A NOTE ON THE REAL PART OF THE RIEMANN ZETA-FUNCTION

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Dedicated to Herman J. J. te Riele on the occasion of his retirement from the CWI in January 2012

Abstract. We consider the real part Re \( \zeta(s) \) of the Riemann zeta-function \( \zeta(s) \) in the half-plane Re \( (s) \geq 1 \). We show how to compute accurately the constant \( \sigma_0 \approx 1.19 \) which is defined to be the supremum of \( \sigma \) such that Re \( \zeta(\sigma + it) \) can be negative (or zero) for some real \( t \). We also consider intervals where Re \( \zeta(1 + it) \leq 0 \) and show that they are rare. The first occurs for \( t \approx 682112.9 \), and has length \( \approx 0.05 \). We list the first 50 such intervals.

1. Introduction

In this note we consider the real part Re \( \zeta(s) \) of the Riemann zeta-function \( \zeta(s) \) in the half-plane \( H = \{ s \in \mathbb{C} \mid \text{Re}(s) \geq 1 \} \). As usual, we write \( s = \sigma + it \), so Re \( (s) = \sigma \geq 1 \). We are mainly interested in the regions where Re \( \zeta(s) \leq 0 \). Since \( \lim_{\sigma \to \infty} \zeta(\sigma + it) = 1 \) (uniformly in \( t \)), Re \( \zeta(\sigma + it) \) cannot be zero for arbitrarily large \( \sigma > 1 \). We define

\[ \sigma_0 := \sup\{ \sigma \in \mathbb{R} \mid (\exists t \in \mathbb{R}) \text{Re} \zeta(\sigma + it) = 0 \}. \]

Thus, Re \( \zeta(s) > 0 \) if \( \sigma > \sigma_0 \). In van de Lune [9] it was shown that \( \sigma_0 \) is the (unique) positive real root of the equation

\[ \sum_p \arcsin \left( \frac{1}{p^\sigma} \right) = \frac{\pi}{2}, \]

where \( p \) runs through the primes (we adopt this convention throughout). In [9] it was also shown that \( \sigma_0 > 1.192 \) and that Re \( \zeta(\sigma_0 + it) \) never vanishes.

The main aim of this note is to show how \( \sigma_0 \) can be computed to arbitrarily high precision by an efficient algorithm. We also mention some results on the behaviour of Re \( \zeta(\sigma + it) \) for \( 1 \leq \sigma < \sigma_0 \), and in particular on the line \( \sigma = 1 \).

2. Accurate Computation of the Constant \( \sigma_0 \)

In this section we assume that \( \sigma \geq \sigma_1 > 1 \), where \( \sigma_1 \) is a suitable constant (e.g. 1.1). We show how the constant \( \sigma_0 \) can be computed within a given error bound. There are three main steps.

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(1) Give an algorithm to evaluate the prime zeta-function

\[ P(\sigma) = \sum_p p^{-\sigma}, \]

for real \( \sigma > 1 \).

(2) Using step 1, give an algorithm to evaluate the function \( f(\sigma) \)
defined by

\[ f(\sigma) = \sum_p \arcsin \left( \frac{1}{p^\sigma} \right) - \frac{\pi}{2}. \]

(3) Use a suitable zero-finding algorithm to locate a zero of \( f(\sigma) \)
in a (sufficiently small) interval where \( f(\sigma) \) changes sign, for
example \([1.1, 1.2]\).

Step 1 is easy. From the Euler product for \( \zeta(\sigma) \) and M"obius inversion,
we have a formula essentially known to Euler \([4, 1748]\):

\[ P(\sigma) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r} \log \zeta(r\sigma), \]

which is valid for \( \sigma > 1 \) (see Titchmarsh \([13, eqn. (1.6.1)]\)). The series
converges rapidly in view of the following Lemma.

Lemma 2.1. For \( \sigma \geq 2 \), \( 0 < \log \zeta(\sigma) < 3/2^\sigma \) and \( 0 < P(\sigma) < 3/2^\sigma \).

