Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

J. van de Lune, H.J.J. te Riele, D.T. Winter

On the zeros of the Riemann zeta function in the critical strip; IV

Department of Numerical Mathematics Report NM-R8515 June

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1. INTRODUCTION

This paper is a continuation of three papers by Brent [1], Brent et al. [2]* and van de Lune et al. [7]**. The computations reported there (up to $g_{300,000,000}$) have been extended up to $g_{1,500,000,000}$ in order to show that *the first* 1,500,000,001 *zeros of Riemann's zeta function in the critical strip are simple and lie on the vertical with real part* 1/2. This establishes the truth of the Riemann hypothesis in the rectangle $\{\sigma + it,\ 0 < \sigma < 1,\ 0 < t < 545,439,823.215\}$. Moreover, parts of the computations reported on in [2] have been repeated in a slightly different manner (see below), so that it is now possible to present *exact* (rather than approximate) counts of the numbers of Gram blocks of various types and of the numbers of Gram intervals with various numbers of zeros, for the whole interval $[g_0,\ g_{1,500,000,000})$.

The FORTRAN/COMPASS program described in van de Lune et al. [6], was run on a CDC CYBER 175/750 computer to separate the zeros of Z(t) in the intervals $[g_{300,000,000}, g_{415,000,000})$ and $[g_{1,445,000,000},\ g_{1,500,000,000})$. A vectorized version ([9], [14]) of that program was run on a CYBER 205 vector computer to separate the zeros of Z(t) in the interval $[g_{415,000,000}, g_{1,445,000,000})$. Finally, the program was run a little further beyond $g_{1,500,000,000}$ to yield 5 Gram blocks of lengths 1, 1, 2, 1 and 1 in $[g_{1,500,000,000}, g_{1,500,000,006})$ which all satisfy Rosser's rule. Applying Theorem 3.2 of [1] we completed the proof of our claim that the first 1,500,000,001 complex zeros of the Riemann zeta function have real part 1/2 (and that all these zeros are simple). The total CPU time used amounted to approximately 900 hours on the CYBER 175/750 and about 1000 hours on the CYBER 205.

---

* A check of the types of exceptions to Rosser's rule given in Table 1 of [2] revealed that $B_{164,689,301}$ is of type 3 rather than 1. Consequently, the following corrections should be made in [2]: In Table 1 replace 164,689,301(1) by 164,689,301(3); in Table 2, last column, replace the first and the third frequencies 53 and 1 by 52 and 2, respectively. In Table 4, last line, replace the frequencies 146,878,417, 26,048,007, 341,855 and 1 by 146,878,418, 26,048,006, 341,854 and 2, respectively. The last sentence on p. 684 (starting with "Note that ...") and the one on p. 687 on lines 3 and 4 from above should be adapted in the sense that a second exception of type 3 has been found, and hence the second Gram interval containing four zeros (leaving unchanged the remark that no type 4 exception had yet been found).

** Correction in [7]: In Table 1, line 2 from below, replace 266527881(2) by 266527881(1). On p. 762, line 4 from below, replace *"second"* by *"third"*.

During the computations in the interval $[g_{156,800,000}, g_{200,000,000})$ reported on in [2], the counts of the numbers of Gram blocks and of the numbers of Gram intervals with 0,1,... zeros were not exact, since small shifts were made in the argument of $Z_A(t)$ in case the value of $Z_A(t)$ in a Gram point g_n was too small to yield the correct sign of $Z(g_n)$ with certainty. Since all subsequent computations were carried out without shifts, i.e., $Z_B(g_n)$ was computed immediately in case $Z_A(g_n)$ was too small, we have rerun and verified our computations in the interval $[g_{156,800,000}, g_{200,000,000})$ on the CYBER 205. The total amount of CYBER 205 CPU time needed for this check was about 21 hours.

Moreover, with the aid of our CYBER 205 program we checked some of the computations reported on in [1] and [7], viz., those concerning the intervals $[g_{1,000}, g_{5,000,000})$ and $[g_{200,000,000}, g_{210,000,000})$. We found exact agreement with the corresponding results in [1] and [7]. The CPU time needed on the CYBER 205 was about 2800 seconds and 5 hours, respectively.

2. COMPUTATION OF Z(t) AND ERROR ANALYSIS

In principle, our method of computing Z(t) is exactly as described in [6] and Section 3 of [2]. In order to run our program on the CYBER 205, the most time consuming part of our FORTRAN/COMPASS CYBER 175/750 program, i.e., the summation of $\sum_{k=1}^{m} k^{-\frac{1}{2}} \cos[t.\log(k) - \theta(t)]$ in $Z_A(t)$ (cf. [1], formula (2.6)), was vectorized by invoking so-called Q8-calls. Details are given in [9] and [14]. As a result, our CYBER 205 program ran about 7 times as fast as the CYBER 175/750 version.

In [6] we have given a rigorous error analysis of our computation of Z(t) on the CYBER 175/750, for t in the interval (3.5x10**7, 3.72x10**8), covering the range of zero # 81,000,000 to zero # 1,000,000,000 of $\zeta(s)$ in the critical strip. In Section 3 of [2] the resulting bounds for the error in the computed value $\tilde{Z}(t)$ of Z(t) are given for methods A and B.

For the computations reported on in the present report we have extended this error analysis to the interval (3.5x10**7, 5.6x10**8), in order to cover the range of zero # 81,000,000 to zero # 1,500,000,000, both for the CYBER 175/750 and the CYBER 205 (several spot checks of the CYBER 205 computations were carried out on the CYBER 175/750). For the CYBER 205 we took into account that the single- and double-precision floating point arithmetic works with 47- and 94- bit binary fractions, respectively. Since this machine works with truncated arithmetic, we took ε_S = 2**(-46) and

ε_D = 2**(-93) in our error analysis (cf. [6, p. 12]). This analysis is completely analogous to the one given in [6] and rather tedious. Therefore, we omit the details. The dominant terms in the upper bounds for the errors are, in the case of method A, the error caused by the interpolation in the cosine table, and, in the case of method B, the inherent error caused by using the Riemann-Siegel formula with the first two terms of its asymptotic expansion. These two terms do not depend on the computer used. The other, non-dominant, terms are quite small compared with the dominant terms. For both our CYBER 205 and our CYBER 175/750 programs we found the following bounds for the error in the computed value $\tilde{Z}(t)$ of Z(t):

$$|\tilde{Z}(t) - Z(t)| < \begin{cases} 3\text{x}10\text{**}(-7)\tau^{1/4} & \text{for method A,} \\ 5.5\text{x}10\text{**}(-3)\tau^{-5/4} & \text{for method B,} \end{cases}$$

for any t (=2πτ) in the interval (3.5x10**7,5.6x10**8). A safe upper bound for this error is 3x10**(-5) for method A and 2.1x10**(-11) for method B. In our program we actually used the very conservative fixed bounds 10**(-4) and 2.5x10**(-7), respectively. Not a single Gram point was found for which method B could not determine the sign of Z(t).

3. STATISTICS

Here we present, for the *whole* interval $[g_0, g_{1,500,000,000})$, some statistics concerning Gram blocks, the numbers of zeros in Gram intervals and the exceptions to Rosser's rule (summarizing the statistics up to $g_{300,000,000}$ given in [1], [2] and [7]). As stated already in Section 1, the counts given here are exact, since no shifts were made when computing Z(t) in the Gram points.

