A NOTE ON *ALMOST FLAT* NUMBERS

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Dedicated to Herman J. J. te Riele on the occasion of his retirement from the CWI in January 2012

In this note we present a solution of the following

Problem. *For $n \in \mathbb{N}$ let $\beta(n)$ be the product of all prime divisors of n (not counting multiplicities).*

So, in the notation of Hardy & Wright, if $n = \prod p^e$ then $\beta(n) = \prod p$ with $\beta(1) = 1$.

A positive integer n is called flat *iff $n = \beta(n)$ (or, equivalently, iff $|\mu(n)| = 1$, where $\mu(n)$ is the Möbius function).*

A positive integer is called almost flat *iff $n/\beta(n)$ is prime.*

(A) Show that the sequence of almost flat numbers has a positive natural density (denoted by d_1), and indicate how this density can be computed to any degree of accuracy. (This is the case $k = 1$ in the next, more general problem (B).)

(B) Let $\omega(n)$ denote the number of different prime divisors of $n \in \mathbb{N}$ (not counting multiplicities), and let k be any (fixed) positive integer.

Show that the sequence of all $n \in \mathbb{N}$ such that $q := q(n) := n/\beta(n)$ is flat with $\omega(q) = k$, has positive natural density (denoted by d_k), and indicate how this density can be computed to any degree of accuracy.

(B1) Solve this problem for $k = 2$.

(B2) How to proceed for $k \geq 3$? Compute d_3, d_4 and d_5.

Solution of (A), the case $k = 1$.

The generating Dirichlet series of the almost flat numbers is obtained by expanding

$$(1) \qquad \sum_{q \text{ prime}} \frac{1}{q^{2s}} \frac{\prod_{p \text{ prime}} \left(1 + \frac{1}{p^s}\right)}{1 + \frac{1}{q^s}} = \frac{\zeta(s)}{\zeta(2s)} \sum_{q \text{ prime}} \frac{1}{q^s(q^s+1)}, \qquad (s = \sigma + it, \sigma > 1).$$

Invoking the well-known Wiener-Ikehara Tauberian theorem (which applies indeed see [3, p. 259–266]), we find that the required natural density d_1 exists and equals

$$d_1 = \frac{6}{\pi^2} \sum_{q \text{ prime}} \frac{1}{q(q+1)}. \tag{2}$$

The sum of the last series may be approximated by observing that

$$\sigma_1 := \sum_{q \text{ prime}} \frac{1}{q(q+1)} = \sum_{q \text{ prime}} \frac{1}{q^2} \frac{1}{1+\frac{1}{q}} = \sum_{q \text{ prime}} \frac{1}{q^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{q^n} = \tag{3}$$
$$= \sum_{n=0}^{\infty} (-1)^n \sum_{q \text{ prime}} \frac{1}{q^{2+n}} = \sum_{n=0}^{\infty} (-1)^n P(n+2)$$

where, for $s > 1$,

$$P(s) := \sum_{p \text{ prime}} \frac{1}{p^s} = \sum_{r=1}^{\infty} \frac{\mu(r)}{r} \log \zeta(rs) \tag{4}$$

(see Titchmarsh [4, p. 12, formula (1.6.1)]) and that, also for $s > 1$

$$\left| \frac{\mu(r)}{r} \log \zeta(rs) \right| \leq \frac{1}{r} \log \zeta(rs) < \frac{1}{r} \log\left(1 + \frac{1}{2^{rs}} + \int_2^{+\infty} \frac{dx}{x^{rs}}\right) <$$
$$< \frac{1}{r} \log\left(1 + \frac{3}{2^{rs}}\right) < \frac{3}{2^{rs}}$$

so that, for $n \geq 2$,

$$P(n) < \sum_{r=1}^{\infty} \frac{1}{r} \log \zeta(rs) < \sum_{r=1}^{\infty} \frac{3}{2^{rs}} < \frac{3}{2^n - 1} \leq \frac{4}{2^n}. \tag{5}$$

A combination of these ingredients is sufficient for a high precision computation of $\sum_{n=2}^{\infty} (-1)^n P(n)$.