Proof. For \( \sigma \geq 2 \), we have

\[ 0 < \zeta(\sigma) - 1 < 2^{-\sigma} + 3^{-\sigma} + \int_{3}^{\infty} x^{-\sigma} dx = 2^{-\sigma} + 3^{-\sigma} + \frac{3^{1-\sigma}}{\sigma-1} < 3/2^\sigma, \]

so

\[ 0 < \log \zeta(\sigma) < \zeta(\sigma) - 1 < 3/2^\sigma. \]

The upper bound on \( P(\sigma) \) follows similarly, using \( P(\sigma) < \zeta(\sigma) - 1 \). \(\square\)

Using (1) and Lemma 2.1, we have

\[ P(\sigma) = \log \zeta(\sigma) + \sum_{r=2}^{\infty} \frac{\mu(r)}{r} \log \zeta(r\sigma), \]

where the \( r \)-th term in the sum is bounded in absolute value by \( 3/2^{\sigma+1} \).
Thus, we can evaluate \( P(\sigma) \) accurately, for given \( \sigma > 1 \), using any
good algorithm for the evaluation of \( \zeta(\sigma) \), for example Euler-Maclaurin
summation. If (1) is used to compute \( P(\sigma) \), \( P(3\sigma) \), \( P(5\sigma) \), \ldots, then
we should take care to compute the relevant terms \( \log \zeta(r\sigma) \) only once.

For step 2, we observe that the arcsin series defining \( f(\sigma) \) converges
slowly and irregularly, since it is a sum over primes which to first order
behaves like \( \sum_p p^{-\sigma} \). The well-known “trick” is to express \( f(\sigma) \)
as a double series and reverse the order of summation, obtaining an expression which is mathematically equivalent but computationally far superior. For some similar examples, see Wrench [15, 1961].

For $|x| < 1$ we have

$$\arcsin(x) = \sum_{k=0}^{\infty} c_k x^{2k+1},$$

where

$$c_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \frac{1}{2k + 1} = \frac{(2k)!}{(2k)! (2k+1)!} \quad \text{for } k \geq 0.$$  

Note that all $c_k$ are positive so that $f(\sigma)$ is strictly convex. It is also clear that $f(\sigma)$ is strictly decreasing for $\sigma > 1$. From the expression for $c_k$, we see that, for $k \geq 1$,

$$(2) \quad c_k \leq \frac{1}{2(2k + 1)}.$$  

For $\sigma > 1$ it is easy to justify interchanging the order of summation in

$$f(\sigma) = \sum_p \sum_{k=0}^{\infty} c_k \left( \frac{1}{p^\sigma} \right)^{2k+1} = \frac{\pi}{2},$$

obtaining

$$(3) \quad f(\sigma) = \sum_{k=0}^{\infty} c_k \sum_p \frac{1}{p^{(2k+1)\sigma}} - \frac{\pi}{2} = \sum_{k=0}^{\infty} c_k P\left( (2k+1)\sigma \right) - \frac{\pi}{2}.$$  

From Lemma 2.1 and the inequality (2), we see that

$$0 < \sum_{k=K+1}^{\infty} c_k P\left( (2k+1)\sigma \right) < 2^{-(2K+3)\sigma},$$

so it is easy to determine $K$ such that we can truncate the series in (3) to a finite sum over $k \leq K$ with a rigorous error bound.

If desired, we can substitute (1) into (3) and interchange the order of summation, obtaining

$$(4) \quad f(\sigma) = \sum_{j \geq 0} d_j \log \zeta((2j+1)\sigma) - \frac{\pi}{2},$$

where

$$d_j = \sum_{k \geq 0, r \geq 1, (2k+1)r = 2j+1} \frac{c_k \mu(r)}{r}.$$
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From the inequality $c_k \leq 1/(2k + 1)$ (valid for $k \geq 0$), it follows that $|d_j| \leq 1$. Using Lemma 2.1, we can determine where to safely truncate the series (4).

