

A Lévy input model with additional state-dependent services

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Abstract

We consider a queuing model with the workload evolving between consecutive i.i.d. exponential timers $\{e_q^{(i)}\}_{i=1,2,\dots}$ according to a spectrally positive Lévy process $Y(t)$ which is reflected at 0. When the exponential clock $e_q^{(i)}$ ends, the additional state-dependent service requirement modifies the workload so that the latter is equal to $F_i(Y(e_q^{(i)}))$ at epoch $e_q^{(1)} + \dots + e_q^{(i)}$ for some random nonnegative i.i.d. functionals F_i . In particular, we focus on the case when $F_i(y) = (B_i - y)^+$, where $\{B_i\}_{i=1,2,\dots}$ are i.i.d. nonnegative random variables. We analyse the steady-state workload distribution for this model.

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1. Introduction

In this paper we focus on a particular queuing system with additional state-dependent services. There has been considerable work on queues with state-dependent service and arrival processes; see for example the survey by Dshalalow [16] for several references. The model under consideration involves a reflected Lévy process connected to the evolution of the workload. Special cases of Lévy processes are the compound Poisson process, the Brownian motion, linear drift processes, and independent sums of the above. For papers that deal with queuing systems driven by Lévy processes, see e.g. [2–4,9,10,15,24–26] and references therein.

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Specifically, in this paper we consider a storage/workload model in which the workload evolves according to a spectrally positive Lévy process $Y(t)$, reflected at zero. That is, let $X(t)$ be a spectrally positive Lévy process (a Lévy process without negative jumps) modelling the input minus the output of the process and define $-\inf_{s \leq 0} X(s) = 0$ and $X(0) = x \geq 0$. Then we have that $Y(t) = X(t) - \inf_{s \leq t} X(s)$ (where $Y(0) = x$ for some initial workload $x \geq 0$). In addition, at arrival epochs of an independent Poisson process with rate q the workload is “reset” to a certain level, depending on the workload level before the arrival. Specifically, if t is the i th arrival epoch of the Poisson process, the workload $V(t)$ equals $F_i(V(t-))$ for some random nonnegative i.i.d. functionals F_i .

The main goal of our paper is to derive the stationary distribution of the workload $V(t)$ for the above-described queuing model. We first identify the stationary distribution of the workload at embedded exponential epochs and then extend this result to an arbitrary time by using renewal arguments. We also identify the tail behaviour of the steady-state workload.

This model unifies and extends several related models in various directions. First of all, if X is a compound Poisson process and if F_i is the identity function, then our model reduces to the workload process of the M/G/1 queue. Kella et al. [25] consider a model with workload removal, which fits into our model by taking $F_i(x) = 0$ and by letting the spectrally positive Lévy process X be a Brownian motion superposed with an independent compound Poisson component. The added generality allows one to analyse more elaborate mechanisms of workload control, where the exponential times can be seen as review times, during which the workload can be changed to a different level as desired. Allowing for general functions F_i opens up the possibility of optimising such controls, although we do not consider this problem here. Instead of considering the classical compound Poisson input process and a linear output process for the queuing model, one can consider the case in which the Lévy process is nondecreasing, i.e. a subordinator; see for example [4,9]. Other related papers are Kella and Whitt [26] and Kaspi et al. [23]. The former paper considers reflected Lévy processes with additional jumps (called secondary jumps) that are independent of the workload process. The latter paper allows for workload-dependent corrections, as well as workload-dependent release rates, but assumes that the Lévy process is degenerate.

Apart from the wish to unify and extend clearing models, we were also challenged by introducing a continuous-time analogue of the alternating service model considered in [38–41,43] that gave rise to the Lindley-type equation $W \stackrel{D}{=} (B - A - W)^+$. In particular, we focus on the case when $F_i(y) = (B_i - y)^+$, where $\{B_i\}_{i=1,2,\dots}$ are i.i.d. nonnegative random variables. Hence, at exponential epochs the controlling mechanism leaves only a portion of the workload depending on the size of the workload just prior to the exponential timer. In particular, if $F_i(y) = (B_i - y)^+$, then this mechanism keeps the workload below a generic random size B , decreasing it when it is relatively large at the exponential epoch and increasing it when it is much smaller than B . This can be viewed as a continuous-time analogue of the above mentioned Lindley-type equation. In the context of workload control mentioned above, this example can be interpreted as a mechanism that keeps existing storage below an upper bound B .

Our results focus on qualitative and quantitative properties of the steady-state workload distribution. We first establish Harris recurrence for the Markov chain embedded at workload adjustment points, yielding the convergence of the workload processes to an invariant distribution. We derive an equation for the invariant distribution of the embedded chain, as well as the invariant distribution of the original process. We use this equation to obtain expressions for the invariant distribution for an example that generalises [4,9,25]. We also investigate the tail

behaviour of the steady-state distribution under both light-tailed and heavy-tailed assumptions. All these results have the common theme that they rely on recently obtained results in the fluctuation theory of spectrally positive Lévy processes. Some of the results could be derived also for other input processes, other than spectrally positive Lévy process, as long as the density of the reflected process at an exponential time could be identified. In this paper, we decided to build a unifying theory and focus only on the spectrally positive case which is the most important for queuing applications.

The paper is organised as follows. In Section 2 we introduce a few basic facts concerning spectrally positive Lévy processes. In Section 3 we determine the steady-state workload distribution using the embedded workload process and the fact that Poisson Arrivals See Time Averages (PASTA). Later on, in Section 4 we present some special cases. Finally, in Section 5 we focus on the tail behaviour of the steady-state workload.

2. Preliminaries

Throughout this paper we exclude the case of processes X with monotone paths. Let the dual process of $X(t)$ be given by $\hat{X}(t) = -X(t)$. The process $\{\hat{X}(s), s \leq t\}$ is a spectrally negative Lévy process and has the same law as the time-reversed process $\{X((t-s)-) - X(t), s \leq t\}$. Following standard conventions, let $\underline{X}(t) = \inf_{s \leq t} X(s)$, $\bar{X}(t) = \sup_{s \leq t} X(s)$ and similarly $\hat{\underline{X}}(t) = \inf_{s \leq t} \hat{X}(s)$, and $\hat{\bar{X}}(t) = \sup_{s \leq t} \hat{X}(s)$. It is well known and easy to check via time reversal that when $X(0) = 0$, $(X(t) - \underline{X}(t), -\underline{X}(t)) \stackrel{\mathcal{D}}{=} (\bar{X}(t), \bar{X}(t) - X(t))$ are identically distributed for every fixed $t \geq 0$; see e.g. [28, Lemma 3.5, p. 74]. Moreover,

$$-\underline{X}(t) \stackrel{\mathcal{D}}{=} \hat{\bar{X}}(t), \quad \bar{X}(t) \stackrel{\mathcal{D}}{=} -\hat{\underline{X}}(t).$$

Since the jumps of \hat{X} are all non-positive, the moment generating function $E[e^{\theta \hat{X}(t)}]$ exists for all $\theta \geq 0$ and is given by $E[e^{\theta \hat{X}(t)}] = e^{t\psi(\theta)}$ for some function $\psi(\theta)$ that is well defined at least on the positive half-axis where it is strictly convex with the property that $\lim_{\theta \rightarrow \infty} \psi(\theta) = +\infty$. Moreover, ψ is strictly increasing on $[\Phi(0), \infty)$, where $\Phi(0)$ is the largest root of $\psi(\theta) = 0$. We shall denote the right-inverse function of ψ by $\Phi : [0, \infty) \rightarrow [\Phi(0), \infty)$.

