On bounded rank positive semidefinite matrix completions of extreme partial correlation matrices

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Abstract

We study a new geometric graph parameter $\mathrm{egd}(G)$, defined as the smallest integer $r \geq 1$ for which any partial symmetric matrix which is completable to a correlation matrix and whose entries are specified at the positions of the edges of G, can be completed to a matrix in the convex hull of correlation matrices of rank at most r. This graph parameter is motivated by its relevance to the bounded rank Grothendieck constant: $\mathrm{egd}(G) \leq r$ if and only if the rank-r Grothendieck constant of G is equal to 1. The parameter $\mathrm{egd}(G)$ is minor monotone. We identify several classes of forbidden minors for $\mathrm{egd}(G) \leq r$ and give the full characterization for the case $r = 2$. We show an upper bound for $\mathrm{egd}(G)$ in terms of a new tree-width-like parameter $\mathrm{la}_{\boxtimes}(G)$, defined as the smallest r for which G is a minor of the strong product of a tree and K_r. We show that, for $G \neq K_{3,3}$ 2-connected on at least 6 nodes, $\mathrm{egd}(G) \leq 2$ if and only if $\mathrm{la}_{\boxtimes}(G) \leq 2$.

Keywords: matrix completion, semidefinite programming, correlation matrix, Gram representation, graph minor, tree-width, Grothendieck constant.

1 Introduction

In this paper we investigate a new graph invariant $\mathrm{egd}(G)$, motivated by its relevance to bounded rank Grothendieck inequalities and to bounded

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rank semidefinite matrix completions. This new geometric graph parameter has also some close connections to some Colin de Verdière spectral graph parameters and to some topological tree-width-like graph parameters.

We start with some notation. Throughout, $\mathcal{S}_n$ will denote the set of $n \times n$ symmetric matrices, $\mathcal{S}_n^+$ is the cone of positive semidefinite (psd) matrices and $\mathcal{S}_n^{++}$ the cone of positive definite matrices. A psd matrix with an all ones diagonal is called a *correlation matrix*. The set

$$\mathcal{E}_n = \{X \in \mathcal{S}_n^+ : X_{ii} = 1 \ \forall i \in [n]\}$$

of all $n \times n$ correlation matrices is known as the *elliptope*. For an integer $r \geq 1$, define also the (in general non-convex) bounded rank elliptope

$$\mathcal{E}_{n,r} = \{X \in \mathcal{E}_n : \operatorname{rank} X \leq r\}.$$

Given a graph $G = (V = [n], E)$, π_E denotes the projection from $\mathcal{S}_n$ onto the subspace $\mathbb{R}^E$ indexed by the edge set of G, and we define the projected elliptope:

$$\mathcal{E}(G) = \pi_E(\mathcal{E}_n).$$

The elements of $\mathcal{E}(G)$ can be seen as the partial symmetric matrices with entries specified at positions corresponding to edges of G that can be completed to a correlation matrix.

For any integer $r \geq 1$, we have the following chain of inclusions:

$$\pi_E(\mathcal{E}_{n,r}) \subseteq \pi_E(\operatorname{conv}(\mathcal{E}_{n,r})) \subseteq \pi_E(\mathcal{E}_n) = \mathcal{E}(G). \tag{1}$$

Hence a natural question is to determine what is the smallest value of $r \geq 1$ for which equality holds in the above chain of inclusions. Equality between the sets on the left and on the right side of (1) has been considered in [17], where the following graph parameter is introduced and studied.

Definition 1.1 *The* Gram dimension *of a graph* $G = ([n], E)$*, denoted by* $\operatorname{gd}(G)$*, is defined as the smallest integer* $r \geq 1$ *such that*

$$\mathcal{E}(G) = \pi_E(\mathcal{E}_{n,r}).$$

Here we investigate when equality holds at the right inclusion of (1), which leads to the following graph parameter.

Definition 1.2 *The* extreme Gram dimension *of a graph* $G = ([n], E)$*, denoted by* $\operatorname{egd}(G)$*, is the smallest integer* $r \geq 1$ *for which*

$$\mathcal{E}(G) = \pi_E(\operatorname{conv}(\mathcal{E}_{n,r})).$$

Equivalently, using the Krein–Milman theorem, $\text{egd}(G)$ *is the smallest integer* $r \geq 1$ *for which*

$$\text{ext}\,\mathcal{E}(G) \subseteq \pi_E(\mathcal{E}_{n,r}),$$

where $\text{ext}\,\mathcal{E}(G)$ *is the set of extreme points of* $\mathcal{E}(G)$*. We denote by* $\mathscr{G}_r$ *the class of graphs* G *with* $\text{egd}(G) \leq r$*.*

Alternatively, $\text{gd}(G)$ and $\text{egd}(G)$ can be defined using the following notion of Gram representation, which also clarifies the origin of the names for the graph parameters.

Definition 1.3 *Given a graph* $G = (V, E)$ *and a vector* $x \in \mathbb{R}^E$*, a* Gram representation *of* x *in* $\mathbb{R}^r$ *is a set of unit vectors* $p_1, \ldots, p_n \in \mathbb{R}^r$ *such that*

$$x_{ij} = p_i^{\mathsf{T}} p_j \quad \forall \{i,j\} \in E.$$

The Gram dimension of $x \in \mathcal{E}(G)$*, denoted by* $\text{gd}(G, x)$*, is the smallest integer* $r \geq 1$ *for which* x *has such a Gram representation in* $\mathbb{R}^r$*. Therefore, the (extreme) Gram dimension of* G *can be reformulated as*

$$\text{gd}(G) = \max_{x \in \mathcal{E}(G)} \text{gd}(G, x), \qquad \text{egd}(G) = \max_{x \in \text{ext}\,\mathcal{E}(G)} \text{gd}(G, x). \tag{2}$$

It is shown in [17] that the graph parameter $\text{gd}(G)$ is minor monotone, and the full list of forbidden minors is identified for graphs with $\text{gd}(G) \leq r$ for the values $r = 2, 3$ and 4. Moreover it is shown there that there are tight connections between the Gram dimension and results about Euclidean graph realizations of Belk and Connelly [2, 3] and the parameter $\nu^=(G)$ of van der Holst [13].

While the Gram dimension $\text{gd}(G)$ permits to give an upper bound on the rank of optimal solutions to any semidefinite program with aggregated sparsity pattern G (see [17]), the extreme Gram dimension permits to upper bound the rank of optimal solutions to optimization programs over the elliptope.

Our new parameter is also related to the celebrated Grothendieck constant. Recall the inclusion $\pi_E(\text{conv}(\mathcal{E}_{n,r})) \subseteq \mathcal{E}(G)$. Then the smallest constant $\kappa \geq 1$ for which

$$\mathcal{E}(G) \subseteq \kappa \cdot \pi_E(\text{conv}\,\mathcal{E}_{n,r})$$

is known as the *rank-r Grothendieck constant of* G, denoted as $\kappa(r, G)$. For $r = 1$, this constant has been introduced and studied by Grothendieck [11] for bipartite graphs (although in a different language), and for general graphs by Alon et al. [1]. The general case $r \geq 1$ is studied by Briët et al. [4], the case $r = 2$ is motivated by its application to ground states in the n-vector model in statistical physics.

The rank-r Grothendieck constant is equal to the integrality gap between two optimization problems: a semidefinite program with a rank constraint:

$$\max \sum_{\{i,j\} \in E(G)} A_{ij} X_{ij} \;\; \text{s.t.} \;\; X \in \mathcal{E}_n, \;\; \operatorname{rank} X \leq r, \tag{3}$$

and its semidefinite relaxation where we remove the rank constraint. The former problem corresponds to optimization over $\pi_E(\operatorname{conv}(\mathcal{E}_{n,r}))$ and the latter to optimization over $\mathcal{E}(G)$. Moreover, problem (3) is hard: For $r = 1$ it is an $\mathcal{N}P$-hard quadratic problem with ± 1-variables (modeling the maximum cut problem) and, for any $r \geq 2$, membership in $\pi_E(\operatorname{conv}(\mathcal{E}_{n,r}))$ is $\mathcal{N}P$-hard [7]. It follows from the definitions that a graph has extreme Gram dimension at most r if and only if its rank-r Grothendieck constant is 1:

$$\operatorname{egd}(G) \leq r \iff \kappa(r, G) = 1.$$

The graph parameter $\operatorname{egd}(G)$ is relevant to problem (3) since, for a graph G satisfying $\kappa(r, G) = 1$, problem (3) can be solved in polynomial time. For $r = 1$ it is known that $\kappa(1, G) = 1$ if and only if G is a forest [16].

The connections described above motivate our study of the graph parameter $\operatorname{egd}(G)$, which also fits within the growing literature on geometric graph parameters defined in terms of rank properties of symmetric matrices (see e.g. [10], the surveys [8, 9, 19] and further references therein).

Contribution of the paper. We show that the graph parameter $\operatorname{egd}(G)$ is minor monotone. As a consequence the class $\mathscr{G}_r$ of graphs with $\operatorname{egd}(G) \leq r$ can be characterized by finitely many forbidden minors. One of the main contributions is a complete characterization of the class $\mathscr{G}_2$ (Theorem 4.1).

On the one hand, we identify three families of graphs F_r, G_r, H_r which are forbidden minors for the class $\mathscr{G}_{r-1}$. This gives all the minimal forbidden minors for $r \leq 2$. The graphs G_r were already considered in [5, 15].

On the other hand we show an upper bound for the extreme Gram dimension in terms of a tree-width-like parameter. This graph parameter, which we denote as $\operatorname{la}_{\boxtimes}(G)$, is defined as the smallest integer r for which G is a minor of the strong product $T \boxtimes K_r$ of a tree T and the complete graph K_r. We call it the *strong largeur d'arborescence* of G, in analogy with the *largeur d'arborescence* $\operatorname{la}_{\square}(G)$ introduced by Colin de Verdière [5], using the Cartesian product instead of the strong product. We show that $\text{egd}(G) \leq \text{la}_{\boxtimes}(G)$.

Our main result is that, for a graph $G \neq K_{3,3}$ 2-connected on at least 6 nodes, $\text{egd}(G) \leq 2$ if and only if $\text{la}_{\boxtimes}(G) \leq 2$ if and only if G does not have F_3 or H_3 as a minor. We also characterize the graphs with $\text{la}_{\boxtimes}(G) \leq 2$ and recover the characterization of [14] for the graphs with $\text{la}_{\square}(G) \leq 2$.

The results and techniques in the paper come in two flavours: in Section 3 they rely mostly on the geometry of faces of the elliptope and linear algebraic tools to construct suitable extreme points of the projected elliptope and, in Section 4, they are purely graph theoretic.

Outline of the paper. Section 2 contains preliminaries about graphs, some properties of the new parameter $\text{egd}(G)$, and basic facts about the geometry of the faces of the elliptope. In Section 3.1 we show that for any graph G we have that $\text{egd}(G) \leq \text{la}_{\boxtimes}(G)$. In Section 3.2 we compute the extreme Gram dimension of the three graph classes F_r, G_r and H_r. In Section 3.3 we consider the graphs K_5 and $K_{3,3}$ which play a special role within the class $\mathscr{G}_2$. Section 4 is devoted to proving the characterization of the class $\mathscr{G}_2$. In Section 4.2 we characterize the chordal graphs in $\mathscr{G}_2$ (Theorem 4.3). In Section 4.3 we show that any graph with no minor F_3 or K_4 admits a chordal extension avoiding these two minors (Theorem 4.6) and in Section 4.4 we show the analogous result for graphs with no F_3 and H_3 minor (Theorem 4.11). Finally in Section 5 we characterize the graphs with $\text{la}_{\boxtimes}(G) \leq 2$ and we explain the links to results about $\text{la}_{\square}(G)$ and point out connections with the graph parameter $\nu(G)$ of Colin de Verdière [5].

2 Preliminaries

2.1 Preliminaries about graphs

We recall some definitions about graphs. Let $G = (V, E)$ be a graph, we also denote its node set by $V(G)$ and its edge set by $E(G)$. A *component* is a maximal connected subgraph of G. A *cutset* is a set $U \subseteq V$ for which $G \backslash U$ (deleting the nodes in U) has more connected components than G, U is a *cut node* if $|U| = 1$, and G is *2-connected* if it is connected and has no cut node. For $W \subseteq V$, $G[W]$ is the subgraph induced by W. Given $\{u, v\} \notin E(G)$, $G + \{u, v\}$ is the graph obtained by adding the edge $\{u, v\}$ to G.

