Gauß’s Lattice Point Problem(s) Revisited
An invitation

Dedicated to Herman J. J. te Riele on the occasion of his retirement from the CWI in January 2012

Walter M. Lioen¹ and Jan van de Lune²

1 Introduction to the original Problem

In analytic number theory there is an abundance of unsolved problems. One of them is Gauß’s Lattice Point Problem for the circle (see Gauß [6, pp. 269–291 (in particular p. 280)] or [7, p. 657]). Hejhal once wrote that this problem might very well be more difficult than the Riemann Hypothesis (cf. [10]).

Let’s recall what this problem is all about: For real $t \geq 0$ let $P(t)$ denote the number of lattice Points $(x,y)$ in the circular disc $x^2 + y^2 \leq t$ (note that the radius of this disc is $\sqrt{t}$), and let $A(t)$ be the Area ($= \pi t$) of the disc. The problem is to estimate the ‘Error’ $E(t) := P(t) - A(t)$ as $t \to \infty$.

In [20, 19, 5] we find some history of this subject. The ultimate goal is to determine the infimum ($\theta$) of all $\alpha$ satisfying $E(t) = O(t^\alpha)$. Till today this is still an unsolved problem. It is clear that all (local) extremes of $E(t)$ occur at the points $t = n(\pm 0)$, so that we may restrict ourselves to the determination of $P(n)$ with $n \in \mathbb{N}$. In the past various (numerical) attempts have been made to get an impression of what $\theta$ might be. See [3, 12, 15, 17].

Various methods have been applied: Gauß’s original root method[6], Tromp’s step method[17]. So far Tromp’s method has by far been superior in speed.

Writing $|E(t)| \leq C \epsilon^{\theta^2 \epsilon^2}$, the best bounds on $\theta$ are $\frac{1}{2} \leq \theta \leq \frac{13}{36} \approx 0.361111$ (cf. [11]). It was Van der Corput[2] who was the first to prove that $\theta < \frac{1}{3}$.

Experimental results suggest that $|E(t)| = o(t^{1/4} \log t)$ (as conjectured in [15] and confirmed in [17]).

¹Walter.Lioen@sara.nl, SARA, P.O. Box 94613, 1090 GP Amsterdam, The Netherlands.
²j.vandelune@hccnet.nl, Langebuorren 49, 9074 CH Hallum, The Netherlands (formerly at CWI, Amsterdam).
We propose to introduce a new method for the computation of $P(n)$, to wit: a fast sieve method based on [8, pp. 241–243, Section 16.9, formula (16.9.5), Theorem 278], which we already announced in [14, Section 7].

We have already written the main features of a program in Delphi Object Pascal and a Fortran version is in progress.

2 Generalization

Now we change notation: $P_2(t)$ will now denote the $P(t)$ of Section 1. Similarly $A_2(t) = A(t)$, $E_2(t) = E(t)$ and $\theta_2 = \theta$. The index 2 refers to the dimension (of the plane). We now define (in 3 dimensional space) $P_3(t)$ as the number of lattice points $(x, y, z)$ satisfying $x^2 + y^2 + z^2 \leq t$, $V_3(t) :=$ the Volume of the pertinent sphere $= \frac{4}{3} \pi t^3$, and $E_3(t) := P_3(t) - V_3(t)$. Of course, also here the problem is to estimate the size of $E_3(t)$ as $t \to \infty$. Writing $|E_3(t)| \leq C_3 t^{\theta_3 + \epsilon}$ the moment of writing the best bound on $\theta_3$ is $\theta_3 \leq \frac{17}{28} \approx 0.607143$ (cf. [1]). Previous theoretical results can be found in [9, 18]. For earlier numerical work see [3, 16].

Since $P_2(t)$ can now be computed very fast, it seems worthwhile to have a go at $E_3(t)$. Summation over horizontal slices of the sphere yields $P_3(n) = P_2(n) + 2 \sum_{k=1}^{\sqrt{n}} P_2(n-k^2)$. However, here is a nasty catch: the values of $P_2(n)$ must be saved, which is rather demanding on fast memory. A simple back of the envelope calculation suggests that at least memory-wise $n \leq 10^{10}$ is feasible using readily available 128–256 GB machines.

3 Invitation

We would like to invite the golden-ager (hopefully with a lot of time) to join the crowd in an attempt to extend the computations on $E_3(t)$.

References and related literature


