A Nonextremal Camion Basis

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ABSTRACT

We construct a 3×21 matrix A and Camion basis B of A such that B does not correspond to an extreme point of the convex hull of basic solutions of $Ax = b$ for any $b \in \mathbb{R}^3$. Computer algebra methods played a critical role in finding both the matrix A and an analytic proof that B is not extremal.

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1. OVERVIEW

A column basis B of a rank m matrix $A \in \mathbb{R}^{m \times n}$ is a *Camion basis* if there are nonsingular diagonal matrices D_m and D_n such that $D_m B^{-1} A D_n$ is nonnegative. Camion bases have many geometric and combinatorial interpretations: they correspond to simplicial regions of hyperplane arrangements [6; 2, §4.4] and mutations of realizable oriented matroids [5], and arise from depth-first-search trees of graphs (see [4]). Camion [3] first showed that every real matrix has at least one Camion basis. Shannon [6] proved that every matrix $A \in \mathbb{R}^{m \times n}$ of rank m has at least n Camion bases, and every column of A is contained in at least m of these bases. The notion of Camion bases has been generalized to oriented matroids, and the existence of Camion bases is a central open problem in oriented matroid theory [2, §7.3].

An interesting construction for Camion bases involves the basic solutions of the linear system $Ax = b$, where $b \in \mathbb{R}^m$ is in general position with respect to the columns of A. Given any column basis B, we write $x(B, b) \in \mathbb{R}^n$ for the corresponding basic solution. Let $C(A, b)$ denote the convex hull in $\mathbb{R}^n$ of the set of all basic solutions of $Ax = b$. Bland and Cho [1] showed that every vertex of $C(A, b)$ gives rise to a Camion basis of A.

PROPOSITION 1 [1]. *If a basic solution $x = x(B, b)$ of $Ax = b$ is a vertex of the convex polytope $C(A, b)$, then the corresponding basis B is a Camion basis of A.*

This raises the natural question whether each Camion basis of a real matrix can be obtained in this way. The answer is affirmative in the special cases $m \leqslant 2$ and $n - m \leqslant 2$ [4, §5.2]. It is the objective of this note to show that the answer is negative in general.

THEOREM 2. *There exists a matrix $\tilde{A} \in \mathbb{R}^{3 \times 21}$ of rank three and a Camion basis B of $\tilde{A}$ such that, for all $b \in \mathbb{R}^3$ in general position with respect to the columns of $\tilde{A}$, the basic solution $x(B, b)$ is not a vertex of $C(\tilde{A}, b)$.*

The proof of Proposition 1 given in [1] is based on the following lemma, which is also used in our proof of Theorem 2. Two vectors x and y being *consistent* means that there are no coordinates i and j with $x_i y_i < 0 < x_j y_j$.

LEMMA 3 [1]. *If $x(B, b)$ is a vertex of $C(A, b)$, then every column in $B^{-1} A$ is consistent with $B^{-1} b$.*

To derive Proposition 1 from Lemma 3, we first choose a nonsingular diagonal matrix D_m such that $D_m B^{-1} b$ is nonnegative. By consistency, each column of $D_m B^{-1}A$ is either nonnegative or nonpositive, and we can choose a nonsingular diagonal matrix D_n such that $D_m B^{-1}AD_n$ is nonnegative.

Fix $m = 3$. A matrix $A \in \mathbb{R}^{3\times n}$ is in *standard form* if $A = [I, N]$, where I is the 3×3 identity matrix. We assume that the matrix $N \in \mathbb{R}^{3\times(n-3)}$ is nonnegative, which implies that I is a Camion basis of A. Let $W(A)$ denote the set of all vectors $b \in \mathbb{R}^3$ for which $x(I, b) = (b, 0)$ is a vertex of $C(A, b)$. This is a *semialgebraic set* (i.e., it is defined by polynomial inequalities), whose structure seems rather complicated in general.

Our method for finding and verifying the example of Theorem 2 was facilitated by numeric and symbolic computation. To gain insight into the problem, we generated random nonnegative matrices of rank three of the form $[I, N]$. Random vectors b were tested for extremality of $x(I, b)$ using MATLAB, a package for matrix computations, and successes and failures were plotted. We found a 3×6 matrix $A = [I, N]$ such that a large open region Δ of $\mathbb{R}^3_+$ appeared to contain no vector b for which the Camion basis I of A is extremal. Plots of the semialgebraic set $W(A)$ were obtained using the computer algebra system MAPLE. The plots were consistent with the empirical observation that $W(A)$ and Δ appear to be disjoint. This was verified analytically; $W(A)$ excludes Δ. Replacing N in A by a row permutation N^* of N gives an excluded region Δ^* that is obtained from Δ by permuting the coordinates. The six Δ^*'s corresponding to all of the row permutations have as their union the entire nonnegative orthant. The 3×21 example of $\tilde{A}$ was produced by appending all six row permutations of N to I, resulting in exclusion of the entire nonnegative orthant, implying by Lemma 3, that I cannot be extremal for $\tilde{A}$. Details follow in the next section.

