# ANALYSIS OF (s,S) INVENTORY MODELS

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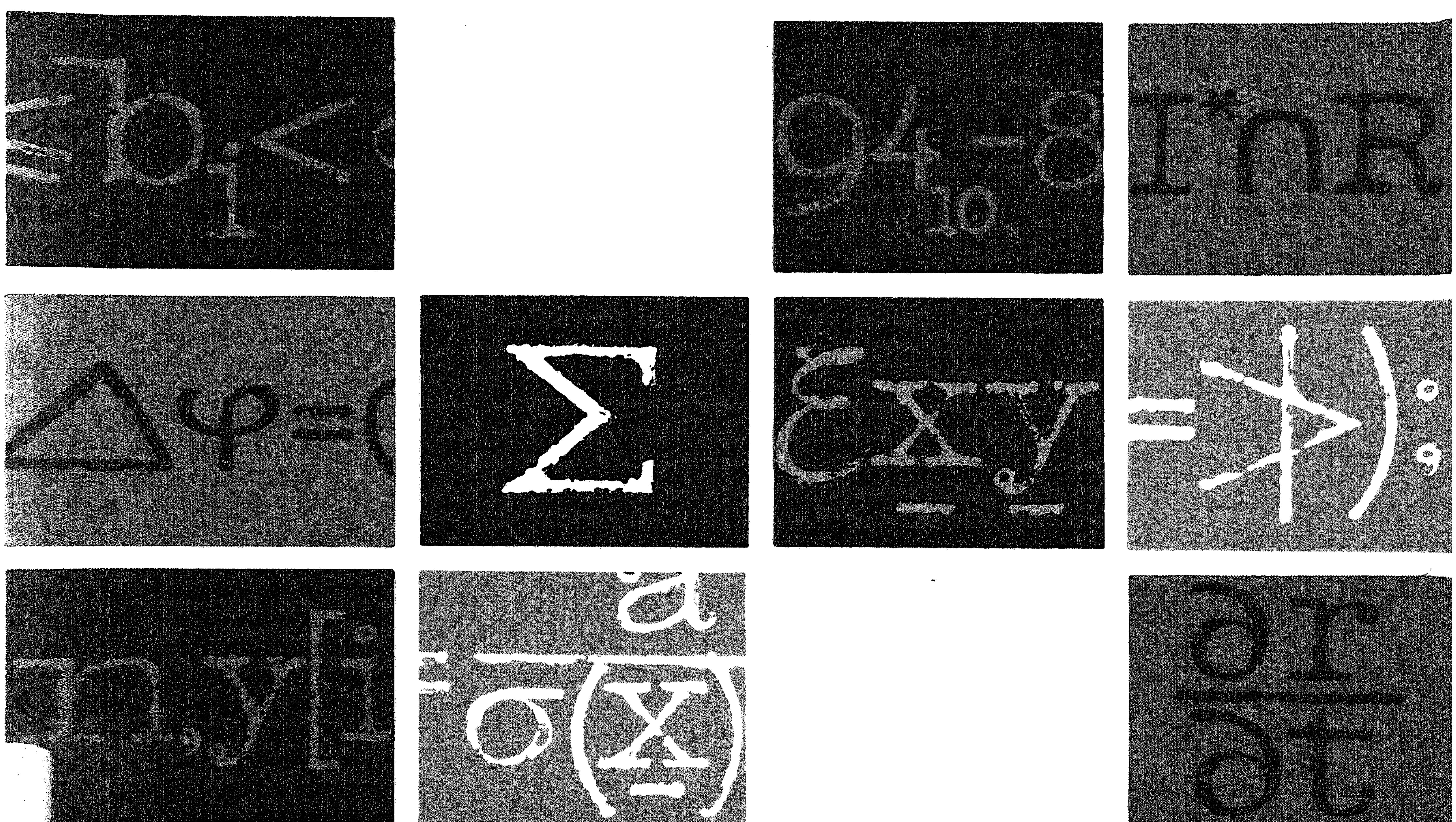

MATHEMATICAL CENTRE TRACTS 40

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CONTENTS

# INTRODUCTION

This book deals with inventory models with a single product and a single stocking point. Both periodic review and continuous review inventory models are considered. In these models much attention will be paid to the familiar (s,S) ordering policy commonly used in practice. We are concerned with both questions of the optimality of (s,S) policies in the infinite period inventory model and with the determination of a number of characteristics for certain (s,S) inventory systems. The optimality proofs will be based on results in the theory of Markovian decision processes. Further, throughout this book we make use of renewal theory. We review, therefore, in chapter I some results in Markovian decision theory and renewal theory. Chapter I is included to help make this book self-contained. A brief description will now be given of the other three chapters. A more detailed description of the contents of each chapter can be found at the beginning of the chapter. The chapters II, III and IV may be read independently of each other.