Denote by σ the Gaussian coefficient and by ν the Lévy measure of \hat{X} (note that σ is also a Gaussian coefficient of X and that $\Pi(A) = \nu(-A)$ is a jump measure of X). Throughout this paper we assume that the following (regularity) condition is satisfied:

$$\sigma > 0 \quad \text{or} \quad \int_{-1}^0 x \nu(dx) = \infty \quad \text{or} \quad \nu(dx) \ll dx, \tag{2.1}$$

where $\ll dx$ means absolute continuity with respect to the Lebesgue measure. Under this assumption the one-dimensional distributions of X and of the reflected process Y are absolutely continuous (see [34, p. 106–107] and [36]). Moreover, we assume that

$$P_x(\tau_0^- < \infty) = 1, \tag{2.2}$$

where

$$\tau_0^- = \inf\{t \geq 0 : X(t) \leq 0\}.$$

Finally, P_x denotes the probability measure P under the condition that $X(0) = x$ (we will skip the subscript when $x = 0$), and E_x indicates the expectation with respect to P_x .

2.1. Scale functions

For $q \geq 0$, there exists a function $W^{(q)} : [0, \infty) \rightarrow [0, \infty)$, called the q -scale function, that is continuous and increasing with Laplace transform

$$\int_0^\infty e^{-\theta y} W^{(q)}(y) dy = (\psi(\theta) - q)^{-1}, \quad \theta > \Phi(q). \tag{2.3}$$

The domain of $W^{(q)}$ is extended to the entire real axis by setting $W^{(q)}(y) = 0$ for $y < 0$. We mention here some properties of the function $W^{(q)}$ that have been obtained in the literature and which we will need later on.

On $(0, \infty)$ the function $y \mapsto W^{(q)}(y)$ is right- and left-differentiable and, as shown in [31], under Condition (2.1), it holds that $y \mapsto W^{(q)}(y)$ is continuously differentiable for $y > 0$.

Closely related to $W^{(q)}$ is the function $Z^{(q)}$ given by

$$Z^{(q)}(y) = 1 + q \int_0^y W^{(q)}(z) dz.$$

The name “ q -scale function” for $W^{(q)}$ and $Z^{(q)}$ is justified as these functions are harmonic for the process \hat{X} killed upon entering $(-\infty, 0)$. Here we give a few examples of scale functions. For further examples of scale functions, see e.g. [13,22,30].

Example 1. If $X(t) = \sigma B(t) - \mu t$ is a Brownian motion with drift μ (a standard model for small service requirements) then

$$W^{(q)}(x) = \frac{1}{\sigma^2 \delta} [e^{(-\omega+\delta)x} - e^{-(\omega+\delta)x}],$$

where $\delta = \sigma^{-2} \sqrt{\mu^2 + 2q\sigma^2}$ and $\omega = \mu/\sigma^2$.

Example 2. Suppose

$$X(t) = \sum_{i=1}^{N(t)} \sigma_i - pt,$$

where p is the speed of the server and $\{\sigma_i\}$ are i.i.d. service times that are coming according to a Poisson process $N(t)$ with intensity λ . We assume that all σ_i are exponentially distributed with mean $1/\mu$. Then $\psi(\theta) = p\theta - \lambda\theta/(\mu + \theta)$ and the scale function of the dual $W^{(q)}$ is given by

$$W^{(q)}(x) = p^{-1} \left(A_+ e^{q^+(q)x} - A_- e^{q^-(q)x} \right),$$

where $A_\pm = \frac{\mu+q^\pm(q)}{q^+(q)-q^-(q)}$ with $q^+(q) = \Phi(q)$ and $q^-(q)$ is the smallest root of $\psi(\theta) = q$:

$$q^\pm(q) = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}.$$

2.2. Fluctuation identities

The functions $W^{(q)}$ and $Z^{(q)}$ play a key role in the fluctuation theory of reflected processes as shown by the following identity (see [5, Theorem VII.4 on p. 191 and (3) on p. 192] or [29, Theorem 5]).

Lemma 2.1. For $\alpha > 0$ and an independent exponential variable e_q with parameter q ,

$$E\left(e^{-\alpha \bar{X}(e_q)}\right) = \frac{q(\alpha - \Phi(q))}{\Phi(q)(\psi(\alpha) - q)},$$

which is equivalent to

$$P(\bar{X}(e_q) \in dx) = \frac{q}{\Phi(q)} W^{(q)}(dx) - q W^{(q)}(x) dx, \quad x > 0.$$

Moreover, $-\underline{X}(e_q)$ follows an exponential distribution with parameter $\Phi(q)$.

The scale function gives also the density $r^{(q)}(x, y)$ of the q -potential measure

$$R^{(q)}(x, A) := E_x \int_0^{\tau_0^-} e^{-qt} \mathbf{1}_A(X(t)) dt = \int_0^\infty e^{-qt} P_x(X(t) \in A, \tau_0^- > t) dt$$

of the process X killed on exiting $[0, \infty)$ when initiated from x . See also [34].

Lemma 2.2. Under (2.1), we have that

$$r^{(q)}(x, y) = \int_{[(x-y)^+, x]} e^{-\Phi(q)z} \left[W^{(q)'}(y - x + z) - \Phi(q) W^{(q)}(y - x + z) \right] dz.$$

Proof. We start by noting that for all $x, y > 0$ and $q > 0$,

$$R^{(q)}(x, dy) = \frac{1}{q} P_x(X(e_q) \in dy, \underline{X}(e_q) \geq 0).$$

It is well known that when $X(0) = 0$, $X(e_q) - \underline{X}(e_q)$ is independent of $\underline{X}(e_q)$ (see [5, Theorem 5, p. 159]). Keeping in mind that $P = P_0$, this leads to

$$\begin{aligned} R^{(q)}(x, dy) &= \frac{1}{q} P(X(e_q) \in dy - x, \underline{X}(e_q) \geq -x) \\ &= \frac{1}{q} P((X(e_q) - \underline{X}(e_q)) + \underline{X}(e_q) \in dy - x, -\underline{X}(e_q) \leq x) \\ &= \frac{1}{q} \int_{[(x-y)^+, x]} P(-\underline{X}(e_q) \in dz) P(X(e_q) - \underline{X}(e_q) \in dy - x + z). \end{aligned}$$

In the above expression, we integrate over the value of $-\underline{X}(e_q)$ which is nonnegative under P_x (this leads to the condition that $-\underline{X}(e_q) \leq x$ under $P = P_0$) and it is less than $X(e_q) = y$ under P_x (hence, $-\underline{X}(e_q) > x - y$ under P). Note that we always have that $-\underline{X}(e_q) \geq 0$ under P , and thus the above integral is equal to the integral over $[0, x]$ when $y > x$.

Recall, that by duality $X(e_q) - \underline{X}(e_q)$ is equal in distribution to $\bar{X}(e_q)$ which has been identified in Lemma 2.1. In addition, the law of $-\underline{X}(e_q)$ is exponentially distributed with parameter $\Phi(q)$. We may, therefore, rewrite the expression for $R^{(q)}(x, dy)$ as follows:

$$R^{(q)}(x, dy) = \int_{[(x-y)^+, x]} e^{-\Phi(q)z} \left[W^{(q)}(dy - x + z) - \Phi(q) W^{(q)}(y - x + z) \right] dz. \tag{2.4}$$

Under Condition (2.1), $W^{(q)}$ is differentiable and hence the last equality completes the proof. \square

Remark. Lemma 2.2 and similar results can be proven without the assumption made in (2.1), but at the cost of more complex expressions. We would have to use (2.4) instead of the much nicer form for $r^{(q)}(x, y)dy$.

3. Steady-state workload distribution

We consider the workload process at the embedded epochs $(e_q^{(1)} + \dots + e_q^{(n)})-$, just before the additional service arrives, where $\{e_q^{(i)}\}_{i=1,2,\dots}$ are i.i.d. exponentially distributed random variables with intensity q . Note that this process is a Markov chain $\{Z_n, n \in \mathbb{N}\}$ with transition kernel:

$$k(x, dy) = \int P_{f(x)}(Y(e_q) \in dy) dP^F(f), \tag{3.1}$$

where P^F is the law of F .