Table 1 gives the numbers of Gram blocks of length ≤ 9 in the interval $[g_0, g_{1,500,000,000})$ for strings of 10**8 successive Gram intervals. The last line gives the totals for the whole interval. The average Gram block length is 1.2089. The number of Gram blocks of length 1 is slowly decreasing, in favour of the number of Gram blocks of length ≥ 2.

In Table 2 we list the numbers of Gram blocks of type (j,k), $1 \le j \le 9$, $1 \le k \le j$, in the interval $[g_0, g_{1,500,000,000})$, as far as they can be classified according to the definition given in [1, p. 1370]. The numbers in parentheses denote the percentages of the totals given in Table 1. We also specify the numbers of Gram blocks of lengths 2 and 3 which cannot be classified, viz., those of length 2 with "0 0" and "2 2" zero-

pattern, and those of length 3 with "0 1 0", "2 3 0" and "0 3 2" zero-pattern. These, very rare, Gram blocks occur in relation to exceptions to Rosser's rule (see below).

The tendency of the so-called "missing two zeros" (cf. [2, Section 2]) in Gram blocks of lengths ≥ 3 and ≤ 6 to lie in one of the two outer Gram intervals of the Gram block is partly illustrated by Table 3. This table gives the numbers of Gram blocks of lengths 4, 5 and 6 in 3 strings of 10**8 Gram intervals, viz., $[g_0$, $g_{100,000,000})$, $[g_{700,000,000}$, $g_{800,000,000})$ and $[g_{1,400,000,000}$, $g_{1,500,000,000})$. For Gram blocks of lengths ≥ 3 and ≤ 6 the distribution of the various types seems to tend to a certain limit which is fairly well reached in the third string. For Gram blocks of length ≥ 7, however, our numbers are probably still too small to supply any evidence for the existence of a limit distribution (although we are tempted to believe that such distributions exist for Gram blocks of all lengths).

Table 4 (which partly summarizes previous such tables in [1] and [2]) gives the first occurrences of the various types of Gram blocks which have been found. Note the rather "late" first occurrence of type (8,4) and (8,5) Gram blocks and the absence of type (8,1) and (8,8) blocks and of blocks of length 9 of various types.

Table 5 shows the numbers of Gram intervals which contain exactly 0,1,... zeros, for strings of 10**8 Gram intervals. In three cases, the index of the last Gram point of a string (and of the first Gram point of the next string) is not a multiple of 10**8 since this index corresponds to a "bad" Gram point. In these cases, the string was extended up to the first "good" Gram point after the bad Gram point. As a partial check of the counts in this table the following relation is useful (this follows from a simple counting argument of the zeros in consecutive Gram intervals): if i_j is the number of Gram intervals with j zeros, in a given string of Gram blocks $[g_{n1}$, $g_{n2})$ which contains n2 - n1 zeros of Z(t), such that missing pairs of zeros connected with exceptions to Rosser's rule have neither "crossed" the Gram point g_{n1} nor g_{n2}, then we have $i_0 = \sum_{j \geq 2} j.i_j$. Note that the number of Gram intervals with precisely one zero is slowly (and monotonically) decreasing in favour of the number of Gram intervals with 0, 2, 3 and 4 zeros.

We have found 3055 exceptions to Rosser's rule (2973 of length 2 and 82 of length 3), including some formerly unknown types. Table 6 surveys the various types of exceptions which have been found so far (note that a length 2, type 10 exception has been defined, but not yet found). For all 3055 exceptions, the missing two zeros were located either in the preceding or in the succeeding Gram block (which always had length ≤ 3). Our main separation program did not search for the missing two zeros in adjacent Gram

blocks: this was done afterwards with an interactive program. Future investigators may find it useful to implement this search in their main separation program. The simplest case where this would not be successful is given bij the zero-pattern "0 0 1 3" (or "3 1 0 0"), i.e., a Gram block of length 2 without any zeros followed (or preceded) by two Gram blocks of length 1, the nearest containing 1 and the other 3 zeros. We do not have the slightest idea where this phenomenon might possibly occur for the first time.

Table 7 gives the frequencies of the exceptions to Rosser's rule in strings of 10**8 successive Gram intervals. Note the gradual, although not monotonic, increase of the numbers of exceptions of length 2 of types 1,2,5 and 6. For all 3055 exceptions, we have computed the local extreme values of S(t) (cf. [1, p. 1370]) near these exceptions.

Table 8 lists those exceptions for which the corresponding local extreme values exceed 2.2 for length 2, and 2.1 for length 3 exceptions. The most extreme values in this table are marked by an asterisk. It should be noted that, by a different method, Karkoschka and Werner [3] have found 10 of the 21 exceptions to Rosser's rule in the interval $[g_{1,000,000,000}, g_{1,100,000,000})$ for which $|S_{extr}| > 2.1$ (including those three for which $|S_{extr}| > 2.2$).

In Table 9 we give frequency counts of those exceptions for which the $|S_{extr}|$ - values lie beyond 2, in intervals of length 0.05, for three groups of 1000 successive exceptions. There is a tendency of the extreme values to increase, albeit very slowly.

In Figures 1 - 5 we present graphs of the function Z(t) near the first exceptions to Rosser's rule of length 2, types 7, 8 and 9, and length 3, types 3 and 4, respectively. The numbers of zeros in Gram intervals are indicated in parentheses. The scale of the graphs prevents the reader to "see" all zeros, but sufficient magnification of the graphs (as in [7]) would resolve these zeros without any difficulties. The local extreme values of S(t) near these exceptions are -2.033411, 2.092910, -2.060040, -2.012663 and 2.012380, respectively.

Figure 6 shows $B_{542,964,969}$, the first Gram block of length 9 (which is of type (9,3)).

Figure 7 presents a graph of Z(t) in the Gram block $B_{1,048,449,112}$ of length 2 and of type (2,1). This Gram block contains the closest pair of zeros observed by us. Comparing their distance with the quantity $2\pi/\log(t_n)$ which is a natural measure for the distance of two consecutive zeros t_n and t_{n+1} of Z(t), we have $(t_{n+1} - t_n)\log(t_n)/2\pi \tilde{=} 0.00034$. The corresponding value for the two close zeros found by Brent

([1, p.1371]) is about 0.0014.

Our program kept track of large values of Z(t) in Gram points. The largest positive and negative values found are: $Z(g_{1,211,024,724}) \tilde{=} 116.88$ and $Z(g_{1,381,150,789}) \tilde{=} -116.147$. Figures 8 and 9 show the graphs of Z(t) near these extrema. Figure 8 shows a Gram block of length 4 with "2 1 1 0" zero-pattern followed by a Gram block of length 3 with "0 1 2" zero-pattern. The large value of Z lies in the common point of the two Gram blocks. Note that this graph shows two zeros which have a distance of approximately 3.85 nearby Gram intervals. Figure 9 shows a Gram block of length 2 with "0 0" zero-pattern (this is an exception to Rosser's rule of length 2, type 2; the missing two zeros are located in the Gram interval preceding this block) followed by a Gram block of length 4 with "0 1 1 2" zero-pattern. Here, the two zeros before and after the extreme value of Z have a distance of approximately 3.65 nearby Gram intervals.