Given an edge $e = \{u, v\} \in E$, $G \backslash e = (V, E \setminus \{e\})$ is the graph obtained from G by *deleting* the edge e and G/e is obtained by *contracting* the edge e: Replace the two nodes u and v by a new node, adjacent to all the neighbors of u and v. A graph M is a *minor* of G, denoted as $M \preceq G$, if M can be obtained from G by a series of edge deletions and contractions and node deletions. Equivalently, M is a minor of a connected graph G if there is a partition of $V(G)$ into nonempty subsets $\{V_i : i \in V(M)\}$ where each $G[V_i]$ is connected and, for each edge $\{i,j\} \in E(M)$, there exists at least one edge in G between V_i and V_j. Then the collection $\{V_i : i \in V(M)\}$ is called an *M-partition* of G and the V_i's are its *classes*.

A graph parameter $f(\cdot)$ is *minor monotone* if $f(G\backslash e), f(G/e) \leq f(G)$ for any graph G and any edge e of G.

Given a finite list $\mathcal{M}$ of graphs, $\mathcal{F}(\mathcal{M})$ denotes the collection of all graphs that do not admit any graph in $\mathcal{M}$ as a minor. By the celebrated graph minor theorem of Robertson and Seymour [20], any class of graphs which is closed under taking minors is of the form $\mathcal{F}(\mathcal{M})$ for some finite set $\mathcal{M}$ of graphs. Hence, if the graph parameter $f(\cdot)$ is minor monotone, then the class of graphs G with $f(G) \leq k$ is characterized by a finite list of excluded minors, for each fixed k.

A *homeomorph* (or subdivision) of a graph M is obtained by replacing its edges by paths. When M has maximum degree at most 3, G admits M as a minor if and only if it contains a homeomorph of M as a subgraph.

A *clique* in G is a set of pairwise adjacent nodes and $\omega(G)$ denotes the maximum cardinality of a clique in G. A k-clique is a clique of cardinality k.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs, where $V_1 \cap V_2$ is a clique in both G_1 and G_2. Their *clique sum* is the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$, also called their *clique k-sum* when $k = |V_1 \cap V_2|$.

If C is a circuit in G, a *chord* of C is an edge $\{u, v\} \in E$ where u and v are two nodes of C that are not consecutive on C. G is said to be *chordal* if every circuit of length at least 4 has a chord. As is well known, a graph G is chordal if and only if G is a clique sum of cliques.

The *tree-width* $\mathrm{tw}(G)$ of G is the smallest integer k such that G is contained in a clique sum of copies of K_{k+1}. Colin de Verdière [5] introduced the following variation: The *largeur d' arborescence* of a graph G, denoted by $\mathrm{la}_{\square}(G)$, is the smallest integer r for which G is a minor of $T \square K_r$ for some tree T. Here $\square$ denotes the Cartesian product. Then,

$$\mathrm{tw}(G) \leq \mathrm{la}_{\square}(G) \leq \mathrm{tw}(G) + 1,$$

the upper bound is shown in [5] and the lower bound in [12]. We use the notation $\mathrm{la}_{\square}(G)$ (instead of the original notation $\mathrm{la}(G)$) in order to emphasize the analogy with our new graph parameter $\mathrm{la}_{\boxtimes}(G)$, which is based on using the strong product $\boxtimes$ instead of the Cartesian product $\square$.

The *strong product* $G \boxtimes G'$ of $G = (V, E)$ and $G' = (V', E')$ has node set $V \times V'$ and distinct nodes $(i, i'), (j, j') \in V \times V'$ are adjacent in $G \boxtimes G'$ when $i = j$ or $(i, j) \in E$, and $i' = j'$ or $(i', j') \in E'$. Then, $\mathrm{la}_{\boxtimes}(G)$ is the smallest integer r for which G is a minor of $T \boxtimes K_r$ for some tree T. It will serve as an upper bound for our new graph parameter $\mathrm{egd}(G)$ (see Section 3.1).

The graph parameters $\mathrm{tw}(G)$, $\mathrm{la}_{\square}(G)$ and $\mathrm{la}_{\boxtimes}(G)$ are minor monotone and satisfy:

$$\mathrm{tw}(G)/2 \leq \mathrm{la}_{\boxtimes}(G) \leq \mathrm{la}_{\square}(G) \leq \mathrm{tw}(G) + 1.$$

If G is the clique k-sum of G_1 and G_2, then $f(G) = \max\{f(G_1), f(G_2)\}$ when $f(G) = \mathrm{tw}(G)$; the same holds for the parameters $f(G) = \mathrm{la}_{\square}(G)$ and $\mathrm{la}_{\boxtimes}(G)$ when $k \leq 1$.

Some more notation. Throughout $[n] = \{1, \ldots, n\}$. For a set $A \subseteq \mathbb{R}^n$, $\langle A \rangle$ denotes the vector space spanned by A and $\mathrm{conv}A$ denotes the convex hull of A. For a matrix $X \in \mathcal{S}_n$, $X \succeq 0$ means that X is positive semidefinite. For $U \subseteq [n]$, $X[U]$ denotes the principal submatrix of X with row and column indices in U and, for $j \in [n]$, $X[\cdot, j]$ denotes the j-th column of X.

2.2 Basic properties of the extreme Gram dimension

Here we investigate the behavior of the graph parameter $\mathrm{egd}(G)$ under some simple graph operations: taking minors and clique sums.

Lemma 2.1 *The graph parameter* $\mathrm{egd}(G)$ *is minor monotone, i.e., for any edge e of G,* $\mathrm{egd}(G \backslash e), \mathrm{egd}(G/e) \leq \mathrm{egd}(G)$.

Proof. Let $G = ([n], E)$ and $e \in E$. The inequality $\mathrm{egd}(G \setminus e) \leq \mathrm{egd}(G)$ follows from the definition. We show that $\mathrm{egd}(G/e) \leq \mathrm{egd}(G) = r$. Say, $e = \{n-1, n\}$ and set $G/e = ([n-1], E^{'})$. Let $x \in \mathcal{E}(G/e)$. Then $x = \pi_{E'}(X)$ for some $X \in \mathcal{E}_{n-1}$. Let $X[\cdot, n-1]$ be the last column of X and set

$$Y = \begin{pmatrix} X & X[\cdot, n-1] \\ X[\cdot, n-1]^{\mathsf{T}} & 1 \end{pmatrix} \in \mathcal{S}_n$$

and $y = \pi_E(Y)$. Then $Y \in \mathcal{E}_n$ and thus $y \in \mathcal{E}(G)$. As $\mathrm{egd}(G) = r$, there exist $Y_1, \ldots, Y_m \in \mathcal{E}_{n,r}$ and scalars $\lambda_i \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that $y = \pi_E(\sum_{i=1}^m \lambda_i Y_i)$. The condition $Y_{n-1,n} = 1$ implies that $(Y_i)_{n-1,n} = 1$ and thus $Y_i[\cdot, n-1] = Y_i[\cdot, n]$ for all $i \in [m]$. Now, let $\hat{Y}_i$ be obtained from Y_i by removing its n-th row and column. Then, $\hat{Y}_i \in \mathcal{E}_{n-1}$, $\mathrm{rank}\, \hat{Y}_i \leq r$, and $x = \pi_{E'}(\sum_{i=1}^m \lambda_i \hat{Y}_i) \in \pi_{E'}(\mathrm{conv}(\mathcal{E}_{n-1,r}))$. This shows $\mathrm{egd}(G/e) \leq r$. $\square$

The following easy, but useful fact about psd completions is well known.

Lemma 2.2 *Given two psd matrices $X_1 \in \mathcal{S}^+_{V_1}$ and $X_2 \in \mathcal{S}^+_{V_2}$ such that $X_1[V_1 \cap V_2] = X_2[V_1 \cap V_2]$, there exists a common psd completion $X \in \mathcal{S}^+_V$, i.e., such that $X[V_i] = X_i$ ($i = 1, 2$), with* $\operatorname{rank} X = \max\{\operatorname{rank} X_1, \operatorname{rank} X_2\}$.

As a direct application, if G is the clique sum of G_1 and G_2, its Gram dimension satisfies: $\operatorname{gd}(G) = \max\{\operatorname{gd}(G_1), \operatorname{gd}(G_2)\}$. For the extreme Gram dimension, the analogous result holds only for clique k-sums with $k \leq 1$.

Lemma 2.3 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. If $|V_1 \cap V_2| \leq 1$ then the clique sum G of G_1, G_2 satisfies* $\operatorname{egd}(G) = \max\{\operatorname{egd}(G_1), \operatorname{egd}(G_2)\}$.

Proof. Let $x \in \mathcal{E}(G)$ and $r = \max\{\operatorname{egd}(G_1), \operatorname{egd}(G_2)\}$, we show that $x \in \pi_E(\operatorname{conv}(\mathcal{E}_{n,r}))$. For $i = 1, 2$, the vector $x_i = \pi_{E_i}(x)$ belongs to $\pi_{E_i}(\operatorname{conv}(\mathcal{E}_{|V_i|,r}))$. Hence, $x_i = \pi_{E_i}(\sum_{j=1}^{m_i} \lambda_{i,j} X^{i,j})$ for some $X^{i,j} \in \mathcal{E}_{|V_i|,r}$ and $\lambda_{i,j} \geq 0$ with $\sum_j \lambda_{i,j} = 1$. As $|V_1 \cap V_2| \leq 1$, any two matrices $X^{1,j}$ and $X^{2,k}$ share at most one diagonal entry, equal to 1 in both matrices. By Lemma 2.2, $X^{1,j}$ and $X^{2,k}$ have a common completion $Y^{j,k} \in \mathcal{E}_{n,r}$. This implies that $x = \pi_E(\sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \lambda_{1,j}\lambda_{2,k} Y^{jk})$, which shows $x \in \pi_E(\operatorname{conv}(\mathcal{E}_{n,r}))$. $\square$

Therefore, the class $\mathscr{G}_r$ is closed under taking disjoint unions and clique 1-sums of graphs. It is *not* closed under clique k-sums when $k \geq 2$. E.g. the graph F_3 from Figure 1 is a clique 2-sum of triangles, however $\operatorname{egd}(F_3) = 3$ (Theorem 3.6) while triangles have extreme Gram dimension 2 (Lemma 2.6).

2.3 The geometry of the elliptope

Recall that, for a convex set K, a set $F \subseteq K$ is a *face* of K if for all $x \in F$, $x = ty + (1-t)z$ with $y, z \in K$ and $t \in (0, 1)$ implies $y, z \in F$. For $x \in K$ the smallest face $F(x)$ of K containing x is well defined, it is the unique face of K containing x in its relative interior. A point $x \in K$ is an *extreme point* of K if $F(x) = \{x\}$. Moreover, z is said to be a *perturbation* of $x \in K$ if $x \pm \epsilon z \in K$ for some $\epsilon > 0$, then the segment $[x - \epsilon z, x + \epsilon z]$ is contained in $F(x)$ and the dimension of $F(x)$ is equal to the dimension of the linear space $\mathcal{P}(x)$ of perturbations of x.

We recall some facts about the faces of the elliptope that we need here. For a matrix $X \in \mathcal{E}_n$, the smallest face $F(X)$ of $\mathcal{E}_n$ containing X is given by

$$F(X) = \{Y \in \mathcal{E}_n : \ker X \subseteq \ker Y\}. \tag{4}$$

Therefore, two matrices in the relative interior of a face F of $\mathcal{E}_n$ have the same rank, while $\operatorname{rank} X > \operatorname{rank} Y$ if X is in the relative interior of F and

Y lies on the boundary of F. Here is the explicit description of the space $\mathcal{P}(X)$ of perturbations of a matrix $X \in \mathcal{E}_n$.