For step 3, we can use a zero-finding algorithm which needs only function (not derivative) evaluations, and gives a guaranteed bound on the final result. For example, the method of bisection could be used, but would be slow, taking about $\log_2(1/\varepsilon)$ function evaluations to obtain a solution with error bounded by $\varepsilon$. In the secant method, a sequence $(x_n)$, converging to a zero of $f$ under suitable conditions, is obtained by computing the approximation $x_n$ by linear interpolation using the two points $(x_{n-1}, f(x_{n-1}))$ and $(x_{n-2}, f(x_{n-2}))$. It converges with order $(1 + \sqrt{5})/2 \approx 1.618$, but does not always give a guaranteed bound on the error. A combination of bisection and linear interpolation, as in the algorithms of Dekker [3] or Brent [2], can give convergence about as fast as the secant method, but with the final result bracketed in a short interval where the function $f$ changes sign.

3. Computational results

The second and third authors independently wrote programs implementing the ideas of §2, using Magma in one case and Mathematica 4 and 8 in the other case. The programs used different strategies to obtain a final interval where $f$ changes sign (in one case taking advantage of the strict convexity of $f$). The output of the programs agrees to at least 500D. We give here the correctly rounded result to 100D:

$$\sigma_0 \approx 1.1923473371861932028975044274255978834011192308379994301371949299052458648483013924084998638378836244.$$ Programs and higher precision values are available from the authors.

4. The distribution of Re $\zeta(\sigma + it)$ for $\sigma \geq 1$

Assuming that the limit exists, we define

$$d(\sigma) = \lim_{T \to +\infty} \frac{1}{T} m\{t \in [0, T] \mid \text{Re} \zeta(\sigma + it) < 0\},$$

where $m$ denotes Lebesgue measure. Informally, $d(\sigma)$ is the probability that $\zeta(s)$ has negative real part on a given vertical line $\text{Re} \ (s) = \sigma$.

The results of Section 2 show that $d(\sigma) = 0$ for $\sigma \geq \sigma_0 \approx 1.19$. Here we briefly discuss the region $1 \leq \sigma < \sigma_0$.

At least for those values of $t$ that are accessible to computation, Re $\zeta(\sigma + it)$ is “usually” positive for $\sigma \geq 1$. The function $d(\sigma)$ is conjectured to be continuous and monotonic decreasing from a positive
value at $\sigma = 1$ to zero at $\sigma = \sigma_0$. Even on the line $\sigma = 1$, $\text{Re}\,\zeta(\sigma + it)$ is usually positive [11]. We can prove that $d(1) < 1/4$, but a Monte Carlo computation suggests that the true value is much smaller. Based on $5 \times 10^{11}$ pseudo-random trials, we estimate $d(1) = (3.80 \pm 0.01) \times 10^{-7}$. Similarly, we estimate $d(1.01) = (1.10 \pm 0.01) \times 10^{-7}$ and $d(1.02) \approx (2.66 \pm 0.04) \times 10^{-8}$, so it can be seen that $d(\sigma)$ decreases rapidly as we move to the right of $\sigma = 1$.

Although $\zeta(s)$ has a simple pole at $s = 1$, the Laurent series

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$$

shows that $\text{Re}\,\zeta(1 + it)$ has a positive limit $\gamma = 0.577\ldots$ (Euler’s constant) as $t \to 0$.

On any fixed vertical line $\sigma > 1$, both $\zeta(\sigma + it)$ and $1/\zeta(\sigma + it)$ are bounded, in fact $\zeta(2\sigma)/\zeta(\sigma) < |\zeta(\sigma + it)| \leq \zeta(\sigma)$. However, the situation is different on the line $\sigma = 1$, as both $\zeta(1 + it)$ and $1/\zeta(1 + it)$ are unbounded. Their true order of growth is unknown. It follows from Titchmarsh [13, Theorem 11.9] and the continuity of $\text{Re}\,\zeta(1 + it)$ that $\text{Re}\,\zeta(1 + it)$ attains all real values. Nevertheless, the “usual” values are quite small. As a special case of [13, Theorem 7.2] we have the mean value theorem

$$\lim_{T \to \infty} \frac{1}{T} \int_{1}^{T} |\zeta(1 + it)|^2 \, dt = \zeta(2) = \frac{\pi^2}{6}.$$

Using ideas as in the proof of [13, Theorem 7.2], we can prove that

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \text{Re}\,\zeta(1 + it) \, dt = 1.$$

Thus, informally, we can say that the typical value of $\text{Re}\,\zeta(1 + it)$ is close to 1. The values have a distribution with mean 1 and variance $\pi^2/6 - 1 \approx 0.645$.

