al. [1]. The general case $r \geq 1$ is studied by Briët et al. [4], the case $r = 2$ is motivated by its application to ground states in the $n$-vector model in statistical physics.

The rank-$r$ Grothendieck constant is equal to the integrality gap between two optimization problems: a semidefinite program with a rank constraint:

$$\max \sum_{(i,j) \in E(G)} A_{ij} X_{ij} \text{ s.t. } X \in \mathcal{E}_n, \text{ rank } X \leq r, \quad (3)$$

and its semidefinite relaxation where we remove the rank constraint. The former problem corresponds to optimization over $\pi_E(\text{conv}(\mathcal{E}_{n,r}))$ and the latter to optimization over $\mathcal{E}(G)$. Moreover, problem (3) is hard: For $r = 1$ it is an $NP$-hard quadratic problem with $\pm 1$-variables (modeling the maximum cut problem) and, for any $r \geq 2$, membership in $\pi_E(\text{conv}(\mathcal{E}_{n,r}))$ is $NP$-hard [7]. It follows from the definitions that a graph has extreme Gram dimension at most $r$ if and only if its rank-$r$ Grothendieck constant is 1:

$$\text{egd}(G) \leq r \iff \kappa(r, G) = 1.$$ 

The graph parameter $\text{egd}(G)$ is relevant to problem (3) since, for a graph $G$ satisfying $\kappa(r, G) = 1$, problem (3) can be solved in polynomial time. For $r = 1$ it is known that $\kappa(1, G) = 1$ if and only if $G$ is a forest [16].

The connections described above motivate our study of the graph parameter $\text{egd}(G)$, which also fits within the growing literature on geometric graph parameters defined in terms of rank properties of symmetric matrices (see e.g. [10], the surveys [8, 9, 19] and further references therein).

**Contribution of the paper.** We show that the graph parameter $\text{egd}(G)$ is minor monotone. As a consequence the class $\mathcal{G}_r$ of graphs with $\text{egd}(G) \leq r$ can be characterized by finitely many forbidden minors. One of the main contributions is a complete characterization of the class $\mathcal{G}_2$ (Theorem 4.1).

On the one hand, we identify three families of graphs $F_r, G_r, H_r$ which are forbidden minors for the class $\mathcal{G}_{r-1}$. This gives all the minimal forbidden minors for $r \leq 2$. The graphs $G_r$ were already considered in [5, 15].

On the other hand we show an upper bound for the extreme Gram dimension in terms of a tree-width-like parameter. This graph parameter, which we denote as $\text{la}_\Box(G)$, is defined as the smallest integer $r$ for which $G$ is a minor of the strong product $T \boxtimes K_r$ of a tree $T$ and the complete graph $K_r$. We call it the *strong largeur d’arborescence* of $G$, in analogy with the *largeur d’arborescence* $\text{la}_\Box(G)$ introduced by Colin de Verdière [5].
using the Cartesian product instead of the strong product. We show that
\( \text{egd}(G) \leq \text{lag}(G) \).

Our main result is that, for a graph \( G \neq K_{3,3} \) 2-connected on at least 6
nodes, \( \text{egd}(G) \leq 2 \) if and only if \( \text{lag}(G) \leq 2 \) if and only if \( G \) does not have
\( F_3 \) or \( H_3 \) as a minor. We also characterize the graphs with \( \text{lag}(G) \leq 2 \) and
recover the characterization of [14] for the graphs with \( \text{lag}(G) \leq 2 \).

The results and techniques in the paper come in two flavours: in Section 3
they rely mostly on the geometry of faces of the ellipote and linear algebraic
tools to construct suitable extreme points of the projected ellipote and, in
Section 4 they are purely graph theoretic.

Outline of the paper. Section 2 contains preliminaries about graphs,
some properties of the new parameter \( \text{egd}(G) \), and basic facts about the
geometry of the faces of the ellipote. In Section 3.1 we show that for any
graph \( G \) we have that \( \text{egd}(G) \leq \text{lag}(G) \). In Section 3.2 we compute the
extreme Gram dimension of the three graph classes \( F_r, G_r \) and \( H_r \). In
Section 3.3 we consider the graphs \( K_5 \) and \( K_{3,3} \) which play a special role
within the class \( \mathcal{G}_2 \). Section 4 is devoted to proving the characterization
of the major class \( \mathcal{G}_2 \). In Section 4.2 we characterize the chordal graphs in \( \mathcal{G}_2 \)
(Theorem 4.3). In Section 4.3 we show that any graph with no minor \( F_3 \)
S or \( K_4 \) admits a chordal extension avoiding these two minors (Theorem 4.6)
and in Section 4.4 we show the analogous result for graphs with no \( F_3 \)
and \( H_3 \) minor (Theorem 4.11). Finally in Section 5 we characterize the graphs
with \( \text{lag}(G) \leq 2 \) and we explain the links to results about \( \text{lag}(G) \) and point
out connections with the graph parameter \( \nu(G) \) of Colin de Verdière [5].

2 Preliminaries

2.1 Preliminaries about graphs

We recall some definitions about graphs. Let \( G = (V, E) \) be a graph, we also
denote its node set by \( V(G) \) and its edge set by \( E(G) \). A component is a
maximal connected subgraph of \( G \). A cutset is a set \( U \subseteq V \) for which \( G\setminus U \)
deleting the nodes in \( U \) has more connected components than \( G \), \( U \) is a cut
node if \( |U| = 1 \), and \( G \) is 2-connected if it is connected and has no cut node.
For \( W \subseteq V \), \( G[W] \) is the subgraph induced by \( W \). Given \( \{u, v\} \notin E(G) \),
\( G + \{u, v\} \) is the graph obtained by adding the edge \( \{u, v\} \) to \( G \).

Given an edge \( e = \{u, v\} \in E \), \( G\setminus e = (V, E \setminus \{e\}) \) is the graph obtained
from \( G \) by deleting the edge \( e \) and \( G/e \) is obtained by contracting the edge \( e \):
Replace the two nodes \( u \) and \( v \) by a new node, adjacent to all the neighbors
of \( u \) and \( v \). A graph \( M \) is a **minor** of \( G \), denoted as \( M \preceq G \), if \( M \) can be obtained from \( G \) by a series of edge deletions and contractions and node deletions. Equivalently, \( M \) is a minor of a connected graph \( G \) if there is a partition of \( V(G) \) into nonempty subsets \( \{V_i : i \in V(M)\} \) where each \( G[V_i] \) is connected and, for each edge \( \{i, j\} \in E(M) \), there exists at least one edge in \( G \) between \( V_i \) and \( V_j \). Then the collection \( \{V_i : i \in V(M)\} \) is called an \( M \)-**partition** of \( G \) and the \( V_i \)’s are its **classes**.

A graph parameter \( f(\cdot) \) is **minor monotone** if \( f(G \setminus e), f(G/e) \leq f(G) \) for any graph \( G \) and any edge \( e \) of \( G \).

Given a finite list \( \mathcal{M} \) of graphs, \( \mathcal{F}(\mathcal{M}) \) denotes the collection of all graphs that do not admit any graph in \( \mathcal{M} \) as a minor. By the celebrated graph minor theorem of Robertson and Seymour [20], any class of graphs which is closed under taking minors is of the form \( \mathcal{F}(\mathcal{M}) \) for some finite set \( \mathcal{M} \) of graphs. Hence, if the graph parameter \( f(\cdot) \) is minor monotone, then the class of graphs \( G \) with \( f(G) \leq k \) is characterized by a finite list of excluded minors, for each fixed \( k \).

A **homeomorph** (or subdivision) of a graph \( M \) is obtained by replacing its edges by paths. When \( M \) has maximum degree at most 3, \( G \) admits \( M \) as a minor if and only if it contains a homeomorph of \( M \) as a subgraph.

A **clique** in \( G \) is a set of pairwise adjacent nodes and \( \omega(G) \) denotes the maximum cardinality of a clique in \( G \). A \( k \)-clique is a clique of cardinality \( k \).

Let \( G_1 = (V_1, E_1) \), \( G_2 = (V_2, E_2) \) be two graphs, where \( V_1 \cap V_2 \) is a clique in both \( G_1 \) and \( G_2 \). Their clique sum is the graph \( G = (V_1 \cup V_2, E_1 \cup E_2) \), also called their **clique \( k \)-sum** when \( k = |V_1 \cap V_2| \).

If \( C \) is a circuit in \( G \), a **chord** of \( C \) is an edge \( \{u, v\} \in E \) where \( u \) and \( v \) are two nodes of \( C \) that are not consecutive on \( C \). \( G \) is said to be **chordal** if every circuit of length at least 4 has a chord. As is well known, a graph \( G \) is chordal if and only if \( G \) is a clique sum of cliques.

The **tree-width** \( \text{tw}(G) \) of \( G \) is the smallest integer \( k \) such that \( G \) is contained in a clique sum of copies of \( K_{k+1} \). Colin de Verdière [5] introduced the following variation: The **largeur d’arbrescence** of a graph \( G \), denoted by \( \text{la}_\square(G) \), is the smallest integer \( r \) for which \( G \) is a minor of \( T \square K_r \) for some tree \( T \). Here \( \square \) denotes the Cartesian product. Then,

\[
\text{tw}(G) \leq \text{la}_\square(G) \leq \text{tw}(G) + 1,
\]

the upper bound is shown in [5] and the lower bound in [12]. We use the notation \( \text{la}_\square(G) \) (instead of the original notation \( \text{la}(G) \)) in order to emphasize the analogy with our new graph parameter \( \text{la}_\boxtimes(G) \), which is based on using the strong product \( \boxtimes \) instead of the Cartesian product \( \square \).
The strong product $G \circ G'$ of $G = (V, E)$ and $G' = (V', E')$ has node set $V \times V'$ and distinct nodes $(i, i'), (j, j') \in V \times V'$ are adjacent in $G \circ G'$ when $i = j$ or $(i, j) \in E$, and $j' = j'$ or $(i', j') \in E'$. Then, $\text{la}_G(G)$ is the smallest integer $r$ for which $G$ is a minor of $T \circ K_r$ for some tree $T$. It will serve as an upper bound for our new graph parameter $\text{egd}(G)$ (see Section 3.1).

The graph parameters $\text{tw}(G)$, $\text{la}_[\square](G)$ and $\text{la}_G(G)$ are minor monotone and satisfy:

\[ \text{tw}(G)/2 \leq \text{la}_G(G) \leq \text{la}_[\square](G) \leq \text{tw}(G) + 1. \]

If $G$ is the clique $k$-sum of $G_1$ and $G_2$, then $f(G) = \max\{f(G_1), f(G_2)\}$ when $f(G) = \text{tw}(G)$; the same holds for the parameters $f(G) = \text{la}_[\square](G)$ and $\text{la}_G(G)$ when $k \leq 1$.

**Some more notation.** Throughout $[n] = \{1, \ldots, n\}$. For a set $A \subseteq \mathbb{R}^n$, $\langle A \rangle$ denotes the vector space spanned by $A$ and $\text{conv} A$ denotes the convex hull of $A$. For a matrix $X \in \mathcal{S}_n$, $X \succeq 0$ means that $X$ is positive semidefinite. For $U \subseteq [n]$, $X[U]$ denotes the principal submatrix of $X$ with row and column indices in $U$ and, for $j \in [n]$, $X[\cdot, j]$ denotes the $j$-th column of $X$.

### 2.2 Basic properties of the extreme Gram dimension

Here we investigate the behavior of the graph parameter $\text{egd}(G)$ under some simple graph operations: taking minors and clique sums.

**Lemma 2.1** The graph parameter $\text{egd}(G)$ is minor monotone, i.e., for any edge $e$ of $G$, $\text{egd}(G \setminus e), \text{egd}(G/e) \leq \text{egd}(G)$.

