A NOTE ON A RELATION BETWEEN THE RIEMANN HYPOTHESIS AND THE SUMMATORY FUNCTION OF $(|\mu(n)| - 6/\pi^2)$

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Dedicated to Herman J. J. te Riele on the occasion of his retirement from the CWI in January 2012

1. Introduction

The numerical computations reported on in this note were motivated by a combination of the following considerations (A) and (B):

(A) The absolute values of the Möbius function, $|\mu(n)|$, can be bulk-computed very quickly by means of a sieve (one of the fastest we know of).

(B) **Theorem.** *If* $\Delta(x) := \sum_{n=1}^{\lfloor x \rfloor} \left(|\mu(n)| - \frac{6}{\pi^2}\right) = \mathcal{O}(x^{\frac{1}{4}+\varepsilon})$ *for every* $\varepsilon > 0$, *then the Riemann Hypothesis is true.*

Before proving (B), we make a few historical remarks.

Writing $M(x) := \sum_{n=1}^{\lfloor x \rfloor} \mu(n)$, it can be shown [1] that the Riemann Hypothesis is equivalent to $M(x) = \mathcal{O}(x^{\frac{1}{2}+\varepsilon})$. In 1911 Axer [2] showed that assuming the slightly stronger Stieltjes Hypothesis, namely that $M(x) = \mathcal{O}(x^{\frac{1}{2}})$, it follows that $\Delta(x) = \mathcal{O}(x^{\frac{2}{5}})$. Although the Stieltjes Hypothesis has to date not been disproved, there are some indications that it might be false. In particular, it was proved that the (slightly stronger still) Mertens Hypothesis, $|M(x)| \leq x^{\frac{1}{2}}$, is false, as in 1985 Odlyzko and te Riele [3] showed that $\limsup M(x)/x^{\frac{1}{2}} > 1.06$, $\liminf M(x)/x^{\frac{1}{2}} < -1.009$, and in 2006 Kotnik and te Riele [4] improved this to $\limsup M(x)/x^{\frac{1}{2}} > 1.218$, $\liminf M(x)/x^{\frac{1}{2}} < -1.229$. In the light of these developments, the condition assumed by Axer in deriving his result may be questioned. This potential problem was overcome in 1980 by Montgomery and Vaughan [5], who developed a method allowing to show rather straightforwardly that the Riemann Hypothesis itself implies the stronger result $\Delta(x) = \mathcal{O}(x^{\frac{1}{3}+\varepsilon})$, and with a more involved argument they were able to strengthen the exponent $\frac{1}{3}$ further to $\frac{9}{28} = 0.321428...$ Their result was published in 1981, and building upon their work, the exponent $\frac{9}{28}$ was improved in the same year by Graham [6] to $\frac{8}{25} = 0.32$, then in 1985 by Baker and Pintz [7] to $\frac{7}{22} = 0.318181...$, and in 1993 by Jia [8] to $\frac{17}{54} = 0.314814...$

It is interesting to note that in none of these works, the authors conjecture the actual order of magnitude of $\Delta(x)$, in the sense of the smallest value of the exponent ν for which $\Delta(x) = \mathcal{O}(x^{\nu+\varepsilon})$ is true. Moreover, we were also unable to find any work on the converse implication, i.e. on a value of ν that would imply the Riemann Hypothesis, as is the case of our theorem (B), of which we now give a proof.

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Theorem. *If* $\Delta(x) = \mathcal{O}(x^{\frac{1}{4}+\varepsilon})$, *then the Riemann Hypothesis is true.*

Proof. Let $s := \sigma + it$ be a complex variable. For $\sigma > 1$ we have

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \prod_p \left(1 + \frac{1}{p^s}\right) = \prod_p \frac{\left(1+\frac{1}{p^s}\right)\left(1-\frac{1}{p^s}\right)}{1-\frac{1}{p^s}} = \prod_p \frac{1-\frac{1}{p^{2s}}}{1-\frac{1}{p^s}} = \frac{\zeta(s)}{\zeta(2s)}$$

so that

$$\sum_{n=1}^{\infty} \frac{|\mu(n)| - \frac{6}{\pi^2}}{n^s} = \frac{\zeta(s)}{\zeta(2s)} - \frac{6}{\pi^2}\zeta(s) = \zeta(s)\left(\frac{1}{\zeta(2s)} - \frac{6}{\pi^2}\right) =: \psi(s) \ .$$

At $s = 1$, $\psi(s)$ is analytic, as the simple pole of $\zeta(s)$ is cancelled by the simple zero of $\frac{1}{\zeta(2s)} - \frac{6}{\pi^2}$. Since $\zeta(2s) \neq 0$ for $\sigma > \frac{1}{2}$, it follows that $\psi(s)$ is analytic for $\sigma > \frac{1}{2}$.