For $n \geq 2$ we may approximate $P(n)$ by evaluating $\sum_{r=1}^{R} \frac{\mu(r)}{r} \log \zeta(rn)$ for a sufficiently large R.

If we want an accuracy of ε then it suffices to take R such that $\sum_{r=R+1}^{\infty} \frac{1}{r} \log \zeta(rn) < \varepsilon$. It is easily seen that

$$R = \left\lfloor \frac{1}{n} \frac{\log\left(\frac{4}{\varepsilon}\right)}{\log 2} \right\rfloor \tag{6}$$

suffices. Using Mathematica we find

$\sigma_1 \approx 0.3302299262\ 6420324101\ 5094588086\ 7447606442\ 5941947407\ \ldots$

so that (recall that $d_1 = \frac{6}{\pi^2}\sigma_1$)

$d_1 \approx 0.2007557220\ 1926598699\ 6250723114\ 4047658535\ 3555535256\ \ldots$

Solution of (B1), the case $k = 2$.

Similarly as in (A) the required density d_2 equals

$$d_2 = \frac{6}{\pi^2} \sum_{p<q} \frac{1}{p(p+1)q(q+1)} \tag{7}$$

(p and q denoting primes).

The last series may also be written as

$$\sigma_2 := \sum_{p<q} \frac{1}{p(p+1)q(q+1)} = \frac{1}{2}\Big(\sum_{p<q} \frac{1}{p(p+1)q(q+1)} + \sum_{q<p} \frac{1}{q(q+1)p(p+1)}\Big) =$$

$$= \frac{1}{2} \sum_{p \neq q} \frac{1}{p(p+1)q(q+1)} = \frac{1}{2}\Big(\sum_{p,q} \frac{1}{p(p+1)q(q+1)} - \sum_{p=q} \frac{1}{p(p+1)q(q+1)}\Big) =$$

$$= \frac{1}{2}\Big[\Big(\sum_p \frac{1}{p(p+1)}\Big)^2 - \sum_p \frac{1}{p^2(p+1)^2}\Big] = \frac{\sigma_1^2}{2} - \frac{1}{2} \sum_p \frac{1}{p^4\left(1+\frac{1}{p}\right)^2}$$

(with $\sigma_1 = \frac{\pi^2}{6} d_1$) and this in its turn may be reduced to a form "only" containing $P(n)$'s, so that we can compute σ_2 (and hence d_2) to any degree of accuracy. Indeed, it is easily verified that

$$\sigma_2 = \frac{1}{2}\sigma_1^2 - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} n P(n+3). \tag{8}$$

Using *Mathematica* we find that

$\sigma_2 \approx 0.0727869325\ 3120878610\ 0250493970\ 5431864431\ 8060075841\ \ldots$

so that

$d_2 \approx 0.0221245744\ 7327116398\ 0012002355\ 9483175788\ 6781598850\ \ldots$

Solution of (B2), the case $k \geq 3$.

For $k \geq 3$ we make use of the well-known (Girard-) Newton formulas. We briefly recall some pertinent details :

We consider (formally) the equation $f(x) = \sum_n (-1)^n \sigma_n x^n = 0$ with roots $\frac{1}{\alpha_k}$. In our application we will have $\alpha_n = \frac{1}{p_n(p_n+1)}$ where p_n is the n-th prime. Then we have

$$\sum_n (-1)^n \sigma_n x^n = \prod_n (1 - \alpha_n x) = 0, \quad \text{with} \quad \sigma_n = \sum_{j_1<j_2<\cdots<j_n} \alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_n}$$

(we assume here that $\sigma_0 = 1$). We define

$$S_k := \sum_n \alpha_n^k.$$

Although the (Girard-) Newton formulas usually express the sums S_n in terms of the coefficients σ_n, we will turn things around and express the (elementary symmetric functions) σ_n in terms of the (exponential sums) S_n. In order to do so we compute $f'(x)$ in two different ways :

$$f'(x) = \sum_n (-1)^n n \sigma_n x^{n-1}$$

and

$$f'(x) = -f(x) \sum_n \frac{\alpha_n}{1-\alpha_n x} = -f(x) \sum_n \sum_{k=0}^{\infty} \alpha_n^{k+1} x^k = -f(x) \sum_{k=0}^{\infty} S_{k+1} x^k.$$