**Proof.** Let $G = ([n], E)$ and $e \in E$. The inequality $\text{egd}(G \setminus e) \leq \text{egd}(G)$ follows from the definition. We show that $\text{egd}(G/e) \leq \text{egd}(G) = r$. Say, $e = \{n-1, n\}$ and set $G/e = ([n-1], E')$. Let $x \in \mathcal{E}(G/e)$. Then $x = \pi_{E'}(X)$ for some $X \in \mathcal{E}_{n-1}$. Let $X[\cdot, n-1]$ be the last column of $X$ and set

\[ Y = \begin{pmatrix} X & X[\cdot, n-1] \end{pmatrix}^{\top} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{S}_n \]

and $y = \pi_E(Y)$. Then $Y \in \mathcal{E}_n$ and thus $y \in \mathcal{E}(G)$. As $\text{egd}(G) = r$, there exist $Y_1, \ldots, Y_m \in \mathcal{E}_{n,r}$ and scalars $\lambda_i \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that $y = \pi_E(\sum_{i=1}^m \lambda_i Y_i)$. The condition $Y_{n-1,n} = 1$ implies that $(Y_i)_{n-1,n} = 1$ and thus $Y_{[\cdot, n-1]} = Y_{i,[\cdot, n]}$ for all $i \in [m]$. Now, let $\hat{Y}_i$ be obtained from $Y_i$ by removing its $n$-th row and column. Then, $\hat{Y}_i \in \mathcal{E}_{n-1}$, rank $\hat{Y}_i \leq r$, and $x = \pi_{E'}(\sum_{i=1}^m \lambda_i \hat{Y}_i) \in \pi_{E'}(\mathcal{E}_{n-1,r})$. This shows $\text{egd}(G/e) \leq r$. \hfill $\square$

The following easy, but useful fact about psd completions is well known.
Lemma 2.2 Given two psd matrices $X_1 \in \mathcal{S}_{V_1}^+$ and $X_2 \in \mathcal{S}_{V_2}^+$ such that $X_1[V_1 \cap V_2] = X_2[V_1 \cap V_2]$, there exists a common psd completion $X \in \mathcal{S}_V^+$, i.e., such that $X[V_i] = X_i$ ($i = 1, 2$), with rank $X = \max\{\text{rank } X_1, \text{rank } X_2\}$.

As a direct application, if $G$ is the clique sum of $G_1$ and $G_2$, its Gram dimension satisfies: $\text{gd}(G) = \max\{\text{gd}(G_1), \text{gd}(G_2)\}$. For the extreme Gram dimension, the analogous result holds only for clique $k$-sums with $k \leq 1$.

Lemma 2.3 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. If $|V_1 \cap V_2| \leq 1$ then the clique sum $G$ of $G_1, G_2$ satisfies $\text{egd}(G) = \max\{\text{egd}(G_1), \text{egd}(G_2)\}$.

**Proof.** Let $x \in \mathcal{E}(G)$ and $r = \max\{\text{egd}(G_1), \text{egd}(G_2)\}$, we show that $x \in \pi_E(\text{conv}(\mathcal{E}_{n,r}))$. For $i = 1, 2$, the vector $x_i = \pi_{E_i}(x)$ belongs to $\pi_{E_i}(\text{conv}(\mathcal{E}_{|V_i|, r}))$. Hence, $x_i = \pi_{E_i}(\sum_{j=1}^{m_i} \lambda_{i,j} X^i_{i,j})$ for some $X^i_{i,j} \in \mathcal{E}_{|V_i|, r}$ and $\lambda_{i,j} \geq 0$ with $\sum_j \lambda_{i,j} = 1$. As $|V_1 \cap V_2| \leq 1$, any two matrices $X^1_{i,j}$ and $X^2_{i,j}$ share at most one diagonal entry, equal to 1 in both matrices. By Lemma 2.2, $X^1_{i,j}$ and $X^2_{i,j}$ have a common completion $Y^j_{i,k} \in \mathcal{E}_{n,r}$. This implies that $x = \pi_E(\sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \lambda_{1,j} \lambda_{2,k} Y^j_{i,k})$, which shows $x \in \pi_E(\text{conv}(\mathcal{E}_{n,r}))$.

Therefore, the class $\mathcal{G}_r$ is closed under taking disjoint unions and clique 1-sums of graphs. It is not closed under clique $k$-sums when $k \geq 2$. E.g. the graph $F_3$ from Figure 1 is a clique 2-sum of triangles, however $\text{egd}(F_3) = 3$ (Theorem 3.6) while triangles have extreme Gram dimension 2 (Lemma 2.6).

### 2.3 The geometry of the elliptope

Recall that, for a convex set $K$, a set $F \subseteq K$ is a face of $K$ if for all $x \in F$, $x = ty + (1 - t)z$ with $y, z \in K$ and $t \in (0, 1)$ implies $y, z \in F$. For $x \in K$ the smallest face $F(x)$ of $K$ containing $x$ is well defined, it is the unique face of $K$ containing $x$ in its relative interior. A point $x \in K$ is an extreme point of $K$ if $F(x) = \{x\}$. Moreover, $z$ is said to be a perturbation of $x \in K$ if $x \pm \varepsilon z \in K$ for some $\varepsilon > 0$, then the segment $[x - \varepsilon z, x + \varepsilon z]$ is contained in $F(x)$ and the dimension of $F(x)$ is equal to the dimension of the linear space $P(x)$ of perturbations of $x$.

We recall some facts about the faces of the elliptope that we need here. For a matrix $X \in \mathcal{E}_n$, the smallest face $F(X)$ of $\mathcal{E}_n$ containing $X$ is given by

$$F(X) = \{Y \in \mathcal{E}_n : \ker X \subseteq \ker Y\}. \quad (4)$$

Therefore, two matrices in the relative interior of a face $F$ of $\mathcal{E}_n$ have the same rank, while rank $X > \text{rank } Y$ if $X$ is in the relative interior of $F$ and
Y lies on the boundary of $F$. Here is the explicit description of the space $\mathcal{P}(X)$ of perturbations of a matrix $X \in \mathcal{E}_n$.

**Proposition 2.4** ([13], see also [6, §31.5]) Let $X \in \mathcal{E}_n$ with rank $r$. Let $u_1, \ldots, u_n \in \mathbb{R}^r$ be a Gram representation of $X$, let $U$ be the $r \times n$ matrix with columns $u_1, \ldots, u_n$ and set $\mathcal{U}_V = \langle u_1u_1^T, \ldots, u_nu_n^T \rangle \subseteq S_r$. The space of perturbations $\mathcal{P}(X)$ at $X$ is given by

$$\mathcal{P}(X) = U^TU_U = \{U^TRU : R \in S_r, \langle R, u_iu_i^T \rangle = 0 \ \forall i \in [n]\} \quad (5)$$

and the dimension of the smallest face $F(X)$ of $\mathcal{E}_n$ containing $X$ is

$$\dim F(X) = \dim \mathcal{P}(X) = \left(\frac{r+1}{2}\right) - \dim \mathcal{U}_V. \quad (6)$$

In particular, $X$ is an extreme point of $\mathcal{E}_n$ if and only if

$$\left(\frac{r+1}{2}\right) = \dim \mathcal{U}_V. \quad (7)$$

Hence, if $X \in \text{ext} \mathcal{E}_n$ with rank $X = r$ then

$$\left(\frac{r+1}{2}\right) \leq n. \quad (8)$$

**Example 2.5** Let $e_1, \ldots, e_r \in \mathbb{R}^r$ be the standard unit vectors. The matrix with Gram representation $\{e_i : i \in [r]\} \cup \{(e_i + e_j)/\sqrt{2} : 1 \leq i < j \leq r\}$ is an extreme point of $\mathcal{E}_n$, since $\mathcal{U}_V$ is full dimensional in $S_r$, where $n = \left(\frac{r+1}{2}\right)$.

It is known that for any $r$ satisfying (8) there exists an extremal matrix in $\mathcal{E}_n$ of rank $r$ [13]. This implies:

**Lemma 2.6** The extreme Gram dimension of the complete graph $K_n$ is

$$\text{egd}(K_n) = r_{\max}(n) := \max \left\{ r \in \mathbb{Z}_+ : \left(\frac{r+1}{2}\right) \leq n \right\} = \left\lfloor \frac{\sqrt{8n+1} - 1}{2} \right\rfloor.$$ 

Hence, $\text{egd}(G) \leq r_{\max}(n)$ for any graph $G$ on $n$ nodes.

Next we establish some tools which will be useful to study the extreme points of the projected elliptope $\mathcal{E}(G)$.
Lemma 2.7 Let $x \in \mathcal{E}(G)$, let $X \in \mathcal{E}_n$ be a rank $r$ completion of $x$ with Gram representation $\{u_1, \ldots, u_n\}$ in $\mathbb{R}^r$ and let $U$ be the $r \times n$ matrix with columns $u_1, \ldots, u_n$. Set

$$U_{ij} = \frac{u_i u_j^T + u_j u_i^T}{2}, \quad \mathcal{U}_V = \langle U_{ii} : i \in V \rangle, \quad \mathcal{U}_E = \langle U_{ij} : \{i, j\} \in E \rangle \subseteq S_r. \quad (9)$$

If $x$ is an extreme point of $\mathcal{E}(G)$, then $\mathcal{U}_E \subseteq \mathcal{U}_V$.

Proof. Assume that $\mathcal{U}_E \not\subseteq \mathcal{U}_V$. Then there exists a matrix $R \in \mathcal{U}_V \setminus \mathcal{U}_E$. As $R \in \mathcal{U}_V$, the matrix $Z = U^T R U = (\langle R, U_{ij} \rangle)_{i,j=1}^n \in S_n$ is a perturbation of $X$ (recall (5) and (9)). As $R \not\in \mathcal{U}_E$, $Z_{ij} \neq 0$ for some edge $\{i, j\} \in E$. Now, $X \pm \epsilon Z \in \mathcal{E}_n$ for some $\epsilon > 0$. Hence, $x$ can be written as the convex combination $(\pi_E(X + \epsilon Z) + \pi_E(X - \epsilon Z))/2$, where $\pi_E(X \pm \epsilon Z)$ are distinct points of $\mathcal{E}(G)$. This contradicts the assumption that $x$ is an extreme point of $\mathcal{E}(G)$. $\square$

Given $x \in \mathcal{E}(G)$, its fiber is the set

$$\text{fib}(x) = \{X \in \mathcal{E}_n : \pi_E(X) = x\}$$

of all psd completions of $x$ in $\mathcal{E}_n$. The following lemma is an easy result from convex analysis.

Lemma 2.8 For $x \in \mathcal{E}(G)$, $x$ is an extreme point of $\mathcal{E}(G)$ if and only if its fiber $\text{fib}(x)$ is a face of $\mathcal{E}_n$. Moreover, if $x$ is an extreme point of $\mathcal{E}(G)$, then any extreme point of $\text{fib}(x)$ is an extreme point of $\mathcal{E}_n$.

3 The extreme Gram dimension of some graphs

3.1 An upper bound for the extreme Gram dimension

In this section we show that the extreme Gram dimension is upper bounded by the strong largeur d’arborescence: $\text{egd}(G) \leq \text{lag}_2(G)$. As we will see in the next section, the class of graphs with $\text{lag}_2(G) \leq 2$ plays a crucial role in the characterization of the class $\mathcal{G}_2$.

Theorem 3.1 For any tree $T$, $\text{egd}(T \boxtimes K_r) \leq r$.

Corollary 3.2 For any graph $G$, $\text{egd}(G) \leq \text{lag}_2(G)$. 

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The main ingredient for the proof of Theorem 3.1 is the following technical lemma.