Proposition 2.4 *([18], see also [6, §31.5]) Let $X \in \mathcal{E}_n$ with rank r. Let $u_1, \ldots, u_n \in \mathbb{R}^r$ be a Gram representation of X, let U be the $r \times n$ matrix with columns $u_1, \ldots, u_n$ and set $\mathcal{U}_V = \langle u_1 u_1^{\mathsf{T}}, \ldots, u_n u_n^{\mathsf{T}} \rangle \subseteq \mathcal{S}_r$. The space of perturbations $\mathcal{P}(X)$ at X is given by*

$$\mathcal{P}(X) = U^{\mathsf{T}} \mathcal{U}_V^{\perp} U = \{U^{\mathsf{T}} R U : R \in \mathcal{S}_r, \langle R, u_i u_i^{\mathsf{T}} \rangle = 0 \ \forall i \in [n]\} \tag{5}$$

and the dimension of the smallest face $F(X)$ of $\mathcal{E}_n$ containing X is

$$\dim F(X) = \dim \mathcal{P}(X) = \binom{r+1}{2} - \dim \ \mathcal{U}_V. \tag{6}$$

In particular, X is an extreme point of $\mathcal{E}_n$ if and only if

$$\binom{r+1}{2} = \dim \ \mathcal{U}_V. \tag{7}$$

Hence, if $X \in \operatorname{ext} \mathcal{E}_n$ with $\operatorname{rank} X = r$ then

$$\binom{r+1}{2} \leq n. \tag{8}$$

Example 2.5 *Let $e_1, \ldots, e_r \in \mathbb{R}^r$ be the standard unit vectors. The matrix with Gram representation $\{e_i : i \in [r]\} \cup \{(e_i + e_j)/\sqrt{2} : 1 \leq i < j \leq r\}$ is an extreme point of $\mathcal{E}_n$, since $\mathcal{U}_V$ is full dimensional in $\mathcal{S}_r$, where $n = \binom{r+1}{2}$.*

It is known that for any r satisfying (8) there exists an extremal matrix in $\mathcal{E}_n$ of rank r [18]. This implies:

Lemma 2.6 *The extreme Gram dimension of the complete graph K_n is*

$$\operatorname{egd}(K_n) = r_{max}(n) := \max\left\{ r \in \mathbb{Z}_+ : \binom{r+1}{2} \leq n \right\} = \left\lfloor \frac{\sqrt{8n+1} - 1}{2} \right\rfloor.$$

Hence, $\operatorname{egd}(G) \leq r_{max}(n)$ for any graph G on n nodes.

Next we establish some tools which will be useful to study the extreme points of the projected elliptope $\mathcal{E}(G)$.

Lemma 2.7 *Let $x \in \mathcal{E}(G)$, let $X \in \mathcal{E}_n$ be a rank r completion of x with Gram representation $\{u_1, \dots, u_n\}$ in $\mathbb{R}^r$ and let U be the $r \times n$ matrix with columns $u_1, \dots, u_n$. Set*

$$U_{ij} = \frac{u_i u_j^\mathsf{T} + u_j u_i^\mathsf{T}}{2}, \ \mathcal{U}_V = \langle U_{ii} : i \in V \rangle, \ \mathcal{U}_E = \langle U_{ij} : \{i,j\} \in E \rangle \subseteq \mathcal{S}_r. \quad (9)$$

If x is an extreme point of $\mathcal{E}(G)$, then $\mathcal{U}_E \subseteq \mathcal{U}_V$.

Proof. Assume that $\mathcal{U}_E \not\subseteq \mathcal{U}_V$. Then there exists a matrix $R \in \mathcal{U}_V^\perp \setminus \mathcal{U}_E^\perp$. As $R \in \mathcal{U}_V^\perp$, the matrix $Z = U^\mathsf{T} R U = (\langle R, U_{ij} \rangle)_{i,j=1}^n \in \mathcal{S}_n$ is a perturbation of X (recall (5) and (9)). As $R \notin \mathcal{U}_E^\perp$, $Z_{ij} \neq 0$ for some edge $\{i,j\} \in E$. Now, $X \pm \epsilon Z \in \mathcal{E}_n$ for some $\epsilon > 0$. Hence, x can be written as the convex combination $(\pi_E(X + \epsilon Z) + \pi_E(X - \epsilon Z))/2$, where $\pi_E(X \pm \epsilon Z)$ are distinct points of $\mathcal{E}(G)$. This contradicts the assumption that x is an extreme point of $\mathcal{E}(G)$. $\square$

Given $x \in \mathcal{E}(G)$, its *fiber* is the set

$$\text{fib}(x) = \{X \in \mathcal{E}_n : \pi_E(X) = x\}$$

of all psd completions of x in $\mathcal{E}_n$. The following lemma is an easy result from convex analysis.

Lemma 2.8 *For $x \in \mathcal{E}(G)$, x is an extreme point of $\mathcal{E}(G)$ if and only if its fiber $\text{fib}(x)$ is a face of $\mathcal{E}_n$. Moreover, if x is an extreme point of $\mathcal{E}(G)$, then any extreme point of $\text{fib}(x)$ is an extreme point of $\mathcal{E}_n$.*

3 The extreme Gram dimension of some graphs

3.1 An upper bound for the extreme Gram dimension

In this section we show that the extreme Gram dimension is upper bounded by the strong largeur d'arborescence: $\text{egd}(G) \leq \text{la}_\boxtimes(G)$. As we will see in the next section, the class of graphs with $\text{la}_\boxtimes(G) \leq 2$ plays a crucial role in the characterization of the class $\mathscr{G}_2$.

Theorem 3.1 *For any tree T, $\text{egd}(T \boxtimes K_r) \leq r$.*

Corollary 3.2 *For any graph G, $\text{egd}(G) \leq \text{la}_\boxtimes(G)$.*

Proof. If $\text{la}_{\boxtimes}(G) = r$, then G is a minor of $T \boxtimes K_r$ for some tree T and thus $\text{egd}(G) \leq \text{egd}(T \boxtimes K_r) \leq r$, by Lemma 2.1 and Theorem 3.1. $\Box$

The main ingredient for the proof of Theorem 3.1 is the following technical lemma.

Lemma 3.3 *Let $\{u_1, \ldots, u_{2r}\}$ be a set of vectors, denote its rank by ρ. Let $\mathcal{U}$ denote the linear span of the matrices $U_{ij} = (u_i u_j^\mathsf{T} + u_j u_i^\mathsf{T})/2$ for all $i, j \in \{1, \ldots, r\}$ and all $i, j \in \{r+1, \ldots, 2r\}$. If $\rho \geq r+1$ then $\dim \mathcal{U} < \binom{\rho+1}{2}$.*

Proof. Let $I \subseteq \{1, \ldots, r\}$ for which $\{u_i : i \in I\}$ is a maximum linearly independent subset of $\{u_1, \ldots, u_r\}$ and let $J \subseteq \{r+1, \ldots, 2r\}$ such that the set $\{u_i : i \in I \cup J\}$ is maximum linearly independent; thus $|I| + |J| = \rho$. Set $K = \{1, \ldots, r\} \setminus I$, $L = \{r+1, \ldots, 2r\} \setminus J$, and $J' = J \setminus \{k\}$, where k is some given (fixed) element of J. For any $l \in L$, there exists scalars $a_{l,i} \in \mathbb{R}$ such that

$$u_l = \sum_{i \in I \cup J'} a_{l,i} u_i + a_{l,k} u_k. \tag{10}$$

Set

$$A_l = \sum_{i \in I \cup J'} a_{l,i} U_{ik} \quad \text{for} \quad l \in L.$$

Then, define the set $\mathcal{W}$ consisting of the matrices U_{ii} for $i \in I \cup J$, U_{ij} for all pairs (i, j) in $I \cup J'$, U_{kj} for all $j \in J'$, and A_l for all $l \in L$. Then, $|W| = \rho + \binom{\rho-1}{2} + r - 1 = \binom{\rho}{2} + r = \binom{\rho+1}{2} + r - \rho \leq \binom{\rho+1}{2} - 1$. In order to conclude the proof it suffices to show that $\mathcal{W}$ spans the space $\mathcal{U}$.

Clearly, $\mathcal{W}$ spans all matrices U_{ij} with $i, j \in \{1, \ldots, r\}$. Fix $l \in L$. Using (10) we obtain that $U_{lk} = A_l + a_{k,l} U_{kk}$ lies in the span of $\mathcal{W}$. Moreover, for $j \in J'$, $U_{lj} = \sum_{i \in I \cup J'} a_{l,i} U_{ij} + a_{l,k} U_{kj}$ also lies in the span of $\mathcal{W}$. Finally, for $l' \in L$, $U_{ll'} = \sum_{i,j \in I \cup J'} a_{l,i} a_{l',j} U_{ij} + a_{l',k} A_l + a_{l,k} A_{l'} + a_{l,k} a_{l',k} U_{kk}$ is also spanned by $\mathcal{W}$. This concludes the proof. $\Box$

Proof. (of Theorem 3.1). Let $G = T \boxtimes K_r$, where T is a tree on $[t]$ and let $G = (V, E)$ with $|V| = n$. So the node set of G is $V = \cup_{i=1}^t V_i$, where the V_i's are pairwise disjoint sets, each of cardinality r. By definition of the strong product, for any edge $\{i, j\}$ of T, the set $V_i \cup V_j$ induces a clique in G, denoted as C_{ij}. Then, G is the union of the cliques C_{ij} over all edges $\{i, j\}$ of T. We show that $\text{egd}(G) \leq r$. For this, pick an extreme element $x \in \text{ext}\, \mathcal{E}(G)$. Then $x = \pi_E(X)$ for some $X \in \mathcal{E}_n$. As C_{ij} is a clique in G, the principal submatrix $X^{ij} := X[C_{ij}]$ is fully determined from x. In order to show that x has a psd completion of rank at most r, it suffices to show that $\text{rank}\, X^{ij} \leq r$ for all edges $\{i, j\}$ of T (then apply Lemma 2.2).

Pick an edge $\{i,j\}$ of T and set $\rho = \operatorname{rank} X^{ij}$. Assume that $\rho \geq r+1$; we show that there exists a nonzero perturbation Z of X^{ij} such that

$$\begin{array}{l} Z_{hk} = 0 \ \forall (h,k) \in (V_i \times V_i) \cup (V_j \times V_j), \\ Z_{hk} \neq 0 \text{ for some } (h,k) \in V_i \times V_j. \end{array} \tag{11}$$

This permits to reach a contradiction: As Z is a perturbation of X^{ij}, there exists $\epsilon > 0$ for which $X^{ij} + \epsilon Z,\ X^{ij} - \epsilon Z \succeq 0$. By construction, C_{ij} is the only maximal clique of G containing the edges $\{h,k\}$ of G with $h \in V_i$ and $k \in V_j$. Hence, one can find a psd completion X' (resp., X'') of the matrix $X^{ij} + \epsilon Z$ (resp., $X^{ij} - \epsilon Z$) and the matrices $X^{i'j'}$ for all edges $\{i',j'\} \neq \{i,j\}$ of T. Now, $x = \frac{1}{2}(\pi_E(X') + \pi_E(X''))$, where $\pi_E(X'), \pi_E(X'')$ are distinct elements of $\mathcal{E}(G)$, contradicting the fact that x is an extreme point of $\mathcal{E}(G)$.

We now construct the desired perturbation Z of X^{ij} satisfying (11). For this let u_h ($h \in V_i \cup V_j$) be a Gram representation of X^{ij} in $\mathbb{R}^\rho$ and let $\mathcal{U} \subseteq \mathcal{S}_\rho$ denote the linear span of the matrices U_{hk} for all $h,k \in V_i$ and all $h,k \in V_j$. Applying Lemma 3.3, as $\rho \geq r+1$, we deduce that $\dim \mathcal{U} < \binom{\rho+1}{2}$. Hence there exists a nonzero matrix $R \in \mathcal{S}_\rho$ lying in $\mathcal{U}^\perp$. Define the matrix $Z \in \mathcal{S}_{2r}$ by $Z_{hk} = \langle R, U_{hk} \rangle$ for all $h,k \in V_i \cup V_j$. By construction, Z is a perturbation of X^{ij} and it satisfies $Z_{hk} = 0$ whenever the pair (h,k) is contained in V_i or in V_j. As $R \neq 0$, $Z \neq 0$ and thus $Z_{hk} \neq 0$ for some $h \in V_i$ and $k \in V_j$. Thus (11) holds and the proof is completed. □

3.2 Three graph classes with extreme Gram dimension r

In this section we construct three classes of graphs F_r, G_r, H_r, whose extreme Gram dimension is equal to r. Therefore, they are forbidden minors for the class $\mathscr{G}_{r-1}$ of graphs with extreme Gram dimension at most $r-1$. As we will see in the next section, this gives all the forbidden minors for the characterization of the class $\mathscr{G}_2$.

The graphs G_r were already considered by Colin de Verdière [5] in relation to the graph parameter $\nu(G)$, to which we will come back in Section 5. Each of the graphs $G = F_r, G_r, H_r$ has $\binom{r+1}{2}$ nodes and thus extreme Gram dimension $\operatorname{egd}(G) \leq r$; moreover, $\operatorname{egd}(G/e) \leq r-1$ after contracting an edge (use Lemma 2.6). To show equality $\operatorname{egd}(G) = r$, we will rely on the following result, which follows directly from Lemma 2.8.