Comparing coefficients we find that (see [1, Chap. 8, p. 166], [2, p. 140] or [5, p. 99])

$$(-1)^m m\, \sigma_m = -\sum_{n=0}^{m-1} (-1)^n \sigma_n S_{m-n} \tag{9}$$

which leads directly to the recurrence (with $\sigma_0 = 1$)

$$\sigma_m = \frac{(-1)^{m+1}}{m}(S_m - \sigma_1 S_{m-1} + \sigma_2 S_{m-2} - \cdots + (-1)^{m-1}\sigma_{m-1} S_1), \qquad (m \geq 1).$$

In this way we easily obtain, for example,

$$\sigma_1 = S_1$$

$$\sigma_2 = \frac{1}{2}(S_1^2 - S_2)$$

$$\sigma_3 = \frac{1}{6}(S_1^3 - 3S_1 S_2 + 2S_3)$$

$$\sigma_4 = \frac{1}{24}(S_1^4 - 6S_1^2 S_2 + 3S_2^2 + 8S_1 S_3 - 6S_4)$$

$$\sigma_5 = \frac{1}{120}(S_1^5 - 10S_1^3 S_2 + 15 S_1 S_2^2 + 20 S_1^2 S_3 - 20 S_2 S_3 - 30 S_1 S_4 + 24 S_5).$$

For $k = 3$ we have to deal with the sum

$$\sigma_3 := \sum_p \frac{1}{p(p+1)} \sum_{p<q} \frac{1}{q(q+1)} \sum_{q<r} \frac{1}{r(r+1)}, \qquad (p, q, r \text{ primes}).$$

In this case we have :

$$\alpha_n = \frac{1}{p_n(p_n+1)}, \quad \sigma_3 = \sum_{n=1}^{\infty} \alpha_n \sum_{n<m} \alpha_m \sum_{m<r} \alpha_r = \tag{10}$$

$$= \sum_n \frac{1}{p_n(p_n+1)} \sum_{n<m} \frac{1}{p_m(p_m+1)} \sum_{m<r} \frac{1}{p_r(p_r+1)}$$

and

$$\sigma_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3).$$

Note that we can compute the $S_n = \sum_p \frac{1}{p^n(p+1)^n}$ by the formula

(11)

$$S_n = \sum_p \frac{1}{p^n(1+p)^n} = \sum_p \frac{1}{p^{2n}}\Big(1+\frac{1}{p}\Big)^{-n} = \sum_p \frac{1}{p^{2n}} \sum_{k=0}^{\infty} \binom{-n}{k} \frac{1}{p^k} =$$
$$= \sum_{k=0}^{\infty} \binom{-n}{k} \sum_p \frac{1}{p^{2n+k}} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} P(2n+k).$$

Proceeding similarly for $k > 3$ we find (using Mathematica)

$d_1 = 0.2007557220\ 1926598699\ 6250723114\ 4047658535\ 3555535256\ \dots$
$d_2 = 0.0221245744\ 7327116398\ 0012002355\ 9483175788\ 6781598850\ \dots$
$d_3 = 0.0010728279\ 2166161493\ 7597184179\ 0511299854\ 7080207983\ \dots$
$d_4 = 0.0000267593\ 5151889275\ 7741972284\ 4743787780\ 5157715943\ \dots$
$d_5 = 0.0000003834\ 9005273872\ 2348794555\ 0178910921\ 5013442743\ \dots$
$d_6 = 0.0000000034\ 4999551430\ 8580387444\ 6993630085\ 9120389312\ \dots$
$d_7 = 0.0000000000\ 2082589566\ 1766505646\ 3194316856\ 4945749335\ \dots$
$d_8 = 0.0000000000\ 0008875408\ 1001607125\ 3428410234\ 4454925913\ \dots$
$d_9 = 0.0000000000\ 0000027791\ 2994465580\ 9631134694\ 5089946028\ \dots$
$d_{10} = 0.0000000000\ 0000000066\ 0331441112\ 0947527899\ 3022631397\ \dots$

References

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