Lemma 3.1. We have that $P_x(Y(e_q) \in dy) = h(x, y)dy + e^{-\Phi(q)x} W^{(q)}(0)\delta_0(dy)$, where

$$h(x, y) = qr^{(q)}(x, y) + e^{-\Phi(q)x} \left[\frac{q}{\Phi(q)} W^{(q)'}(y) - qW^{(q)}(y) \right], \tag{3.2}$$

and where $r^{(q)}(x, y)$ is given in Lemma 2.2.

Proof. Define $\kappa_0 = \inf\{t \geq 0 : Y(t) = 0\}$, and observe that

$$\begin{aligned} P_x(Y(e_q) \in dy) &= P_x(Y(e_q) \in dy, \kappa_0 > e_q) + P_x(Y(e_q) \in dy, \kappa_0 < e_q) \\ &= P_x(X(e_q) \in dy, \tau_0^- > e_q) + P(Y(e_q) \in dy) P_x(\tau_0^- < e_q) \\ &= qr^{(q)}(x, y)dy + P(\bar{X}(e_q) \in dy) P(-\underline{X}(e_q) > y) \\ &= qr^{(q)}(x, y)dy + e^{-\Phi(q)x} \left[\frac{q}{\Phi(q)} W^{(q)'}(y) - qW^{(q)}(y) \right] \mathbf{1}_{\{y>0\}} dy \\ &\quad + e^{-\Phi(q)x} W^{(q)}(0)\delta_0(dy), \end{aligned}$$

where in the second equality we use the lack of memory of the exponential distribution and the fact that X is spectrally positive; hence $X(\kappa_0) = Y(\kappa_0) = 0$, so the last equality follows from Lemma 2.1. \square

By PASTA, the stationary distribution π of Z_n is the same as the stationary distribution of the workload at an arbitrary moment, if one of them exists and is unique. Therefore, we have the following main result.

Theorem 3.1. Suppose that a unique stationary distribution π exists; then also $V(\cdot)$ has a unique stationary distribution, and it is equal to π . Let $V(\infty)$ be a random variable with such a distribution. Then, for a bounded function g ,

$$\begin{aligned} Eg(V(\infty)) &= \int_{[0,\infty)} \pi(dx) \int dP^F(f) \int_0^\infty g(y)dy \left\{ qr^{(q)}(f(x), y) \right. \\ &\quad \left. + e^{-\Phi(q)f(x)} \left[\frac{q}{\Phi(q)} W^{(q)'}(y) - qW^{(q)}(y) \right] \right\} \\ &\quad + g(0)W^{(q)}(0) \int_0^\infty \int dP^F(f) e^{-\Phi(q)f(x)} \pi(dx) \end{aligned}$$

$$\begin{aligned}
 &= \int_{[0,\infty)} \pi^+(dx) \int_0^\infty g(y)dy \left\{ qr^{(q)}(x, y) + e^{-\Phi(q)x} \left[\frac{q}{\Phi(q)} W^{(q)'}(y) \right. \right. \\
 &\quad \left. \left. - qW^{(q)}(y) \right] \right\} + g(0)W^{(q)}(0) \int_0^\infty e^{-\Phi(q)x} \pi^+(dx),
 \end{aligned}$$

where the distribution π^+ satisfies $\int \pi(dx)g(f(x))dP^F(f) = \int \pi^+(dx)g(x)$; hence, it is a stationary distribution of a Markov chain right after workload correction, and it satisfies the following balance equation:

$$\begin{aligned}
 &\int_0^\infty g(y)d\pi^+(y) + g(0)\pi^+(0) \\
 &= \int_{[0,\infty)} \iint_{[0,\infty)} g(f(y))P_x(Y(e_q) \in dy)dP^F(f)d\pi^+(x)
 \end{aligned} \tag{3.3}$$

with $P_x(Y(e_q) \in dy)$ given in Lemma 3.1 and where $r^{(q)}(x, y)$ is given in Lemma 2.2.

Proof. The proof follows from rewriting the distribution of the workload process just before a correction and using Lemma 3.1; see also (3.1). Specifically, before a correction one starts with the stationary distribution π , then a correction takes place according to a function f which is a realisation of the functional F , and finally a reflected Lévy process evolves for the next exponential time horizon. \square

Remark. When X is a compound Poisson process with negative drift p , for the process V with absolutely continuous stationary distribution one can identify Rice’s formula relating the density of $V(\infty)$ with the intensity of up- and down crossings of a fixed level; for details see [8,35].

We now turn to the question of the existence and uniqueness of a stationary distribution π , and also to the question whether this is a limit law for the continuous-time workload process $V(t)$, $t \geq 0$ and the embedded workload process $Z_n, n \geq 1$ as t or $n \rightarrow \infty$. If this convergence holds (by PASTA they are equivalent) we call the distribution arising in the limit a limiting distribution.

We give some positive results in this direction that should cover most applications. Both are based on a comparison property. Namely, we assume there exists a sequence (A_n, B_n) in the positive orthant indexed by $n \geq 1$, such that

$$F_n(x) \leq F_n^a(x) := A_n x + B_n. \tag{3.4}$$

This equality can of course w.l.o.g. be assumed to hold a.s. for every x and every n , enabling coupling arguments in what follows.

We start with a useful lemma.

Lemma 3.2. *If $Z_n = z$, then*

$$Z_{n+1} \stackrel{D}{=} \max\{F(z) + X(e_q), \bar{X}(e_q)\}. \tag{3.5}$$

Proof. If $Z_n = z$, we see that $Z_{n+1} \stackrel{\mathcal{D}}{=} Y(e_q)$, with $Y(0) = F(z)$. If $Y(0) = F(z)$, we further have

$$Y(t) \stackrel{\mathcal{D}}{=} \max\{F(z) + X(t), \bar{X}(t)\}.$$

Indeed, since, for every $t \geq 0$, $(X(t) - \underline{X}(t), -\underline{X}(t)) \stackrel{\mathcal{D}}{=} (\bar{X}(t), \bar{X}(t) - X(t))$, it follows that $(X(t), X(t) - \underline{X}(t)) \stackrel{\mathcal{D}}{=} (X(t), \bar{X}(t))$ and thus

$$\begin{aligned} x + X(t) - \inf_{0 \leq s \leq t} (x + X(s))^- &= \max\{x + X(t), X(t) - \underline{X}(t)\} \\ &\stackrel{\mathcal{D}}{=} \max\{x + X(t), \bar{X}(t)\}. \quad \square \end{aligned}$$

We now state our stability result that applies to the case where the driving Lévy process is not a subordinator.

Theorem 3.2. *Suppose that (3.4) holds, and suppose that the Lévy process $X(\cdot)$ is not a subordinator. Then there exists a unique stationary and limiting distribution if one of the three conditions holds:*

1. $A_n = 0$;
2. $A_n = 1$ and $E[B_1] + E[X(1)]/q < 0$;
3. $E[\log A_n] < 0$ and $E[\log B_n] < \infty$.

Proof. The idea of the proof is as follows: we investigate the workload process $V^a(\cdot)$, where we take $F_n^a(\cdot)$, $n \geq 1$, at the instants where the workload is corrected. We show that 0 is a regeneration point for that process under the conditions on (A_n, B_n) in the theorem. For the general process, we can construct the random functions F_n and F_n^a such that $F_n(x) \leq F_n^a(x)$ for every $x \geq 0$, $n \geq 1$. This yields $0 \leq V(t) \leq V_a(t)$, so that the $V(\cdot)$ process hits 0 whenever $V^a(\cdot)$ does.

In view of this obvious construction it actually suffices to prove the claim for the process $V^a(\cdot)$. Note that it suffices to show stability for its embedded chain (Z_n^a) . Let $(X_1, M_1) \stackrel{\mathcal{D}}{=} (X(e_q), \bar{X}(e_q))$, and let (X_n, M_n) for $n \geq 1$ be an i.i.d. sequence. In view of Lemma 3.2, we see that $Z_{n+1}^a \stackrel{\mathcal{D}}{=} \max\{A_n Z_n^a + B_n + X_n, M_n\}$. Without loss of generality, we can assume that this equality in distribution is actually driving the chain (Z_n^a) .