Lemma 3.3 Let \( \{u_1, \ldots, u_r\} \) be a set of vectors, denote its rank by \( \rho \). Let \( \mathcal{U} \) denote the linear span of the matrices \( U_{ij} = (u_i^Tu_j + u_j^Tu_i^T)/2 \) for all \( i, j \in \{1, \ldots, r\} \) and all \( i, j \in \{r+1, \ldots, 2r\} \). If \( \rho \geq r+1 \) then \( \dim \mathcal{U} < (\rho+1)^2/2 \).

Proof. Let \( I \subseteq \{1, \ldots, r\} \) for which \{\( u_i : i \in I \)\} is a maximum linearly independent subset of \{\( u_1, \ldots, u_r \)\} and let \( J \subseteq \{r+1, \ldots, 2r\} \) such that the set \{\( u_i : i \in I \cup J \)\} is maximum linearly independent; thus \( |I| + |J| = \rho \). Set \( K = \{1, \ldots, r\} \setminus I \), \( L = \{r+1, \ldots, 2r\} \setminus J \), and \( J' = J \setminus \{k\} \), where \( k \) is some given (fixed) element of \( J \). For any \( l \in L \), there exists scalars \( a_{l,i} \in \mathbb{R} \) such that

\[
\begin{align*}
    u_l &= \sum_{i \in I \cup J'} a_{l,i}u_i + a_{l,k}u_k. \\
\end{align*}
\]

Set

\[
A_l = \sum_{i \in I \cup J'} a_{l,i}U_{ik} \quad \text{for } l \in L.
\]

Then, define the set \( \mathcal{W} \) consisting of the matrices \( U_{ii} \) for \( i \in I \cup J \), \( U_{ij} \) for all pairs \((i, j)\) in \( I \cup J' \), \( U_{kj} \) for all \( j \in J' \), and \( A_l \) for all \( l \in L \). Then, \( |\mathcal{W}| = \rho + (\rho^{-1}) + r - 1 = \binom{\rho}{2} + r = (\rho + 1)^2/2 + r - \rho \leq (\rho + 1)^2/2 - 1 \). In order to conclude the proof it suffices to show that \( \mathcal{W} \) spans the space \( \mathcal{U} \).

Clearly, \( \mathcal{W} \) spans all matrices \( U_{ij} \) with \( i, j \in \{1, \ldots, r\} \). Fix \( l \in L \). Using (10) we obtain that \( U_{ik} = A_l + a_{l,k}U_{kk} \) lies in the span of \( \mathcal{W} \). Moreover, for \( j \in J' \), \( U_{ij} = \sum_{i \in I \cup J'} a_{l,i}U_{ij} + a_{l,k}U_{kj} \) also lies in the span of \( \mathcal{W} \). Finally, for \( l' \in L \), \( U_{i_{l'}} = \sum_{i,j \in I \cup J'} a_{l',i}U_{ij} + a_{l',k}A_l + a_{l,k}A_{l'} + a_{l,k}a_{l',k}U_{kk} \) is also spanned by \( \mathcal{W} \). This concludes the proof.

Proof. (of Theorem 3.1). Let \( G = T \boxtimes K_r \), where \( T \) is a tree on \([t]\) and let \( G = (V, E) \) with \( |V| = n \). So the node set of \( G \) is \( V = \bigcup_{i=1}^{t} V_i \), where the \( V_i \)'s are pairwise disjoint sets, each of cardinality \( r \). By definition of the strong product, for any edge \{\( i, j \)\} of \( T \), the set \( V_i \cup V_j \) induces a clique in \( G \), denoted as \( C_{ij} \). Then, \( G \) is the union of the cliques \( C_{ij} \) over all edges \{\( i, j \)\} of \( T \). We show that \( \text{egd}(G) \leq r \). For this, pick an extreme element \( x \in \text{ext} \mathcal{E}(G) \). Then \( x = \pi_E(X) \) for some \( X \in \mathcal{E}_n \). As \( C_{ij} \) is a clique in \( G \), the principal submatrix \( X^{ij} := X[C_{ij}] \) is fully determined from \( x \). In order to show that \( x \) has a psd completion of rank at most \( r \), it suffices to show that \( \text{rank} X^{ij} \leq r \) for all edges \{\( i, j \)\} of \( T \) (then apply Lemma 2.2).
Pick an edge \{i, j\} of T and set \( \rho = \text{rank} X_{ij} \). Assume that \( \rho \geq r + 1 \); we show that there exists a nonzero perturbation \( Z \) of \( X_{ij} \) such that
\[
Z_{hk} = 0 \quad \forall (h, k) \in (V_i \times V_i) \cup (V_j \times V_j),
\]
\[
Z_{hk} \neq 0 \quad \text{for some} \quad (h, k) \in V_i \times V_j.
\]
This permits to reach a contradiction: As \( Z \) is a perturbation of \( X_{ij} \), there exists \( \epsilon > 0 \) for which \( X_{ij} + \epsilon Z, \ X_{ij} - \epsilon Z \succeq 0 \). By construction, \( C_{ij} \) is the only maximal clique of \( G \) containing the edges \{h, k\} of \( G \) with \( h \in V_i \) and \( k \in V_j \). Hence, one can find a psd completion \( X' \) (resp., \( X'' \)) of the matrix \( X_{ij} + \epsilon Z \) (resp., \( X_{ij} - \epsilon Z \)) and the matrices \( X'_{ij}j' \) for all edges \{i', j'\} \neq \{i, j\} of \( T \). Now, \( x = \frac{1}{2}(\pi_{E}(X') + \pi_{E}(X'')) \), where \( \pi_{E}(X'), \pi_{E}(X'') \) are distinct elements of \( \mathcal{E}(G) \), contradicting the fact that \( x \) is an extreme point of \( \mathcal{E}(G) \).

We now construct the desired perturbation \( Z \) of \( X_{ij} \) satisfying \((11)\). For this let \( u_h \) \( (h \in V_i \cup V_j) \) be a Gram representation of \( X_{ij} \) in \( \mathbb{R}^p \) and let \( \mathcal{U} \subseteq \mathcal{S}_\rho \) denote the linear span of the matrices \( U_{hk} \) for all \( h, k \in V_i \) and all \( h, k \in V_j \). Applying Lemma \([3,3,\,\square]\), as \( \rho \geq r + 1 \), we deduce that \( \dim \mathcal{U} < (\rho+1)^2 \).

Hence there exists a nonzero matrix \( R \in S_\rho \) lying in \( \mathcal{U}^\perp \). Define the matrix \( Z \in S_{2r} \) by \( Z_{hk} = \langle R, U_{hk} \rangle \) for all \( h, k \in V_i \cup V_j \). By construction, \( Z \) is a perturbation of \( X_{ij} \) and it satisfies \( Z_{hk} = 0 \) whenever the pair \((h, k)\) is contained in \( V_i \) or in \( V_j \). As \( R \neq 0 \), \( Z \neq 0 \) and thus \( Z_{hk} \neq 0 \) for some \( h \in V_i \) and \( k \in V_j \). Thus \((11)\) holds and the proof is completed. \( \square \)

### 3.2 Three graph classes with extreme Gram dimension \( r \)

In this section we construct three classes of graphs \( F_r \), \( G_r \), \( H_r \), whose extreme Gram dimension is equal to \( r \). Therefore, they are forbidden minors for the class \( \mathcal{G}_{r-1} \) of graphs with extreme Gram dimension at most \( r - 1 \). As we will see in the next section, this gives all the forbidden minors for the characterization of the class \( \mathcal{G}_2 \).

The graphs \( G_r \) were already considered by Colin de Verdière [5] in relation to the graph parameter \( \nu(G) \), to which we will come back in Section \([5,\,\square]\).

Each of the graphs \( G = F_r, G_r, H_r \) has \( \binom{r+1}{2} \) nodes and thus extreme Gram dimension \( \text{egd}(G) \leq r \); moreover, \( \text{egd}(G/e) \leq r - 1 \) after contracting an edge (use Lemma \([2,\,\square]\)). To show equality \( \text{egd}(G) = r \), we will rely on the following result, which follows directly from Lemma \([2,8]\).

**Lemma 3.4** Suppose that there exists \( x \in \mathcal{E}(G) \) such that \( \text{fib}(x) = \{X\} \), where \( X \in \text{ext} \mathcal{E}_n \) and \( \text{rank} X = r \). Then \( \text{egd}(G) \geq r \).
To use this lemma we need tools permitting to show existence of a unique completion for a vector $x \in \mathcal{E}(G)$. We introduce below such a tool: 'forcing a non-edge with a minimally singular clique', based on the following property of psd matrices:

$$
\begin{pmatrix}
A & b \\
b^T & \alpha
\end{pmatrix} \succeq 0 \implies b^T u = 0 \forall u \in \ker A.
$$

**Lemma 3.5** Let $x \in \mathcal{E}(G)$, let $C \subseteq V$ be a clique of $G$ and $\{i, j\} \notin E(G)$ with $i \notin C$, $j \in C$. Set $x[C] = (x_{ij})_{i,j \in C} \in \mathcal{E}[C]$ (setting $x_{ii} = 1$ for all $i$). Assume that $i$ is adjacent to all nodes of $C \setminus \{j\}$ and that $x[C]$ is minimally singular (i.e., $x[C]$ is singular but any principal submatrix of $x[C]$ is nonsingular). Then the $(i,j)$-th entry $X_{ij}$ is uniquely defined in any completion $X \in \text{fib}(x)$ of $x$.

**Proof.** Let $X \in \text{fib}(x)$. The principal submatrix $X[C \cup \{i\}]$ has the block form shown in (12) where all entries are specified (from $x$) except the entry $b_j = X_{ij}$. As $x[C]$ is singular there exists a nonzero vector $u$ in the kernel of $x[C]$. Moreover, $u_j \neq 0 \forall j \in C$, since $x[C \setminus \{j\}]$ is nonsingular. Hence the condition $b^T u = 0$ permits to derive the value of $X_{ij}$ from $x$. \qed

When applying Lemma 3.5 we will say that "the clique $C$ forces the pair $\{i,j\}$". The lemma will be used in an iterative manner: Once a non-edge $\{i,j\}$ has been forced, we know the value $X_{ij}$ in any psd completion $X$ and thus we can replace $G$ by $G + \{i,j\}$ and search for a new forced pair in the extended graph $G + \{i,j\}$.

### 3.2.1 The class $F_r$

For $r \geq 2$ the graph $F_r$ has $r + \binom{r}{2} = \binom{r+1}{2}$ nodes, denoted as $v_i$ (for $i \in [r]$) and $v_{ij}$ (for $1 \leq i < j \leq r$); it consists of a clique $K_r$ on the nodes $\{v_1, \ldots, v_r\}$ together with the cliques $C_{ij}$ on $\{v_i, v_j, v_{ij}\}$ for all $1 \leq i < j \leq r$. The graphs $F_3$ and $F_4$ are illustrated in Figure 1.

For $r = 2$, $F_2 = K_3$ has extreme Gram dimension 2. More generally:

**Theorem 3.6** For $r \geq 2$, $\text{egd}(F_r) = r$. Moreover, $F_r$ is a minimal forbidden minor for the class $\mathcal{G}_{r-1}$.

**Proof.** First we show that $\text{egd}(F_r) \geq r$. For this we label the nodes $v_1, \ldots, v_r$ by the standard unit vectors $e_1, \ldots, e_r \in \mathbb{R}^r$ and $v_{ij}$ by the vector $(e_i + e_j)/\sqrt{2}$. Consider the Gram matrix $X$ of these $n = \binom{r+1}{2}$ vectors and its projection $x = \pi_{E(F_r)}(X) \in \mathcal{E}(F_r)$. Then, $X$ is an extreme point of $\mathcal{E}_n$. 