Now suppose that for every $\varepsilon > 0$ we have

$$\Delta(x) := \sum_{n=1}^{\lfloor x \rfloor} \left(|\mu(n)| - \frac{6}{\pi^2}\right) = \mathcal{O}(x^{\frac{1}{4}+\varepsilon}) \quad \text{as } x \to \infty \ .$$

Then for $\sigma > 1$

$$\begin{aligned}
\psi(s) &= \sum_{n=1}^{\infty} \frac{|\mu(n)| - \frac{6}{\pi^2}}{n^s} = \sum_{n=1}^{\infty} \int_{n-0}^{n+1-0} \frac{1}{x^s}\, d\Delta(x) \\
&= \int_{1-0}^{\infty} \frac{1}{x^s}\, d\Delta(x) = \frac{\Delta(x)}{x^s}\bigg|_{1-0}^{\infty} - \int_{1-0}^{\infty} \Delta(x)\, dx^{-s} \\
&= 0 - 0 - \int_{1-0}^{\infty} \Delta(x)\,(-s)\,x^{-s-1} dx = s \int_1^{\infty} \frac{\Delta(x)}{x^{s+1}}\, dx \ .
\end{aligned}$$

Since by assumption $\Delta(x) = \mathcal{O}(x^{\frac{1}{4}+\varepsilon})$, we find that $\psi(s) = s\int_1^{\infty} \frac{\Delta(x)}{x^{s+1}}\, dx$ is analytic for $\sigma > \frac{1}{4} + \varepsilon$, and as ε may be taken arbitrarily small, it follows that $\psi(s)$ is analytic for $\sigma > \frac{1}{4}$.

Now we proceed by contradiction: Suppose that the Riemann Hypothesis is not true, and let $\rho = \frac{1}{2} + \alpha + i\beta$ be the zero of $\zeta(s)$ with the smallest $\beta > 0$ and the largest α with $0 < \alpha < \frac{1}{2}$. (I.e., first take the smallest $\beta > 0$ and then the largest corresponding α.)

Now consider the point $s = \frac{\rho}{2} = \frac{1}{4} + \frac{\alpha}{2} + i\frac{\beta}{2}$. Then $\zeta(2s) = \zeta(\rho) = 0$, so that $\frac{1}{\zeta(2s)}$ has a pole at $s = \frac{\rho}{2}$. Clearly, $\frac{1}{\zeta(2s)} - \frac{6}{\pi^2}$ then also has a pole there. Since $\frac{\rho}{2} = \frac{1}{4} + \frac{\alpha}{2} + i\frac{\beta}{2}$ and $\psi(s) = \zeta(s)(\frac{1}{\zeta(2s)} - \frac{6}{\pi^2})$ is regular for $\sigma > \frac{1}{4}$, it follows that $\zeta(s)$ must have a zero of at least the same order at $s = \frac{1}{4} + \frac{\alpha}{2} + i\frac{\beta}{2}$. But then $\zeta(s)$ also has a zero at $s = (1 - (\frac{1}{4} + \frac{\alpha}{2})) + i\frac{\beta}{2} = \frac{3}{4} - \frac{\alpha}{2} + i\frac{\beta}{2}$. Since $\frac{1}{2} < \frac{3}{4} - \frac{\alpha}{2} < 1$ and $0 < \frac{\beta}{2} < \beta$, this contradicts our assumption about the minimality of β. □

2. The computational results

For $x \geq 1$ we define

$$q(x) := \frac{\Delta(x)}{x^{\frac{1}{4}}} \ .$$

Thus, the sufficient condition for the validity of the Riemann Hypothesis can also be written as $q(x) = \mathcal{O}(x^{\varepsilon})$, and to investigate this condition numerically, we need to perform a systematic search for large extrema of $|q(x)|$. Since all positive maxima and negative minima of $q(x)$ occur at $x \in \mathbb{N}$, for our purpose we may restrict our computations of $q(x)$ to integer values of x.