We first show that the chain (Z_n^a) returns to a compact set of the form $[0, N]$ in finite expected time. Once this is established, we will show that it is possible for the original process $V^a(\cdot)$ to reach 0 before the next workload correction. We will actually establish Harris ergodicity with minimal extra effort.

First assume $A_n = 0$. Fix $N > 0$ such that $P(B_1 + M_1 \leq N) > 0$ and define $\tau_N = \inf\{n \geq 1 : Z_n \leq N\}$. It is easy to construct i.i.d. random variables $\{C_n, n \in \mathbb{N}\}$ such that $C_1 \stackrel{\mathcal{D}}{=} B_1 + M_1$ and $Z_n \leq C_n$, $n \in \mathbb{N}$. Observe that

$$\begin{aligned} P(\tau_N > k \mid Z_0 = x) &= P(Z_1^a > N, \dots, Z_k^a > N \mid Z_0^a = x) \\ &\leq P(C_1 > N, \dots, C_k > N) = P(C_1 > N)^k, \end{aligned}$$

which implies that

$$E[\tau_N \mid Z_0 = x] < \infty. \tag{3.6}$$

We now turn to showing a similar boundedness property for parts 2 and 3 of the theorem. We first consider part 2. Let $Z_1^a = x$. Observe that

$$E[Z_2^a \mid Z_1^a = x] \leq E[\max\{A_1x + B_1 + X_1, M_1\}] \\ = E[A_1x + B_1 + X_1] + E[(M_1 - (A_1x + B_1 + X_1))^+].$$

The second term converges to 0 as $x \rightarrow \infty$ so for every ϵ we can take N large enough such that $E[(M_1 - (A_1x + B_1 + X_1))^+] < \epsilon$ for $x > N$. Thus, for $x > N$,

$$E[Z_2^a \mid Z_1^a = x] \leq E[A_1^q]x + E[B_1] + E[X_1] + \epsilon.$$

If $A_1 = 1$ (more generally, if $E[A_1] \leq 1$), stability, and in particular the statement of part 2 follows by taking ϵ strictly smaller than $-(E[X_1] + EB_1) = -(E[X(1)]/q + E[B_1])$.

Part 3 does not follow automatically, since $E[A_1] \leq 1$ does not necessarily imply $E[\log A_1] < 0$. For that, we proceed with an indirect argument. Note that the chain (Z'_n) governed by the recursion $Z'_{n+1} = A_n Z'_n + B_n + M_n$ converges to a finite limit a.s. since also $E[\log(B_n + M_n)] < \infty$ (see [20]). Consequently, there exists N such that the chain (Z'_n) returns to $[0, N]$ after finite expected time. Since $X_n \leq M_n$ for every $n \geq 1$ we see that $Z_n \leq Z_n^a \leq Z'_n$ for every $n \geq 1$, implying that $E[\tau_N \mid Z_0^a = x] < \infty$ for every $x \geq 0$.

The above results show that the embedded chain (Z_n^a) , (and by coupling/comparison also (Z_n)) always returns to a compact set of the form $[0, N]$ after finite expected time. We now show that this implies Harris ergodicity once we find a constant $p > 0$ and a probability measure $Q(\cdot)$ such that for any Borel set D

$$P(Z_1 \in D \mid Z_0 = x) \geq pQ(D), \quad x \in [0, N]. \tag{3.7}$$

Let $N > 0$ be arbitrary and let $x \in [0, N]$. We construct p and $Q(\cdot)$ as follows. Recall that $\kappa_0 = \inf\{t \geq 0 : Y(t) = 0\}$, let N_2 be large enough such that $P(A_1N + B_1 \leq N_2) > 0$.

$$P(Z_1 \in D \mid Z_0 = x) \geq P_x(Z_1 \in D, A_1N + B_1 \leq N_2) \\ \geq P(A_1N + B_1 \leq N_2)P_{N_2}(\kappa_0 < e_q; Z_1 \in D) \\ \geq P(A_1N + B_1 \leq N_2)P_{N_2}(\kappa_0 < e_q)P_0(Y(e_q) \in D) \\ \geq P(A_1N + B_1 \leq N_2)P_{N_2}(\kappa_0 < e_q)P_0(Y(e_q) \in D) \\ := pQ(D).$$

Since the paths of $Y(\cdot)$ are non-monotone, we have that $p > 0$, implying (3.7). Note that the above computation establishes that 0 is a regeneration point (but the process may leave that point instantaneously), and that we provided a relatively simple example where the reference measure Q can be found explicitly. \square

We have established positive Harris recurrence by exploiting the fact that 0 is a regeneration point. Note that the ability to reach 0 also plays an important role in the stability analysis of [23]. Hitting 0 is possible since $X(\cdot)$ was assumed not to be a subordinator. If this assumption does not hold, it is still possible to obtain stability conditions, for example by using a contraction argument as surveyed in [14], which comes at the expense of additional technical conditions. We have tried to strike a balance between generality and conciseness. Specific cases that may not be covered by our general results may be dealt with by using one of the techniques reviewed in [19]. This is done in particular in [23].

Theorem 3.3. *If $X(\cdot)$ is a subordinator then the conditions in the above theorem are sufficient for stability if, in addition $F_n(\cdot)$ is contracting on the average, i.e., $|F_n(x) - F_n(y)| \leq K|x - y|$*

a.s. such that $E[\log K] < 0$, and there exists some $\epsilon > 0$ such that $E[K^\epsilon] < \infty$, $E[F_n(0)^\epsilon] < \infty$, and $E[X(1)^\epsilon] < \infty$.

Proof. The result is an immediate consequence of Theorem 5.1 in [14] specialised to the state space $[0, \infty)$ equipped with the L_1 norm. The moment conditions that are formulated above imply the algebraic tail conditions imposed in [14] by using Markov’s inequality. \square

In Section 4 we analyse more specific examples.

4. Computational examples

We now turn to analysing a few specific examples. We find that there are several solution strategies. One can either solve the equations given in Section 3 directly, or one can also take a less direct route, using Laplace transforms. We shall consider examples of both strategies.

To this end, we start with the following simple, but very useful observation. Using PASTA, V has the same distribution as Z which is the generic random variable following the equilibrium distribution π of the Markov chain $\{Z_n, n \in \mathbb{N}\}$ of the workload process embedded at times $(e_q^{(1)} + \dots + e_q^{(n)})$ —(i.e. right before the “correction”). The following useful lemma follows immediately from Lemma 3.2.

Lemma 4.1. *The following equality holds in distribution:*

$$Z \stackrel{\mathcal{D}}{=} \max\{F(Z) + X(e_q), \bar{X}(e_q)\}. \tag{4.1}$$

Example 3. The most trivial example is when $F(y) = B \geq 0$. In this simple case, there is no need to use the formula derived for the transition density $k(x, y)$. Using the above lemma, we see that

$$\begin{aligned} V(\infty) &\stackrel{\mathcal{D}}{=} \max\{B + X(e_q), \bar{X}(e_q)\} \\ &= \bar{X}(e_q) + \max\{B + X(e_q) - \bar{X}(e_q), 0\} \\ &\stackrel{\mathcal{D}}{=} \bar{X}(e_q) + \max\{B - e_{\Phi(q)}, 0\}. \end{aligned}$$

In the last equation, which follows from the Wiener–Hopf factorisation, $e_{\Phi(q)}$ is a random variable which is exponentially distributed with rate $\Phi(q)$, which is independent of everything else. Observe that

$$\begin{aligned} E[e^{-s \max\{B - e_{\Phi(q)}, 0\}}] &= P(e_{\Phi(q)} > B) + E[e^{-s(B - e_{\Phi(q)})}; e_{\Phi(q)} < B] \\ &= P(e_{\Phi(q)} > B) + E[e^{-s(B - e_{\Phi(q)})}] - E[e^{-s(B - e_{\Phi(q)})}; e_{\Phi(q)} \geq B] \\ &= E[e^{-\Phi(q)B}] + E[e^{-sB}] \frac{\Phi(q)}{\Phi(q) - s} - \frac{\Phi(q)}{\Phi(q) - s} E[e^{-\Phi(q)B}] \\ &= \frac{\Phi(q)E[e^{-sB}] - sE[e^{-\Phi(q)B}]}{\Phi(q) - s}. \end{aligned}$$

Combining this with Lemma 2.1, we obtain

$$E[e^{-sV(\infty)}] = \frac{q(s - \Phi(q))}{\Phi(q)(\psi(s) - q)} \frac{\Phi(q)E[e^{-sB}] - sE[e^{-\Phi(q)B}]}{\Phi(q) - s}. \tag{4.2}$$

This is an extension of various results in the literature focusing on clearing models (i.e. systems with workload removal) where $B = 0$. See for example [11,25] and references therein. Note that V is distributed as a convolution of $\bar{X}(e_q)$ and $\max\{F(V) - e_{\Phi(q)}, 0\} = \max\{B - e_{\Phi(q)}, 0\}$ (see [26, Proposition 6.1] for a similar decomposition result).