In chapter II the periodic review inventory model with backlogging of unfilled demand and a fixed lead time is considered. In this chapter various quantities for the dynamic (s,S) model are determined and further optimality questions for the infinite period inventory model are studied. A unified proof of the existence of optimal (s,S) policies for both the total discounted cost and the average cost criteria is given.

In the chapters III and IV two powerful techniques for analysing (s,S) inventory systems are applied.

Chapter III is devoted to a probabilistic analysis of an (s,S) inventory model in which the demand epochs are generated by a renewal process, the demands are independent random variables with a common distribution, excess demand is backlogged, and the lead time is a constant. The time dependent and the asymptotic behaviour of a number of stochastic processes arising in the (s,S) inventory model are determined. Further, certain long-run averages for the (s,S) policy are given.

Chapter IV discusses a generalization of the classical formula for the long-run average cost for decision processes with a regeneration point. This generalized formula is applied to an (s,S) inventory model.

I am indebted to professor G. de Leve, professor J.Th. Runnenburg and my colleague Arie Hordijk, who read the manuscript and made a number of helpful comments. Finally, thanks are due to mrs. S.M.T. Hillebrand for the fine typing of the manuscript.

# CHAPTER I SOME RESULTS IN MARKOVIAN DECISION THEORY AND RENEWAL THEORY

This chapter is an expository chapter on some aspects of both Markovian decision theory and renewal theory.

In section 1.1 we discuss a number of known results in the theory of Markovian decision processes. This discussion is also meant to provide insight into this important subject of dynamic programming. In subsection 1.1.1 we define the model. Subsections 1.1.2 and 1.1.3 deal with the total discounted cost and the average cost criteria. For the sake of completeness, the results of the subsections 1.1.2 and 1.1.3 that are needed in the sequel will be proved in subsection 1.1.4.

In section 1.2 we give a number of known results in renewal theory that will be needed in the next chapters.

## 1.1. MARKOVIAN DECISION PROCESSES

### *1.1.1. Model*

We are concerned with a dynamic system which at times $t = 1,2,\ldots$ is observed to be in one of a possible number of states. Let $I$ denote the space of all possible states. We assume $I$ to be countable. If at time $t$ the system is observed in state $i$, then an action $a$ must be chosen from a given set $A(i)$. We assume that for each $i \in I$ the set of actions $A(i)$ is countable.

If the system is in state $i$ at time $t$ and action $a$ is chosen, then, regardless of the history of the system, two things occur:

(i) we incur a known cost $w_{ia}$,

(ii) at time $t+1$ the system will be in state $j$ with probability $q_{ij}(a)$.

Thus, both the costs and the transition probabilities are functions only of the last state and the subsequently taken action. We assume that "the laws of motion" $q_{ij}(a)$ are known, and, of course, satisfy

$$0 \leq q_{ij}(a) \leq 1 \quad \text{and} \quad \sum_{j \in I} q_{ij}(a) = 1 \text{ for all } a \in A(i);\ i,j \in I.$$

A policy, to be denoted by $R$, is a prescription for taking actions at each point of time. We shall permit a policy for taking an action at time $t$ to be a function of the entire "history" of the system up to time $t$. We will allow actions to be taken which are determined by a random mechanism, the random mechanism will be a function of the "history". To be specific, let $x_t$ be the observed state at time $t$ and let $a_t$ be the observed action taken at time t. A *policy* R for controlling the system is a set of non-negative functions $\{D_a(H_{t-1},x_t),\ a \in A(x_t),\ t \geq 1\}$ satisfying

$$\sum_{a \in A(x_t)} D_a(H_{t-1},x_t) = 1$$

for every $H_{t-1}$ and $x_t$, t = 1,2,..., where $H_{t-1}$ denotes the history of the system up to time t-1, i.e. $H_{t-1} = (x_1,a_1,\dots,x_{t-1},a_{t-1})$. The interpretation being: if $H_{t-1}$ denotes the history of the system up to time t-1 and $x_t$ is the state of the system at time t, then a random mechanism is to be used which assigns the probability $D_a(H_{t-1},x_t)$ of taking action a at time t.