Lemma 3.4 *Suppose that there exists $x \in \mathcal{E}(G)$ such that $\operatorname{fib}(x) = \{X\}$ where $X \in \operatorname{ext} \mathcal{E}_n$ and $\operatorname{rank} X = r$. Then $\operatorname{egd}(G) \geq r$.*

To use this lemma we need tools permitting to show existence of a *unique* completion for a vector $x \in \mathcal{E}(G)$. We introduce below such a tool: 'forcing a non-edge with a minimally singular clique', based on the following property of psd matrices:

$$\begin{pmatrix} A & b \\ b^{\mathsf{T}} & \alpha \end{pmatrix} \succeq 0 \Longrightarrow b^{\mathsf{T}} u = 0 \ \forall u \in \ker A. \tag{12}$$

Lemma 3.5 *Let $x \in \mathcal{E}(G)$, let $C \subseteq V$ be a clique of G and $\{i,j\} \notin E(G)$ with $i \notin C$, $j \in C$. Set $x[C] = (x_{ij})_{i,j \in C} \in \mathcal{E}_{|C|}$ (setting $x_{ii} = 1$ for all i). Assume that i is adjacent to all nodes of $C \setminus \{j\}$ and that $x[C]$ is minimally singular (i.e., $x[C]$ is singular but any principal submatrix of $x[C]$ is nonsingular). Then the (i,j)-th entry X_{ij} is uniquely defined in any completion $X \in \mathrm{fib}(x)$ of x.*

Proof. Let $X \in \mathrm{fib}(x)$. The principal submatrix $X[C \cup \{i\}]$ has the block form shown in (12) where all entries are specified (from x) except the entry $b_j = X_{ij}$. As $x[C]$ is singular there exists a nonzero vector u in the kernel of $x[C]$. Moreover, $u_j \neq 0 \ \ \forall j \in C$, since $x[C \setminus \{j\}]$ is nonsingular. Hence the condition $b^{\mathsf{T}} u = 0$ permits to derive the value of X_{ij} from x. □

When applying Lemma 3.5 we will say that "the clique C forces the pair $\{i,j\}$". The lemma will be used in an iterative manner: Once a non-edge $\{i,j\}$ has been forced, we know the value X_{ij} in any psd completion X and thus we can replace G by $G + \{i,j\}$ and search for a new forced pair in the extended graph $G + \{i,j\}$.

3.2.1 The class F_r

For $r \geq 2$ the graph F_r has $r + \binom{r}{2} = \binom{r+1}{2}$ nodes, denoted as v_i (for $i \in [r]$) and v_{ij} (for $1 \leq i < j \leq r$); it consists of a clique K_r on the nodes $\{v_1, \ldots, v_r\}$ together with the cliques C_{ij} on $\{v_i, v_j, v_{ij}\}$ for all $1 \leq i < j \leq r$. The graphs F_3 and F_4 are illustrated in Figure 1.

For $r = 2$, $F_2 = K_3$ has extreme Gram dimension 2. More generally:

Theorem 3.6 *For $r \geq 2$, $\mathrm{egd}(F_r) = r$. Moreover, F_r is a minimal forbidden minor for the class $\mathscr{G}_{r-1}$.*

Proof. First we show that $\mathrm{egd}(F_r) \geq r$. For this we label the nodes $v_1, \ldots, v_r$ by the standard unit vectors $e_1, \ldots, e_r \in \mathbb{R}^r$ and v_{ij} by the vector $(e_i + e_j)/\sqrt{2}$. Consider the Gram matrix X of these $n = \binom{r+1}{2}$ vectors and its projection $x = \pi_{E(F_r)}(X) \in \mathcal{E}(F_r)$. Then, X is an extreme point of $\mathcal{E}_n$

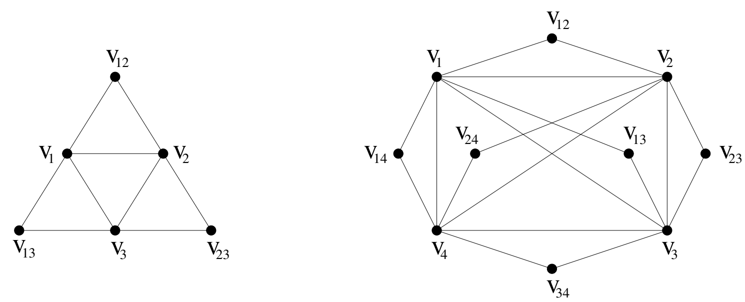

Figure 1: The graphs F_3 and F_4.

(Example 2.5). We now show that X is the only psd completion of x which, in view of Lemma 3.4, implies that $\text{egd}(F_r) \geq r$. For this we use Lemma 3.5. Observe that, for each $1 \leq i < j \leq r$, the matrix $x[C_{ij}]$ is minimally singular. First, for any $k \in [r] \setminus \{i,j\}$, the clique C_{ij} forces the non-edge $\{v_k, v_{ij}\}$ and then, for any other $1 \leq i' < j' \leq r$, the clique C_{ij} forces the non-edge $\{v_{ij}, v_{i'j'}\}$. Hence, in any psd completion of x, all the entries indexed by non-edges are uniquely determined, i.e., $\text{fib}(x) = \{X\}$.

Next, we show minimality. Let e be an edge of F_r, we show that $\text{egd}(H) \leq r-1$ where $H = F_r \backslash e$. If e is an edge of the form $\{v_i, v_{ij}\}$, then H is the clique 1-sum of an edge and a graph on $\binom{r+1}{2} - 1$ nodes and thus $\text{egd}(H) \leq r-1$ follows using Lemmas 2.3 and 2.6. Suppose now that e is contained in the central clique K_r, say $e = \{v_1, v_2\}$. We show that H is contained in a graph of the form $T \boxtimes K_{r-1}$ for some tree T. We choose T to be the star $K_{1,r-1}$ and we give a suitable partition of the nodes of F_r into sets $V_0 \cup V_1 \cup \ldots \cup V_{r-1}$, where each V_i has cardinality at most $r-1$, V_0 is assigned to the center node of the star $K_{1,r-1}$ and $V_1, \ldots, V_{r-1}$ are assigned to the $r-1$ leaves of $K_{1,r-1}$. Namely, set $V_0 = \{v_{12}, v_3, \ldots, v_r\}$, $V_1 = \{v_1, v_{13}, \ldots, v_{1r}\}$, $V_2 = \{v_2, v_{23}, \ldots, v_{2r}\}$ and, for $k \in \{3, \ldots, r-1\}$, $V_k = \{v_{kj} : k+1 \leq j \leq r\}$. Then, in the graph H, each edge is contained in one of the sets $V_0 \cup V_k$ for $1 \leq k \leq r-1$. This shows that H is a subgraph of $K_{1,r-1} \boxtimes K_{r-1}$ and thus $\text{egd}(H) \leq r-1$ (by Theorem 3.1). □

As an application of Theorem 3.6 we get:

Corollary 3.7 *If the tree T has a node of degree at least $(r-1)/2$ then* $\text{egd}(T \boxtimes K_r) = r$.

Proof. Directly from Theorem 3.6, as $T \boxtimes K_r$ contains a subgraph F_r. □

3.2.2 The class G_r

Consider an equilateral triangle and subdivide each side into $r-1$ equal segments. Through these points draw line segments parallel to the sides of the triangle. This construction creates a triangulation of the big triangle into $(r-1)^2$ congruent equilateral triangles. The graph G_r corresponds to the edge graph of this triangulation. The graph G_5 is illustrated in Figure 2.

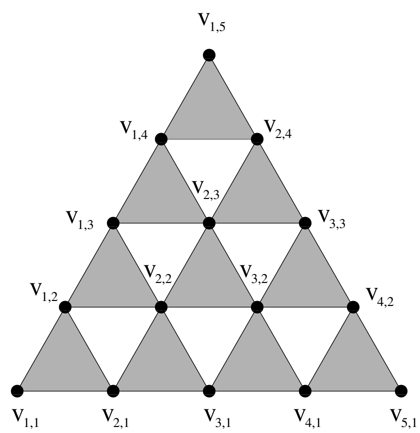

Figure 2: The graph G_5.

The graph G_r has $\binom{r+1}{2}$ vertices, which we denote $v_{i,l}$ for $l \in [r]$ and $i \in [r-l+1]$ (with $v_{1,l}, \ldots, v_{r-l+1,l}$ at level l, see Figure 2). Note that $G_2 = K_3 = F_2$, $G_3 = F_3$, but $G_r \neq F_r$ for $r \geq 4$. Using the following lemma we can construct some points of $\mathcal{E}(G_r)$ with a unique completion.

Lemma 3.8 *Consider a labeling of the nodes of G_r by vectors $w_{i,l}$ satisfying the following property (P_r): For each triangle $C_{i,l} = \{v_{i,l}, v_{i+1,l}, v_{i,l+1}\}$ of G_r, the set $\{w_{i,l}, w_{i+1,l}, w_{i,l+1}\}$ is minimally linearly dependent. (These triangles are shaded in Figure 2). Let X be the Gram matrix of the vectors $w_{i,l}$ and let $x = \pi_{E(G_r)}(X)$ be its projection. Then X is the unique completion of x.*

Proof. For $r = 2$, $G_2 = K_3$ and there is nothing to prove. Let $r \geq 3$ and assume that the claim holds for $r-1$. Consider a labeling $w_{i,l}$ of G_r satisfying (P_r) and the corresponding vector $x \in \mathcal{E}(G_r)$. We show, using Lemma 3.5, that the entries Y_{uv} of a psd completion Y of x are uniquely determined for all $\{u, v\} \notin E(G_r)$. For this, denote by H, R, L the sets of nodes lying on the 'horizontal' side, the 'right' side and the 'left' side of G_r, respectively (refer to the drawing of G_r of Figure 2). Observe that each of

$G_r\backslash H$, $G_r\backslash R$, $G_r\backslash L$ is a copy of G_{r-1}. As the induced vector labelings on each of these graphs satisfies the property (P_{r-1}), we can conclude using the induction assumption that the entry Y_{uv} is uniquely determined whenever the pair $\{u,v\}$ is contained in the vertex set of one of $G_r\backslash H$, $G_r\backslash R$, or $G_r\backslash L$. The only non-edges $\{u,v\}$ that are not yet covered arise when u is a corner of G_r and v lies on the opposite side, say $u = v_{1,1}$ and $v = v_{r-l+1,l} \in R$. If $l \neq 1, r$ then the clique $C_{1,1} = \{v_{1,1}, v_{2,1}, v_{1,2}\}$ forces the pair $\{u,v\}$ (since $\{v, v_{1,2}\} \in E(G_r\backslash H)$ and $\{v, v_{2,1}\} \in E(G_r\backslash L)$). If $l = r$ then the clique $C_{1,r-1} = \{v_{1,r-1}, v_{2,r-1}, v_{1,r}\}$ forces the pair $\{u,v\}$ (since $\{u, v_{1,r-1}\} \in E(G_r\backslash R)$ and the value at the pair $\{u, v_{2,r-1}\}$ has just been specified). Analogously for the case $l = 1$. This concludes the proof. $\square$

Theorem 3.9 *For $r \geq 2$, $\mathrm{egd}(G_r) = r$. Moreover, G_r is a minimal forbidden minor for the class $\mathscr{G}_{r-1}$.*

Proof. First we show that $\mathrm{egd}(G_r) \geq r$. For this, choose a vector labeling of the nodes of G_r satisfying the conditions of Lemma 3.8: Label the nodes $v_{1,1}, \ldots, v_{r,1}$ at level $l = 1$ by the standard unit vectors $w_{1,1} = e_1, \ldots, w_{r,1} = e_r$ in $\mathbb{R}^r$ and define inductively $w_{i,l+1} = \frac{w_{i,l}+w_{i+1,l}}{\|w_{i,l}+w_{i+1,l}\|}$ for $l = 1, \ldots, r-1$. By Lemma 3.8 their Gram matrix X is the unique completion of its projection $x = \pi_{E(G_r)}(X) \in \mathcal{E}(G_r)$. Moreover, X is extreme in $\mathcal{E}_n$ since $\mathcal{U}_V$ is full-dimensional in $\mathcal{S}_r$. This shows $\mathrm{egd}(G_r) \geq r$, by Lemma 3.4.

We now show that $\mathrm{egd}(G_r\backslash e) \leq r-1$. For this we use the inequalities: $\mathrm{egd}(G_r\backslash e) \leq \mathrm{la}_{\boxtimes}(G_r\backslash e) \leq \mathrm{la}_{\square}(G_r\backslash e) \leq r-1$, where the leftmost inequality follows from Corollary 3.2 and the rightmost one is shown in [14]. $\square$

Corollary 3.10 *The graph parameter $\mathrm{egd}(G)$ is unbounded for the class of planar graphs.*

Corollary 3.11 *Let T be a tree which contains a path with $2r-2$ nodes. Then, $\mathrm{egd}(T \boxtimes K_r) = r$.*

Proof. It is shown in [5] that G_r is a minor of the Cartesian product of two paths P_r and P_{2r-2} (with, respectively, r and $2r-2$ nodes). Hence, $G_r \preceq P_{2r-2} \square P_r \preceq T \boxtimes K_r$ and thus $r = \mathrm{egd}(G_r) \leq \mathrm{egd}(T \boxtimes K_r)$. $\square$

3.2.3 The class H_r

In this section we consider a third class of graphs H_r for every $r \geq 3$. In order to explain the general definition we first describe the base case $r = 3$.