Example 4. We now consider an example where it seems more natural to solve the equations developed in Section 3 directly. Consider the case when $F(y) = (B - y)^+$ with B being exponentially distributed with intensity β (note that this model reminds the hysteretic control developed in [3,4]). Moreover, we assume that $X(t) = \sum_{i=1}^{N(t)} \sigma_i - pt$ is a compound Poisson process with exponentially distributed service times σ_i with intensity μ (see also the setup of Example 2). Note that

$$\psi(\theta) = p\theta - \lambda \int_0^\infty (1 - e^{-\theta z}) \mu e^{-\mu z} dz = p\theta - \lambda \frac{\theta}{\mu(\mu + \theta)}$$

and recall that $\Phi(q) = q^+(q)$. Thus,

$$\begin{aligned} Ee^{-\alpha V(\infty)} &= q \int_{[0, \infty)} \pi^+(dx) \int_0^\infty dy e^{-\alpha y} r^{(q)}(x, y) \\ &\quad + \frac{q\tilde{\pi}^+(\Phi(q))(\alpha - \Phi(q))}{\Phi(q)(\psi(\alpha) - q)} + \frac{1}{p}(A_+ - A_-)\tilde{\pi}^+(\Phi(q)) \\ &= H(\alpha, \pi^+) + \frac{q\tilde{\pi}^+(\Phi(q))(\alpha - \Phi(q))}{\Phi(q)(\psi(\alpha) - q)} + \frac{1}{p}(A_+ - A_-)\tilde{\pi}^+(\Phi(q)), \end{aligned}$$

where

$$\begin{aligned} H(\theta, u) &= \frac{q}{p} A^- \left\{ \tilde{u}(q^+(q)) \frac{q^+(q) - q^-(q)}{(\theta - q^+(q))(\theta - q^-(q))} - \tilde{u}(\theta) \frac{2q^+(q)}{\theta^2 - q^+(q)^2} \right. \\ &\quad \left. + \tilde{u}(\theta + q^+(q) - q^-(q)) \frac{2q^-(q)}{\theta^2 - q^-(q)^2} \right\} \end{aligned}$$

and $\tilde{u}(\theta) = \int_{[0, \infty)} e^{-\theta x} u(dx)$. To complete the computations we have to find the LST $\tilde{\pi}^+$ of the stationary distribution π^+ . By the memoryless property of the exponential distribution of B we have that $\pi^+(dx) = \beta e^{-\beta x} dx$, $x \geq 0$. Hence,

$$\tilde{\pi}^+(\theta) = \pi^+(0) + \frac{\beta}{\beta + \theta}.$$

We will find now $\pi^+(0)$ using Theorem 3.1:

$$\begin{aligned} \pi^+(0) &= \pi^+(0)\beta \int_0^\infty e^{-\beta t} dt \int_t^\infty \left[\frac{q}{\Phi(q)} W^{(q)'}(y) - qW^{(q)}(y) \right] dy \\ &\quad + \beta^2 \int_0^\infty e^{-\beta x} dx \int_0^\infty e^{-\beta t} dt \int_t^\infty \left\{ qr^{(q)}(x, y) \right. \\ &\quad \left. + e^{-\Phi(q)x} \left[\frac{q}{\Phi(q)} W^{(q)'}(y) - qW^{(q)}(y) \right] \right\} dy. \end{aligned}$$

Now let $\epsilon_\beta(dx) = \beta e^{-\beta x} dx$ and

$$G(q) = \frac{\beta q}{p(\beta - q^-(q))} \frac{q^-(q) - q^+(q)}{q^-(q)q^+(q)}.$$

Then,

$$\pi^+(0) = \frac{G(q) \frac{\beta}{\beta + q^+(q)} + H(0, \epsilon_\beta) - H(\beta, \epsilon_\beta)}{1 - G(q)}.$$

More general cases can be handled at the cost of more cumbersome computations. For example, we can add a compound Poisson process with phase-type jumps to the Lévy process, and we can allow B to have a phase-type distribution. See [12] for similar computations in a discrete-time setting.

If one cannot expect to obtain distributions in closed form, one can still aim to obtain Laplace transforms. Using similar arguments as in Example 3, we obtain the key equation (abbreviating $V = V(\infty)$)

$$E[e^{-sV}] = \frac{q(s - \Phi(q))}{\Phi(q)(\psi(s) - q)} \frac{\Phi(q)E[e^{-sF(V)}] - sE[e^{-\Phi(q)F(V)}]}{\Phi(q) - s}. \tag{4.3}$$

This equation is, of course, too complicated to solve for an arbitrary F , but nevertheless seems to be useful.

Example 5. Suppose that $F(x) = \delta x$, $\delta \in (0, 1)$. This case is a generalisation of a model for the throughput behaviour of a data connection under the Transmission Control Protocol (TCP) where typically the Lévy process is a simple deterministic drift; see for example [1,21,32,33] and references therein.

Eq. (4.3) reduces to

$$E[e^{-sV}] = \frac{q}{q - \psi(s)} E[e^{-s\delta V}] + \frac{qs}{\Phi(q)(\psi(s) - q)} E[e^{-\Phi(q)\delta V}]. \tag{4.4}$$

This is an equation of the form $v(s) = g(s)v(\delta s) + h(s)$, which, since $v(0) = 1$, has as (formal) solution

$$v(s) = \prod_{j=0}^{\infty} g(\delta^j s) + \sum_{k=0}^{\infty} h(\delta^k s) \prod_{j=0}^{k-1} g(\delta^j s).$$

Specialising to our situation we obtain

$$\begin{aligned} v(s) &= \prod_{j=0}^{\infty} \frac{q}{q - \psi(\delta^j s)} + v(\delta \Phi(q)) \sum_{k=0}^{\infty} \frac{qs\delta^k}{\Phi(q)(\psi(s\delta^k) - q)} \prod_{j=0}^{k-1} \frac{q}{q - \psi(\delta^j s)} \\ &= \prod_{j=0}^{\infty} \frac{q}{q - \psi(\delta^j s)} + v(\delta \Phi(q)) \frac{s}{\Phi(q)} \sum_{k=0}^{\infty} \delta^k \prod_{j=0}^k \frac{q}{q - \psi(\delta^j s)}. \end{aligned}$$

Since $\psi(0) = 0$, it easily follows that the infinite products converge, and the final expression for $v(s)$ yields an equation from which the only remaining unknown constant $v(\delta \Phi(q))$ can be solved explicitly.

5. Tail behaviour

In this section, we consider the tail behaviour of $V = V(\infty)$ (assuming this random variable exists) under a variety of assumptions on the tail behaviour of the Lévy measure ν . The treatment in this section can be seen as the continuous-time analogue of the results in [42]. There, a similar result is shown where e_q is geometrically distributed, $X(\cdot)$ is replaced by a (general) random walk, and $F(y) = (B - y)^+$, with B identical to the increments of the random walk. In [42] we modify ideas from [20] in the light-tailed case and develop stochastic lower and upper bounds in the heavy-tailed case. Here we take a different approach which is based on the well-developed fluctuation theory of spectrally one-sided Lévy processes. Before we present our main results, we first state some lemmas.