Let C denote the class of all policies. An important subclass of C is the class of the memoryless policies (cf. [9] and chapter 7 in [12]). A memoryless policy R is a policy such that $D_a(H_{t-1},x_t = i) = D_{ia}^{(t)}$ independent of $H_{t-1}$ for all i, a, t. A subclass of the class of memoryless policies is the class of the stationary deterministic policies. A stationary deterministic policy is a memoryless policy for which $D_{ia}^{(t)} = D_{ia}$ independent of t, and, in addition, $D_{ia}$ = 1 or 0 for all i, a. In words, when R is a stationary deterministic policy, then to each state i corresponds an action $a_i \in A(i)$ such that R prescribes action $a_i$ when the system is in state i.

Give an initial state $i \in I$ and the use of policy $R \in C$, let the random variable *) $\underline{x}_t$ be the state at time t and let the random variable $\underline{a}_t$ be the decision to be taken at time t. The stochastic process $\{\underline{x}_t, \underline{a}_t\}$ is called a *Markovian decision process*. The term Markovian is employed because of the special assumptions regarding the laws of motion. However, the process $\{\underline{x}_t, \underline{a}_t\}$ is not necessarily a Markov process, because the policy R may be such that the prescription for taking actions is dependent upon the entire history of the process. When R is a memoryless policy, then $\{\underline{x}_t\}$ is a Markov chain, not necessarily stationary. However, when R is a stationary deterministic policy, then $\{\underline{x}_t\}$ is a Markov chain with stationary transition probabilities.

In order to indicate the dependence of the probabilities on the policy R, the notation $P_R\{E|\underline{x}_1 = i\}$ will denote the probability of an event E occurring when the initial state $\underline{x}_1 = i$ and policy R is used.

Common measures of effectiveness of a policy governing a Markovian decision process are the total expected discounted cost and the average expected cost per unit time. The latter criterion involves far more difficulties than the former one.

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*) Throughout this book random variables will be underlined.

A typical example of a Markovian decision model is given by an inventory system for a single product, where the inventory level is under periodic review. After each review, the action taken is that of ordering a certain amount of the product. The laws of motion of the system are determined by the pattern of demand for the product between the times of review. The costs involved, such as ordering costs, inventory costs and shortage costs, determine the function $w_{ia}$.

### *1.1.2. The discounted cost criterion*

This criterion involves a fixed discount factor $\alpha$, where $0 < \alpha < 1$, with the interpretation that a unit cost incurred at time $t = n$ has a value of $\alpha^{n-1}$ at time $t = 1$.

For any $i \in I$ and $R \in C$, let

$$V_\alpha(i;R) = \sum_{t=1}^{\infty} \alpha^{t-1} \sum_{j\in I} \sum_{a\in A(j)} P_R\{\underline{x}_t = j,\ \underline{a}_t = a \mid \underline{x}_1 = i\} w_{ja},$$

provided it exists ($\pm\,\infty$ are admitted). Note that if the cost function $w_{ia}$ is uniformly bounded, say by M, then $V_\alpha(i;R)$ exists and is uniformly bounded by $M/(1-\alpha)$.

The quantity $V_\alpha(i;R)$ represents the total expected discounted cost when the initial state is i and policy R is used.

A policy $R^* \in C$ is called *optimal* under the discounted cost criterion if

$$V_\alpha(i;R^*) \leq V_\alpha(i;R) \quad \text{for all } i \in I \text{ and } R \in C.$$

A number of known results for the total discounted cost criterion are summarized in the following theorem.

*Theorem 1.1.1.*

Suppose there exists a finite number M such that $|w_{ia}| \leq M$ for all $a \in A(i)$ and $i \in I$. Then

(a) The "optimality equation"

$$(1.1.1) \qquad u(i) = \inf_{a\in A(i)} \{w_{ia} + \alpha \sum_{j\in I} q_{ij}(a)\, u(j)\}, \qquad i \in I$$

has the unique bounded solution

$$u^*(i) = \inf_{R\in C} V_\alpha(i;R), \qquad i \in I.$$

Moreover, for every $\varepsilon > 0$ there exists a stationary deterministic policy R such that $V_\alpha(i;R) \leq u^*(i) + \varepsilon$ for all $i \in I$.