In [9, Table 1], van de Lune gave a list of values of $t > 0$ such that $\text{Re}\,\zeta(1 + it) < 0$ and is (approximately) a local minimum. The list was not claimed to be exhaustive. The smallest $t$ listed was $t = 682112.92$ with $\text{Re}\,\zeta(1 + it) \approx -0.003$. We have shown, using the “maximum slope principle” [10], that this is very close to the smallest $t$ for which $\text{Re}\,\zeta(1 + it) \leq 0$. More precisely, $\text{Re}\,\zeta(1 + it) > 0$ for $0 < t < 682112.8913$, and there is a local minimum of $-0.0027652$ at $t \approx 682112.9169$. In applying the maximum slope principle we used the bound

$$\left|\frac{d}{dt} \text{arg}\,\zeta(1 + it)\right| = \left|\frac{\text{Re}\,\zeta'(1 + it)}{\zeta(1 + it)}\right| \leq \frac{3}{4} \log(t^2 + 4) + 7 \text{ for } t \geq 10.$$
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Table 1. First 50 negative local minima of Re$\zeta(1 + it)$

<table>
<thead>
<tr>
<th>$t$</th>
<th>Re$\zeta$</th>
<th>length</th>
<th>$t$</th>
<th>Re$\zeta$</th>
<th>length</th>
</tr>
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<td>0.0529</td>
<td>8350473.4853</td>
<td>−0.0019</td>
<td>0.0451</td>
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<td>0.0655</td>
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<tr>
<td>1466782.0667</td>
<td>−0.0013</td>
<td>0.0391</td>
<td>8452317.9526</td>
<td>−0.0090</td>
<td>0.0900</td>
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<tr>
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<tr>
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<td>−0.0189</td>
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<td>12276788.1573</td>
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<tr>
<td>4052438.9330</td>
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<td>4197235.0783</td>
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</tbody>
</table>

Table 1 lists the first 50 local minima of Re$\zeta(1 + it)$ for which $t > 0$ and Re$\zeta(1 + it) \leq 0$ (no minima are exactly zero). The values in the table are rounded to 4 decimal places. The columns headed “length” give the lengths of the intervals containing $t$ in which Re$\zeta$ is negative. To 8 decimal places, the first interval, of length 0.05291225, is (682112.89133824, 682112.94425049). The sum of the lengths of the first 50 intervals is 6.48390168, giving an estimate $d(1) \approx 3.85 \times 10^{-7}$. This is close to our Monte Carlo estimate $d(1) \approx 3.80 \times 10^{-7}$.

In this brief note we refrain from commenting on the region $\sigma \in [1/2, 1)$, but refer the interested reader to the literature, such as Bohr and Jessen [1], Titchmarsh [13, §11.13], Tsang [14], Joyner [6], Laurinčikas [8], Steuding [12] and Kühn [7].
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References

[1] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zeta-
            Constructive Aspects of the Fundamental Theorem of Algebra (B. Dejon and
            French translation 1796).
            tionsbehandling (BIT) 8, 3 (1968), 187–202.
            Mathematics, ETH Zürich, Switzerland, March 2011.
[8] A. Laurinčikas, Limit theorems for the Riemann zeta-function, Mathematics
[9] J. van de Lune, Some observations concerning the zero-curves of the real and
            imaginary parts of Riemann’s zeta function. Afdeling Zuivere Wiskunde [De-
            partment of Pure Mathematics], Report ZW 201/83. Mathematisch Centrum,
            zeros of partial sums of Riemann’s zeta function, Computational Methods in
[11] D. C. Milioto, A method for zeroing-in on $\text{Re} \zeta(\sigma + it) < 0$ in
[14] K. Tsang, The Distribution of the Values of the Riemann Zeta-function, PhD

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