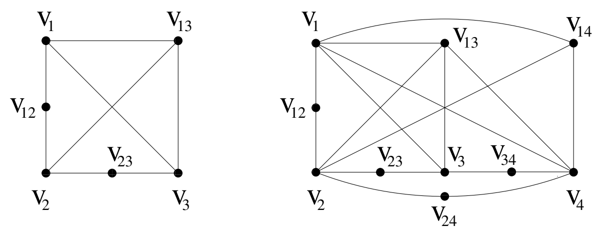

Figure 3: The graphs H_3 and H_4.

The graph H_3 is shown in Figure 3. It is obtained by taking a complete graph K_4, with vertices v_1, v_2, v_3 and v_{13}, and subdividing two adjacent edges: here we insert node v_{12} between v_1 and v_2 and node v_{23} between nodes v_2 and v_3.

Lemma 3.12 $\operatorname{egd}(H_3) = 3$ *and* H_3 *is a minimal forbidden minor for* $\mathscr{G}_2$.

Proof. As H_3 has 6 nodes, $\operatorname{egd}(H_3) \leq 3$. To show equality, we use the following vector labeling for the nodes of H_3: Label the nodes v_1, v_2, v_3 by the standard unit vectors $e_1, e_2, e_3 \in \mathbb{R}^3$ and v_{ij} by $(e_i+e_j)/\sqrt{2}$ for $1 \leq i < j \leq 3$. Let $X \in \mathcal{E}_6$ be their Gram matrix and set $x = \pi_{E(H_3)}(X) \in \mathcal{E}(H_3)$. Then X has rank 3 and X is an extreme point of $\mathcal{E}_6$. We now show that X is the unique completion of x in $\mathcal{E}_6$. For this let $Y \in \operatorname{fib}(x)$. Consider its principal submatrices Z, Z' indexed by $\{v_1, v_2, v_3, v_{13}\}$ and $\{v_1, v_2, v_{12}\}$, of the form:

$$Z = \begin{pmatrix} 1 & a & 0 & \sqrt{2}/2 \\ a & 1 & b & 0 \\ 0 & b & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & 1 \end{pmatrix} \qquad Z' = \begin{pmatrix} 1 & a & \sqrt{2}/2 \\ a & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \end{pmatrix},$$

where $a, b \in \mathbb{R}$. Then, $\det(Z) = -(a+b)^2/2$ implies $a+b = 0$, and $\det(Z') = a(1-a)$ implies $a \geq 0$. Similarly, $b \geq 0$ using the principal submatrix of Y indexed by $\{v_2, v_3, v_{23}\}$. This shows $a = b = 0$ and thus the entries of Y at the positions $\{v_1, v_2\}$ and $\{v_2, v_3\}$ are uniquely specified. Remains to show that the entries are uniquely specified at the non-edges containing v_{12} or v_{23}. For this we use Lemma 3.5: First the clique $\{v_2, v_3, v_{23}\}$ forces the pairs $\{v_1, v_{23}\}$ and $\{v_{13}, v_{23}\}$ and then the clique $\{v_1, v_2, v_{12}\}$ forces the pairs $\{v_{23}, v_{12}\}$, $\{v_{13}, v_{12}\}$, and $\{v_3, v_{12}\}$. Thus we have shown $Y = X$, which concludes the proof that $\operatorname{egd}(H_3) = 3$.

Lastly, we verify that $\operatorname{egd}(H_3 \backslash e) \leq 2$ for each edge $e \in E(H_3)$. If deleting the edge e creates a cut node, then the result follows using Lemma 2.3.

Otherwise, $H_3 \backslash e$ is contained in $T \boxtimes K_2$, where T is a path (for $e = \{v_2, v_{13}\}$) or a claw $K_{1,3}$ (for $e = \{v_1, v_{13}\}$ or $\{v_3, v_{13}\}$), and the result follows from Theorem 3.1. $\square$

We now describe the graph H_r, or rather a class $\mathcal{H}_r$ of such graphs. Any graph $H_r \in \mathcal{H}_r$ is constructed in the following way. Its node set is $V = V_0 \cup V_3 \cup \ldots \cup V_r$, where $V_0 = \{v_{ij} : 3 \leq i < j \leq r\}$ and, for $i \in \{3, \ldots, r\}$, $V_i = \{v_1, v_2, v_{12}, v_i, v_{1i}, v_{2i}\}$. So H_r has $n = \binom{r+1}{2}$ nodes. Its edge set is defined as follows: On each set V_i we put a copy of H_3 (with index i playing the role of index 3 in the description of H_3 above) and, for each $3 \leq i < j \leq r$, we have the edges $\{v_i, v_{ij}\}$ and $\{v_j, v_{ij}\}$ as well as exactly one edge, call it e_{ij}, from the set

$$F_{ij} = \{\{v_i, v_j\}, \{v_i, v_{1j}\}, \{v_j, v_{1i}\}, \{v_{1i}, v_{1j}\}\}. \tag{13}$$

Figure 3 shows the graph H_4 for the choice $e_{34} = \{v_4, v_{13}\}$.

Theorem 3.13 *For any graph $H_r \in \mathcal{H}_r$ ($r \geq 3$),* $\text{egd}(H_r) = r$.

Proof. We label the nodes $v_1, \ldots, v_r$ by $e_1, \ldots, e_r \in \mathbb{R}^r$ and v_{ij} by $(e_i + e_j)/\sqrt{2}$. Let $X \in \mathcal{E}_n$ be their Gram matrix and $x = \pi_{E(H_r)}(X) \in \mathcal{E}(H_r)$. Then X is an extreme point of $\mathcal{E}_n$, we show that $\text{fib}(x) = \{X\}$. For this let $Y \in \text{fib}(x)$. We already know that $Y[V_i] = X[V_i]$ for each $i \in \{3, \ldots, r\}$. Indeed, as the subgraph of H_r induced by V_i is H_3, this follows from the way we have chosen the labeling and from the proof of Lemma 3.12. Hence we may now assume that we have a complete graph on each V_i and it remains to show that the entries of Y are uniquely specified at the non-edges that are not contained in some set V_i ($3 \leq i \leq r$). For this note that the vectors labeling the set $C_{ij} = \{v_i, v_j, v_{ij}\}$ are minimally linearly dependent. Using Lemma 3.5, one can verify that all remaining non-edges are forced using these sets C_{ij} and thus $Y = X$. This shows that $\text{egd}(H_r) \geq r$. $\square$

In contrast to the graphs F_r and G_r, we do not know whether $H_r \in \mathcal{H}_r$ is a *minimal* forbidden minor for $\mathscr{G}_{r-1}$ for $r \geq 4$.

3.3 Two special graphs: $K_{3,3}$ and K_5

In this section we consider the graphs $K_{3,3}$ and K_5 which will play a special role in the characterization of the class $\mathscr{G}_2$. First we compute the extreme Gram dimension of $K_{3,3}$. Note that its Gram dimension is $\text{gd}(K_{3,3}) = 4$ as $K_{3,3}$ contains a K_4-minor but it contains no K_5 and $K_{2,2,2}$-minor [17].

Theorem 3.14 $\text{egd}(K_{3,3}) = 2$.

The proof can be sketched as follows: Let $x \in \mathcal{E}(K_{3,3})$. First we show that any matrix $X \in \text{fib}(x)$ has rank at most 3 (Lemma 3.15). Next we show two technical lemmas which will be used to show that fib(x) contains at least two distinct elements. Therefore fib(x) must contain a matrix of rank at most 2 (see the paragraph after (4)) and thus $\text{egd}(K_{3,3}) \leq 2$.

Lemma 3.15 *For $x \in \text{ext}\,\mathcal{E}(K_{3,3})$, any $X \in \text{fib}(x)$ has rank at most 3.*

Proof. Let $x \in \text{ext}\,\mathcal{E}(K_{3,3})$ and let $X \in \text{fib}(x)$ with $\text{rank}\, X \geq 4$. Let $u_1, \ldots, u_6$ be a Gram representation of X and choose a subset $\{u_i : i \in I\}$ of linearly independent vectors with $|I| = 4$. Let E_I denote the set of edges of $K_{3,3}$ induced by I and set

$$\mathcal{U}_I = \{U_{ii} : i \in I\} \cup \{U_{ij} : \{i,j\} \in E_I\}.$$

Then $\mathcal{U}_I$ consists of linearly independent elements. Moreover, as any four nodes induce at least three edges in $K_{3,3}$, we have that $|\mathcal{U}_I| \geq 4+3 = 7$. By Lemma 2.7, $\mathcal{U}_I$ is contained in $\mathcal{U}_V$. We arrive at a contradiction since $\mathcal{U}_V$ has dimension 6 while $\mathcal{U}_I$ has dimension at least 7. □

Next we state two technical lemmas.

Lemma 3.16 *Let $X, Z \in \mathcal{S}_n$ with $X \succeq 0$ and satisfying:*

$$Xz = 0 \Longrightarrow z^{\mathsf{T}} Z z \geq 0, \quad Xz = 0, z^{\mathsf{T}} Z z = 0 \Longrightarrow Zz = 0. \qquad (14)$$

Then $X + tZ \succeq 0$ for some $t > 0$.

Proof. Up to an orthogonal transformation we may assume $X = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, where D is a diagonal matrix with positive diagonal entries. Correspondingly, write Z in block form: $Z = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}$. The conditions (14) show that $C \succeq 0$ and that the kernel of C is contained in the kernel of B. This implies that $X + tZ \succeq 0$ for some $t > 0$. □

Lemma 3.17 *Let $x \in \text{ext}\,\mathcal{E}(K_{3,3})$, let $X \in \text{fib}(x)$ be an extreme matrix of $\mathcal{E}_6$ of rank 3, with Gram representation $\{u_1, \ldots, u_6\} \subseteq \mathbb{R}^3$. Let $V_1 = \{1,2,3\}$ and $V_2 = \{4,5,6\}$ be the bipartition of the node set of $K_{3,3}$. There exist matrices $Y_1, Y_2 \in \mathcal{S}_3$ such that $Y_1 + Y_2 \succ 0$ and*

$$\langle Y_k, U_{ii} \rangle = 0 \ \forall i \in V_k \ \forall k \in \{1,2\} \quad \textit{and} \quad \exists k \in \{1,2\} \ \exists i,j \in V_k \ \langle Y_k, U_{ij} \rangle \neq 0.$$

Proof. Define $\mathcal{U}_k = \langle U_{ii} : i \in V_k\rangle \subseteq \mathcal{W}_k = \langle U_{ij} : i,j \in V_k\rangle \subseteq \mathcal{S}_3$ for $k = 1,2$. By assumption, $\dim \mathcal{U}_V = 6$, thus $\dim \mathcal{U}_1 = \dim \mathcal{U}_2 = 3$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \{0\}$. Moreover, as $\mathcal{U}_1^\perp \cap \mathcal{U}_2^\perp = \mathcal{U}_V^\perp = \{0\}$, we have that $\mathcal{S}_3 = \mathcal{U}_1^\perp \oplus \mathcal{U}_2^\perp$ and $\mathcal{W}_1^\perp \cap \mathcal{W}_2^\perp = \{0\}$.

Assume for contradiction that $\mathcal{S}_3^{++}$ is contained in $\mathcal{W}_1^\perp \oplus \mathcal{W}_2^\perp$. This implies that $\mathcal{W}_1^\perp \oplus \mathcal{W}_2^\perp = \mathcal{S}_3 = \mathcal{U}_1^\perp \oplus \mathcal{U}_2^\perp$ and thus $\mathcal{W}_k = \mathcal{U}_k$ as $\mathcal{U}_k \subseteq \mathcal{W}_k$. Hence, $\mathcal{W}_1 \cap \mathcal{W}_2 = \{0\}$.