(b) A policy $R^* \in C$ is optimal if and only if

$$V_\alpha(i;R^*) = \inf_{a\in A(i)} \{w_{ia} + \alpha \sum_{j\in I} q_{ij}(a)\, V_\alpha(j;R^*)\} \text{ for all } \quad i \in I.$$

(c) If an optimal policy exists, then there exists also an optimal stationary deterministic policy.

(d) If A(i) is finite for all $i \in I$, then there exists an optimal stationary deterministic policy.

A proof of this theorem, in a more generalized setting, is given in [6]; theorem 1.1.1 (b), (c) and (d) can be generalized to an arbitrary state and action space. Another proof of theorem 1.1.1 (d) is given in [8,12]. An elementary proof of theorem 1.1.1 is given in [22] by exploiting a result *) given in [9]. Finally, an elementary proof of most of theorem 1.1.1 can be found in [39].

*Remark 1.1.1.*

1. If an optimal policy exists, then we can replace inf by min in the right-hand side of (1.1.1), and, moreover, any stationary deterministic policy which, when in state i, prescribes an action which minimizes the right-hand side of (1.1.1) is optimal.
2. If A(i) is not finite for some i, then it is easy to give a counter-example showing that an optimal policy may not exist (see, for instance, [6,22]).
3. Consider the case that A(i) is finite for each $i \in I$ and the finite cost function $w_{ia}$ is not uniformly bounded. A counterexample in [23] shows that an optimal policy may not exist. In [8] a counterexample demon-

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*) For every $i_0 \in I$ and $R_0 \in C$ there exists a memoryless policy R such that $P_R\{\underline{x}_t=j,\ \underline{a}_t=a|\underline{x}_1=i_0\}=P_{R_0}\{\underline{x}_t=j,\ \underline{a}_t=a|\underline{x}_1=i_0\}$ for all $a\in A(j), j\in I$ and $t=1,2,\ldots$ .

strates that an optimal policy may exist, whereas no optimal stationary deterministic policy exists (see also [23]).

4. If I is finite and $A(i)$ is finite for each $i \in I$, then an optimal stationary deterministic policy can be computed by finite algorithms (Howard's policy improvement method or linear programming) [5,12,25,39].

### *1.1.3. The average cost criterion*

This criterion does not involve a discounting of costs and will be defined as follows. For any $i \in I$ and $R \in C$, let

$$(1.1.2) \qquad g(i;R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \sum_{j\in I} \sum_{a\in A(j)} P_R\{\underline{x}_t = j, \underline{a}_t = a \mid \underline{x}_1 = i\} w_{ja},$$

provided it exists ($\pm\infty$ are admitted). When the limit exists $g(i;R)$ represents the average expected cost per unit time when the initial state is i and policy R is used.

A policy $R^* \in C$ is called *optimal* under the average cost criterion if

$$g(i;R^*) \leq g(i;R) \qquad \text{for all } i \in I \text{ and } R \in C.$$

Sufficient conditions for the existence of an optimal stationary deterministic policy are stated in the following theorem.

*Theorem 1.1.2.*

Suppose there exists a set of finite numbers $\{g, v(i), i \in I\}$ such that

$$(1.1.3) \qquad g + v(i) = \min_{a\in A(i)} \{w_{ia} + \sum_{j\in I} q_{ij}(a)\, v(j)\} \quad \text{for all } i \in I\ ^{*)},$$

and

$$\lim_{n\to\infty} \frac{1}{n} E_R(v(\underline{x}_n) \mid \underline{x}_1 = i) = 0 \qquad \text{for all } i \in I \text{ and all } R \in C,$$

than any stationary deterministic policy $R^*$ which, when in state i, prescribes an action which minimizes the right-hand side of (1.1.3) is optimal. Moreover

$$g = g(i;R^*) = \min_{R\in C} g(i;R) \qquad \text{for all } i \in I,$$

*) Note this assumption includes that the infimum of the quantity between brackets is attained for each $i \in I$.

and for policy $R^*$ the lim inf in (1.1.2) can be replaced by lim.

