SEQUENCING AND COOPERATION

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In machine scheduling the first problem is to find a timetable that is optimal with respect to some efficiency criterion. If the jobs come from different clients the solution of the optimization problem is not the end of the story. In addition, we have to decide how the minimal total cost must be distributed among the parties involved. In this note, cost allocation problems will be considered to arise from one-machine scheduling problems with an additive and weakly increasing cost function. We will show that the cooperative games related to these cost allocation problems have a nonempty core. Furthermore, we give a rule that assigns a core element of the associated cost saving game to each scheduling problem of this kind and an initial order of the jobs.

Traditionally, a scheduling problem is the task to find a timetable that is optimal with respect to some efficiency criterion. An efficiency criterion defines an order on the set of feasible timetables and the problem is to find an optimal one. The efficiency criterion is usually given by a function of the job completion times and the timetable is better if the function value is lower.

In practice, the jobs to be processed often come from different clients, and they are only interested in an efficient processing of their own jobs. Hence, there is a controversy between social benefit and individual benefits. In this note, we deal with the problem of reconciling these two visions.

More concretely, we are going to investigate the following:

a. there are n jobs to be processed on *one* machine; each job can start at time $t = 0$;
b. different jobs come from different clients;
c. each client is using an efficiency criterion represented by a weakly monotonic function of the completion time of *his* job;
d. the total efficiency function is the sum of the efficiency functions of the clients;
e. there is an initial order $\sigma_0 : N \rightarrow \{1, \ldots, n\}$ and client i can derive from this order the *right* to be processed in the $\sigma_0(i)$th time slot, i.e., in the time period $[t_i, t_i + p_i]$ with $t_i = \sum_{1 \leq \sigma_0(j) < \sigma_0(i)} p_j$.

An efficiency function that satisfies properties c and d is called an *additive* and *regular* efficiency function.

The problem is the distribution of the cost savings that can be made by changing the initial order into another order. One could avoid the problem and propose an equal split of the benefits. We will, however, follow a more sophisticated approach, taking into account the "virtual cost savings that coalitions of clients could have made."

So, we have a set N of n clients each having one job to be processed on one and the same machine. The type of client i is determined by the weakly monotonic

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efficiency function $f_i:\mathbf{R}_+ \rightarrow \mathbf{R}$ and the processing time $p_i > 0$ of the job. There is an initial order $\sigma_0: N \rightarrow \{1, \ldots, n\}$ which gives each client a ranking number. Any order $\sigma:N \rightarrow \{1, \ldots, n\}$ determines a, what is called *semi-active timetable* $\tau = (t_1, \ldots, t_n)$ wherein $t_i := \sum_{\sigma(j)<\sigma(i)} p_j$ is the time that job i can start. Then, assuming no pre-emption or interruption, the completion time of job i is $C_i = t_i + p_i$. As the efficiency functions f_i are weakly monotonic it makes no sense to look at other (not semi-active) timetables.

If we consider a coalition of clients $S \subset N$, this group of clients can also generate cost savings by changing their processing order. In this paper we assume:

f. coalitions S are only allowed to change position within groups that are connected in S with respect to the initial order σ_0; if there are, for example, five clients and the initial order is $1 < 2 < \cdots < 5$, coalition (1, 2, 4, 5) can only switch the position of jobs 1 and 2 or jobs 4 and 5 (or both), but a switch $2 \leftrightarrow 4$ is not allowed.

Now we can define $v(S)$ as the maximal cost savings that coalition S can produce by changing positions within σ_0-connected groups (= σ_0-components). Then we have a *cooperative game* $v:2^N \rightarrow \mathbf{R}$ with $S \rightarrow v(S)$. Notice that the game v depends on the initial order σ_0, that $v(N)$ is the total benefit to be distributed among the clients and that one-person coalitions cannot generate any cost savings, i.e., $v(i) = 0$ for all $i \in N$.

A natural question is whether the total cost savings $v(N)$ can be distributed among the clients in such a way that each coalition at least obtains the benefit they can generate by themselves. A distribution is lacking in stability if this is not the case. A coalition S that obtains less than the profit they can produce by themselves may be tempted to split off and follow their own way of action. Therefore, the question is: Is there a vector $x \in \mathbf{R}^N$ such that

$$\sum_{i\in N} x_i = v(N) \quad \text{and} \quad \sum_{i\in S} x_i \geq v(S) \quad \text{for all } S \subset N?$$

In the theory of cooperative games such a distribution x is called a *core allocation* of the cooperative game $v:2^N \rightarrow \mathbf{R}$.

In this note we show that the cost saving games generated by scheduling problems of the type we described before have core allocations, or more precisely, we will give a simple rule which assigns a core allocation of the associated game to each scheduling problem of this type. As turns out, it will only be necessary to compute the values of $2n - 1$ coalitions to find this core allocation.

In the literature there are two papers dealing with the issues we are talking about. In Tijs et al. (1984) the cost for job i to be processed on the jth place is given and coalitions are allowed to make any switch of position. The cost saving games arising from such *permutation situations* have been proved to have core allocations. In Curiel, Pederzoli and Tijs (1989) the efficiency functions are linear in the completion time and a rule is given (the equal gain splitting rule) which assigns to each *sequencing situation* a core allocation of the associated cost saving game. The allocation rule we give in this paper is an extension of the EGS rule to more general situations. The *sequencing games* of Curiel, Pederzoli and Tijs are convex games, i.e., $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all coalitions $S, T \subset N$. The existence of core allocations is implied by this fact. The games we are going to consider are, in general, not convex.