4. DISCUSSION

Our statistical material suggests that the zeros of Z(t) are distributed among the Gram intervals according to some hitherto unknown probabilistic law. E.g., the counts of the Gram blocks of lengths 3, 4, 5 and 6 of various types seem to tend to a certain (discrete) convex distribution (cf. the percentages in Tables 2 and 3) which depends on the length of the Gram block. It would be interesting to have a probabilistic model which could explain or at least support this phenomenon.

Several results in the literature (cf. [4], [5], [8], [10], [11], [12], [13]) have been derived from the existence of a number A, say, for which all complex zeros $\sigma + it$ of $\zeta(s)$ with $|t| < A$ have real part 1/2. Since our computations have increased this bound considerably, it might be worthwile to trace the consequences of this new bound. We have not pursued this.

History has shown that proving the truth of the Riemann hypothesis is an extremely delicate task. Nevertheless, we hope that our computational results may stimulate interest and efforts which might lead to a better understanding of the Riemann hypothesis and of the distribution of the complex zeros of the Riemann zeta function.

ACKNOWLEDGEMENTS

We wish to thank Joke Blom for her help in preparing the figures. The computat-

ions on the CDC CYBER 205 computer were carried out shortly after its installation at the Academic Computer Centre Amsterdam (SARA). On the one hand, this was a big advantage since this allowed us to use a substantial portion of its computational capacity. On the other hand, some hardware problems urged us to repeat several parts of our computations a number of times. We are much obliged to SARA and to CDC for the support during this stage of our work.

REFERENCES

[1] BRENT, R.P., *On the zeros of the Riemann zeta function in the critical strip*, Math. Comp., v. 33, 1979, pp. 1361-1372.

[2] BRENT, R.P., J. van de LUNE, H.J.J. te RIELE & D.T. WINTER, *On the zeros of the Riemann zeta function in the critical strip. II*, Math. Comp., v. 39, 1982, pp. 681-688.

[3] KARKOSCHKA, E. & P. WERNER, *Einige Ausnahmen zur Rosserschen Regel in der Theorie der Riemannschen Zetafunktion*, Computing, v. 27, 1981, pp. 57-69.

[4] KATAI, I., *On oscillations of number-theoretic functions*, Acta Arithm., v. 13, 1967, pp. 107-122.

[5] LEHMAN, R.S., *On the difference* $\pi(x) - \mathrm{li}(x)$, Acta Arithm., v. 11, 1966, pp. 397-410.

[6] LUNE, J. van de, H.J.J. te RIELE & D.T. WINTER, *Rigorous high speed separation of zeros of Riemann's zeta function*, Report NW 113/81, October 1981, Mathematical Centre, Amsterdam.

[7] LUNE, J. van de & H.J.J. te RIELE, *On the zeros of the Riemann zeta function in the critical strip. III*, Math. Comp., v. 41, 1983, pp. 759-767.

[8] PINTZ, J., *On the sign changes of* $M(x) = \sum_{n \le x} \mu(n)$, Analysis, v. 1, pp. 191-195.

[9] RIELE, H.J.J. te, D.T. WINTER & J. van de LUNE, *Numerical verification of the Riemann hypothesis on the CYBER 205*, to appear in: A. EMMEN (ed.), Proc. International Supercomputer Applications Symp., North-Holland, 1985.

[10] ROSSER, J.B. & L. SCHOENFELD, *Approximate formulas for functions of prime numbers*, Ill. J. Math., v. 6, 1962, pp. 64-94.

[11] ROSSER, J.B. & L. SCHOENFELD, *Sharper bounds for the Chebyshev functions* $\theta(x)$ *and* $\psi(x)$, Math. Comp., v. 29, 1975, pp. 243-269.

[12] SCHOENFELD, L., *Sharper bounds for the Chebyshev functions* $\theta(x)$ *and* $\psi(x)$. *II*, Math. Comp., v. 30, 1976, pp. 337-360.

[13] SCHOENFELD, L., *An improved estimate for the summatory function of the Moebius function*, Acta Arithm., v. 15, 1969, pp. 221-233.

[14] WINTER, D.T. & H.J.J. te RIELE, *Optimization of a program for the verification of the Riemann hypothesis*, Supercomputer, v. 5, 1985, pp. 29-32.

TABLE 1

Number of Gram blocks of given length k:
J'(k,n) = J(k,n + 10**8) - J(k,n)

n/10**8	k = 1	2	3	4	5	6	7	8	9
0	71004697	10493487	2169610	340360	25813	1436	54	0	0
1	69951462	10553967	2328243	431251	42822	2600	159	6	0
2	69601860	10569849	2375444	463716	51246	3223	234	5	0
3	69374447	10583331	2404468	484378	56775	3708	260	4	0
4	69216943	10583988	2426955	499662	61697	4156	300	6	0
5	69092437	10593934	2439977	511447	64960	4481	312	12	1
6	68982347	10593521	2455933	521450	69010	4895	352	16	0
7	68901963	10597074	2465035	529346	71551	5128	387	21	0
8	68827274	10598124	2474834	537063	73568	5532	357	23	1
9	68753961	10604699	2483174	542402	76147	5623	416	15	1
10	68700299	10601103	2491646	548805	77819	5859	426	13	0
11	68643888	10605270	2495792	554834	79666	6194	451	26	0
12	68592095	10607592	2502851	558808	81373	6446	445	35	0
13	68551250	10607492	2509106	562524	82815	6476	461	22	2
14	68509616	10609975	2513530	565833	84424	6788	497	22	1
Totals	1036704539	158803406	36536598	7651879	999686	72545	5111	226	6
%	83.55	12.80	2.94	0.62	0.08	0.01	0.00		

TABLE 2

Number of Gram blocks of type (j,k), j=1,2,...,9, k=1,2,...,j,
in the interval $[g_0, g_{1,500,000,000})$

j	k = 1	2	3	4	5	6	7	8	9
1	1036704539								
2	79405034	79395174	+ 2973 " 0 0 " - blocks						
(%)	(50)	(50)	+ 225 " 2 2 " - blocks						
3	17330591	1867470	17338450	+ 82 " 0 1 0 " - blocks					
(%)	(47)	(5)	(48)	+ 2 " 2 3 0 " - blocks					
				+ 3 " 0 3 2 " - blocks					
4	3520937	306187	305141	3519614					
(%)	(46)	(4)	(4)	(46)					
5	412874	69153	35262	69257	413140				
(%)	(41)	(7)	(4)	(7)	(41)				
6	16505	14789	5097	5082	14556	16516			
(%)	(23)	(20)	(7)	(7)	(20)	(23)			
7	98	1559	743	305	718	1585	103		
(%)	(2)	(31)	(15)	(6)	(14)	(31)	(2)		
8	0	20	78	18	10	75	25	0	
9	0	0	2	1	0	1	2	0	0

TABLE 3

Number of Gram blocks of type (j,k), j=4,5,6, k=1,2,...,j, in the intervals $[g_{n.10**8}, g_{(n+1).10**8})$, for n=0, 7 and 14

j	n	total	k = 1	2	3	4	5	6
4	Ø	340360	152448	17793	17972	152147		
		(%)	(45)	(5)	(5)	(45)		
	7	529346	243856	20724	20649	244117		
		(%)	(46)	(4)	(4)	(46)		
	14	565833	261995	21151	20879	261808		
		(%)	(46)	(4)	(4)	(46)		
5	Ø	25813	8549	3570	1579	3539	8576	
		(%)	(33)	(14)	(6)	(14)	(33)	
	7	71551	29813	4779	2452	4698	29809	
		(%)	(42)	(7)	(3)	(7)	(42)	
	14	84424	35934	4892	2651	4930	36017	
		(%)	(43)	(6)	(3)	(6)	(43)	
6	Ø	1436	93	473	157	129	518	66
		(%)	(6)	(33)	(11)	(9)	(36)	(5)
	7	5128	1166	1042	339	348	1006	1227
		(%)	(23)	(20)	(7)	(7)	(20)	(24)
	14	6788	1875	1121	415	400	1114	1863
		(%)	(28)	(17)	(6)	(6)	(16)	(27)