As $\dim \mathcal{U}_k = 3$, we have $\dim\langle u_i \mid i \in V_k\rangle \geq 2$ for $k = 1,2$. Say, $\{u_1, u_2\}$ and $\{u_4, u_5\}$ are linearly independent. As $\dim\langle u_i : i \in [6]\rangle = 3$, there exists a non-zero vector $\lambda \in \mathbb{R}^4$ such that $0 \neq w = \lambda_1 u_1 + \lambda_2 u_2 = \lambda_3 u_4 + \lambda_4 u_5$. Hence we obtain that $ww^\mathsf{T} \in \mathcal{W}_1 \cap \mathcal{W}_2$, contradicting $\mathcal{W}_1 \cap \mathcal{W}_2 = \{0\}$.

Hence we have shown that $\mathcal{S}_3^{++} \not\subseteq \mathcal{W}_1^\perp \oplus \mathcal{W}_2^\perp$. So there exists a positive definite matrix Y which does not belong to $\mathcal{W}_1^\perp \oplus \mathcal{W}_2^\perp$. Write $Y = Y_1 + Y_2$, where $Y_k \in \mathcal{U}_k^\perp$ for $k = 1,2$. We may assume, say, that $Y_1 \notin \mathcal{W}_1^\perp$. Thus Y_1, Y_2 satisfy the lemma. □

Proof. (of Theorem 3.14). Let $x \in \operatorname{ext} \mathcal{E}(K_{3,3})$, let $X \in \operatorname{fib}(x)$ be an extreme point of $\mathcal{E}_6$ of rank 3, and let $\{u_1, \ldots, u_6\}$ be its Gram representation. Let Y_1 and Y_2 be the matrices provided by Lemma 3.17 and define the matrix $Z \in \mathcal{S}_6$ by $Z_{ij} = \langle Y_k, U_{ij}\rangle$ for $i, j \in V_k$, $k \in \{1,2\}$, and $Z_{ij} = 0$ for $i \in V_1$, $j \in V_2$. By Lemma 3.17, Z is a nonzero matrix with zero diagonal entries and zeros at the positions corresponding to the edges of $K_{3,3}$.

Next we show that $X + tZ \succeq 0$ for some $t > 0$, using Lemma 3.16. For this it is enough to verify that (14) holds. Assume $Xz = 0$, i.e., $a := \sum_{i \in V_1} z_i u_i = -\sum_{j \in V_2} z_j u_j$. Then,

$$z^\mathsf{T} Z z = \sum_{k=1,2} \sum_{i,j \in V_k} z_i z_j \langle Y_k, U_{ij}\rangle = \langle Y_1 + Y_2, aa^\mathsf{T}\rangle \geq 0,$$

since $Y_1 + Y_2 \succ 0$. Moreover, $z^\mathsf{T} Z z = 0$ implies $a = 0$ and thus $Zz = 0$ since, for $i \in V_k$, $(Zz)_i = \sum_{j \in V_k} \langle Y_k, U_{ij}\rangle z_j = \pm\langle Y_k, (u_i a^\mathsf{T} + a u_i^\mathsf{T})/2\rangle$. Hence, the matrix $X + tZ$ also belongs to the fiber of x. Combining with Lemma 3.15, we deduce that $\operatorname{fib}(x)$ contains a matrix of rank at most 2. □

We know that both graphs $K_{3,3}$ and K_5 belong to the class $\mathscr{G}_2$. We now show that they are in some sense maximal for this property.

Lemma 3.18 *Let $G \neq K_{3,3}, K_5$ be a 2-connected graph that contains K_5 or $K_{3,3}$ as a subgraph. Then G contains H_3 as a minor and thus* $\operatorname{egd}(G) \geq 3$.

Proof. The proof is based on the following observations. If G is a 2-connected graph containing strictly K_5 or $K_{3,3}$ as a subgraph, then G has a minor H which is one of the following graphs: (a) H is K_5 with one more node adjacent to two nodes of K_5, (b) H is $K_{3,3}$ with one more edge added, (c) H is $K_{3,3}$ with one more node adjacent to two adjacent nodes of $K_{3,3}$. Then H contains a H_3 subgraph in cases (a) and (b), and a H_3 minor in case (c) (easy verification). Hence, $\text{egd}(G) \geq \text{egd}(H_3) = 3$. □

4 Forbidden minor characterization of $\mathscr{G}_2$

In this section we characterize the class $\mathscr{G}_2$ of graphs with extreme Gram dimension at most 2. We show that $G \in \mathscr{G}_2$ if and only if G is a clique 0- and 1-sum of some graphs which either (i) have at most 5 nodes, or are (ii) $K_{3,3}$, or (iii) a minor of $T \boxtimes K_2$ for some tree T.

4.1 The main result

Here we formulate several characterizations for the class $\mathscr{G}_2$ and we outline the proof. By Theorem 3.1 we know that

$$\text{la}_{\boxtimes}(G) \leq 2 \Longrightarrow G \in \mathscr{G}_2 \tag{15}$$

and, by Theorem 3.6 and Lemma 3.12, we know that

$$G \in \mathscr{G}_2 \Longrightarrow G \text{ has no minors } F_3,\ H_3. \tag{16}$$

The three graph properties involved in (15), (16) are not equivalent in general: $K_5, K_{3,3} \in \mathscr{G}_2$ (Theorem 3.14), but $\text{la}_{\boxtimes}(K_5) = \text{la}_{\boxtimes}(K_{3,3}) = 3$ (see Section 5). However, these two graphs are exceptional since they cannot occur as proper subgraphs of a 2-connected graph with no H_3 minor (Lemma 3.18). As the class $\mathscr{G}_2$ is closed under taking clique 0- and 1-sums, it suffices to characterize the 2-connected graphs in $\mathscr{G}_2$. We show the following result:

Theorem 4.1 *Let G be a 2-connected graph on $n \geq 6$ nodes and assume that $G \neq K_{3,3}$. Then the following assertions are equivalent.*

(i) *$G \in \mathscr{G}_2$, i.e.,* $\text{egd}(G) \leq 2$.

(ii) *G has no minors F_3 or H_3.*

(iii) $\text{la}_{\boxtimes}(G) \leq 2$*, i.e., G is a minor of $T \boxtimes K_2$ for some tree T.*

In the rest of Section 4 we prove the implication (ii) $\Longrightarrow$ (iii). The proof is in two steps. First we consider the chordal case and show:

(1) The chordal case: If $G \in \mathcal{F}(F_3, H_3)$ is chordal, then G is a contraction minor of $T \boxtimes K_2$ for some tree T (Section 4.2, Theorem 4.3).

Then we reduce the general case to the chordal case and show:

(2) Reduction to the chordal case: Any graph $G \in \mathcal{F}(F_3, H_3)$ is subgraph of a chordal graph $G' \in \mathcal{F}(F_3, H_3)$.

For case (2) we first exclude K_4 instead of H_3 (Section 4.3, Theorem 4.6) and then we derive from it the general result (Section 4.4, Theorem 4.11).

4.2 The case of chordal graphs

Our goal in this section is to characterize the 2-connected chordal graphs G with $\text{egd}(G) \leq 2$. By Lemma 3.18, if $G \neq K_5$ has $\text{egd}(G) \leq 2$, then $\omega(G) \leq 4$. Denote by $\mathcal{C}$ the family of all 2-connected chordal graphs with $\omega(G) \leq 4$. Any graph $G \in \mathcal{C}$ is a clique 2- or 3-sum of K_3's and K_4's. Note that F_3 belongs to $\mathcal{C}$ and has $\text{egd}(F_3) = 3$. On the other hand, any graph $G = T \boxtimes K_2$ where T is a tree, belongs to $\mathcal{C}$ and has $\text{egd}(G) = 2$. These graphs are "special clique 2-sums" of K_4's, as they satisfy the following property: every 4-clique has at most two edges which are cutsets and these two edges are not adjacent. This motivates the following definitions, useful in the proof of Theorem 4.3 below.

Definition 4.2 *Let $G \in \mathcal{C}$ (i.e., G is chordal 2-connected with $\omega(G) \leq 4$).*

(i) *An edge of G is called* free *if it belongs to exactly one maximal clique (i.e., its endpoints do not form a cutset) and* non-free *otherwise.*

(ii) *A 3-clique in G is called* free *if it contains at least one free edge.*

(iii) *A 4-clique in G is called* free *if it does not have two adjacent non-free edges. A free 4-clique can be partitioned as $\{a, b\} \cup \{c, d\}$, called its* two sides*, where only $\{a, b\}$ and $\{c, d\}$ can be non-free (i.e., cutsets).*

(iv) *G is called free if all its maximal cliques are free.*

For instance, F_3, $K_5 \backslash e$ (the clique 3-sum of two K_4's) are not free. Hence any free graph in $\mathcal{C}$ is a clique 2-sum of free K_3's and free K_4's. Note also that $\text{la}_{\boxtimes}(K_5 \setminus e) = 3$. We now show that for a graph $G \in \mathcal{C}$ the property of being free is equivalent to having $\text{la}_{\boxtimes}(G) \leq 2$ and also to having $egd(G) \leq 2$.

Theorem 4.3 *Let $G \in \mathcal{C} \setminus \{K_5 \backslash e\}$. The following assertions are equivalent.*

(i) *G has no minors F_3 or H_3.*

(ii) *G does not contain F_3 as a subgraph.*

(iii) *G is free.*

(iv) *G is a contraction minor of $T \boxtimes K_2$ for some tree T.*

(v) $\mathrm{egd}(G) \leq 2$.

Proof. (i) $\Rightarrow$ (ii) is clear and (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) follow from earlier results.
Proof of (ii) $\Longrightarrow$ (iii): Assume that (ii) holds. First we show that G does not contain a subgraph $K_5 \backslash e$. For this assume that $G[U] = K_5 \backslash e$ for some $U \subseteq V(G)$. As $G \neq K_5 \backslash e$ and G is 2-connected chordal, there is a node $u \notin U$ which is adjacent to two adjacent nodes of U and then one can find a F_3 subgraph in G. Therefore, G is a clique 2-sum of K_3's and K_4's. We now show that each of them is free.

Suppose first that $C = \{a, b, c\}$ is a maximal 3-clique which is not free. Then, there exist nodes $u, v, w \notin C$ such that $\{a, b, u\}$, $\{a, c, v\}$, $\{b, c, w\}$ are cliques in G. Moreover, u, v, w are pairwise distinct (if $u = v$ then $C \cup \{u\}$ is a clique, contradicting maximality of C) and we find a F_3 subgraph in G.

Suppose now that $C = \{a, b, c, d\}$ is a 4-clique which is not free and, say, both edges $\{a, b\}$ and $\{a, c\}$ are non-free. Then, there exist nodes $u, v \notin C$ such that $\{a, b, u\}$ and $\{a, c, v\}$ are cliques. Moreover, $u \neq v$ (else we find a $K_5 \backslash e$ subgraph) and thus we find a F_3 subgraph in G. Thus (iii) holds.

Proof of (iii) $\Longrightarrow$ (iv): Assume that G is free, $G \neq K_4, K_3$ (else we are done). When all maximal cliques are 4-cliques, it is easy to show using induction on $|V(G)|$ that $G = T \boxtimes K_2$, where T is a tree and each side of a 4-clique of G corresponds to a node of T.

Assume now that G has a maximal 3-clique $C = \{a, b, c\}$. Say, $\{b, c\}$ is free and $\{a, b\}$ is a cutset. Write $V(G) = V' \cup V'' \cup \{a, b\}$, where V'' is the (vertex set of the) component of $G \backslash \{a, b\}$ containing c, and V' is the union of the other components. Now replace node a by two new nodes a', a'' and replace C by the 4-clique $C' = \{a', a'', b, c\}$. Moreover, replace each edge $\{u, a\}$ by $\{u, a'\}$ if $u \in V'$ and by $\{u, a''\}$ if $u \in V''$. Let G' be the graph obtained in this way. Then $G' \in \mathcal{C}$ is free, G' has one less maximal 3-clique than G, and $G = G'/\{a', a''\}$. Iterating, we obtain a graph $\widehat{G}$ which is a clique 2-sum of free K_4's and contains G as a contraction minor. By the above, $\widehat{G} = T \boxtimes K_2$ and thus G is a contraction minor of $T \boxtimes K_2$. $\square$

4.3 Structure of the graphs with no F_3 or K_4 minor

In this section we investigate the class $\mathcal{F}(F_3, K_4)$. We start with two technical lemmas.