The values of $q(n)$ for all $n \le N^2$ may be computed by a sieve-program, the functioning of which can be briefly outlined as follows:

(1) Precompute (by a sieve) all primes $p \le N$ and store their squares p^2.
[Sieving over the odd numbers we can reach $n = (2N+1)^2$.]

(2) Set $\texttt{nDone} = 0$ and declare a sieve block $\texttt{SB}$ as an array of length $\texttt{LSB} = N$ of integers.
[Or, equivalently, of Booleans.]

(3) Set $\texttt{MaxQ} = \texttt{MinQ} = 0$ and $\texttt{SB} = \{1, 1, 1, \ldots, 1\}$.
[Or, equivalently, $\texttt{SB} = \{\texttt{True}, \texttt{True}, \ldots, \texttt{True}\}$.]

(4) For $p^2 \mid n$ set $\texttt{SB}[n] = 0$.
[Or, equivalently, $\texttt{SB}[n] = \texttt{False}$. SB now contains the values of $|\mu(n)|$. Note that a complete factorization of n is not necessary.]

(5) For $\texttt{nDone} < n \le \texttt{nDone} + \texttt{LSB}$, compute $q(n)$.
If $q(n) > \texttt{MaxQ}$ then replace MaxQ by $q(n)$,
else if $q(n) < \texttt{MinQ}$ then replace MinQ by $q(n)$.

(6) Output the values MaxQ, MinQ, their pertaining values of n, and possibly some other relevant results.
[The maximal and the minimal value of $q(n)$ in the current sieve block.]

(7) If $n < N^2$ then increment nDone by LSB and go to (3).
[Note that we use the same sieve block SB repeatedly.]

We wrote such a program using Delphi 6 (Borland Inc., Scotts Valley, CA, USA) and ran it on a PC with 2 GB of RAM. Tables 1 and 2 below present the maximal and the minimal value of $q(x)$ in 15 contiguous intervals spanning the range $1 \le x \le 5 \times 10^{14}$ (with $\texttt{LSB} = 53361000 = 2^3 \times 3^2 \times 5^3 \times 7^2 \times 11^2$).

x-range	x	$\sum_{n=1}^{\lfloor x \rfloor} \lvert\mu(n)\rvert$	$\Delta(x)$	$q(x)$
[1, 10)	7	6	+1.744...	+1.0725...
[10, 100)	43	29	+2.859...	+1.1165...
[100, 1000)	115	73	+3.088...	+0.9430...
$[10^3, 10^4)$	1663	1017	+6.017...	+0.9422...
$[10^4, 10^5)$	47523	28905	+14.480...	+0.9807...
$[10^5, 10^6)$	351115	213474	+21.675...	+0.8904...
$[10^6, 10^7)$	2015403	1225252	+33.895...	+0.8995...
$[10^7$, LSB)	10143015	6166263	+49.286...	+0.8733...
[LSB, $10 \times$ LSB)	413384223	251307591	+118.359...	+0.8300...
[$10 \times$ LSB, $10^2 \times$ LSB)	4804033147	2920502173	+224.733...	+0.8536...
[$10^2 \times$ LSB, $10^3 \times$ LSB)	29109682663	17696565352	+334.792...	+0.8105...
[$10^3 \times$ LSB, $10^4 \times$ LSB)	183141684519	111336794166	+667.699...	+1.0206...
[$10^4 \times$ LSB, $10^5 \times$ LSB)	987483328243	600317878538	+670.064...	+0.6721...
[$10^5 \times$ LSB, $10^6 \times$ LSB)	12693019531903	7716430579185	+1378.655...	+0.7304...
[$10^6 \times$ LSB, $10^7 \times$ LSB)	214455677199819	130373418319553	+3324.378...	+0.8687...