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Figure 1: The graphs $F_3$ and $F_4$.

(Example 2.5). We now show that $X$ is the only psd completion of $x$ which, in view of Lemma 3.4, implies that $\text{egd}(F_r) \geq r$. For this we use Lemma 3.5. Observe that, for each $1 \leq i < j \leq r$, the matrix $x[C_{ij}]$ is minimally singular. First, for any $k \in [r] \setminus \{i,j\}$, the clique $C_{ij}$ forces the non-edge $\{v_k, v_{ij}\}$ and then, for any other $1 \leq i' < j' \leq r$, the clique $C_{ij}$ forces the non-edge $\{v_{ij}, v_{i'j'}\}$. Hence, in any psd completion of $x$, all the entries indexed by non-edges are uniquely determined, i.e., $\text{fib}(x) = \{X\}$.

Next, we show minimality. Let $e$ be an edge of $F_r$, we show that $\text{egd}(H) \leq r - 1$ where $H = F_r \setminus e$. If $e$ is an edge of the form $\{v_i, v_{ij}\}$, then $H$ is the clique 1-sum of an edge and a graph on $\left(\frac{r+1}{2}\right) - 1$ nodes and thus $\text{egd}(H) \leq r - 1$ follows using Lemmas 2.3 and 2.6. Suppose now that $e$ is contained in the central clique $K_r$, say $e = \{v_1, v_2\}$. We show that $H$ is contained in a graph of the form $T \boxtimes K_{r-1}$ for some tree $T$. We choose $T$ to be the star $K_{1, r-1}$ and we give a suitable partition of the nodes of $F_r$ into sets $V_0 \cup V_1 \cup \ldots \cup V_{r-1}$, where each $V_i$ has cardinality at most $r - 1$, $V_0$ is assigned to the center node of the star $K_{1, r-1}$ and $V_1, \ldots, V_{r-1}$ are assigned to the $r - 1$ leaves of $K_{1, r-1}$. Namely, set $V_0 = \{v_{12}, v_3, \ldots, v_r\}$, $V_1 = \{v_1, v_{13}, \ldots, v_{1r}\}$, $V_2 = \{v_2, v_{23}, \ldots, v_{2r}\}$ and, for $k \in \{3, \ldots, r - 1\}$, $V_k = \{v_{kj} : k + 1 \leq j \leq r\}$. Then, in the graph $H$, each edge is contained in one of the sets $V_0 \cup V_k$ for $1 \leq k \leq r - 1$. This shows that $H$ is a subgraph of $K_{1, r-1} \boxtimes K_{r-1}$ and thus $\text{egd}(H) \leq r - 1$ (by Theorem 3.1).

As an application of Theorem 3.6 we get:

**Corollary 3.7** If the tree $T$ has a node of degree at least $(r - 1)/2$ then $\text{egd}(T \boxtimes K_r) = r$.

**Proof.** Directly from Theorem 3.6 as $T \boxtimes K_r$ contains a subgraph $F_r$. \hfill $\Box$

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3.2.2 The class \( G_r \)

Consider an equilateral triangle and subdivide each side into \( r - 1 \) equal segments. Through these points draw line segments parallel to the sides of the triangle. This construction creates a triangulation of the big triangle into \( (r - 1)^2 \) congruent equilateral triangles. The graph \( G_r \) corresponds to the edge graph of this triangulation. The graph \( G_5 \) is illustrated in Figure 2.

![Figure 2: The graph \( G_5 \).](image)

The graph \( G_r \) has \( \left( \frac{r+1}{2} \right) \) vertices, which we denote \( v_{i,l} \) for \( l \in [r] \) and \( i \in [r - l + 1] \) (with \( v_{1,l}, \ldots, v_{r-l+1,l} \) at level \( l \), see Figure 2). Note that \( G_2 = K_3 = F_2 \), \( G_3 = F_3 \), but \( G_r \neq F_r \) for \( r \geq 4 \). Using the following lemma we can construct some points of \( E(G_r) \) with a unique completion.

**Lemma 3.8** Consider a labeling of the nodes of \( G_r \) by vectors \( w_{i,l} \) satisfying the following property \((P_r)\): For each triangle \( C_{i,l} = \{v_{i,l}, v_{i+1,l}, v_{i,l+1}\} \) of \( G_r \), the set \( \{w_{i,l}, w_{i+1,l}, w_{i,l+1}\} \) is minimally linearly dependent. (These triangles are shaded in Figure 2). Let \( X \) be the Gram matrix of the vectors \( w_{i,l} \) and let \( x = \pi_{E(G_r)}(X) \) be its projection. Then \( X \) is the unique completion of \( x \).

**Proof.** For \( r = 2 \), \( G_2 = K_3 \) and there is nothing to prove. Let \( r \geq 3 \) and assume that the claim holds for \( r - 1 \). Consider a labeling \( w_{i,l} \) of \( G_r \) satisfying \((P_r)\) and the corresponding vector \( x \in E(G_r) \). We show, using Lemma 3.5 that the entries \( Y_{uv} \) of a psd completion \( Y \) of \( x \) are uniquely determined for all \( \{u,v\} \not\in E(G_r) \). For this, denote by \( H, R, L \) the sets of nodes lying on the ‘horizontal’ side, the ‘right’ side and the ‘left’ side of \( G_r \), respectively (refer to the drawing of \( G_r \) of Figure 2). Observe that each of
\( G_r \setminus H, G_r \setminus R, G_r \setminus L \) is a copy of \( G_{r-1} \). As the induced vector labelings on each of these graphs satisfies the property \( (P_{r-1}) \), we can conclude using the induction assumption that the entry \( Y_{uv} \) is uniquely determined whenever the pair \( \{u, v\} \) is contained in the vertex set of one of \( G_r \setminus H, G_r \setminus R, \) or \( G_r \setminus L \). The only non-edges \( \{u, v\} \) that are not yet covered arise when \( u \) is a corner of \( G_r \) and \( v \) lies on the opposite side, say \( u = v_{1,1} \) and \( v = v_{r-l+1,l} \in R \). If \( l \neq 1, r \) then the clique \( C_{1,1} = \{v_{1,1}, v_{2,1}, v_{1,2}\} \) forces the pair \( \{u, v\} \) (since \( \{v, v_{1,2}\} \in E(G_r \setminus H) \) and \( \{v, v_{2,1}\} \in E(G_r \setminus L) \)). If \( l = r \) then the clique \( C_{1,r-1} = \{v_{1,r-1}, v_{2,r-1}, v_{1,1}\} \) forces the pair \( \{u, v\} \) (since \( \{u, v_{1,r-1}\} \in E(G_r \setminus R) \) and the value at the pair \( \{u, v_{2,r-1}\} \) has just been specified). Analogously for the case \( l = 1 \). This concludes the proof. \( \Box \)

**Theorem 3.9** For \( r \geq 2 \), \( \text{egd}(G_r) = r \). Moreover, \( G_r \) is a minimal forbidden minor for the class \( \mathcal{G}_{r-1} \).

**Proof.** First we show that \( \text{egd}(G_r) \geq r \). For this, choose a vector labeling of the nodes of \( G_r \) satisfying the conditions of Lemma 3.8. Label the nodes \( v_{1,1}, \ldots, v_{r,1} \) at level \( l = 1 \) by the standard unit vectors \( w_{1,1} = e_1, \ldots, w_{r,1} = e_r \) in \( \mathbb{R}^r \) and define inductively \( w_{i,l+1} = \frac{w_{i,l} + w_{r+1,l}}{\|w_{i,l} + w_{r+1,l}\|} \) for \( l = 1, \ldots, r-1 \). By Lemma 3.8 their Gram matrix \( X \) is the unique completion of its projection \( x = \pi_{E(G_r)}(X) \in E(G_r) \). Moreover, \( X \) is extreme in \( E_n \) since \( U_Y \) is full-dimensional in \( S_r \). This shows \( \text{egd}(G_r) \geq r \), by Lemma 3.4.

We now show that \( \text{egd}(G_r \setminus e) \leq r - 1 \). For this we use the inequalities: \( \text{egd}(G_r \setminus e) \leq \text{lag}(G_r \setminus e) \leq \text{lac}(G_r \setminus e) \leq r - 1 \), where the leftmost inequality follows from Corollary 3.2 and the rightmost one is shown in [14]. \( \Box \)

**Corollary 3.10** The graph parameter \( \text{egd}(G) \) is unbounded for the class of planar graphs.

**Corollary 3.11** Let \( T \) be a tree which contains a path with \( 2r - 2 \) nodes. Then, \( \text{egd}(T \boxtimes K_r) = r \).

**Proof.** It is shown in [5] that \( G_r \) is a minor of the Cartesian product of two paths \( P_r \) and \( P_{2r-2} \) (with, respectively, \( r \) and \( 2r - 2 \) nodes). Hence, \( G_r \subseteq P_{2r-2} \boxtimes P_r \subseteq T \boxtimes K_r \) and thus \( r = \text{egd}(G_r) \leq \text{egd}(T \boxtimes K_r) \). \( \Box \)

### 3.2.3 The class \( H_r \)

In this section we consider a third class of graphs \( H_r \) for every \( r \geq 3 \). In order to explain the general definition we first describe the base case \( r = 3 \).
The graph $H_3$ is shown in Figure 3. It is obtained by taking a complete graph $K_4$, with vertices $v_1, v_2, v_3$ and $v_{13}$, and subdividing two adjacent edges: here we insert node $v_{12}$ between $v_1$ and $v_2$ and node $v_{23}$ between nodes $v_2$ and $v_3$.

**Lemma 3.12** $\text{egd}(H_3) = 3$ and $H_3$ is a minimal forbidden minor for $\mathcal{F}_2$.

**Proof.** As $H_3$ has 6 nodes, $\text{egd}(H_3) \leq 3$. To show equality, we use the following vector labeling for the nodes of $H_3$: Label the nodes $v_1, v_2, v_3$ by the standard unit vectors $e_1, e_2, e_3 \in \mathbb{R}^3$ and $v_{ij}$ by $(e_i+e_j)/\sqrt{2}$ for $1 \leq i < j \leq 3$. Let $X \in \mathcal{E}_6$ be their Gram matrix and set $x = \pi_{E(G)}(X) \in \mathcal{E}(H_3)$. Then $X$ has rank 3 and $X$ is an extreme point of $\mathcal{E}_6$. We now show that $X$ is the unique completion of $x$ in $\mathcal{E}_6$. For this let $Y \in \text{fib}(x)$. Consider its principal submatrices $Z, Z'$ indexed by $\{v_1, v_2, v_3, v_{13}\}$ and $\{v_1, v_2, v_{12}\}$, of the form:

$$Z = \begin{pmatrix}
1 & a & 0 & \sqrt{2}/2 \\
 a & 1 & b & 0 \\
 0 & b & 1 & \sqrt{2}/2 \\
\sqrt{2}/2 & 0 & \sqrt{2}/2 & 1
\end{pmatrix}, \quad Z' = \begin{pmatrix}
1 & a & \sqrt{2}/2 \\
 a & 1 & \sqrt{2}/2 \\
\sqrt{2}/2 & \sqrt{2}/2 & 1
\end{pmatrix},$$

where $a, b \in \mathbb{R}$. Then, $\det(Z) = -(a+b)^2/2$ implies $a+b = 0$, and $\det(Z') = a(1-a)$ implies $a \geq 0$. Similarly, $b \geq 0$ using the principal submatrix of $Y$ indexed by $\{v_2, v_3, v_{23}\}$. This shows $a = b = 0$ and thus the entries of $Y$ at the positions $\{v_1, v_2\}$ and $\{v_2, v_{12}\}$ are uniquely specified. Remains to show that the entries are uniquely specified at the non-edges containing $v_{12}$ or $v_{23}$. For this we use Lemma 3.5. First the clique $\{v_2, v_3, v_{23}\}$ forces the pairs $\{v_1, v_{23}\}$ and $\{v_{13}, v_{23}\}$ and then the clique $\{v_1, v_2, v_{12}\}$ forces the pairs $\{v_{23}, v_{12}\}, \{v_{13}, v_{12}\}$, and $\{v_3, v_{12}\}$. Thus we have shown $Y = X$, which concludes the proof that $\text{egd}(H_3) = 3$.