This theorem is proved in [37] and it can be generalized to an arbitrary state and action space. For the case in which $A(i)$ is finite for each $i$ and the cost function $w_{ia}$ is bounded, theorem 1.1.2 was first proved in [10] under the condition that the functional equation (1.1.3) has a bounded solution. As noted in [36], it follows from the results in [10] and [11] that a sufficient condition for the existence of a bounded solution of (1.1.3) is that: (i) $A(i)$ is finite for each $i$ and $\{w_{ia}\}$ is bounded, (ii) for each stationary deterministic policy the resulting Markov chain $\{\underline{x}_t\}$ is positive recurrent, and (iii) there exists some state (say 0) and a constant $T < \infty$ such that $M_{i0}(R) < T$ for all $i \in I$ and all stationary deterministic policies $R$, where $M_{i0}(R)$ denotes the mean recurrence time from state $i$ to state 0 when using policy $R$. Moreover, it is shown in [36] that the conditions (ii) and (iii) can be replaced by the weaker condition that there exists some state (say 0) and a constant $N < \infty$ such that $|V_\alpha(i) - V_\alpha(0)| \leq N$ for all $i \in I$ and $0 < \alpha < 1$, where $V_\alpha(i) = \min_{R\in C} V_\alpha(i;R)$.

We note that the policy $R^*$ from theorem 1.1.2 also minimizes $\phi(i;R)$, where $\phi(i;R)$ is equal to the right-hand side of (1.1.2) in which lim inf is replaced by lim sup. Further, we note that the solution of (1.1.3) is not unique; if $\{g, v(i)\}$ satisfies (1.1.3) and if $c$ is a constant, then $\{g, v(i) + c\}$ satisfies also (1.1.3).

Recently, a new set of conditions guaranteeing the existence of an optimal stationary deterministic policy has been given in [24].

*Remark 1.1.2.*

The average cost criterion is much more complicated than the total discounted cost criterion. To see this, consider the case that $A(i)$ is finite for all $i \in I$ and the cost function $w_{ia}$ is uniformly bounded. A counterexample due to Maitra (see [10,22]) shows that an optimal policy may not exist. In [10] a counterexample has been given in which an optimal stationary randomized policy exists, whereas no optimal stationary deterministic policy exists (a stationary randomized policy is a memoryless policy for which $D_{ia}^{(t)} = D_{ia}$ independent of $t$ for all $i$ and $a$). An even more surprising counterexample in [17] demonstrates that an optimal nonstationary policy may exist, whereas no optimal stationary randomized policy exists. In the latter counterexample it is remarkable that any stationary randomized policy leads to an ergodic Markov chain $\{\underline{x}_n\}$. In [40] a counterexample shows that an ε-optimal stationary deterministic policy for state i may not exist, that is, a stationary deterministic policy $R_0$ such that $g(i;R_0) \leq \inf_{R\in C} g(i;R) + \varepsilon$ may not exist.

Finally, we note that if I is finite and if A(i) is finite for all $i \in I$, then an optimal stationary deterministic policy exists and such a policy can be determined by Howard's policy improvement method or linear programming [5,12,22,25,39].

### *1.1.4. Optimality proofs*

We shall need in the sequel the first part of theorem 1.1.1 (a), theorem theorem 1.1.1 (b) and theorem 1.1.2. In order to keep our treatment self-contained, we shall give a proof of these statements.

We first present a proof of theorem 1.1.2 as given in [37].

*Theorem 1.1.2.*

Suppose there exists a set of finite numbers $\{g, v(i), i \in I\}$ such that

$$(1.1.3) \qquad g + v(i) = \min_{a\in A(i)} \{w_{ia} + \sum_{j\in I} q_{ij}(a)\, v(j)\} \qquad \text{for all } i \in I$$

and

$$\lim_{n\to\infty} \frac{1}{n} E_R(v(\underline{x}_n) | \underline{x}_1 = i) = 0 \qquad \text{for all } i \in I \text{ and all } R \in C,$$

then any stationary deterministic policy $R^*$ which, when in state i, prescribes an action which minimizes the right-hand side of (1.1.3) is optimal. Moreover

$$g = g(i;R^*) = \min_{R\in C} g(i;R) \qquad \text{for all } i \in I,$$

and for policy $R^*$ the lim inf in (1.1.2) can be replaced by lim.

*Proof*

Let $R^*$ be a stationary deterministic policy which, for each i, prescribes an action which minimizes the right-hand side of (1.1.3).