TABLE 4

First occurrences of Gram blocks of various types

j	k	n	j	k	n
2	1	133	7	1	258,779,892
2	2	125	7	2	13,869,654
			7	3	17,121,221
3	1	3,356	7	4	37,091,042
3	2	2,144	7	5	20,641,464
3	3	4,921	7	6	52,266,282
			7	7	195,610,937
4	1	83,701			
4	2	39,889	8	2	112,154,948
4	3	18,243	8	3	175,330,804
4	4	67,433	8	4	717,574,239
			8	5	454,025,825
5	1	1,833,652	8	6	145,659,810
5	2	243,021	8	7	165,152,519
5	3	601,944			
5	4	68,084	9	3	542,964,969
5	5	455,256	9	4	1,331,284,715
			9	6	978,739,921
6	1	20,046,223	9	7	818,033,831
6	2	2,656,216			
6	3	4,718,714			
6	4	1,181,229			
6	5	2,842,089			
6	6	19,986,469			

TABLE 5

Number of Gram intervals in $[g_{n1}, g_{n2})$ containing exactly m zeros

n1	n2	m = 0	m = 1	m = 2	m = 3	m = 4
0	100000000	13197331	73771910	12864188	166570	1
100000000	200000000	13534327	73106626	13183768	175278	1
200000000	300000000	13641172	72895112	13286261	177454	1
300000000	400000000	13711578	72755501	13354267	178651	3
400000000	500000001	13756913	72666325	13396614	180148	1
500000001	600000000	13795033	72589842	13435215	179909	0
600000000	700000000	13826622	72528206	13463727	181440	5
700000000	800000000	13849738	72481722	13487344	181194	2
800000000	900000000	13871424	72439079	13507575	181917	5
900000000	1000000003	13894145	72393385	13530805	181664	4
1000000003	1100000001	13908465	72365865	13542874	182791	3
1100000001	1200000000	13924978	72332789	13559487	182744	1
1200000000	1300000000	13940406	72302046	13574692	182854	2
1300000000	1400000000	13952735	72278368	13585060	183836	1
1400000000	1500000000	13964556	72254377	13597581	183483	3
0	1500000000	206769423	1089161153	201369458	2699933	33
%		13.78	72.61	13.42	0.18	

TABLE 6 : see next page.

TABLE 7

Number of exceptions to Rosser's rule in the intervals $[g_n, g_{n+100,000,000})$ for n=0(100,000,000)1,400,000,000

	<---------- LENGHT = 2 ---------->										<--LENGHT = 3->			
	TYPE										TYPE			
n/10**8	1	2	3	4	5	6	7	8	9	10	1	2	3	4
0	15	9	1	0	0	0	0	0	0	0	0	0	0	0
1	37	38	1	0	1	2	0	0	0	0	0	0	0	0
2	52	68	0	1	3	2	0	0	0	0	2	1	0	0
3	53	77	2	0	3	2	0	0	0	0	3	2	0	1
4	79	78	0	0	5	3	0	0	0	0	1	0	0	1
5	94	71	0	0	7	5	0	1	0	0	2	5	0	0
6	82	100	4	1	8	12	0	0	0	0	1	3	0	0
7	104	98	1	0	6	7	0	1	0	0	2	2	1	0
8	98	108	1	4	7	7	0	0	0	0	3	4	0	0
9	107	115	1	1	11	13	0	0	1	0	2	4	0	1
10	121	100	1	1	11	8	1	0	0	0	5	2	0	1
11	112	132	0	1	15	7	0	0	0	0	2	8	0	0
12	130	135	1	0	12	13	0	1	1	0	3	1	0	0
13	128	123	0	1	17	14	1	0	0	0	2	8	0	0
14	128	123	2	1	8	16	0	0	0	0	3	6	0	0
Totals	1340	1375	15	11	114	111	2	3	2	0	31	46	1	4

TABLE 6

Various types of exceptions to Rosser's rule and their frequencies in $[g_0, g_{1,500,000,000})$

LENGTH = 2 — Gram block of length 2 with " 0 0 " zero-pattern

g_{n-3}	g_{n-2}	g_{n-1}	g_n	g_{n+1}	g_{n+2}	g_{n+3}	g_{n+4}	g_{n+5}	type	freq. (note)
			0	0	3				1	1340
		3	0	0					2	1375
			0	0	4	0			3	15
	0	4	0	0					4	11
			0	0	2	2			5	114
	2	2	0	0					6	111
			0	0	2	3	0		7	(1) 2
0	3	2	0	0					8	(2) 3
			0	0	4	1	0		9	(3) 2
0	1	4	0	0					10	0

LENGTH = 3 — Gram block of length 3 with "0 1 0" zero-pattern

g_{n-2}	g_{n-1}	g_n	g_{n+1}	g_{n+2}	g_{n+3}	g_{n+4}	g_{n+5}	type	freq. (note)
		0	1	0	3			1	31
	3	0	1	0				2	46
		0	1	0	4	0		3	(4) 1
0	4	0	1	0				4	(5) 4

Notes — B$_n$ for n =

(1)	1,070,232,754	1,368,002,233		
(2)	526,196,236	750,375,860	1,209,834,868	
(3)	983,377,342	1,257,209,100		
(4)	744,719,566			
(5)	368,714,780	437,953,501	958,241,934	1,003,780,082

TABLE 7 : see previous page.

TABLE 8

Exceptions to Rosser's rule for which the absolute local extreme value of S(t) exceeds 2.2 (length 2) resp. 2.1 (length 3)

Notation: n (type) extreme value of S(t) close to B_n, where n is the index of the Gram block B_n containing no zeros

LENGTH = 2	342,331,983	(2)	2.252286	
	566,415,147	(1)	-2.207335	
	1,032,818,120	(1)	-2.219390	
	1,063,458,444	(1)	-2.206212	
	1,081,300,140	(1)	-2.290363	*
	1,126,600,767	(2)	2.218140	
	1,140,009,253	(1)	-2.206885	
	1,257,893,676	(1)	-2.265578	
	1,268,572,909	(2)	2.207575	
	1,316,842,760	(2)	2.216533	
	1,331,951,563	(1)	-2.208338	
	1,333,195,692	(2)	2.313651	*
	1,353,464,414	(1)	-2.208858	
	1,372,703,317	(1)	-2.244361	
	1,389,937,190	(2)	2.203209	
	1,423,302,021	(5)	-2.204753	
	1,430,028,952	(2)	2.200009	
	1,440,874,660	(1)	-2.202507	
	1,496,982,189	(1)	-2.207073	
LENGTH = 3	1,089,751,985	(1)	-2.122474	
	1,339,212,674	(2)	2.106224	