Lastly, we verify that $\text{egd}(H_3 \setminus e) \leq 2$ for each edge $e \in E(H_3)$. If deleting the edge $e$ creates a cut node, then the result follows using Lemma 2.3.
Theorem 3.14 \[ \text{egd}(K) \]

The Gram dimension of \( G \) role in the characterization of the class \( K \). In this section we consider the graphs \( C \) labeling the set \( Y \) to show that the entries of \( V \) may now assume that we have a complete graph on each \( V \). Hence we have chosen the labeling and from the proof of Lemma 3.12. Indeed, as the subgraph of \( H \) playing the role of index 3 in the description of \( H_3 \) above) and, for each \( 3 \leq i < j \leq r \), we have the edges \( \{v_i, v_j\} \) and \( \{v_j, v_i\} \) as well as exactly one edge, call it \( e_{ij} \), from the set

\[ F_{ij} = \{\{v_i, v_j\}, \{v_i, v_{ij}\}, \{v_j, v_{ij}\}, \{v_{ij}, v_{ij}\}\}. \] (13)

Figure 3 shows the graph \( H_4 \) for the choice \( e_{34} = \{v_4, v_{13}\} \).

**Theorem 3.13** For any graph \( H_r \in \mathcal{H}_r \) \((r \geq 3)\), \( \text{egd}(H_r) = r \).

**Proof.** We label the nodes \( v_1, \ldots, v_r \) by \( e_1, \ldots, e_r \in \mathbb{R}^r \) and \( v_{ij} \) by \( (e_i + e_j)/\sqrt{2} \). Let \( X \in \mathcal{E}_n \) be their Gram matrix and \( x = \pi_{E(H_r)}(X) \in \mathcal{E}(H_r) \). Then \( X \) is an extreme point of \( \mathcal{E}_n \), we show that \( \text{fib}(x) = \{X\} \). For this let \( Y \in \text{fib}(x) \). We already know that \( Y[V_i] = X[V_i] \) for each \( i \in \{3, \ldots, r\} \). Indeed, as the subgraph of \( H_r \) induced by \( V_i \) is \( H_3 \), this follows from the way we have chosen the labeling and from the proof of Lemma 3.12. Hence we may now assume that we have a complete graph on each \( V_i \) and it remains to show that the entries of \( Y \) are uniquely specified at the non-edges that are not contained in some set \( V_i \) \((3 \leq i \leq r)\). For this note that the vectors labeling the set \( C_{ij} = \{v_i, v_j, v_{ij}\} \) are minimally linearly dependent. Using Lemma 3.3, one can verify that all remaining non-edges are forced using these sets \( C_{ij} \) and thus \( Y = X \). This shows that \( \text{egd}(H_r) \geq r \). \( \Box \)

In contrast to the graphs \( F_r \) and \( G_r \), we do not know whether \( H_r \in \mathcal{H}_r \) is a minimal forbidden minor for \( \mathcal{G}_{r-1} \) for \( r \geq 4 \).

### 3.3 Two special graphs: \( K_{3,3} \) and \( K_5 \)

In this section we consider the graphs \( K_{3,3} \) and \( K_5 \) which will play a special role in the characterization of the class \( \mathcal{G}_2 \). First we compute the extreme Gram dimension of \( K_{3,3} \). Note that its Gram dimension is \( \text{gd}(K_{3,3}) = 4 \) as \( K_{3,3} \) contains a \( K_4 \)-minor but it contains no \( K_5 \) and \( K_{2,2,2} \)-minor [17].

**Theorem 3.14** \( \text{egd}(K_{3,3}) = 2 \).
The proof can be sketched as follows: Let \( x \in \mathcal{E}(K_{3,3}) \). First we show that any matrix \( X \in \text{fib}(x) \) has rank at most 3 (Lemma 3.15). Next we show two technical lemmas which will be used to show that \( \text{fib}(x) \) contains at least two distinct elements. Therefore \( \text{fib}(x) \) must contain a matrix of rank at most 2 (see the paragraph after (4)) and thus \( \text{egd}(K_{3,3}) \leq 2 \).

**Lemma 3.15** For \( x \in \text{ext} \mathcal{E}(K_{3,3}) \), any \( X \in \text{fib}(x) \) has rank at most 3.

**Proof.** Let \( x \in \text{ext} \mathcal{E}(K_{3,3}) \) and let \( X \in \text{fib}(x) \) with rank \( X \geq 4 \). Let \( u_1, \ldots, u_6 \) be a Gram representation of \( X \) and choose a subset \( \{u_i : i \in I\} \) of linearly independent vectors with \( |I| = 4 \). Let \( E_I \) denote the set of edges of \( K_{3,3} \) induced by \( I \) and set

\[
U_I = \{U_{ii} : i \in I\} \cup \{U_{ij} : \{i,j\} \in E_I\}.
\]

Then \( U_I \) consists of linearly independent elements. Moreover, as any four nodes induce at least three edges in \( K_{3,3} \), we have that \( |U_I| \geq 4 + 3 = 7 \). By Lemma 2.7 \( U_I \) is contained in \( U_V \). We arrive at a contradiction since \( U_V \) has dimension 6 while \( U_I \) has dimension at least 7. \( \Box \)

Next we state two technical lemmas.

**Lemma 3.16** Let \( X, Z \in \mathcal{S}_n \) with \( X \succeq 0 \) and satisfying:

\[
Xz = 0 \implies z^TZz \geq 0, \quad Xz = 0, \quad z^TZz = 0 \implies Zz = 0. \tag{14}
\]

Then \( X + tZ \succeq 0 \) for some \( t > 0 \).

**Proof.** Up to an orthogonal transformation we may assume \( X = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \), where \( D \) is a diagonal matrix with positive diagonal entries. Correspondingly, write \( Z \) in block form: \( Z = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \). The conditions (14) show that \( C \succeq 0 \) and that the kernel of \( C \) is contained in the kernel of \( B \). This implies that \( X + tZ \succeq 0 \) for some \( t > 0 \). \( \Box \)

**Lemma 3.17** Let \( x \in \text{ext} \mathcal{E}(K_{3,3}) \), let \( X \in \text{fib}(x) \) be an extreme matrix of \( \mathcal{E}_6 \) of rank 3, with Gram representation \( \{u_1, \ldots, u_6\} \subseteq \mathbb{R}^3 \). Let \( V_1 = \{1, 2, 3\} \) and \( V_2 = \{4, 5, 6\} \) be the bipartition of the node set of \( K_{3,3} \). There exist matrices \( Y_1, Y_2 \in \mathcal{S}_3 \) such that \( Y_1 + Y_2 \succ 0 \) and

\[
\langle Y_k, U_{ii} \rangle = 0 \quad \forall i \in V_k \quad \forall k \in \{1, 2\} \quad \text{and} \quad \exists k \in \{1, 2\} \; \exists i, j \in V_k \; \langle Y_k, U_{ij} \rangle \neq 0.
\]

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Proof. Define $U_k = \langle U_{ii} : i \in V_k \rangle \subseteq W_k = \langle U_{ij} : i, j \in V_k \rangle \subseteq \mathcal{S}_3$ for $k = 1, 2$. By assumption, $\dim U_1 = 6$, thus $\dim U_1 = \dim U_2 = 3$ and $U_1 \cap U_2 = \{0\}$. Moreover, as $U_1^\perp \cap U_2^\perp = U_1^\perp = \{0\}$, we have that $\mathcal{S}_3 = U_1^\perp \oplus U_2^\perp$ and $W_1^\perp \cap W_2^\perp = \{0\}$.

Assume for contradiction that $\mathcal{S}_3^{++}$ is contained in $W_1^\perp \oplus W_2^\perp$. This implies that $W_1^\perp \oplus W_2^\perp = \mathcal{S}_3 = U_1^\perp \oplus U_2^\perp$ and thus $W_k = U_k$ as $U_k \subseteq W_k$. Hence, $W_1 \cap W_2 = \{0\}$.

As $\dim U_k = 3$, we have $\dim \langle u_i | i \in V_k \rangle \geq 2$ for $k = 1, 2$. Say, $\{u_1, u_2\}$ and $\{u_4, u_5\}$ are linearly independent. As $\dim \langle u_i | i \in [6] \rangle = 3$, there exists a non-zero vector $\lambda \in \mathbb{R}^4$ such that $0 \neq w = \lambda_1 u_1 + \lambda_2 u_2 = \lambda_3 u_4 + \lambda_4 u_5$. Hence we obtain that $w w^T \in W_1 \cap W_2$, contradicting $W_1 \cap W_2 = \{0\}$.

Hence we have shown that $\mathcal{S}_3^{++} \not\subseteq W_1^\perp \oplus W_2^\perp$. So there exists a positive definite matrix $Y$ which does not belong to $W_1^\perp \oplus W_2^\perp$. Write $Y = Y_1 + Y_2$, where $Y_k \in U_k^\perp$ for $k = 1, 2$. We may assume, say, that $Y_1 \notin W_1^\perp$. Thus $Y_1, Y_2$ satisfy the lemma. \hfill \square

Proof. (of Theorem 3.14). Let $x \in \text{ext} \mathcal{E}(K_{3,3})$, let $X \in \text{fib}(x)$ be an extreme point of $\mathcal{E}$ of rank 3, and let $\{u_1, \ldots, u_6\}$ be its Gram representation. Let $Y_1$ and $Y_2$ be the matrices provided by Lemma 3.17 and define the matrix $Z \in \mathcal{S}_6$ by $Z_{ij} = \langle Y_k, U_{ij} \rangle$ for $i, j \in V_k$, $k \in \{1, 2\}$, and $Z_{ij} = 0$ for $i \in V_1$, $j \in V_2$. By Lemma 3.17, $Z$ is a nonzero matrix with zero diagonal entries and zeros at the positions corresponding to the edges of $K_{3,3}$.

Next we show that $X + tZ \geq 0$ for some $t > 0$, using Lemma 3.16. For this it is enough to verify that \cite{14} holds. Assume $X z = 0$, i.e., $a := \sum_{i \in V_1} z_i u_i = -\sum_{j \in V_2} z_j u_j$. Then,

$$z^T Z z = \sum_{k=1,2} \sum_{i,j \in V_k} z_i z_j \langle Y_k, U_{ij} \rangle = \langle Y_1 + Y_2, aa^T \rangle \geq 0,$$

since $Y_1 + Y_2 \succ 0$. Moreover, $z^T Z z = 0$ implies $a = 0$ and thus $Z z = 0$ since, for $i \in V_k$, $(Z z)_i = \sum_{j \in V_k} \langle Y_k, U_{ij} \rangle z_j = \pm \langle Y_k, (u_i a^T + a u_i^T)/2 \rangle$. Hence, the matrix $X + tZ$ also belongs to the fiber of $x$. Combining with Lemma 3.15, we deduce that $\text{fib}(x)$ contains a matrix of rank at most 2. \hfill \square

We know that both graphs $K_{3,3}$ and $K_5$ belong to the class $\mathcal{F}_2$. We now show that they are in some sense maximal for this property.

Lemma 3.18 Let $G \neq K_{3,3}, K_5$ be a 2-connected graph that contains $K_5$ or $K_{3,3}$ as a subgraph. Then $G$ contains $H_3$ as a minor and thus $\text{egd}(G) \geq 3$.