Lemma 4.4 *Let G and M be two 2-connected graphs, let $\{x, y\} \notin E(G)$ be a cutset in G, and let $r \geq 2$ be the number of components of $G \setminus \{x, y\}$.*

(i) *Assume that $G \in \mathcal{F}(M)$, but the graph $G + \{x, y\}$ has a M-minor with M-partition $\{V_i : i \in V(M)\}$. If $x \in V_i$ and $y \in V_j$, then $M \setminus \{i, j\}$ has at least r components and thus $i \neq j$.*

(ii) *Assume that M does not have two adjacent nodes forming a cutset in M. If $G \in \mathcal{F}(M)$, then $G + \{x, y\} \in \mathcal{F}(M)$.*

Proof. (i) Let $C_1, \ldots, C_r \subseteq V(G)$ be the node sets of the components of $G \setminus \{x, y\}$. As G is 2-connected, there is an $x-y$ path P_s in $G[C_s \cup \{x, y\}]$ for each $s \in [r]$. Now let U be a component of $M \setminus \{i, j\}$. By the definition of the M-partition, the graph $G[\bigcup_{k \in U} V_k]$ is connected. As $x, y \notin \bigcup_{k \in U} V_k$, we deduce that $\bigcup_{k \in U} V_k \subseteq C_s$ for some $s \in [r]$. In other words, every component of $M \setminus \{i, j\}$ corresponds to exactly one component of $G \setminus \{x, y\}$. Assume for contradiction that $M \setminus \{i, j\}$ has less than r components. Then there is at least one component C_s which does not correspond to any component of $M \setminus \{i, j\}$. That is, $(\bigcup_{k \neq i,j} V_k) \cap C_s = \emptyset$, so that $C_s \subseteq V_i \cup V_j$. Hence the path P_s is contained in $G[V_i \cup V_j]$, thus $\{V_i : i \in V(M)\}$ remains an M-partition of G and we find a M-minor in G, a contradiction. Therefore, $M \setminus \{i, j\}$ has at least $r \geq 2$ components. This implies that $\{i, j\}$ is a cutset of M and thus $i \neq j$ since M is 2-connected.

(ii) Assume $G + \{x, y\}$ has a M-minor, with corresponding M-partition $\{V_i : i \in V(M)\}$. By (i), the nodes x and y belong to two distinct classes V_i and V_j and $\{i, j\}$ is a cutset in M. This implies that $\{i, j\} \notin E(M)$ and thus M is a minor of G, a contradiction. $\square$

Lemma 4.5 *Let $G \in \mathcal{F}(K_4)$ be a 2-connected graph and let $\{x, y\} \notin E(G)$. If there are at least three (internally vertex) disjoint paths from x to y, then $\{x, y\}$ is a cutset and $G \setminus \{x, y\}$ has at least 3 components.*

Proof. If P_1, P_2, P_3 are distinct vertex disjoint paths from x to y, then $P_1 \setminus \{x, y\}$, $P_2 \setminus \{x, y\}$ and $P_3 \setminus \{x, y\}$ lie in distinct components of $G \setminus \{x, y\}$, for if not one would find a homeomorph of K_4. $\square$

We now show the main result of this section.

Theorem 4.6 *Let $G \in \mathcal{F}(F_3, K_4)$ be a 2-connected graph. Then there exists a chordal graph $Q \in \mathcal{F}(F_3, K_4)$ containing G as a subgraph.*

Proof. Let $G \in \mathcal{F}(F_3, K_4)$ be 2-connected. If $\{x, y\} \notin E(G)$ is such that there exist at least three disjoint paths in G from x to y, then we can add the edge $\{x, y\}$ without creating a K_4 or F_3-minor: $G + \{x, y\} \in \mathcal{F}(F_3, K_4)$, this follows from Lemma 4.4 (applied to $M = F_3$ and K_4) and Lemma 4.5. So we can add edges iteratively until obtaining a graph $\widehat{G} \in \mathcal{F}(F_3, K_4)$ containing G as a subgraph and satisfying:

$$\forall \{x, y\} \notin E(\widehat{G}) \text{ there are at most two disjoint } x - y \text{ paths in } \widehat{G}. \quad (17)$$

If $\widehat{G}$ is chordal we are done. So consider a chordless circuit C in $\widehat{G}$. Note that any circuit C' distinct from C, which meets C in at least two nodes, meets C in exactly two nodes that are adjacent (easy consequence of (17)). Call an edge of C *busy* if it is contained in some circuit $C' \neq C$. If $e_1 \neq e_2$ are two busy edges of C and $C_i \neq C$ is a circuit containing e_i, then C_1, C_2 are (internally) disjoint (use (17)). Hence C can have at most two busy edges, for otherwise one would find a F_3-minor in $\widehat{G}$.

We now show how to triangulate C without creating a K_4 or F_3-minor: If C has two busy edges denoted, say, $\{v_1, v_2\}$ and $\{v_k, v_{k+1}\}$ (possibly $k = 2$), then we add the edges $\{v_1, v_i\}$ for $i \in \{3, \ldots, k\}$ and the edges $\{v_k, v_i\}$ for $i \in \{k + 2, \ldots, |C|\}$, see Figure 4 a). If C has only one busy edge $\{v_1, v_2\}$, add the edges $\{v_1, v_i\}$ for $i \in \{3, \ldots, |C| - 1\}$, see Figure 4 b). (If C has no busy edge then $G = C$, triangulate from any node and we are done).

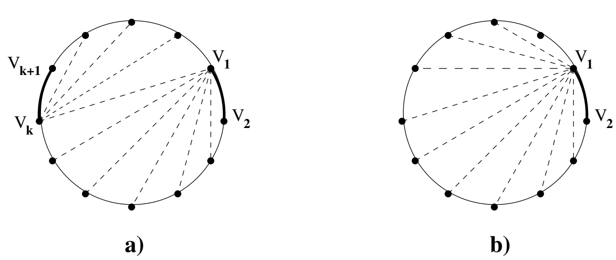

Figure 4: Triangulating a chordless circuit with a) two or b) one busy edge.

Let Q denote the graph obtained from $\widehat{G}$ by triangulating all its chordless circuits in this way. Hence Q is a chordal extension of $\widehat{G}$ (and thus of G). We show that $Q \in \mathcal{F}(F_3, K_4)$. First we see that $Q \in \mathcal{F}(K_4)$ by applying iteratively Lemma 4.4 (ii) (for $M = K_4$): For each $i \in \{3, \ldots, k\}$, $\{v_1, v_i\}$

is a cutset of $\widehat{G}$ and of $\widehat{G} + \{\{v_1, v_j\} : j \in \{3, \ldots, i-1\}\}$ (and analogously for the other added edges $\{v_k, v_i\}$). Hence Q is a clique 2-sum of triangles. We now verify that each triangle is free which will conclude the proof, using Theorem 4.3.

For this let $\{a, b, c\}$ be a triangle in Q. First note that if (say) $\{a, b\} \in E(Q) \setminus E(\widehat{G})$, then a, b, c lie on a common chordless circuit C of $\widehat{G}$. Indeed, let C be a chordless circuit of $\widehat{G}$ containing a, b and assume $c \notin C$. By (17), $\widehat{G} \setminus \{a, b\}$ has at most two components and thus there is a path from c to one of the two paths composing $C \setminus \{a, b\}$. Together with the triangle $\{a, b, c\}$ this gives a homeomorph of K_4 in Q, contradicting $Q \in \mathcal{F}(K_4)$, just shown above. Hence the triangle $\{a, b, c\}$ lies in C and thus has a free edge.

Suppose now that $\{a, b, c\}$ is a triangle contained in $\widehat{G}$. If it is not free then there is a F_3 on $\{a, b, c, x, y, z\}$ where x (resp., y, and z) is adjacent to a, b (resp., a, c, and b, c). Say $\{x, a\} \in E(Q) \setminus E(\widehat{G})$ (as there is no F_3 in $\widehat{G}$). Then x, a, b lie on a chordless circuit C of $\widehat{G}$ and $\{x, b\} \in E(\widehat{G})$ (since $\{a, b\}$ is a busy edge). Moreover, $c, y, z \notin C$ for otherwise we get a K_4-minor in Q. Then delete the edge $\{x, a\}$ and replace it by the $\{x, a\}$-path along C. Do the same for any other edge of $E(Q) \setminus E(\widehat{G})$ connecting y and z to $\{a, b, c\}$. After that we get a F_3-minor in $\widehat{G}$, a contradiction. □

4.4 Structure of the graphs with no F_3 or H_3 minor

Here we investigate the graphs $G \in \mathcal{F}(F_3, H_3)$. By the results in Section 4.3 we may assume that G contains some homeomorph of K_4. Figure 5 shows a homeomorph of K_4, where the original nodes are denoted as 1,2,3,4 and called its *corners*, and the wiggled lines correspond to subdivided edges (i.e., to paths P_{ij} between the corners $i \neq j \in [4]$).

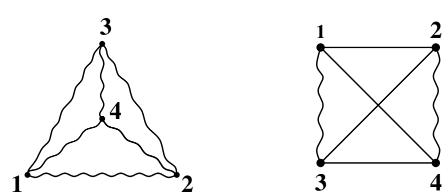

Figure 5: A homeomorph of K_4 and its two sides (cf. Lemma 4.7)

To help the reader visualize F_3 and H_3 we use Figure 6. Notice the special role of node 5 in H_3 (denoted by a square) and of the (dashed) triangle $\{1, 2, 3\}$.

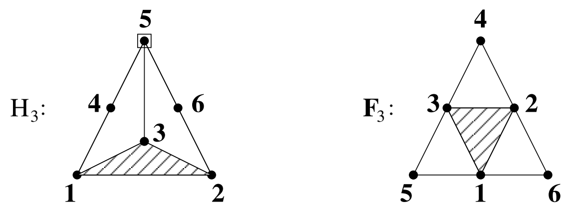

Figure 6: The graphs H_3 and F_3.

The starting point of the proof is to investigate the structure of homeomorphs of K_4 in a graph of $\mathcal{F}(H_3)$.

Lemma 4.7 *Let G be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then there is a partition of the corner nodes of H into $\{1,3\}$ and $\{2,4\}$ for which the following holds.*

(i) *Only the paths P_{13} and P_{24} can have more than 2 nodes.*

(ii) *Every component of $G \setminus H$ is connected to P_{13} or to P_{24}.*

Then P_{13} and P_{24} are called the two sides *of H (cf. Figure 5).*

Proof. We use the graphs from Figure 7 which all contain a subgraph H_3.
Case 1: $H = K_4$. If $G \setminus H$ has a unique component C then $|C| \geq 2$ as $n \geq 6$. If C is connected to two nodes of H, then the conclusion of the lemma holds. Otherwise, C is connected to at least three nodes in H and then the graph from Figure 7 a) is a minor of G, a contradiction.

If there are at least two components in $G \setminus H$, then they cannot be connected to two adjacent edges of H for, otherwise, the graph of Figure 7 b) is a minor of G, a contradiction. Hence the lemma holds.
Case 2: $H \neq K_4$. Say, P_{13} has at least 3 nodes. Then the edges $\{1,i\}$, $\{3,i\}$ $(i = 2,4)$ cannot be subdivided (else H is a homeomorph of H_3). So (i) holds. We now show (ii). Indeed, if a component of $G \setminus H$ is connected to both P_{13} and P_{24}, then at least one of the graphs in Figure 7 c) and d) will be a minor of G, a contradiction. □

Lemma 4.7 implies that there is no path with at least 3 nodes between the sides of a K_4-homeomorph. We now show that, moreover, there is no additional edge between the two sides. More precisely:

Lemma 4.8 *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then there exists no edge between the two sides of H except between their endpoints.*

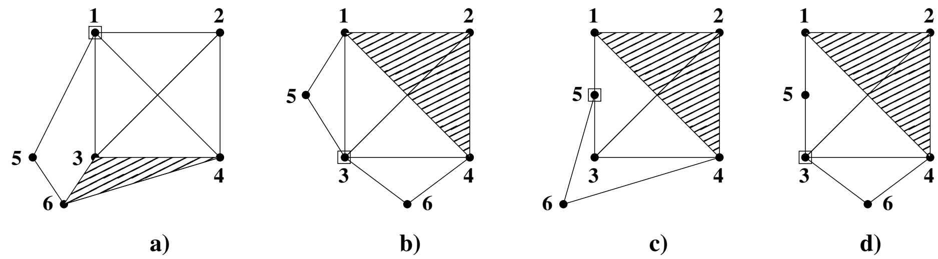

Figure 7: Bad subgraphs in the proof of Lemma 4.7.

Proof. Say, P_{13} and P_{24} are the two sides of H. Assume for a contradiction that $\{a, b\} \in E(G)$, where a lies on P_{13} and b on P_{24}.

Assume first that a is an internal node of P_{13} and b is an internal node of P_{24}. If $|V(H)| = 6$, then $H = K_{3,3}$ and Lemma 3.18 implies that G has a H_3 minor, a contradiction. Hence, $|V(H)| > 6$ and we can assume w.l.o.g. that the path from 1 to a within P_{13} has at least 3 nodes. Then G contains a homeomorph of K_4 with corner nodes $1, b, 4, a$, where the two paths from 1 to a and from 1 to b (via 2) have at least 3 nodes, giving a H_3 minor and thus a contradiction.