2. THE EXAMPLE

We consider the matrix $A = [I, N]$, where

$$N = \begin{bmatrix} \frac{1}{1000} & \frac{3}{20} & \frac{1}{40} \\ \frac{9}{10} & \frac{4}{5} & \frac{1}{2} \\ \frac{2}{25} & \frac{1}{100} & \frac{1}{2} \end{bmatrix}.$$

Let $\Pi_1, \Pi_2, \ldots, \Pi_6$ be all six 3×3 permutation matrices. We claim that the 3×21 matrix

$$\tilde{A} = [I, \Pi_1 N, \Pi_2 N, \Pi_3 N, \Pi_4 N, \Pi_5 N, \Pi_6 N]$$

satisfies $W(\tilde{A}) = \emptyset$. In order to prove this claim (and hence Theorem 2), we observe

$$\begin{aligned} W(\tilde{A}) &\subseteq W(\Pi_1 A) \cap W(\Pi_2 A) \cap W(\Pi_3 A) \\ &\quad \cap W(\Pi_4 A) \cap W(\Pi_5 A) \cap W(\Pi_6 A) \\ &= \Pi_1 W(A) \cap \Pi_2 W(A) \cap \Pi_3 W(A) \\ &\quad \cap \Pi_4 W(A) \cap \Pi_5 W(A) \cap \Pi_6 W(A), \qquad (*) \end{aligned}$$

which is easily verified from the definition of the operator $W(\cdot)$. Let Δ denote the triangle in $\mathbb{R}^3$ with vertices $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(1, 0, 0)$, and $(\frac{1}{2}, 0, \frac{1}{2})$.

LEMMA 4. *For the matrix A above, the set* $W(A)$ *is disjoint from the triangle* Δ.

Proof of Theorem 2 from Lemma 4. The set $W(A)$ is invariant under scaling by positive real numbers, which means $W(A)$ is disjoint from the triangular cone $\mathbb{R}_+\Delta$. By $(*)$, the set $W(\tilde{A})$ is disjoint from $\bigcup_{i=1}^{6} \Pi_i(\mathbb{R}_+\Delta)$. However, this union equals the entire nonnegative cone $\mathbb{R}_+^3$. Therefore, by Lemma 3, the set $W(\tilde{A})$ is empty, as desired. ∎

It remains to prove Lemma 4. Denoting the columns of A by $a_1, \dots, a_6$, the Camion bases of A are $I = [a_1, a_2, a_3]$, $B_1 = [a_1, a_4, a_5]$, $B_2 = [a_1, a_4, a_6]$, $B_3 = [a_2, a_4, a_6]$, $B_4 = [a_3, a_5, a_6]$, $B_5 = [a_1, a_2, a_5]$, and $B_6 = [a_2, a_3, a_4]$. Let $L(b)$ denote the 3×6 matrix consisting of the last three rows of the 6×6 matrix $[x(B_1, b), x(B_2, b), \dots, x(B_6, b)]$. Each entry of $L(b)$ is a linear function of $b = (b_1, b_2, b_3)$. The 3×3 minor of $L(b)$ with column indices $\{i < j < k\} \subset \{1, \dots, 6\}$ is abbreviated $D_{ijk}(b)$. This is a homogeneous polynomial of degree three in $b = (b_1, b_2, b_3)$.

Suppose the $b \in W(A)$. Then there exists a vector $f \in \mathbb{R}^6$ such that $f^t \cdot x(I, b) > f^t \cdot x(B_i, b)$ for $i = 1, 2, \dots, 6$. Since A is in standard form, we may suppose $f = (0, 0, 0, c_1, c_2, c_3)$. Then the vector $c = (c_1, c_2, c_3)$ satisfies $c \cdot L(b) < 0$. Therefore there can be no nonnegative vector in the null space of $L(b)$, except the zero vector. Cramer's rule implies that among the four expressions $D_{123}(b)$, $-D_{124}(b)$, $D_{134}(b)$, and $-D_{234}(b)$ at least one is positive and at least one is negative. We claim that this is not possible for any point $b \in \Delta$.

In order to see this, we apply the coordinate projection $(u, v, w) \to (u, v)$, which takes the triangle Δ bijectively onto the triangle Δ' in the (u, v)

plane having the vertices $(\frac{1}{3}, \frac{1}{3})$, $(1, 0)$, and $(\frac{1}{2}, 0)$. The four polynomials in question transform into

$$D_{123}(u,v) = \tfrac{400}{1353}(45u + 49v - 45)(864u + 47v - 44)(u + v - 1),$$

$$-D_{124}(u,v) = \tfrac{20}{4961}v(45u + 49v - 45)(-1360u + 519v - 220),$$

$$\begin{aligned} D_{134}(u,v) = \tfrac{20}{363}(&-1382400u^3 - 1308560u^2v + 1452800u^2 \\ &+238779uv^2 - 143260uv - 70400u + 14619v^3 \\ &-28039v^2 + 13420v),\end{aligned}$$

$$-D_{234}(u,v) = \tfrac{500}{1353}(-20u + v)(864u + 47v - 44)(u + v - 1).$$

It remains to verify that all four polynomials are nonnegative for all (u, v) in the triangle Δ'. Verification for three of the four is easy, since the polynomials are products of linear terms. Verification for the remaining polynomial, D_{134}, was carried out by trapping the three roots of the univariate cubic polynomials $D_{134}(u, \alpha)$ in intervals outside of the interior of Δ' for each fixed value of α between 0 and $\frac{1}{3}$. The endpoints of each of the families of intervals are parametrized by a pair of linear functions of α on which D_{134} has opposite signs over all choices of α between 0 and $\frac{1}{3}$. This completes the proof of Lemma 4 and of Theorem 2.

Additional details and plots of the curves $D_{ijk}(u, v) = 0$ can be found in [4].

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