Proof. The proof is based on the following observations. If $G$ is a 2-connected graph containing strictly $K_5$ or $K_{3,3}$ as a subgraph, then $G$ has
a minor \( H \) which is one of the following graphs: (a) \( H \) is \( K_5 \) with one more node adjacent to two nodes of \( K_5 \), (b) \( H \) is \( K_{3,3} \) with one more edge added, (c) \( H \) is \( K_{3,3} \) with one more node adjacent to two adjacent nodes of \( K_{3,3} \). Then \( H \) contains a \( H_3 \) subgraph in cases (a) and (b), and a \( H_3 \) minor in case (c) (easy verification). Hence, \( \text{egd}(G) \geq \text{egd}(H_3) = 3. \) \hfill \( \square \)

4 Forbidden minor characterization of \( \mathcal{G}_2 \)

In this section we characterize the class \( \mathcal{G}_2 \) of graphs with extreme Gram dimension at most 2. We show that \( G \in \mathcal{G}_2 \) if and only if \( G \) is a clique 0- and 1-sum of some graphs which either (i) have at most 5 nodes, or are (ii) \( K_{3,3} \), or (iii) a minor of \( T \otimes K_2 \) for some tree \( T \).

4.1 The main result

Here we formulate several characterizations for the class \( \mathcal{G}_2 \) and we outline the proof. By Theorem 3.1 we know that
\[
\text{lag}(G) \leq 2 \implies G \in \mathcal{G}_2
\] (15)
and, by Theorem 3.6 and Lemma 3.12 we know that
\[
G \in \mathcal{G}_2 \implies G \text{ has no minors } F_3, \ H_3.
\] (16)

The three graph properties involved in (15), (16) are not equivalent in general: \( K_5, K_{3,3} \in \mathcal{G}_2 \) (Theorem 3.14), but \( \text{lag}(K_5) = \text{lag}(K_{3,3}) = 3 \) (see Section 5). However, these two graphs are exceptional since they cannot occur as proper subgraphs of a 2-connected graph with no \( H_3 \) minor (Lemma 3.18). As the class \( \mathcal{G}_2 \) is closed under taking clique 0- and 1-sums, it suffices to characterize the 2-connected graphs in \( \mathcal{G}_2 \). We show the following result:

**Theorem 4.1** Let \( G \) be a 2-connected graph on \( n \geq 6 \) nodes and assume that \( G \neq K_{3,3} \). Then the following assertions are equivalent.

(i) \( G \in \mathcal{G}_2 \), i.e., \( \text{egd}(G) \leq 2. \)

(ii) \( G \) has no minors \( F_3 \) or \( H_3. \)

(iii) \( \text{lag}(G) \leq 2, \) i.e., \( G \) is a minor of \( T \otimes K_2 \) for some tree \( T. \)

In the rest of Section 4, we prove the implication (ii) \( \implies \) (iii). The proof is in two steps. First we consider the chordal case and show:
(1) **The chordal case:** If \( G \in \mathcal{F}(F_3, H_3) \) is chordal, then \( G \) is a contraction minor of \( T \boxtimes K_2 \) for some tree \( T \) (Section 4.2 Theorem 4.3).

Then we reduce the general case to the chordal case and show:

(2) **Reduction to the chordal case:** Any graph \( G \in \mathcal{F}(F_3, H_3) \) is subgraph of a chordal graph \( G' \in \mathcal{F}(F_3, H_3) \).

For case (2) we first exclude \( K_4 \) instead of \( H_3 \) (Section 4.3 Theorem 4.6) and then we derive from it the general result (Section 4.4 Theorem 4.11).

### 4.2 The case of chordal graphs

Our goal in this section is to characterize the 2-connected chordal graphs \( G \) with \( \text{egd}(G) \leq 2 \). By Lemma 3.18 if \( G \neq K_5 \) has \( \text{egd}(G) \leq 2 \), then \( \omega(G) \leq 4 \). Denote by \( \mathcal{C} \) the family of all 2-connected chordal graphs with \( \omega(G) \leq 4 \). Any graph \( G \in \mathcal{C} \) is a clique 2- or 3-sum of \( K_3 \)'s and \( K_4 \)'s. Note that \( F_3 \) belongs to \( \mathcal{C} \) and has \( \text{egd}(F_3) = 3 \). On the other hand, any graph \( G = T \boxtimes K_2 \) where \( T \) is a tree, belongs to \( \mathcal{C} \) and has \( \text{egd}(G) = 2 \). These graphs are “special clique 2-sums” of \( K_4 \)'s, as they satisfy the following property: every 4-clique has at most two edges which are cutsets and these two edges are not adjacent. This motivates the following definitions, useful in the proof of Theorem 4.3 below.

**Definition 4.2** Let \( G \in \mathcal{C} \) (i.e., \( G \) is chordal 2-connected with \( \omega(G) \leq 4 \)).

(i) An edge of \( G \) is called free if it belongs to exactly one maximal clique (i.e., its endpoints do not form a cutset) and non-free otherwise.

(ii) A 3-clique in \( G \) is called free if it contains at least one free edge.

(iii) A 4-clique in \( G \) is called free if it does not have two adjacent non-free edges. A free 4-clique can be partitioned as \( \{a, b\} \cup \{c, d\} \), called its two sides, where only \( \{a, b\} \) and \( \{c, d\} \) can be non-free (i.e., cutsets).

(iv) \( G \) is called free if all its maximal cliques are free.

For instance, \( F_3, K_5 \setminus e \) (the clique 3-sum of two \( K_4 \)'s) are not free. Hence any free graph in \( \mathcal{C} \) is a clique 2-sum of free \( K_3 \)'s and free \( K_4 \)'s. Note also that \( \text{lag}(K_5 \setminus e) = 3 \). We now show that for a graph \( G \in \mathcal{C} \) the property of being free is equivalent to having \( \text{lag}(G) \leq 2 \) and also to having \( \text{egd}(G) \leq 2 \).

**Theorem 4.3** Let \( G \in \mathcal{C} \setminus \{K_5 \setminus e\} \). The following assertions are equivalent.
(i) \( G \) has no minors \( F_3 \) or \( H_3 \).

(ii) \( G \) does not contain \( F_3 \) as a subgraph.

(iii) \( G \) is free.

(iv) \( G \) is a contraction minor of \( T \boxtimes K_2 \) for some tree \( T \).

(v) \( \text{egd}(G) \leq 2 \).

**Proof.** (i) \( \Rightarrow \) (ii) is clear and (iv) \( \Rightarrow \) (v) \( \Rightarrow \) (i) follow from earlier results. 

**Proof of (ii) \( \Rightarrow \) (iii):** Assume that (ii) holds. First we show that \( G \) does not contain a subgraph \( K_5 \setminus e \). For this assume that \( G[U] = K_5 \setminus e \) for some \( U \subseteq V(G) \). As \( G \neq K_5 \setminus e \) and \( G \) is 2-connected chordal, there is a node \( u \notin U \) which is adjacent to two adjacent nodes of \( U \) and then one can find a \( F_3 \) subgraph in \( G \). Therefore, \( G \) is a clique 2-sum of \( K_3 \)'s and \( K_4 \)'s. We now show that each of them is free.

Suppose first that \( C = \{a, b, c\} \) is a maximal 3-clique which is not free. Then, there exist nodes \( u, v, w \notin C \) such that \( \{a, b, u\}, \{a, c, v\}, \{b, c, w\} \) are cliques in \( G \). Moreover, \( u, v, w \) are pairwise distinct (if \( u = v \) then \( C \cup \{u\} \) is a clique, contradicting maximality of \( C \)) and we find a \( F_3 \) subgraph in \( G \).

Suppose now that \( C = \{a, b, c, d\} \) is a 4-clique which is not free and, say, both edges \( \{a, b\} \) and \( \{a, c\} \) are non-free. Then, there exist nodes \( u, v \notin C \) such that \( \{a, b, u\} \) and \( \{a, c, v\} \) are cliques. Moreover, \( u \neq v \) (else we find a \( K_5 \setminus e \) subgraph) and thus we find a \( F_3 \) subgraph in \( G \). Thus (iii) holds.

**Proof of (iii) \( \Rightarrow \) (iv):** Assume that \( G \) is free, \( G \neq K_4, K_3 \) (else we are done). When all maximal cliques are 4-cliques, it is easy to show using induction on \( |V(G)| \) that \( G = T \boxtimes K_2 \), where \( T \) is a tree and each side of a 4-clique of \( G \) corresponds to a node of \( T \).

Assume now that \( G \) has a maximal 3-clique \( C = \{a, b, c\} \). Say, \( \{b, c\} \) is free and \( \{a, b\} \) is a cutset. Write \( V(G) = V' \cup V'' \cup \{a, b\} \), where \( V'' \) is the (vertex set of the) component of \( G \setminus \{a, b\} \) containing \( c \), and \( V' \) is the union of the other components. Now replace node \( a \) by two new nodes \( a', a'' \) and replace \( C \) by the 4-clique \( C' = \{a', a'', b, c\} \). Moreover, replace each edge \( \{u, a\} \) by \( \{u, a'\} \) if \( u \in V' \) and by \( \{u, a''\} \) if \( u \in V'' \). Let \( G' \) be the graph obtained in this way. Then \( G' \in \mathcal{C} \) is free, \( G' \) has one less maximal 3-clique than \( G \), and \( G = G' / \{a', a''\} \). Iterating, we obtain a graph \( \tilde{G} \) which is a clique 2-sum of free \( K_4 \)'s and contains \( G \) as a contraction minor. By the above, \( \tilde{G} = T \boxtimes K_2 \) and thus \( G \) is a contraction minor of \( T \boxtimes K_2 \). \( \square \)
4.3 Structure of the graphs with no $F_3$ or $K_4$ minor

In this section we investigate the class $\mathcal{F}(F_3,K_4)$. We start with two technical lemmas.

**Lemma 4.4** Let $G$ and $M$ be two 2-connected graphs, let $\{x,y\} \notin E(G)$ be a cutset in $G$, and let $r \geq 2$ be the number of components of $G \setminus \{x,y\}$.

(i) Assume that $G \in \mathcal{F}(M)$, but the graph $G + \{x,y\}$ has a $M$-minor with $M$-partition $\{V_i : i \in V(M)\}$. If $x \in V_i$ and $y \in V_j$, then $M \setminus \{i,j\}$ has at least $r$ components and thus $i \neq j$.

(ii) Assume that $M$ does not have two adjacent nodes forming a cutset in $M$. If $G \in \mathcal{F}(M)$, then $G + \{x,y\} \in \mathcal{F}(M)$.

**Proof.** (i) Let $C_1, \ldots, C_r \subseteq V(G)$ be the node sets of the components of $G \setminus \{x,y\}$. As $G$ is 2-connected, there is an $x$–$y$ path $P_s$ in $G[C_s \cup \{x,y\}]$ for each $s \in [r]$. Now let $U$ be a component of $M \setminus \{i,j\}$. By the definition of the $M$-partition, the graph $G[\bigcup_{k \in U} V_k]$ is connected. As $x, y \notin \bigcup_{k \in U} V_k$, we deduce that $\bigcup_{k \in U} V_k \subseteq C_s$ for some $s \in [r]$. In other words, every component of $M \setminus \{i,j\}$ corresponds to exactly one component of $G \setminus \{x,y\}$. Assume for contradiction that $M \setminus \{i,j\}$ has less than $r$ components. Then there is at least one component $C_s$ which does not correspond to any component of $M \setminus \{i,j\}$. That is, $(\bigcup_{k \neq i,j} V_k) \cap C_s = \emptyset$, so that $C_s \subseteq V_i \cup V_j$. Hence the path $P_s$ is contained in $G[V_i \cup V_j]$, thus $\{V_i : i \in V(M)\}$ remains an $M$-partition of $G$ and we find a $M$-minor in $G$, a contradiction. Therefore, $M \setminus \{i,j\}$ has at least $r \geq 2$ components. This implies that $\{i,j\}$ is a cutset of $M$ and thus $i \neq j$ since $M$ is 2-connected.