Assume now that only a is an internal node of P_{13} and, say $b = 2$. If $|V(H)| = 5$, then $G \setminus H$ has at least one component. By Lemma 4.7, this component connects either to the path P_{13} or to the edge $\{2, 4\}$. In both cases, it is easy to verify that one of the graphs in Figure 8 will be a minor of G, a contradiction since all of them have a H_3 subgraph. On the other hand, if $|V(H)| \geq 6$, then one of the paths from 1 to a, from a to 3 (within P_{13}), or P_{24} is subdivided. This implies that G contains a homeomorph of K_4 with corner nodes $a, 1, 2, 4$ or $a, 2, 3, 4$, which thus contains two adjacent subdivided edges, giving a H_3 minor. $\square$

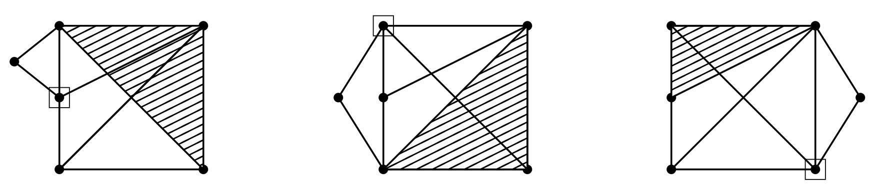

Figure 8: Bad subgraphs in the proof of Lemma 4.8.

Lemmas 4.7 and 4.8 imply directly:

Corollary 4.9 *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then the endnodes of at least one of the two sides of H form a cutset in G. Moreover, if P_{13} is a side of H and its endnodes $\{1,3\}$ do not form a cutset, then $P_{13} = \{1,3\}$ and there is no component of $G \setminus H$ which is connected to P_{13}.*

We now show that one may add edges to G so that all minimal homeomorphs of K_4 are 4-cliques, without creating a F_3 or H_3 minor.

Lemma 4.10 *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathcal{F}(F_3, H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. The graph obtained by adding to G the edges between the endpoints of the sides of H belongs to $\mathcal{F}(F_3, H_3)$.*

Proof. Say P_{13} and P_{24} are the sides of H. Assume $|V(P_{13})| \geq 3$ and $\{1,3\} \notin E(G)$. By Corollary 4.9, $\{1,3\}$ is a cutset in G. We show that $\widehat{G} = G + \{1,3\} \in \mathcal{F}(F_3, H_3)$. First, applying Lemma 4.4 (ii) with $M = H_3$ and $\{x,y\} = \{1,3\}$, we obtain that $\widehat{G} \in \mathcal{F}(H_3)$.

Next, assume for contradiction that $\widehat{G}$ has a F_3 minor, with F_3-partition $\{V_i : i \in [6]\}$, where we use the same labeling as in Figure 6. Applying Lemma 4.4 (i) with $M = F_3$ and $\{x,y\} = \{1,3\}$, we see that the nodes 1 and 3 belong to distinct classes of the F_3-partition, which correspond to a cutset of F_3. Say, $1 \in V_1$ and $3 \in V_2$. Then the nodes 2 and 4 do not lie in $V_1 \cup V_2$ (for otherwise, one would have an F_3-partition in G). Next we show that the nodes 2 and 4 do not belong to the same class of the F_3-partition. Assume for contradiction that $2, 4 \in V_k$. If $\{2,4\}$ is not a cutset in G then, by Corollary 4.9, $P_{24} = \{2,4\}$ and no component of $G \setminus H$ connects to $\{2,4\}$. Hence $V_k = \{2,4\}$ and we can move node 2 to the class V_1, so that we obtain a F_3-partition of G, a contradiction. If $\{2,4\}$ is a cutset of G, then every component of $G \setminus \{2,4\}$ except the one containing 1 and 3 has to lie within V_k, so we can again move node 2 to V_1 and obtain a F_3-partition of G.

Accordingly, the nodes $1, 2, 3$ and 4 belong to distinct classes and we can assume w.l.o.g. that $2 \notin V_6$. Observe that every $1-2$ path in G is either the edge $\{1,2\}$ or meets the nodes 3 or 4. Similarly, every $2-3$ path in G is either the edge $\{2,3\}$ or meets the nodes 1 or 4. An easy case analysis shows that whatever the position of nodes 2 and 4 in the F_3-partition we always find a $1-2$ or a $2-3$ path violating the above conditions. □

We are now ready to show the main result of this section.

Theorem 4.11 *Let $G \neq K_{3,3}$ be 2-connected in $\mathcal{F}(F_3, H_3)$ on $n \geq 6$ nodes. There exists a chordal graph $Q \in \mathcal{F}(F_3, H_3)$ containing G as a subgraph.*

Proof. If $G \in \mathcal{F}(F_3, K_4)$ then we are done by Theorem 4.6. Otherwise, we augment the graph G by adding the edges between the endpoints of the sides of every homeomorph of K_4 contained in G. Let $\widehat{G}$ be the graph obtained in this way. By Lemma 4.10, we know that $\widehat{G} \in \mathcal{F}(F_3, H_3)$. Hence, for each K_4-homeomorph H in $\widehat{G}$, its corners form a 4-clique. Moreover, if C, C' are two distinct 4-cliques of $\widehat{G}$, then $C \cap C'$ is contained in a side of C and C'.

Consider a 4-clique $C = \{1,2,3,4\}$ in $\widehat{G}$, say with sides $\{1,3\}$, $\{2,4\}$ (so each component of $\widehat{G} \setminus C$ connects to $\{1,3\}$ or to $\{2,4\}$, by Lemma 4.7). Pick an edge f between the two sides (i.e., $f = \{i,j\}$ with $i \in \{1,3\}$, $j \in \{2,4\}$) and delete this edge f from $\widehat{G}$. We repeat this process with every 4-clique in $\widehat{G}$ and obtain the graph $G_0 = \widehat{G} \setminus \{f_1, \ldots, f_k\}$, if $\widehat{G}$ has k 4-cliques.

By construction, G_0 belongs to $\mathcal{F}(F_3, K_4)$ and is 2-connected. Hence, we can apply Theorem 4.6 to G_0 and obtain a chordal graph $Q_0 \in \mathcal{F}(F_3, K_4)$ containing G_0 as a subgraph. Hence, Q_0 is a clique 2-sum of free triangles. It suffices now to show that the augmented graph $Q = Q_0 + \{f_i : i \in [k]\}$ is a clique 2-sum of free K_3's and K_4's. Then Q is a chordal graph in $\mathcal{F}(F_3, H_3)$ (by Theorem 4.3) containing $\widehat{G}$ and thus G, and the proof is completed.

For this, consider again a 4-clique $C = \{1,2,3,4\}$ in $\widehat{G}$ with sides $\{1,3\}$ and $\{2,4\}$. Then, each component of $\widehat{G} \setminus C$ connects to $\{1,3\}$ or $\{2,4\}$. We claim that the same holds for each component of $Q_0 \setminus C$. Indeed, a component of $Q_0 \setminus C$ is a union of some components of $\widehat{G} \setminus C$. Thus it connects to two nodes (to 1,3, or to 2,4), or to at least three nodes of C. But the latter case cannot occur since we would then find a K_4 minor in Q_0.

Assume that the edge $f = \{1,4\}$ was deleted from the 4-clique C when making the graph G_0. We now show that adding it back to Q_0 results in a free graph. Indeed, by adding the edge $\{1,4\}$ we only replace the two maximal 3-cliques $\{1,3,4\}$ and $\{1,2,4\}$ by a new maximal 4-clique $\{1,2,3,4\}$, which is free. We iterate this process for each of the edges $f_1, \ldots, f_k$ and obtain that $Q = Q_0 + \{f_i : i \in [k]\}$ is a free graph in $\mathcal{C}$. □

5 Concluding remarks

Colin de Verdière [5] introduced the *largeur d'arborescence* $\text{la}_{\Box}(G)$ as upper bound for his graph parameter $\nu(G)$, which is defined as the maximum corank of a matrix $A \in \mathcal{S}^+_n$ satisfying: $A_{ij} = 0$ if and only if $i \neq j$ and $\{i,j\} \notin E(G)$, and the following non-degeneracy condition (known as the *strong Arnold property*):

$$AX = 0,\ X \in \mathcal{S}_n,\ X_{ij} = 0\ \forall \{i,j\} \in V \cup E \Longrightarrow X = 0.$$

He shows that $\nu(G)$ is minor monotone, $\nu(G) \leq \text{la}_{\square}(G)$, with equality for the graphs G_r: $\nu(G_r) = \text{la}_{\square}(G_r) = r$, as well as

$$\text{la}_{\square}(G) \leq 1 \Longleftrightarrow \nu(G) \leq 1 \Longleftrightarrow G \text{ has no minor } K_3.$$

Kotlov [14] shows:

$$\text{la}_{\square}(G) \leq 2 \Longleftrightarrow \nu(G) \leq 2 \Longleftrightarrow G \text{ has no minors } F_3, K_4.$$

The most work is showing that $\text{la}_{\square}(G) \leq 2$ if $G \in \mathcal{F}(K_4, F_3)$. In fact, this also follows from our characterization of the class $\mathcal{F}(K_4, F_3)$. Indeed, if $G \in \mathcal{F}(K_4, F_3)$ is 2-connected then we have shown that G is subgraph of G' which is a clique 2-sum of free triangles. Now our argument in the proof of Theorem 4.3 also shows that G' is a contraction minor of $T \square K_2$ for some tree T (as each triangle of G' arises as contraction of a 4-clique which can be replaced by a 4-circuit). In this sense our characterization is a refinement of Kotlov's result tailored to our needs.

We now characterize the graphs with $\text{la}_{\boxtimes}(G) \leq 2$. The *wheel* W_5 is obtained from the circuit C_4 by adding a node adjacent to all nodes of C_4.

Theorem 5.1 *For a graph G, $\text{la}_{\boxtimes}(G) \leq 2$ if and only if $G \in \mathcal{F}(F_3, H_3, W_5)$.*

Proof. We already know that $\text{la}_{\boxtimes}(G) \geq \text{egd}(G) = 3$ for $G = F_3, H_3$. Suppose for contradiction that $\text{la}_{\boxtimes}(W_5) \leq 2$. Then $\text{la}_{\boxtimes}(W_5) = \text{la}_{\boxtimes}(H)$ where H is a chordal extension of W_5 and H is a contraction minor of some $T \boxtimes K_2$. As W_5 is not chordal, H contains W_5 with one added chord on its 4-circuit, i.e., H contains $K_5 \setminus e$ and thus $\text{la}_{\boxtimes}(H) \geq \text{la}_{\boxtimes}(K_5 \setminus e) = 3$. Therefore, F_3, H_3, W_5 are forbidden minors for the property $\text{la}_{\boxtimes}(G) \leq 2$. Conversely, assume that $G \in \mathcal{F}(F_3, H_3, W_5)$ is 2-connected, we show that $\text{la}_{\boxtimes}(G) \leq 2$. This is clear if G has $n \leq 4$ nodes, or if G has $n = 5$ nodes and it has a node of degree 2. If G has $n = 5$ nodes and each node has degree at least 3, then one can easily verify that G contains W_5. If G has $n \geq 6$ nodes then $\text{la}_{\boxtimes}(G) \leq 2$ follows from Theorem 4.1 (since $G \neq K_{3,3}$ as $W_5 \preceq K_{3,3}$). $\square$

Summarizing, we have $\nu(G) \leq \text{la}_{\square}(G)$ and $\text{egd}(G) \leq \text{la}_{\boxtimes}(G) \leq \text{la}_{\square}(G)$. Moreover, $\text{egd}(G) = \nu(G)$ if $\nu(G) \leq 2$. Also, $\nu(K_n) = n - 1$ [5] and thus $\nu(K_n) > \text{la}_{\boxtimes}(K_n) \geq \text{egd}(K_n)$ if $n \geq 4$. An interesting open question is whether the inequality $\text{egd}(G) \leq \nu(G)$ holds in general. We point out the analogous inequality: $\nu^{=}(G) \leq \text{gd}(G)$, shown in [17]. The parameter $\nu^{=}(G)$ is the analogue of $\nu(G)$ studied by van der Holst [13] (same definition as $\nu(G)$, but now requiring only that $A_{ij} = 0$ for $\{i, j\} \in E(G)$ and allowing zero entries at positions on the diagonal and at edges), and $\nu^=$ satisfies: $\nu(G) \leq \nu^=(G)$.

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