Fix the initial state $\underline{x}_1 = x_1$ and the policy R to be used. For any $n = 2,3,\dots$, we have

$$E_R\{\sum_{t=2}^{n}[v(\underline{x}_t) - E_R(v(\underline{x}_t)|\underline{x}_1, \underline{a}_1, \ldots, \underline{x}_{t-1}, \underline{a}_{t-1})]\} = 0,$$

where all expectations are understood to be conditioned on $\underline{x}_1 = x_1$. But,

$$E_R\{v(\underline{x}_t)|\underline{x}_1=x_1, \underline{a}_1=a_1, \ldots, \underline{x}_{t-1}=x_{t-1}, \underline{a}_{t-1}=a_{t-1}\} = \sum_{j\in I} v(j)q_{x_{t-1}j}(a_{t-1}) =$$

$$= w_{x_{t-1}a_{t-1}} + \sum_{j\in I} v(j)q_{x_{t-1}j}(a_{t-1}) - w_{x_{t-1}a_{t-1}} \geq$$

$$\geq \min_{a\in A(x_{t-1})} \{w_{x_{t-1}a} + \sum_{j\in I} v(j)q_{x_{t-1}j}(a)\} - w_{x_{t-1}a_{t-1}} =$$

$$= g + v(x_{t-1}) - w_{x_{t-1}a_{t-1}},$$

with equality for $R = R^*$, since $R^*$ is defined to take a minimizing action. Hence

$$0 \leq E_R\{\sum_{t=2}^{n}[v(\underline{x}_t) - g - v(\underline{x}_{t-1}) + w_{\underline{x}_{t-1}\underline{a}_{t-1}}]\}, \qquad n \geq 2,$$

or

$$g \leq \frac{1}{n-1} E_R\, v(\underline{x}_n) - \frac{1}{n-1} E_R\, v(x_1) + \frac{1}{n-1} E_R\{\sum_{t=1}^{n-1} w_{\underline{x}_t\underline{a}_t}\}, \qquad n \geq 2,$$

with equality for $R = R^*$. Letting $n \to \infty$, we obtain the desired results.

Consider next the discounted cost criterion. Assume there exists a constant $M < \infty$ such that $|w_{ia}| \leq M$ for all $a \in A(i)$ and $i \in I$.

The proof of the next lemma follows a proof given in [6].

*Lemma 1.1.1.*

The optimality equation (1.1.1) has a unique bounded solution.

*Proof*

Let B be the set of all real-valued bounded functions on I. A metric $\rho$ on B is defined by

$$\rho(u,v) = \sup_{i\in I} |u(i) - v(i)|, \qquad u, v \in B.$$

The space B is complete in this metric [32]. Define the mapping $T_\alpha : B \to B$ by

$$(T_\alpha u)(i) = \inf_{a \in A(i)} \{w_{ia} + \alpha \sum_{j \in I} q_{ij}(a)\, u(j)\}, \qquad i \in I.$$

By the fixed point theorem (cf.[32]) the optimality equation $T_\alpha u = u$ has a unique bounded solution when the operator $T_\alpha$ is a contraction mapping, i.e. if there exists a number $\beta$ with $0 < \beta < 1$ such that $\rho(T_\alpha u, T_\alpha v) \leq \beta\, \rho(u,v)$ for all $u, v \in B$.

Clearly, the operator $T_\alpha$ is monotone, i.e. $u(i) \leq v(i)$ for all $i \in I$ implies that $(T_\alpha u)(i) \leq (T_\alpha v)(i)$ for all $i \in I$. Moreover, if $v_c$ is an element of B whose value at each $i \in I$ is the constant c, then $(T_\alpha(u+v_c))(i) = (T_\alpha u)(i)+\alpha c$ for all $i \in I$. Hence, since $u(i) \leq v(i) + \rho(u,v)$ for all $i \in I$, we obtain

$$(T_\alpha u)(i) \leq (T_\alpha v)(i) + \alpha\, \rho(u,v) \qquad \text{for all } i \in I.$$

By interchanging the roles of u and v, we obtain

$$(T_\alpha v)(i) \leq (T_\alpha u)(i) + \alpha\, \rho(u,v) \qquad \text{for all } i \in I.$$

Hence $\rho(T_\alpha u, T_\alpha v) \leq \alpha\, \rho(u,v)$ for all $u, v \in B$. This ends the proof.

Let

$$V_\alpha(i) = \inf_{R \in C} V_\alpha(i;R), \qquad i \in I.$$

Since $|w_{ia}| \leq M$ for all i and a, we have $|V_\alpha(i;R)| \leq M/(1-\alpha)$ for all i and R, and hence the function $V_\alpha(i)$, $i \in I$, is bounded.

The proof of the next lemma is due to Ross [39].

*Lemma 1.1.2.*

The function $V_\alpha(i)$, $i \in I$, satisfies the optimality equation (1.1.1).