(ii) Assume $G + \{x,y\}$ has a $M$-minor, with corresponding $M$-partition $\{V_i : i \in V(M)\}$. By (i), the nodes $x$ and $y$ belong to two distinct classes $V_i$ and $V_j$ and $\{i,j\}$ is a cutset in $M$. This implies that $\{i,j\} \notin E(M)$ and thus $M$ is a minor of $G$, a contradiction. \hfill \Box

**Lemma 4.5** Let $G \in \mathcal{F}(K_4)$ be a 2-connected graph and let $\{x,y\} \notin E(G)$. If there are at least three (internally vertex) disjoint paths from $x$ to $y$, then $\{x,y\}$ is a cutset and $G \setminus \{x,y\}$ has at least 3 components.

**Proof.** If $P_1, P_2, P_3$ are distinct vertex disjoint paths from $x$ to $y$, then $P_1 \setminus \{x,y\}, P_2 \setminus \{x,y\}$ and $P_3 \setminus \{x,y\}$ lie in distinct components of $G \setminus \{x,y\}$, for if not one would find a homeomorph of $K_4$. \hfill \Box

We now show the main result of this section.
**Theorem 4.6** Let $G \in F(F_3, K_4)$ be a 2-connected graph. Then there exists a chordal graph $Q \in F(F_3, K_4)$ containing $G$ as a subgraph.

**Proof.** Let $G \in F(F_3, K_4)$ be 2-connected. If $\{x, y\} \notin E(G)$ is such that there exist at least three disjoint paths in $G$ from $x$ to $y$, then we can add the edge $\{x, y\}$ without creating a $K_4$ or $F_3$-minor: $G + \{x, y\} \in F(F_3, K_4)$, this follows from Lemma 4.4 (applied to $M = F_3$ and $K_4$) and Lemma 4.5. So we can add edges iteratively until obtaining a graph $\hat{G} \in F(F_3, K_4)$ containing $G$ as a subgraph and satisfying:

$$\forall \{x, y\} \notin E(\hat{G}) \text{ there are at most two disjoint } x - y \text{ paths in } \hat{G}. \quad (17)$$

If $\hat{G}$ is chordal we are done. So consider a chordless circuit $C$ in $\hat{G}$. Note that any circuit $C'$ distinct from $C$, which meets $C$ in at least two nodes, meets $C$ in exactly two nodes that are adjacent (easy consequence of (17)). Call an edge of $C$ busy if it is contained in some circuit $C' \neq C$. If $e_1 \neq e_2$ are two busy edges of $C$ and $C_i \neq C$ is a circuit containing $e_i$, then $C_1, C_2$ are (internally) disjoint (use (17)). Hence $C$ can have at most two busy edges, for otherwise one would find a $F_3$-minor in $\hat{G}$.

We now show how to triangulate $C$ without creating a $K_4$ or $F_3$-minor: If $C$ has two busy edges denoted, say, $\{v_1, v_2\}$ and $\{v_k, v_{k+1}\}$ (possibly $k = 2$), then we add the edges $\{v_1, v_i\}$ for $i \in \{3, \ldots, k\}$ and the edges $\{v_i, v_{i+1}\}$ for $i \in \{k + 2, \ldots, |C|\}$, see Figure 4 a). If $C$ has only one busy edge $\{v_1, v_2\}$, add the edges $\{v_i, v_1\}$ for $i \in \{3, \ldots, |C| - 1\}$, see Figure 4 b). (If $C$ has no busy edge then $G = C$, triangulate from any node and we are done).

![Figure 4: Triangulating a chordless circuit with a) two or b) one busy edge.](image)

Let $Q$ denote the graph obtained from $\hat{G}$ by triangulating all its chordless circuits in this way. Hence $Q$ is a chordal extension of $\hat{G}$ (and thus of $G$). We show that $Q \in F(F_3, K_4)$. First we see that $Q \in F(K_4)$ by applying iteratively Lemma 4.4 (ii) (for $M = K_4$): For each $i \in \{3, \ldots, k\}$, $\{v_1, v_i\}$

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is a cutset of \( \hat{G} \) and of \( \hat{G} + \{\{v_i, v_j\} : j \in \{3, \ldots, i - 1\}\} \) (and analogously for the other added edges \( \{v_k, v_i\} \)). Hence \( Q \) is a clique 2-sum of triangles. We now verify that each triangle is free which will conclude the proof, using Theorem 4.3.

For this let \( \{a, b, c\} \) be a triangle in \( Q \). First note that if (say) \( \{a, b\} \in E(Q) \setminus E(\hat{G}) \), then \( a, b, c \) lie on a common chordless circuit \( C \) of \( \hat{G} \). Indeed, let \( C \) be a chordless circuit of \( \hat{G} \) containing \( a, b \) and assume \( c \notin C \). By (17), \( \hat{G} \setminus \{a, b\} \) has at most two components and thus there is a path from \( c \) to one of the two paths composing \( C \setminus \{a, b\} \). Together with the triangle \( \{a, b, c\} \) this gives a homeomorph of \( K_4 \) in \( Q \), contradicting \( Q \in F(K_4) \), just shown above. Hence the triangle \( \{a, b, c\} \) lies in \( C \) and thus has a free edge.

Suppose now that \( \{a, b, c\} \) is a triangle contained in \( \hat{G} \). If it is not free then there is a \( F_3 \) on \( \{a, b, c, x, y, z\} \) where \( x \) (resp., \( y \), and \( z \)) is adjacent to \( a, b \) (resp., \( a, c \), and \( b, c \)). Say \( \{x, a\} \in E(Q) \setminus E(\hat{G}) \) (as there is no \( F_3 \) in \( \hat{G} \)). Then \( x, a, b \) lie on a chordless circuit \( C \) of \( G \) and \( \{x, b\} \in E(\hat{G}) \) (since \( \{a, b\} \) is a busy edge). Moreover, \( c, y, z \notin C \) for otherwise we get a \( K_4 \)-minor in \( Q \). Then delete the edge \( \{x, a\} \) and replace it by the \( \{x, a\} \)-path along \( C \). Do the same for any other edge of \( E(Q) \setminus E(\hat{G}) \) connecting \( y \) and \( z \) to \( \{a, b, c\} \). After that we get a \( F_3 \)-minor in \( \hat{G} \), a contradiction.

\[ \square \]

### 4.4 Structure of the graphs with no \( F_3 \) or \( H_3 \) minor

Here we investigate the graphs \( G \in F(F_3, H_3) \). By the results in Section 4.3 we may assume that \( G \) contains some homeomorph of \( K_4 \). Figure 5 shows a homeomorph of \( K_4 \), where the original nodes are denoted as 1,2,3,4 and called its corners, and the wiggled lines correspond to subdivided edges (i.e., to paths \( P_{ij} \) between the corners \( i \neq j \in [4] \)).

![Figure 5](image)

Figure 5: A homeomorph of \( K_4 \) and its two sides (cf. Lemma 4.7)

To help the reader visualize \( F_3 \) and \( H_3 \) we use Figure 6. Notice the special role of node 5 in \( H_3 \) (denoted by a square) and of the (dashed) triangle \( \{1, 2, 3\} \).
Figure 6: The graphs $H_3$ and $F_3$.

The starting point of the proof is to investigate the structure of homeomorphs of $K_4$ in a graph of $\mathcal{F}(H_3)$.

**Lemma 4.7** Let $G$ be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let $H$ be a homeomorph of $K_4$ contained in $G$. Then there is a partition of the corner nodes of $H$ into $\{1, 3\}$ and $\{2, 4\}$ for which the following holds.

(i) Only the paths $P_{13}$ and $P_{24}$ can have more than 2 nodes.

(ii) Every component of $G \setminus H$ is connected to $P_{13}$ or to $P_{24}$.

Then $P_{13}$ and $P_{24}$ are called the two sides of $H$ (cf. Figure 5).

**Proof.** We use the graphs from Figure 7 which all contain a subgraph $H_3$.

**Case 1:** $H = K_4$. If $G \setminus H$ has a unique component $C$ then $|C| \geq 2$ as $n \geq 6$. If $C$ is connected to two nodes of $H$, then the conclusion of the lemma holds. Otherwise, $C$ is connected to at least three nodes in $H$ and then the graph from Figure 7a is a minor of $G$, a contradiction.

If there are at least two components in $G \setminus H$, then they cannot be connected to two adjacent edges of $H$ for, otherwise, the graph of Figure 7b is a minor of $G$, a contradiction. Hence the lemma holds.

**Case 2:** $H \neq K_4$. Say, $P_{13}$ has at least 3 nodes. Then the edges $\{1, i\}$, $\{3, i\}$ ($i = 2, 4$) cannot be subdivided (else $H$ is a homeomorph of $H_3$). So (i) holds. We now show (ii). Indeed, if a component of $G \setminus H$ is connected to both $P_{13}$ and $P_{24}$, then at least one of the graphs in Figure 7c and d) will be a minor of $G$, a contradiction. \qed

Lemma 4.7 implies that there is no path with at least 3 nodes between the sides of a $K_4$-homeomorph. We now show that, moreover, there is no additional edge between the two sides. More precisely:

**Lemma 4.8** Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let $H$ be a homeomorph of $K_4$ contained in $G$. Then there exists no edge between the two sides of $H$ except between their endpoints.
Proof. Say, $P_{13}$ and $P_{24}$ are the two sides of $H$. Assume for a contradiction that $\{a, b\} \in E(G)$, where $a$ lies on $P_{13}$ and $b$ on $P_{24}$.

Assume first that $a$ is an internal node of $P_{13}$ and $b$ is an internal node of $P_{24}$. If $|V(H)| = 6$, then $H = K_{3,3}$ and Lemma 3.18 implies that $G$ has a $H_3$ minor, a contradiction. Hence, $|V(H)| > 6$ and we can assume w.l.o.g. that the path from 1 to $a$ within $P_{13}$ has at least 3 nodes. Then $G$ contains a homeomorph of $K_4$ with corner nodes 1, $a$, 2, 4, where the two paths from 1 to $a$ and from 1 to $b$ (via 2) have at least 3 nodes, giving a $H_3$ minor and thus a contradiction.

Assume now that only $a$ is an internal node of $P_{13}$ and, say $b = 2$. If $|V(H)| = 5$, then $G \setminus H$ has at least one component. By Lemma 4.7, this component connects either to the path $P_{13}$ or to the edge $\{2, 4\}$. In both cases, it is easy to verify that one of the graphs in Figure 8 will be a minor of $G$, a contradiction since all of them have a $H_3$ subgraph. On the other hand, if $|V(H)| \geq 6$, then one of the paths from 1 to $a$, from $a$ to 3 (within $P_{13}$), or $P_{24}$ is subdivided. This implies that $G$ contains a homeomorph of $K_4$ with corner nodes $a, 1, 2, 4$ or $a, 2, 3, 4$, which thus contains two adjacent subdivided edges, giving a $H_3$ minor.

\[\square\]

Figure 8: Bad subgraphs in the proof of Lemma 4.8

Lemmas 4.7 and 4.8 imply directly:
Corollary 4.9  Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let $H$ be a homeomorph of $K_4$ contained in $G$. Then the endnodes of at least one of the two sides of $H$ form a cutset in $G$. Moreover, if $P_{13}$ is a side of $H$ and its endnodes $\{1,3\}$ do not form a cutset, then $P_{13} = \{1,3\}$ and there is no component of $G \setminus H$ which is connected to $P_{13}$.