TABLE 9

Distribution of absolute local extreme values of S(t), beyond 2, in intervals [S1,S2) of length 0.05

exception numbers	S1 = 2.00 S2 = 2.05	2.05 2.10	2.10 2.15	2.15 2.20	2.20 2.25	2.25 2.30	2.30 2.35
1 - 1000	728	193	67	10	1	1	0
1001 - 2000	681	228	66	21	3	1	0
2001 - 3000	673	215	83	17	10	1	1

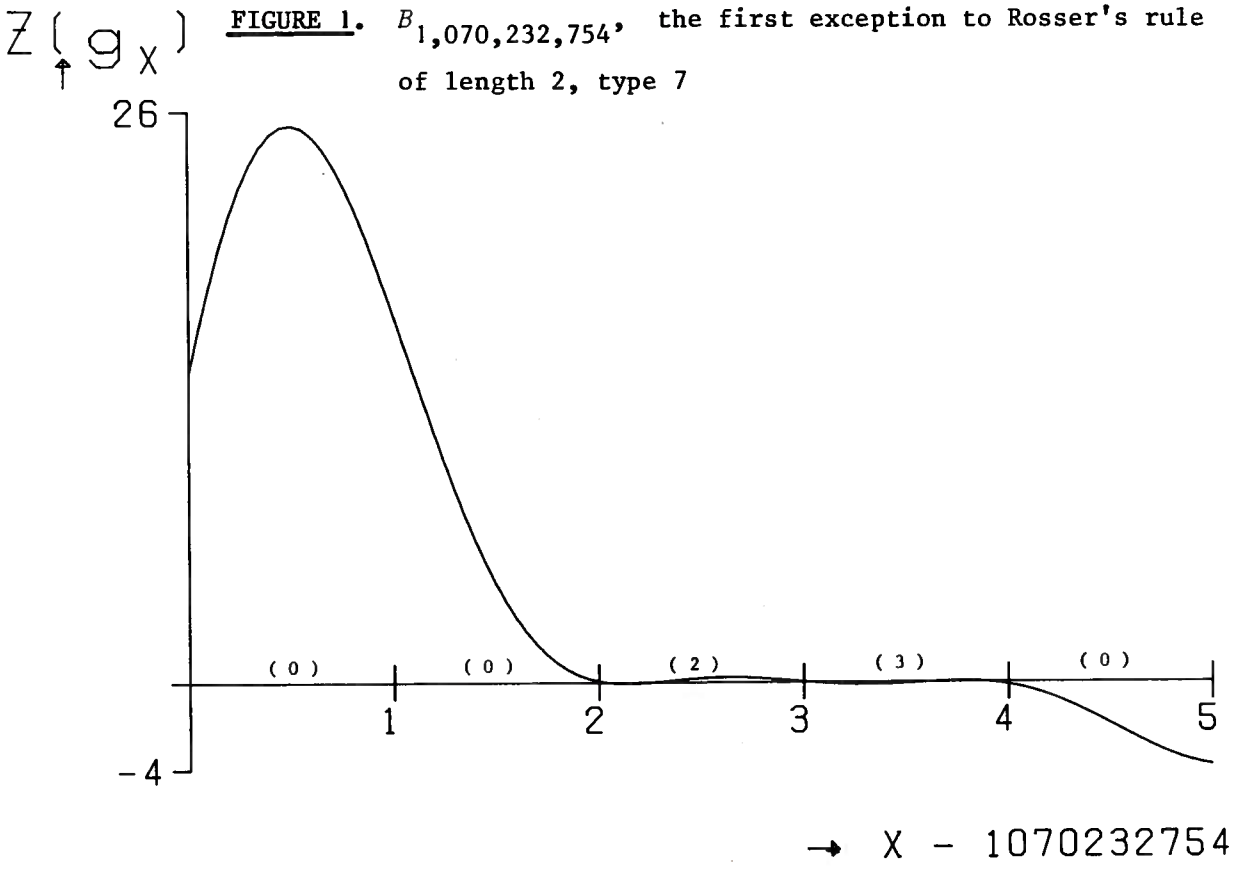


FIGURE 1. $B_{1,070,232,754}$, the first exception to Rosser's rule of length 2, type 7

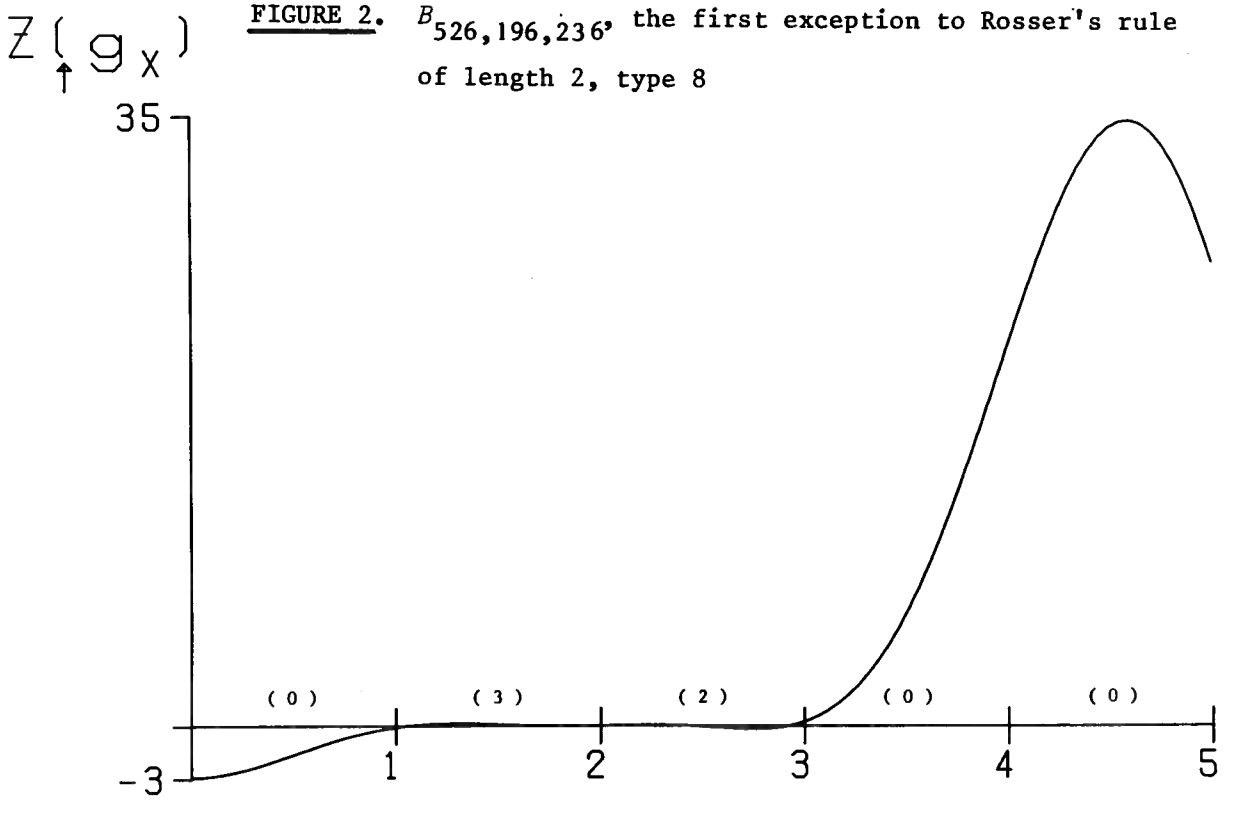


FIGURE 2. $B_{526,196,236}$, the first exception to Rosser's rule of length 2, type 8