Let us complete this Introduction with an example.

Example. Let N consist of three clients with the efficiency functions:

$f_1(C) := 0.5C$, $f_2(C) := 4$ if $C > 3$,

$f_2(C) := 0$ if $C \leq 3$,

$f_3(C) := \max(0, C - 4)$.

So client 1 is paying half of his completion time, client 2 is paying a penalty of 4 if his job is completed after $t = 3$, and client 3 has due date 4 and is paying his tardiness. Let the processing times be $(p_1, p_2, p_3) = (2, 2, 2)$ and $\sigma_0: 1 < 2 < 3$. The cost saving game has the values $v(123) = 4$, $v(12) = 3$, and $v(23) = 2$. This game is not convex as $v(12) + v(23) > v(123) + v(2)$. The β-rule (as we will define below) gives the core allocation (1, 2.5, 0.5).

1. THE β RULE

If (N, v) is a cooperative game and $\sigma_0: N \rightarrow \{1, \ldots, n\}$ is an ordering of the players, the β rule is defined by the formula

$$\beta_i(v) := \tfrac{1}{2}[v(\bar{P}r(i, \sigma_0)) - v(Pr(i, \sigma_0)) + v(\bar{F}(i, \sigma_0)) - v(F(i, \sigma_0))],$$

where $\bar{P}r(i, \sigma_0)$, $Pr(i, \sigma_0)$, $\bar{F}(i, \sigma_0)$, and $F(i, \sigma_0)$ are the coalitions $\{j \in N|\sigma_0(j) \leq \sigma_0(i)\}$, $\{j \in N|\sigma_0(j) < \sigma_0(i)\}$, $\{j \in N|\sigma_0(j) \geq \sigma_0(i)\}$, and $\{j \in N|\sigma_0(j) > \sigma_0(i)\}$, respectively. Every coordinate $\beta_i(v)$ is the average of the marginal of i in the coalition consisting of the players preceding i and in the coalition of players following i with respect to the initial order σ_0. In general, the β rule does not give a core allocation of the game, but it does for games that arise from one-machine scheduling problems satisfying conditions c–f.

Theorem. *If (N, v) is a cooperative game generated by a one-machine scheduling problem with an additive and regular efficiency criterion, the β rule gives a core allocation of the game.*

The proof follows from the next two propositions. The first proposition gives conditions for a cooperative game that guarantees that the β rule gives a core element. The second proposition states that one-machine scheduling problems (with additive and regular efficiency criterion) generate games with these properties.

Proposition 1. *The β-rule gives a core element of a cooperative game (N, v) if*

a. *the game is superadditive (i.e., $v(S) + v(T) \leq v(S \cup T)$ whenever $S \cap T = \emptyset$);*
b. *the game is σ_0-component additive (i.e., $v(S) = \sum_{T \in S/\sigma_0} v(T)$ where S/σ_0 is the collection of components of S under the order σ_0).*

Proof. Take any σ_0-connected coalition T and suppose that $T = \{i \in N | a \leq \sigma_0(i) \leq b\}$. Writing $\bar{P}r(i)$ instead of $\bar{P}r(i, \sigma_0)$, the β rule gives for coalition T

$$\begin{aligned} 2 \sum_{i \in T} \beta_i(v) &= \sum_{i \in T} [v(\bar{P}r(i)) - v(Pr(i)) \\ &\quad + v(\bar{F}(i)) - v(F(i))] \\ &= [v(\bar{P}r(\sigma_0^{-1}(b))) - v(Pr(\sigma_0^{-1}(a))) \\ &\quad + v(\bar{F}(\sigma_0^{-1}(a))) \\ &\quad - v(F(\sigma_0^{-1}(b)))] \geq 2v(T). \end{aligned}$$

The second equality is obtained by canceling equal terms from the first expression and the inequality follows from superadditivity, and

$$Pr(\sigma_0^{-1}(a)) \cup T = \bar{P}r(\sigma_0^{-1}(b))$$

and

$$T \cup F(\sigma_0^{-1}(b)) = \bar{F}(\sigma_0^{-1}(a)).$$

If $T = N (a = 1$ and $b = n)$ we obtain:

$$\begin{aligned} 2 \sum_{i \in N} \beta_i(v) &= v(\bar{P}r(\sigma_0^{-1}(n))) + v(\bar{F}(\sigma_0^{-1}(1))) \\ &= 2v(N). \end{aligned}$$

For non-σ_0-connected coalitions the core inequalities follow from σ_0-component additivity.

Proposition 2. *Every one-machine scheduling problem with an additive and regular cost criterion gives rise to a cost saving game that is superadditive and σ_0-component additive.*

Proof. If S and T are disjoint coalitions we can combine any action of S with any action of T to a feasible action of $S \cup T$. The cost savings of the combined action is the sum of the profits yielded by the action of S and T. This gives the superadditivity of (N, v). From the definition of v it follows immediately that the game is σ_0-component additive.

The proof of the theorem follows from Propositions 1 and 2.

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