We now show that one may add edges to $G$ so that all minimal homeomorphs of $K_4$ are 4-cliques, without creating a $F_3$ or $H_3$ minor.

Lemma 4.10  Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(F_3, H_3)$ on $n \geq 6$ nodes and let $H$ be a homeomorph of $K_4$ contained in $G$. The graph obtained by adding to $G$ the edges between the endpoints of the sides of $H$ belongs to $\mathcal{F}(F_3, H_3)$.

Proof.  Say $P_{13}$ and $P_{24}$ are the sides of $H$. Assume $|V(P_{13})| \geq 3$ and $\{1,3\} \not\in E(G)$. By Corollary 4.9, $\{1,3\}$ is a cutset in $G$. We show that $\tilde{G} = G + \{1,3\} \in \mathcal{F}(F_3, H_3)$. First, applying Lemma 4.4 (ii) with $M = H_3$ and $\{x,y\} = \{1,3\}$, we get that $\tilde{G} \in \mathcal{F}(H_3)$.

Next, assume for contradiction that $\tilde{G}$ has a $F_3$ minor, with $F_3$-partition $\{V_i : i \in [6]\}$, where we use the same labeling as in Figure 6. Applying Lemma 4.4 (i) with $M = F_3$ and $\{x,y\} = \{1,3\}$, we see that the nodes 1 and 3 belong to distinct classes of the $F_3$-partition, which correspond to a cutset of $F_3$. Say, $1 \in V_1$ and $3 \in V_2$. Then the nodes 2 and 4 do not lie in $V_1 \cup V_2$ (for otherwise, one would have an $F_3$-partition in $G$). Next we show that the nodes 2 and 4 do not belong to the same class of the $F_3$-partition. Assume for contradiction that $2, 4 \in V_k$. If $\{2,4\}$ is not a cutset in $G$ then, by Corollary 4.9, $P_{24} = \{2,4\}$ and no component of $G \setminus H$ connects to $\{2,4\}$. Hence $V_k = \{2,4\}$ and we can move node 2 to the class $V_1$, so that we obtain a $F_3$-partition of $G$, a contradiction. If $\{2,4\}$ is a cutset of $G$, then every component of $G \setminus \{2,4\}$ except the one containing 1 and 3 has to lie within $V_k$, so we can again move node 2 to $V_1$ and obtain a $F_3$-partition of $G$.

Accordingly, the nodes 1, 2, 3 and 4 belong to distinct classes and we can assume w.l.o.g. that $2 \not\in V_6$. Observe that every $1 - 2$ path in $G$ is either the edge $\{1,2\}$ or meets the nodes 3 or 4. Similarly, every $2 - 3$ path in $G$ is either the edge $\{2,3\}$ or meets the nodes 1 or 4. An easy case analysis shows that whatever the position of nodes 2 and 4 in the $F_3$-partition we always find a $1 - 2$ or a $2 - 3$ path violating the above conditions. \(\square\)

We are now ready to show the main result of this section.

Theorem 4.11  Let $G \neq K_{3,3}$ be 2-connected in $\mathcal{F}(F_3, H_3)$ on $n \geq 6$ nodes. There exists a chordal graph $Q \in \mathcal{F}(F_3, H_3)$ containing $G$ as a subgraph.
Proof. If \( G \in \mathcal{F}(F_3, K_4) \) then we are done by Theorem 4.6. Otherwise, we augment the graph \( G \) by adding the edges between the endpoints of the sides of every homeomorph of \( K_4 \) contained in \( G \). Let \( \widehat{G} \) be the graph obtained in this way. By Lemma 4.10, we know that \( \widehat{G} \in \mathcal{F}(F_3, H_3) \). Hence, for each \( K_4 \)-homeomorph \( H \) in \( G \), its corners form a 4-clique. Moreover, if \( C, C' \) are two distinct 4-cliques of \( G \), then \( C \cap C' \) is contained in a side of \( C \) and \( C' \).

Consider a 4-clique \( C = \{1, 2, 3, 4\} \) in \( \widehat{G} \), say with sides \( \{1, 3\}, \{2, 4\} \) (so each component of \( \widehat{G} \setminus C \) connects to \( \{1, 3\} \) or to \( \{2, 4\} \), by Lemma 4.7). Pick an edge \( f \) between the two sides (i.e., \( f = \{i, j\} \) with \( i \in \{1, 3\}, j \in \{2, 4\} \)) and delete this edge \( f \) from \( G \). We repeat this process with every 4-clique in \( \widehat{G} \) and obtain the graph \( G_0 = \widehat{G} \setminus \{f_1, \ldots, f_k\} \), if \( \widehat{G} \) has \( k \) 4-cliques.

By construction, \( G_0 \) belongs to \( \mathcal{F}(F_3, K_4) \) and is 2-connected. Hence, we can apply Theorem 4.6 to \( G_0 \) and obtain a chordal graph \( Q_0 \in \mathcal{F}(F_3, K_4) \) containing \( G_0 \) as a subgraph. Hence, \( Q_0 \) is a clique 2-sum of free triangles. It suffices now to show that the augmented graph \( Q = Q_0 + \{f_i : i \in [k]\} \) is a clique 2-sum of free \( K_3 \)'s and \( K_4 \)'s. Then \( Q \) is a chordal graph in \( \mathcal{F}(F_3, H_3) \) (by Theorem 4.3) containing \( \widehat{G} \) and thus \( G \), and the proof is completed.

For this, consider again a 4-clique \( C = \{1, 2, 3, 4\} \) in \( \widehat{G} \) with sides \( \{1, 3\} \) and \( \{2, 4\} \). Then, each component of \( \widehat{G} \setminus C \) connects to \( \{1, 3\} \) or \( \{2, 4\} \). We claim that the same holds for each component of \( Q_0 \setminus C \). Indeed, a component of \( Q_0 \setminus C \) is a union of some components of \( G \setminus C \). Thus it connects to two nodes (to 1, 3, or to 2, 4), or to at least three nodes of \( C \). But the latter case cannot occur since we would then find a \( K_4 \) minor in \( Q_0 \).

Assume that the edge \( f = \{1, 4\} \) was deleted from the 4-clique \( C \) when making the graph \( G_0 \). We now show that adding it back to \( Q_0 \) results in a free graph. Indeed, by adding the edge \( \{1, 4\} \) we only replace the two maximal 3-cliques \( \{1, 3, 4\} \) and \( \{1, 2, 4\} \) by a new maximal 4-clique \( \{1, 2, 3, 4\} \), which is free. We iterate this process for each of the edges \( f_1, \ldots, f_k \) and obtain that \( Q = Q_0 + \{f_i : i \in [k]\} \) is a free graph in \( C \). \( \square \)

5 Concluding remarks

Colin de Verdière [5] introduced the largeur d’arborescence \( \lambda(G) \) as upper bound for his graph parameter \( \nu(G) \), which is defined as the maximum corank of a matrix \( A \in \mathcal{S}_n^+ \) satisfying: \( A_{ij} = 0 \) if and only if \( i \neq j \) and \( \{i, j\} \not\in E(G) \), and the following non-degeneracy condition (known as the strong Arnold property):

\[
AX = 0, \ X \in \mathcal{S}_n, \ X_{ij} = 0 \ \forall \{i, j\} \in V \cup E \implies X = 0.
\]
He shows that \( \nu(G) \) is minor monotone, \( \nu(G) \leq \text{la}_{\square}(G) \), with equality for the graphs \( G_r \): \( \nu(G_r) = \text{la}_{\square}(G_r) = r \), as well as
\[
\text{la}_{\square}(G) \leq 1 \iff \nu(G) \leq 1 \iff G \text{ has no minor } K_3.
\]
Kotlov \cite{p14} shows:
\[
\text{la}_{\square}(G) \leq 2 \iff \nu(G) \leq 2 \iff G \text{ has no minors } F_3, K_4.
\]

The most work is showing that \( \text{la}_{\square}(G) \leq 2 \) if \( G \in \mathcal{F}(K_4, F_3) \). In fact, this also follows from our characterization of the class \( \mathcal{F}(K_4, F_3) \). Indeed, if \( G \in \mathcal{F}(K_4, F_3) \) is 2-connected then we have shown that \( G \) is subgraph of \( G' \) which is a clique 2-sum of free triangles. Now our argument in the proof of Theorem \ref{thm:characterization} also shows that \( G' \) is a contraction minor of \( T \square K_2 \) for some tree \( T \) (as each triangle of \( G' \) arises as contraction of a 4-clique which can be replaced by a 4-circuit). In this sense our characterization is a refinement of Kotlov’s result tailored to our needs.

We now characterize the graphs with \( \text{la}_{\square}(G) \leq 2 \). The wheel \( W_5 \) is obtained from the circuit \( C_4 \) by adding a node adjacent to all nodes of \( C_4 \).

**Theorem 5.1** For a graph \( G \), \( \text{la}_{\square}(G) \leq 2 \) if and only if \( G \in \mathcal{F}(F_3, H_3, W_5) \).

**Proof.** We already know that \( \text{la}_{\square}(G) \geq \text{egd}(G) = 3 \) for \( G = F_3, H_3 \). Suppose for contradiction that \( \text{la}_{\square}(W_5) \leq 2 \). Then \( \text{la}_{\square}(W_5) = \text{la}_{\square}(H) \) where \( H \) is a chordal extension of \( W_5 \) and \( H \) is a contraction minor of some \( T \square K_2 \). As \( W_5 \) is not chordal, \( H \) contains \( W_5 \) with one added chord on its 4-circuit, i.e., \( H \) contains \( K_5 \setminus e \) and thus \( \text{la}_{\square}(H) \geq \text{la}_{\square}(K_5 \setminus e) = 3 \). Therefore, \( F_3, H_3, W_5 \) are forbidden minors for the property \( \text{la}_{\square}(G) \leq 2 \). Conversely, assume that \( G \in \mathcal{F}(F_3, H_3, W_5) \) is 2-connected, we show that \( \text{la}_{\square}(G) \leq 2 \). This is clear if \( G \) has \( n \leq 4 \) nodes, or if \( G \) has \( n = 5 \) nodes and it has a node of degree 2. If \( G \) has \( n = 5 \) nodes and each node has degree at least 3, then one can easily verify that \( G \) contains \( W_5 \). If \( G \) has \( n \geq 6 \) nodes then \( \text{la}_{\square}(G) \leq 2 \) follows from Theorem \ref{thm:la-square}(since \( G \neq K_{3,3} \) as \( W_5 \leq K_{3,3} \)). \( \square \)

Summarizing, we have \( \nu(G) \leq \text{la}_{\square}(G) \) and \( \text{egd}(G) \leq \text{la}_{\square}(G) \leq \text{la}_{\square}(G) \). Moreover, \( \text{egd}(G) = \nu(G) \) if \( \nu(G) \leq 2 \). Also, \( \nu(K_n) = n - 1 \) \cite{p13} and thus \( \nu(K_n) > \text{la}_{\square}(K_n) \geq \text{egd}(K_n) \) if \( n \geq 4 \). An interesting open question is whether the inequality \( \text{egd}(G) \leq \nu(G) \) holds in general. We point out the analogous inequality: \( \nu^=(G) \leq \text{gd}(G) \), shown in \cite{p17}. The parameter \( \nu^=(G) \) is the analogue of \( \nu(G) \) studied by van der Holst \cite{p13} (same definition as \( \nu(G) \), but now requiring only that \( A_{ij} = 0 \) for \( \{i, j\} \in E(G) \) and allowing
zero entries at positions on the diagonal and at edges), and \( \nu = \) satisfies: 
\[ \nu(G) \leq \nu^\pm(G). \]

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**References**