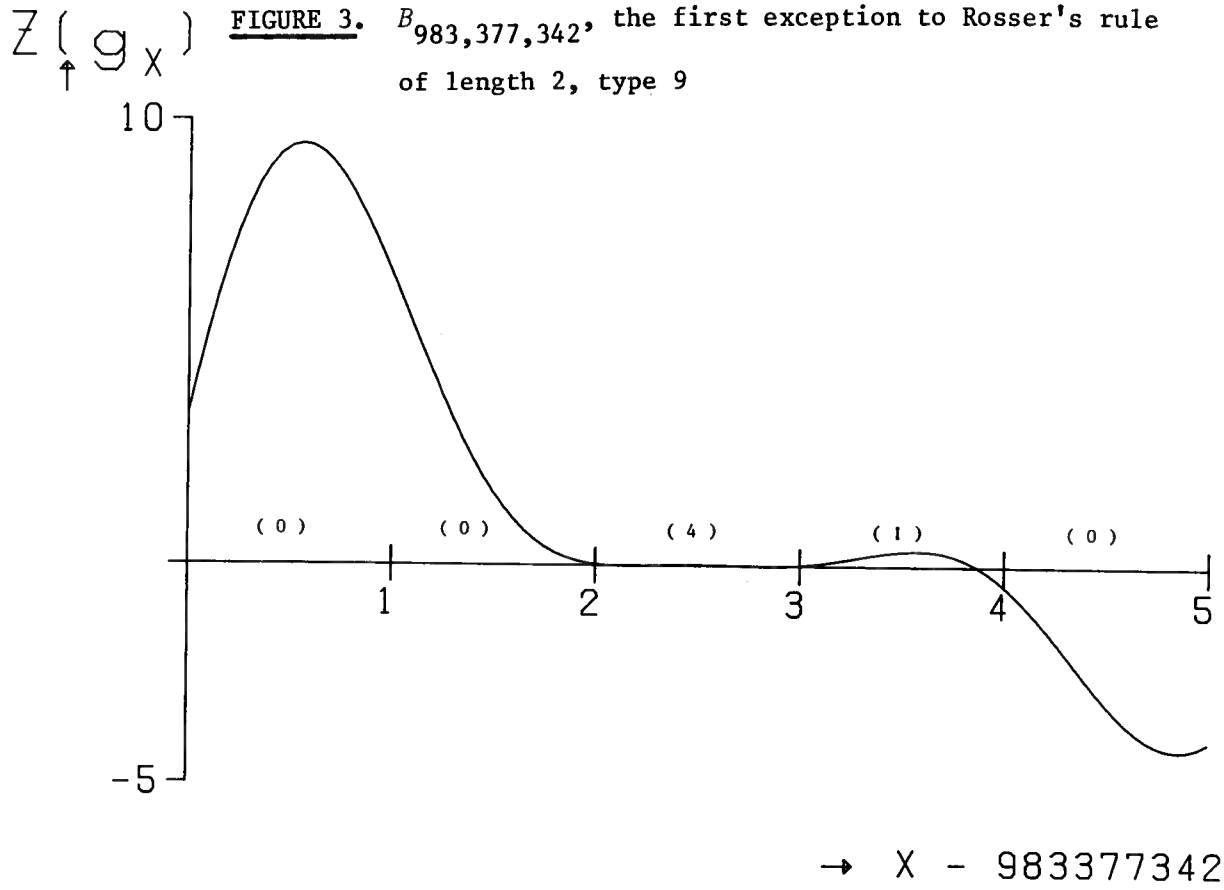


FIGURE 3. $B_{983,377,342}$, the first exception to Rosser's rule of length 2, type 9

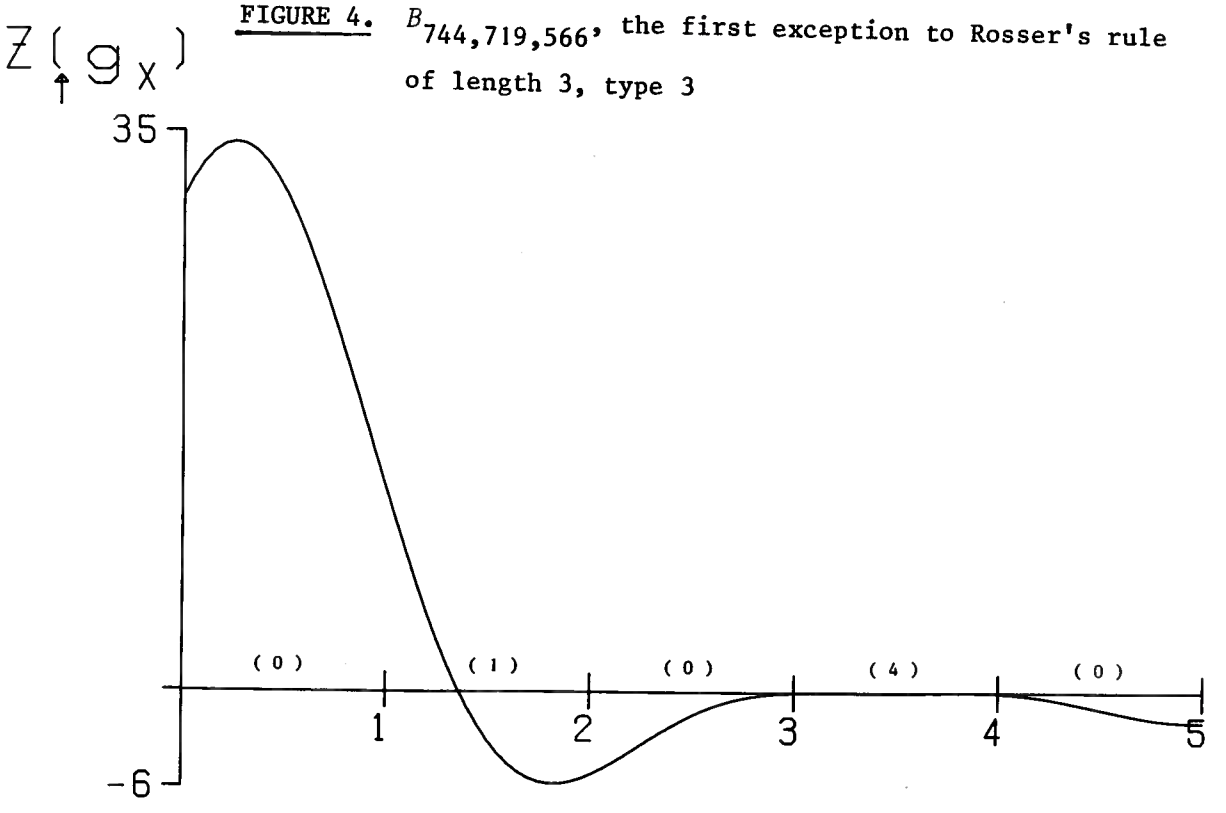


FIGURE 4. $B_{744,719,566}$, the first exception to Rosser's rule of length 3, type 3

FIGURE 5. $B_{368,714,780}$, the first exception to Rosser's rule of length 3, type 4