Again, we will exploit that, by PASTA, V has the same law as Z .

Lemma 5.1. *The following (in)equalities hold:*

$$P(Z > x) = P(X(e_q) + F(Z) > x) + \int_0^\infty (P(\bar{X}(e_q) > x) - P(\bar{X}(e_q) > x + y)) \times P(-\underline{X}(e_q) - F(Z) \in dy), \tag{5.1}$$

$$P(Z > x) \leq P(X(e_q) + F(Z) > x) + P(\bar{X}(e_q) > x)P(-\underline{X}(e_q) > F(Z)), \tag{5.2}$$

$$P(Z > x) \geq P(\bar{X}(e_q) > x)P(-\underline{X}(e_q) > F(Z)). \tag{5.3}$$

Proof. All identities follow from Lemma 3.2, $X(e_q) = \bar{X}(e_q) - (\bar{X}(e_q) - X(e_q))$ and the fact that $\bar{X}(e_q)$ and $\bar{X}(e_q) - X(e_q)$ are independent, recalling that $\bar{X}(e_q) - X(e_q) \stackrel{D}{=} \bar{X}(e_q)$. \square

In addition, we need two standard results from the literature on Lévy processes.

Lemma 5.2. [Kyprianou [28, p. 165]] *The random variable $\bar{X}(e_q)$ has the same law as $H(e_{\kappa(q,0)})$, where $\{(L^{-1}(t), H(t)), t \geq 0\}$ is a ladder height process of X with the Laplace exponent $\kappa(q, \zeta)$ defined by*

$$Ee^{qL^{-1}(t) + \zeta H(t)} = e^{\kappa(q, \zeta)t}.$$

From the above, one can easily derive the following version of the Pollaczek–Khinchine formula.

Lemma 5.3. [Bertoin [5, p. 172]] *The following identity holds:*

$$P(\bar{X}(e_q) > x) = \kappa(q, 0)U^{(q)}(x, \infty),$$

where

$$U^{(q)}(dx) = \int_0^\infty \int_0^\infty e^{-qs} P(H(t) \in dx, L^{-1}(t) \in ds)dt$$

is the renewal function of the ladder height process $\{(L^{-1}(t), H(t)), t < L(e_q)\}$ and $L(\cdot)$ is a local time of X .

We now turn to the tail behaviour of Z . Let $\Pi(A) = \nu(-A)$ be the Lévy measure of the spectrally positive Lévy process X (with support on R_+). First we investigate the case where the Lévy measure is a member of the so-called convolution equivalent class $\mathcal{S}^{(\alpha)}$. To define this

class, take $\alpha \geq 0$. We shall say that measure Π is *convolution equivalent* ($\Pi \in \mathcal{S}^{(\alpha)}$) if for fixed y we have that

$$\lim_{u \rightarrow \infty} \frac{\bar{\Pi}(u - y)}{\bar{\Pi}(u)} = e^{\alpha y}, \quad \text{if } \Pi \text{ is nonlattice,}$$

$$\lim_{n \rightarrow \infty} \frac{\bar{\Pi}(n - 1)}{\bar{\Pi}(n)} = e^{\alpha}, \quad \text{if } \Pi \text{ is lattice with span 1,}$$

and

$$\lim_{u \rightarrow \infty} \frac{\bar{\Pi}^{*2}(u)}{\bar{\Pi}(u)} = 2 \int_0^\infty e^{\alpha y} \Pi(dy),$$

where $*$ denotes convolution and $\bar{\Pi}(u) = \Pi((u, \infty))$. When $\alpha = 0$, then we are in the subclass of subexponential measures and there is no need to distinguish between the lattice and nonlattice cases (see [7]). We start from the following auxiliary result, which is the continuous-time analogue of Lemma 2 in [42].

Lemma 5.4. *Assume that $\Pi \in \mathcal{S}^{(\alpha)}$ and $\psi(\alpha) < q$ for $\psi(\alpha) = \log Ee^{\alpha X(1)}$. Then*

$$P(X(e_q) > x) \sim \frac{q}{(q - \psi(\alpha))^2} \bar{\Pi}(x), \tag{5.4}$$

$$P(\bar{X}(e_q) > x) \sim \frac{q}{(q - \psi(\alpha))^2} \frac{\Phi(q) + \alpha}{\Phi(q)} \bar{\Pi}(x), \tag{5.5}$$

where $f(x) \sim g(x)$ means that $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$.

Remark. Note that for $\alpha = 0$

$$P(\bar{X}(e_q) > x) \sim P(X(e_q) > x) \sim \frac{1}{q} \bar{\Pi}(x).$$

Proof. It is well known that $P(X(t) > x) \sim Ee^{\alpha X(t)} \bar{\Pi}_{X(t)}(x)$ for t fixed as $x \rightarrow \infty$, where $\Pi_{X(t)}$ is a Lévy measure of $X(t)$ (see [18]). Since $X(t)$ is infinitely divisible we have $\Pi_{X(t)}(\cdot) = t\Pi(\cdot)$ and hence $P(X(t) > x) \sim t(Ee^{\alpha X(1)})^t \bar{\Pi}(x)$. Since $X(e_q) \leq \bar{X}(e_q)$ by (5.5) and the dominated convergence theorem we obtain (5.4). We will use similar arguments as in the proof of Lemma 3.5 of Klüppelberg et al. [27]. For $\Pi_H \in \mathcal{S}^{(\alpha)}$ note that

$$P(H(t) > u) \sim t(Ee^{\alpha H(1)})^t \bar{\Pi}_H(u),$$

where Π_H is the Lévy measure of the process $\{H(t), t < e_{\kappa(q,0)}\}$ (see [18]). Using uniform in u Kesten bounds [27]:

$$P(H(t) > u) \leq P(H([t] + 1) > u) \leq K(\epsilon)(Ee^{\alpha H(1)} + \epsilon)^{[t]+1} \bar{\Pi}_H(u)$$

for any $\epsilon > 0$ and some constant $K(\epsilon)$, and the dominated convergence theorem, we derive by Lemma 5.2,

$$\lim_{u \rightarrow \infty} \frac{P(\bar{X}(e_q) > u)}{\bar{\Pi}_H(u)} = \frac{\kappa(q, 0)}{(\kappa(q, 0) - \log Ee^{\alpha H(1)})^2}. \tag{5.6}$$

The Wiener–Hopf factorisation yields that

$$Ee^{\alpha X(e_q)} = Ee^{\alpha H(e_{\kappa(q,0)})} Ee^{\alpha \hat{H}(e_{\hat{\kappa}(q,0)})},$$

where $(\hat{L}^{-1}(t), \hat{H}(t))$ is a downward ladder height process with Laplace exponent $\hat{\kappa}(q, \zeta)$. Since X is spectrally positive, we can choose the process $\hat{H}(t) = -t$ and hence

$$Ee^{\alpha \hat{H}(e_{\hat{\kappa}(q,0)})} = \hat{\kappa}(q, 0) \int_0^\infty e^{-\hat{\kappa}(q,0)t} e^{-\alpha t} dt = \frac{\hat{\kappa}(q, 0)}{\hat{\kappa}(q, \alpha)},$$

where $\hat{\kappa}(q, \alpha) = \Phi(q) + \alpha$. Thus

$$\frac{q}{q - \psi(\alpha)} \frac{\hat{\kappa}(q, \alpha)}{\hat{\kappa}(q, 0)} = \frac{\kappa(q, 0)}{\kappa(q, 0) - \log Ee^{\alpha H(1)}}.$$

Using the well known fact that $q = \kappa(q, 0)\hat{\kappa}(q, 0)$ (see [28, p. 166]) we identify the right-hand side of (5.6) as

$$\lim_{u \rightarrow \infty} \frac{P(\bar{X}(e_q) > u)}{\bar{\Pi}_H(u)} = \frac{q}{(q - \psi(\alpha))^2} \frac{(\Phi(q) + \alpha)^2}{\Phi(q)}.$$