*Proof*

Let $R \in C$ be an arbitrary policy, and let $\pi(a|i)$ be the probability that R chooses action a when in state i at time $t = 1$. Then,

$$V_\alpha(i;R) = \sum_{a \in A(i)} \pi(a|i)\{w_{ia} + \sum_{j \in I} q_{ij}(a)\, W_\alpha(j;R;i;a)\}, \qquad i \in I,$$

where

$$W_\alpha(j;R;i;a) = \sum_{t=2}^{\infty} \alpha^{t-1} \sum_k \sum_{a'} P_R\{\underline{x}_t = k, \underline{a}_t = a' \mid \underline{x}_1 = i, \underline{a}_1 = a, \underline{x}_2 = j\} w_{ka'}.$$

The above relation for $V_\alpha(i;R)$ involves an interchange of the order of summation justified by the assumption of the boundedness of the function $w_{ia}$. It is readily seen that for each i and a there exists a policy $R_{ia} \in C$ such that

$$\alpha V_\alpha(j;R_{ia}) = W_\alpha(j;R;i;a) \qquad \text{for all } j \in I.$$

Hence, since $V_\alpha(j;R_{ia}) \geq V_\alpha(j)$ for all $j \in I$, we obtain

$$\begin{aligned} V_\alpha(i;R) &\geq \sum_{a\in A(i)} \pi(a|i)\{w_{ia} + \alpha \sum_{j\in I} q_{ij}(a)\, V_\alpha(j)\} \geq \\ &\geq \sum_{a\in A(i)} \pi(a|i) \inf_{a'\in A(i)} \{w_{ia'} + \alpha \sum_{j\in I} q_{ij}(a')\, V_\alpha(j)\} = \\ &= \inf_{a\in A(i)} \{w_{ia} + \alpha \sum_{j\in I} q_{ij}(a)\, V_\alpha(j)\} \qquad \text{for all } i \in I. \end{aligned}$$

Since R is arbitrary, this inequality implies

$$\text{(1.1.4)} \qquad V_\alpha(i) \geq \inf_{a\in A(i)} \{w_{ia} + \alpha \sum_{j\in I} q_{ij}(a)\, V_\alpha(j)\} \qquad \text{for all } i \in I.$$

Choose $\varepsilon > 0$. For any $i \in I$, let $a_i \in A(i)$ be such that

$$\text{(1.1.5)} \qquad w_{ia_i} + \alpha \sum_{j\in I} q_{ij}(a_i) V_\alpha(j) \leq \inf_{a\in A(i)} \{w_{ia} + \alpha \sum_{j\in I} q_{ij}(a)\, V_\alpha(j)\} + \varepsilon.$$

Moreover, for each $i \in I$ we choose a policy $R_i \in C$ such that

$$V_\alpha(i;R_i) \leq V_\alpha(i) + \varepsilon.$$

Let the policy $R^*$ be defined as follows. The policy $R^*$ chooses action $a_i$ at time t = 1 when in state i, and, if the next state is j, then policy $R^*$ views the process as originating in state j and follows the policy $R_j$. Then

$$\begin{aligned} \text{(1.1.6)} \qquad V_\alpha(i;R^*) &= w_{ia_i} + \alpha \sum_{j\in I} q_{ij}(a_i)\, V_\alpha(i;R_j) \leq \\ &\leq w_{ia_i} + \alpha \sum_{j\in I} q_{ij}(a_i) V_\alpha(i) + \alpha\, \varepsilon \qquad \text{for all } i \in I. \end{aligned}$$

From the relation $V_\alpha(i) \leq V_\alpha(i;R^*)$ for all $i \in I$, (1.1.6) and (1.1.5) it follows that

(1.1.7) $$V_\alpha(i) \leq \inf_{a \in A(i)} \{w_{ia} + \alpha \sum_{j \in I} q_{ij}(a)\, V_\alpha(j)\} + (\alpha+1)\varepsilon \text{ for all } i \in I.$$

Since $\varepsilon$ was chosen arbitrarily, the lemma follows from (1.1.4) and (1.1.7).

From the lemmas 1.1.1 and 1.1.2 it follows that the optimality equation (1.1.1) has the unique solution $u^*(i) = V_\alpha(i)$, $i \in I$, and this in turn implies theorem 1.1.1 (b).

## 1.2. RENEWAL THEORY

In this section we collect a number of known results in renewal theory that will be needed in the sequel.