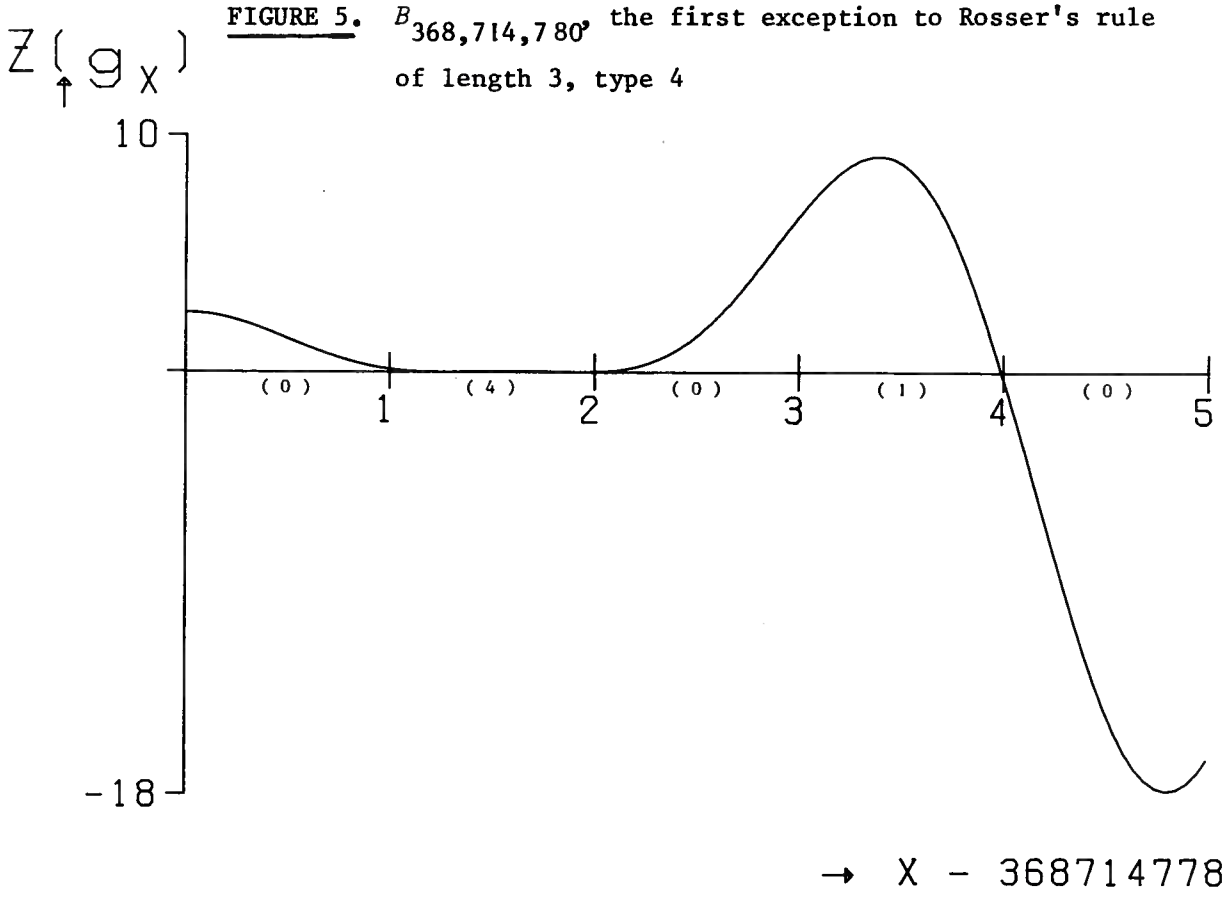
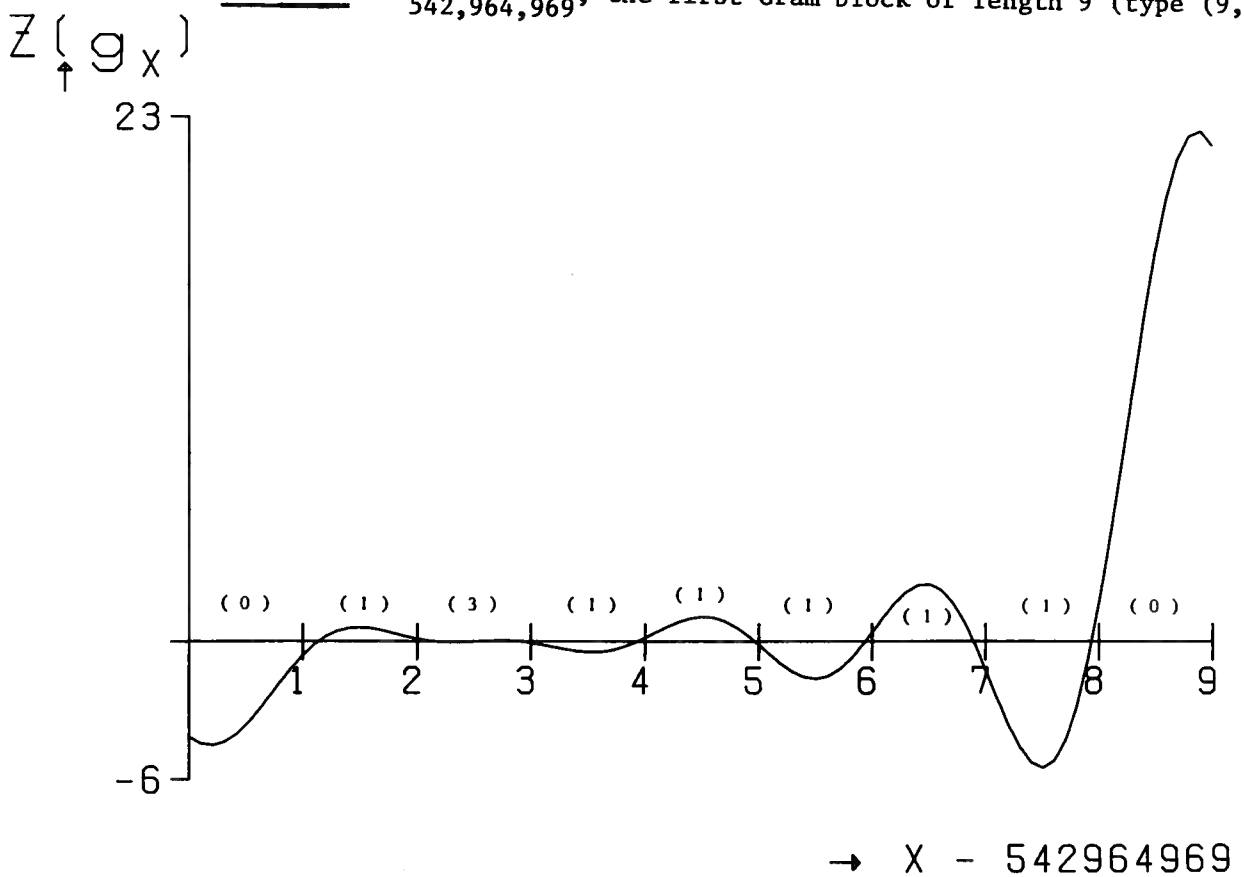


FIGURE 6. $B_{542,964,969}$, the first Gram block of length 9 (type (9,3))