Now using similar arguments like in [37] (see also [28, Th. 7.7 on p. 191] and [28, Th. 7.8 on p. 195]) we derive

$$\bar{\Pi}_H(u) = \int_0^\infty \bar{\Pi}(u + y) \hat{V}(dy),$$

where $\hat{V}(y)$ is the renewal function of the downward ladder height process $\{(\hat{L}^{-1}(t), \hat{H}(t)), t < \hat{\kappa}(q, 0)\} = \{(\hat{L}^{-1}(t), \hat{H}(t)), \hat{L}^{-1}(t) < e_q\}$. Thus

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\bar{\Pi}_H(u)}{\bar{\Pi}(u)} &= \int_0^\infty e^{-\alpha y} \hat{V}(dy) \\ &= \int_0^\infty Ee^{-q\hat{L}^{-1}(t) - \alpha\hat{H}(t)} dt \\ &= \int_0^\infty e^{-\hat{\kappa}(q, \alpha)t} dt = \frac{1}{\hat{\kappa}(q, \alpha)} = \frac{1}{\Phi(q) + \alpha}. \end{aligned}$$

Hence, by [17] also $\Pi_H \in \mathcal{S}^{(\alpha)}$ if and only if $\Pi \in \mathcal{S}^{(\alpha)}$. This completes the proof. \square

It is known [18] that if for independent random variables χ_i ($i = 1, 2$) we have $P(\chi_i > u) \sim c_i \bar{G}(u)$ as $u \rightarrow \infty$ and $G \in \mathcal{S}^{(\alpha)}$, then $P(\chi_1 + \chi_2 > u) \sim (c_1 Ee^{\alpha\chi_2} + c_2 Ee^{\alpha\chi_1})\bar{G}(u)$. This observation and (5.1) in Lemmas 5.1 and 5.4 yield the following main result.

Theorem 5.1. *Assume that $\Pi \in \mathcal{S}^{(\alpha)}$ and $\psi(\alpha) < q$. Moreover, let $F(y) \leq F_0(\geq 0)$ for any y , and assume that there exists a constant $c \geq 0$ such that $P(F(y) > x) \sim P(F_0 > x) \sim c\bar{\Pi}(x)$ as $x \rightarrow \infty$ for each y (If $c = 0$ then $P(F(y) > x) = o(\bar{\Pi}(x))$). Then*

$$\begin{aligned} P(Z > x) &\sim \left\{ cEe^{\alpha X(e_q)} + \frac{q}{(q - \psi(\alpha))^2} Ee^{\alpha F(Z)} + \frac{q}{(q - \psi(\alpha))^2} \frac{\Phi(q) + \alpha}{\Phi(q)} \right. \\ &\quad \left. \times E \left[\left(1 - e^{-\alpha(-\underline{X}(e_q) - F(Z))} \right); -\underline{X}(e_q) - F(Z) > 0 \right] \right\} \bar{\Pi}(x) \end{aligned}$$

as $x \rightarrow \infty$.

The conditions in this theorem are satisfied by both examples $F(y) = 0$ (in which case we take $F_0 = 0, c = 0$) and $F(y) = (B - y)^+$ (in which case $F_0 = B$). If Π is subexponential

($\Pi \in \mathcal{S}^{(0)}$), then

$$P(Z > x) \sim \left(c + \frac{1}{q}\right) \bar{\Pi}(x).$$

We will consider now the Cramér case (light-tailed case). Assume that there exists $\Phi(q)$ such that

$$\psi(\Phi(q)) = q \tag{5.7}$$

and that

$$m(q) := \left. \frac{\partial \kappa(q, \beta)}{\partial \beta} \right|_{\beta = -\Phi(q)} < \infty. \tag{5.8}$$

Note that if $\Pi \in \mathcal{S}(\alpha)$ and $\psi(\alpha) < q$, then condition (5.7) is not satisfied. Moreover, we assume that

$$Ee^{\Phi(q)F(Z)} < \infty. \tag{5.9}$$

Theorem 5.2. *Assume that (5.7)–(5.9) hold and that the support of Π is nonlattice. Then*

$$P(Z > x) \sim Ce^{-\Phi(q)x}$$

as $x \rightarrow \infty$, where

$$C = P(-\underline{X}(e_q) > F(Z))\kappa(q, 0) (\Phi(q)m(q))^{-1}.$$

Proof. We introduce the new probability measure

$$\left. \frac{dP^\theta}{dP} \right|_{\mathcal{F}_t} = e^{\theta X(t) - \psi(\theta)t},$$

where \mathcal{F}_t is a natural filtration of X . On P^θ , the process X is again a spectrally positive Lévy process with the Lévy measure $\Pi_\theta(dx) = e^{\theta x} \Pi(dx)$, which is also nonlattice. Let $U_\theta^{(q)}$ be the renewal function appearing in Lemma 5.3 with P replaced by P^θ . Recall that $L^{-1}(t)$ is a stopping time. Hence, from the optional stopping theorem, we have that

$$\begin{aligned} e^{-\Phi(q)x} U_{\Phi(q)}^{(q)}(dx) &= \int_0^\infty \int_0^\infty e^{-\Phi(q)x} P^{\Phi(q)}(H(t) \in dx, L^{-1}(t) \in ds) dt \\ &= \int_0^\infty \int_0^\infty e^{-\Phi(q)x} e^{-qs + \Phi(q)x} P(H(t) \in dx, L^{-1}(t) \in ds) dt = U^{(q)}(dx). \end{aligned}$$

We follow now Bertoin and Doney [6] (see also [28, Th. 7.6 on p. 185]). From Lemma 5.3 we have

$$\begin{aligned} e^{\Phi(q)x} P(\bar{X}(e_q) > x) &= \kappa(q, 0) \int_x^\infty e^{-\Phi(q)(y-x)} U_{\Phi(q)}^{(q)}(dy) \\ &= \kappa(q, 0) \int_0^\infty e^{-\Phi(q)z} U_{\Phi(q)}^{(q)}(x + dy). \end{aligned}$$

From [28, Th. 5.4 on p. 114] it follows that $U_{\Phi(q)}^{(q)}(dy)$ has a nonlattice support. From the key renewal theorem (see [28, Cor. 5.3 on p. 114]) the measure $U_{\Phi(q)}(x + dy)$ converges weakly to

the Lebesgue measure $\frac{1}{E^{\Phi(q)}H(1)}dy$ (see [28, Th. 7.6 on p. 185]). Thus

$$\lim_{x \rightarrow \infty} e^{\Phi(q)x} P(\bar{X}(e_q) > x) = \frac{\kappa(q, 0)}{\Phi(q)E^{\Phi(q)}H(1)}.$$

Observe that

$$\begin{aligned} E^{\Phi(q)}H(1) &= \int_0^\infty te^{-t}E^{\Phi(q)}H(1)dt = \int_0^\infty e^{-t}E^{\Phi(q)}H(t)dt = \int_0^\infty xU^{(1)}(dx) \\ &= \int_0^\infty e^{-t}dt \int_0^\infty xP^{\Phi(q)}(H(t) \in dx) \\ &= \int_0^\infty e^{-t} \int_0^\infty xe^{\Phi(q)x-qs}P(H(t) \in dx, L^{-1}(t) \in ds)dt \\ &= \int_0^\infty e^{-t}EH(t)e^{\Phi(q)H(t)-qL^{-1}(t)}dt \\ &= \int_0^\infty te^{-t-\kappa(q,-\Phi(q))t}dt \left. \frac{\partial \kappa(q, \beta)}{\partial \beta} \right|_{\beta=-\Phi(q)}. \end{aligned}$$