Table 1: The maximal values of $q(x)$ in 15 intervals spanning the investigated x-range.

x-range	x	$\sum_{n=1}^{\lfloor x \rfloor} \lvert\mu(n)\rvert$	$\Delta(x)$	$q(x)$
[1, 10)	10 – 0	6	+0.528...	+0.2972...
[10, 100)	56	34	–0.043...	–0.0160...
[100, 1000)	380	229	–2.012...	–0.4557...
$[10^3, 10^4)$	1864	1130	–3.176...	–0.4833...
$[10^4, 10^5)$	80156	48715	–14.004...	–0.8323...
$[10^5, 10^6)$	436484	265330	–20.453...	–0.7957...
$[10^6, 10^7)$	1146476	696952	–21.832...	–0.6671...
$[10^7$, LSB)	17199380	10455914	–55.237...	–0.8577...
[LSB, 10 × LSB)	487335681	296264461	–107.180...	–0.7213...
[10 × LSB, 10^2 × LSB)	3620494684	2200996604	–236.522...	–0.9642...
[10^2 × LSB, 10^3 × LSB)	20219949552	12292254976	–354.781...	–0.9408...
[10^3 × LSB, 10^4 × LSB)	379688379896	230822855814	–583.825...	–0.7437...
[10^4 × LSB, 10^5 × LSB)	744078020392	452345193748	–742.189...	–0.7991...
[10^5 × LSB, 10^6 × LSB)	11590475428980	7046164135224	–1426.117...	–0.7729...
[10^6 × LSB, 10^7 × LSB)	154953313738408	94200318939698	–3970.103...	–1.1252...

Table 2: The minimal values of $q(x)$ in 15 intervals spanning the investigated x-range.

The presented computational data of course cannot rigorously resolve the question whether $\Delta(x) = \mathcal{O}(x^{\frac{1}{4}+\varepsilon})$, but they do not seem to contradict this estimate. In particular, the data shown in Fig. 1 could be interpreted as suggesting that perhaps $\Delta(x) = \mathcal{O}(x^{\frac{1}{4}} \log x)$, and even $|\Delta(x)| \leq x^{\frac{1}{4}} \log x$ for all $x > 2$.

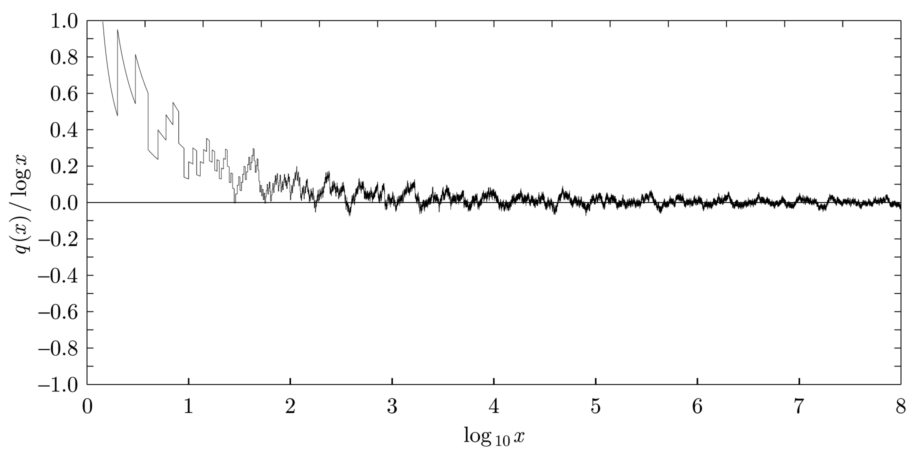

Figure 1: A plot of the function $q(x)/\log x$ in the x-range $[1, 10^8]$.

References

[1] E.C. Titchmarsh and D.R. Heath-Brown. *The Theory of the Riemann Zeta-function, 2nd edition.* Oxford University Press (1986) 370.

[2] A. Axer. Über einige Grenzwertesätze. *Sitzungsber. Kais. Akad. Wiss. Wien* **120/2a** (1911) 1253–1298.

[3] A.M. Odlyzko and H.J.J. te Riele. Disproof of the Mertens conjecture. *J. reine angew. Math.* **357** (1985) 138–160.

[4] T. Kotnik and H.J.J. te Riele. The Mertens conjecture revisited. *Lect. Notes Comp. Sci.* **4076** (2006) 156–167.

[5] H.L. Montgomery and R.C. Vaughan. On the distribution of square-free numbers. *Recent Progress in Analytic Number Theory*. Academic Press (1981) 247–256.
[6] S.W. Graham. The distribution of squarefree numbers. *J. London Math. Soc.* **24** (1981) 54–64.
[7] R.C. Baker and J. Pintz. The distribution of square-free numbers. *Acta Arith.* **46** (1985) 73–79.
[8] C.H. Jia. The distribution of square-free numbers. *Sci. China Ser. A* **36** (1993) 154–169.

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