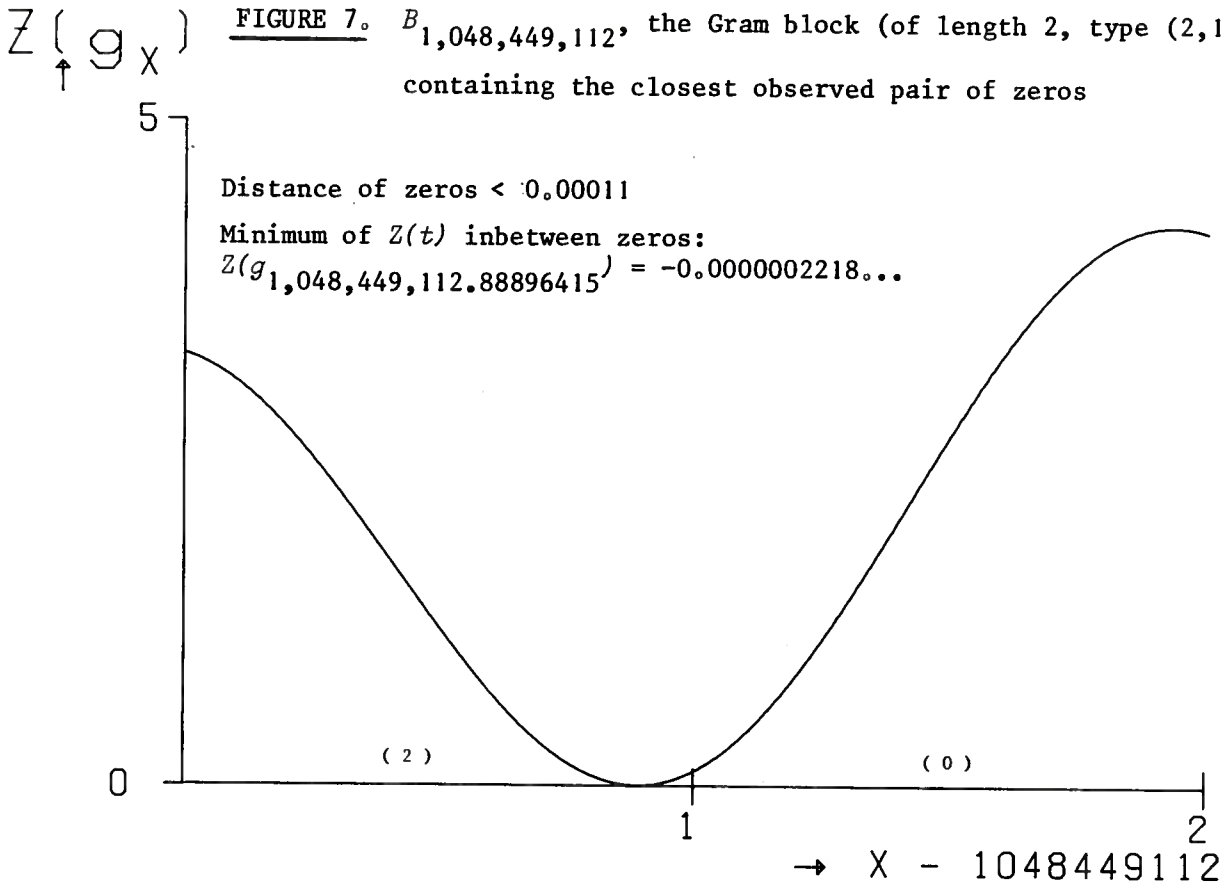


FIGURE 7. $B_{1,048,449,112}$, the Gram block (of length 2, type (2,1)) containing the closest observed pair of zeros

Distance of zeros < 0.00011

Minimum of $Z(t)$ inbetween zeros:
$Z(g_{1,048,449,112.88896415}) = -0.0000002218...$

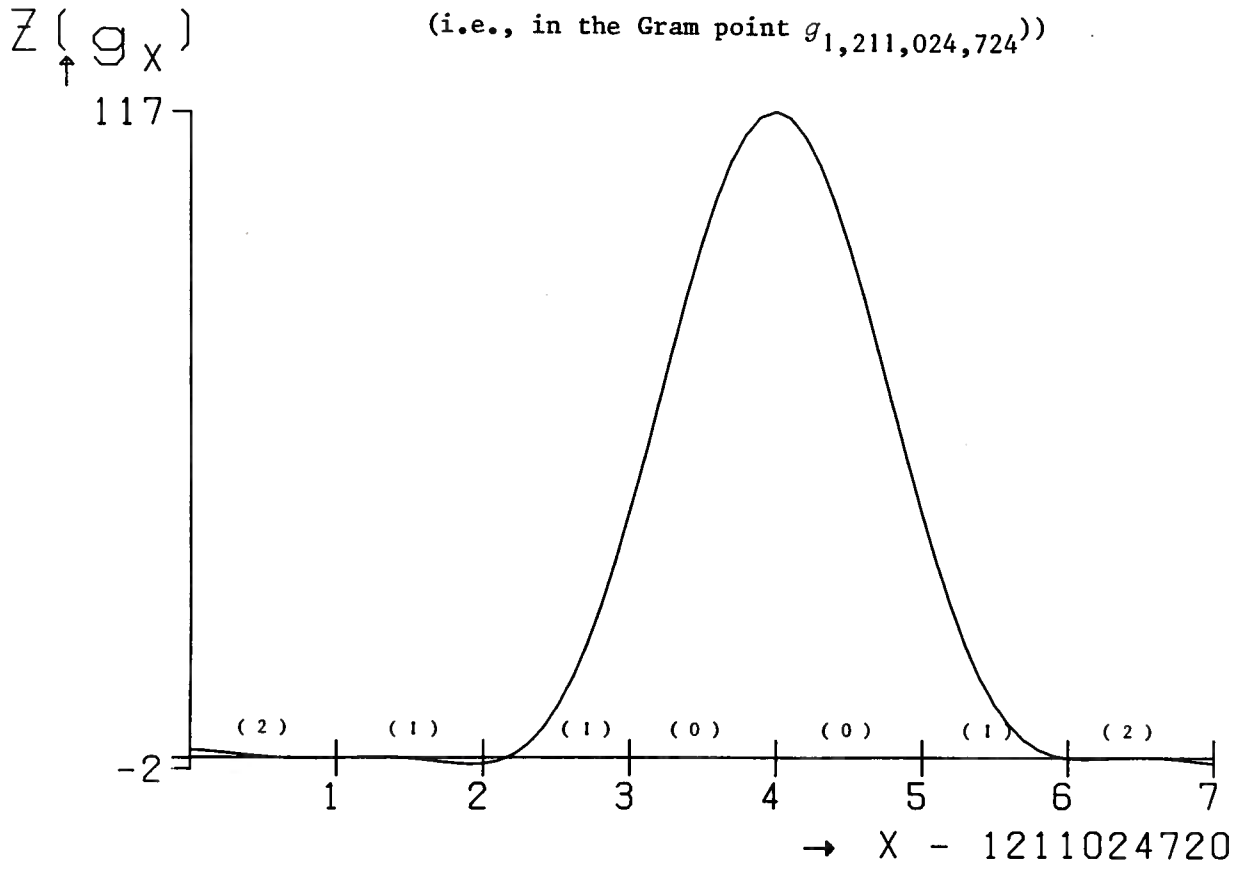


FIGURE 8. $Z(t)$ near its largest found positive value (i.e., in the Gram point $g_{1,211,024,724}$))

FIGURE 9. *Z(t)* near its largest found negative value (i.e., in the Gram point $g_{1,381,150,789}$))