From the Wiener–Hopf factorisation (see [28, p. 167]) it follows that

$$q - \psi(\theta) = \kappa(q, -\theta)\hat{\kappa}(q, \theta).$$

From the convexity of the Laplace exponents ϕ and ψ we have that $\hat{\kappa}(q, \Phi(q)) = 2\Phi(q) > 0$ and hence $\kappa(q, -\Phi(q)) = 0$. Finally,

$$E^{\Phi(q)}H(1) = \left. \frac{\partial \kappa(q, \beta)}{\partial \beta} \right|_{\beta=-\Phi(q)}.$$

Note that by (5.7) and (5.9), $P(X(e_q) > x) = o(e^{-\Phi(q)x})$ and $P(X(e_q) + F(Z) > x) = o(e^{-\Phi(q)x})$. Inequalities (5.2) and (5.3) in Lemma 5.1 complete the proof. \square

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References

- [1] E. Altman, K. Avrachenkov, C. Barakat, R. Núñez-Queija, State-dependent M/G/1 type queueing analysis for congestion control in data networks, *Computer Networks* 39 (2002) 789–808.
- [2] L. Andersen, M. Mandjes, Structural properties of reflected Lévy processes, *Queueing Systems* 63 (2009) 301–322.
- [3] R. Bekker, Queues with Lévy input and hysteretic control, *Queueing Systems* 63 (2009) 281–299.
- [4] R. Bekker, O.J. Boxma, O. Kella, Queues with delays in two-stage strategies and Lévy input, *Journal of Applied Probability* 45 (2008) 314–332.
- [5] J. Bertoin, Lévy Processes, in: *Cambridge Tracts in Mathematics*, vol. 121, Cambridge University Press, 1996.
- [6] J. Bertoin, R.A. Doney, Cramér’s estimate for Lévy processes, *Statistics and Probability Letters* 21 (1994) 363–365.
- [7] J. Bertoin, R.A. Doney, Some asymptotic results for transient random walks, *Advances in Applied Probability* 28 (1996) 207–226.
- [8] K. Borovkov, G. Last, On level crossings for a general class of piecewise-deterministic Markov processes, *Advances in Applied Probability* 40 (3) (2008) 815–834.

- [9] O.J. Boxma, M. Mandjes, O. Kella, On a queuing model with service interruptions, *Probability in the Engineering and Informational Sciences* 22 (2008) 537–555.
- [10] O.J. Boxma, M. Mandjes, O. Kella, On a generic class of Lévy-driven vacation models, *Probability in the Engineering and Informational Sciences* (2010).
- [11] O.J. Boxma, D. Perry, W. Stadje, Clearing models for M/G/1 queues, *Queueing Systems* 38 (2001) 287–306.
- [12] O.J. Boxma, M. Vlasiou, On queues with service and interarrival times depending on waiting times, *Queueing Systems* 56 (2007) 121–132.
- [13] L. Chaumont, A.E. Kyprianou, J.C. Pardo, Some explicit identities associated with positive self-similar Markov processes, *Stochastic Processes and their Applications* (2008) Available online 7 May 2008.
- [14] P. Diaconis, D. Freedman, Iterated random functions, *SIAM Review* 41 (1999) 45–76.
- [15] K. Debicki, M. Mandjes, M. van Uitert, A tandem queue with Lévy input: a new representation of the downstream queue length, *Probability in the Engineering and Informational Sciences* 21 (2007) 83–107.
- [16] J.H. Dshalalow, Queueing systems with state dependent parameters, in: *Frontiers in Queueing: Models and Applications in Science and Engineering*, 1997, pp. 61–116.
- [17] P. Embrechts, C.M. Goldie, On convolution tails, *Stochastic Processes and their Applications* 13 (1982) 263–278.
- [18] P. Embrechts, C.M. Goldie, N. Veraverbeke, Subexponentiality and infinite divisibility, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 49 (1979) 335–347.
- [19] S. Foss, T. Konstantopoulos, An overview of some stochastic stability methods, *Journal of the Operations Research Society of Japan* 47 (2004) 275–303.
- [20] C. Goldie, Implicit renewal theory and tails of solutions of random equations, *The Annals of Applied Probability* 1 (1991) 126–166.
- [21] F. Guillemin, P. Robert, B. Zwart, AIMD algorithms and exponential functionals, *The Annals of Applied Probability* 14 (2004) 90–117.
- [22] F. Hubalek, A.E. Kyprianou, Old and new examples of scale functions for spectrally negative Lévy processes, *Mathematics* (2008) [arXiv:0801.0393v2](https://arxiv.org/abs/0801.0393v2).
- [23] H. Kaspi, O. Kella, D. Perry, Dam processes with state dependent batch sizes and intermittent production processes with state dependent rates, *Queueing Systems: Theory and Applications* 24 (1996) 37–57.
- [24] O. Kella, O. Boxma, M. Mandjes, A Lévy process reflected at a Poisson age process, *Journal of Applied Probability* 43 (2006) 21–230.
- [25] O. Kella, D. Perry, W. Stadje, A stochastic clearing model with a Brownian and a compound Poisson component, *Probability in the Engineering and Informational Sciences* 17 (2003) 1–22.
- [26] O. Kella, W. Whitt, Queues with server vacations and Lévy processes with secondary jump input, *The Annals of Applied Probability* 1 (1991) 104–117.
- [27] C. Klüppelberg, A.E. Kyprianou, R.A. Maller, Ruin probabilities and overshoots for general Lévy insurance risk processes, *The Annals of Applied Probability* 14 (2004) 1766–1801.
- [28] A.E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, in: Universitext, Springer-Verlag, Berlin, 2006.
- [29] A.E. Kyprianou, Z. Palmowski, A martingale review of some fluctuation theory for spectrally negative Lévy processes, in: *Séminaire de Probabilités XXXVIII*, in: *Lecture Notes in Math.*, vol. 1857, Springer, Berlin, 2005, pp. 16–29.
- [30] A.E. Kyprianou, V. Rivero, Special, conjugate and complete scale functions for spectrally negative Lévy processes, *The Electronic Journal of Probability* 13 (2008) 1672–1701.
- [31] A. Lambert, Completely asymmetric Lévy processes confined in a finite interval, *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques* 36 (2000) 251–274.
- [32] K. Maulik, B. Zwart, Tail asymptotics for exponential functionals of Lévy processes, *Stochastic Processes and their Applications* 116 (2006) 156–177.
- [33] K. Maulik, B. Zwart, An extension of the square root law of TCP, *Annals of Operations Research* 170 (1) (2009) 217–232. doi:10.1007/s10479-008-0437-8.
- [34] M. Pistorius, Exit problems of Lévy processes with applications in finance, Ph.D. Thesis, Utrecht University, The Netherlands, 2003.
- [35] M. Rubinovitch, J.W. Cohen, Level crossings and stationary distributions for general dams, *Journal of Applied Probability* 17 (1980) 218–226.
- [36] H.G. Tucker, Absolute continuity of infinitely divisible distributions, *Pacific Journal of Mathematics* 12 (1962) 1125–1129.
- [37] V. Vigon, Votre Lévy rampe-t-il? *Journal of the London Mathematical Society* 65 (2002) 243–256.
- [38] M. Vlasiou, A non-increasing Lindley-type equation, *Queueing Systems* 56 (2007) 41–52.

- [39] M. Vasiou, I.J.B.F. Adan, An alternating service problem, *Probability in the Engineering and Informational Sciences* 19 (2005) 409–426.
- [40] M. Vasiou, I.J.B.F. Adan, Exact solution to a Lindley-type equation on a bounded support, *Operations Research Letters* 35 (2007) 105–113.
- [41] M. Vasiou, I.J.B.F. Adan, J. Wessels, A Lindley-type equation arising from a carousel problem, *Journal of Applied Probability* 41 (2004) 1171–1181.
- [42] M. Vasiou, Z. Palmowski, Tail asymptotics for a random sign Lindley recursion, *Journal of Applied Probability* 47 (1) (2010) 72–83.
- [43] M. Vasiou, B. Zwart, Time-dependent behaviour of an alternating service queue, *Stochastic Models* 23 (2007) 235–263.