Let $\underline{x}_1, \underline{x}_2, \ldots$ be a sequence of mutually independent, non-negative and identically distributed random variables with common probability distribution $F(t) = P\{\underline{x}_n \leq t\}$. It is assumed that

$$F(0) < 1 \text{ and } \mu = \int_0^\infty t\, F(dt) < \infty.$$

Let us first introduce some notation. Let

$$\underline{s}_0 = 0,\ \underline{s}_n = \underline{x}_1 + \ldots + \underline{x}_n \qquad \text{for } n = 1,2,\ldots .$$

The sequence $\{\underline{s}_n, n \geq 0\}$ of random variables is called a *renewal process*. The random variables $\underline{x}_1, \underline{x}_2, \ldots$ are often called the *interarrival times* of the renewal process. We agree to say that at epoch $t \geq 0$ the $n^{th}$ renewal occurs if $\underline{s}_n = t$ for some $n \geq 1$.

Let $F^{(n)}$ be the n-fold convolution of F with itself, i.e.

(1.2.1) $$F^{(0)}(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}$$

and

(1.2.2) $$F^{(n)}(t) = \begin{cases} \int_0^t F(t-y)F^{(n-1)}(dy) & \text{for } t \geq 0;\ n = 1,2,\ldots,^{*)} \\ 0 & \text{for } t < 0;\ n = 1,2,\ldots . \end{cases}$$

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*) In this section any finite interval of integration is closed; the integral is the Lebesgue-Stieltjes integral on $\mathbb{R}$.

Note that $F^{(1)} = F$. Clearly,

$$F^{(n)}(t) = P\{\underline{s}_n \leq t\} \qquad \text{for } n = 0,1,\dots .$$

Let the so-called renewal function $U(t)$ be defined by

$$U(t) = \sum_{n=1}^{\infty} F^{(n)}(t) \qquad \text{for } t \geq 0.^{*)}$$

The following lemma will be frequently used.

*Lemma 1.2.1.*

The function $U(t)$ is finite. Moreover, $F^{(1)}(t) + \dots + F^{(n)}(t)$ converges exponentially fast to $U(t)$ as $n \to \infty$ for each $t \geq 0$.

*Proof*

By $F(0) < 1$ and the right-continuity of $F$, there exists a number $\alpha > 0$ such that $1-F(\alpha) > 0$. Put $\beta = 1-F(\alpha)$. Define the process $\{x_n', n \geq 1\}$ by putting $x_n' = \alpha$ if $\underline{x}_n > \alpha$ and zero otherwise. The random variables $\underline{x}_1', \underline{x}_2', \dots$ are mutually independent and identically distributed with $P\{\underline{x}_n' = \alpha\} = \beta$ and $P\{\underline{x}_n' = 0\} = 1-\beta$. Fix $t \geq 0$. Let $M$ be the largest integer less than or equal to $t/\alpha$. Then

$$F^{(n)}(t) \leq P\{\sum_{i=1}^{n} \underline{x}_i' \leq t\} = \sum_{j=0}^{M} \binom{n}{j} \beta^j (1-\beta)^{n-j} \leq \sum_{j=0}^{M} n^j (1-\beta)^{n-j} \quad \text{for } n > M.$$

This proves the lemma.

The proof given is due to Smith [46]. Another proof can be found in [15].

For any $t \geq 0$, let

$$\underline{n}(t) = \sup\{n \mid \underline{s}_n \leq t\}.$$

Since the event $\{\underline{n}(t) = n\}$ occurs if and only if $\underline{s}_n \leq t$ but $\underline{s}_{n+1} > t$, we have $P\{\underline{n}(t) = n\} = F^{(n)}(t) - F^{(n+1)}(t)$, $(n = 0,1,\dots)$, and hence (cf. [15,46])

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*) In [15] the renewal function includes the term $F^{(0)}(t) \equiv 1$, however we we prefer the usual definition as given here.

$$(1.2.3) \qquad E\,\underline{n}(t) = U(t) \qquad \text{for all } t \geq 0.$$

Thus $U(t)$ can be interpreted as the expected number of renewals in $[0,t]$. The following relation will be intuitively clear from above interpretation. For each number $a \geq 0$, there exists a finite constant $C_a$ such that

$$(1.2.4) \qquad U(t+a) - U(t) \leq C_a \qquad \text{for all } t \geq 0.$$

The proof of (1.2.4) is simple and it can be found in [15].

From (1.2.1), (1.2.2) and the definition of $U(t)$ it follows that $U(t)$ satisfies the so-called integral equation from renewal theory,

$$(1.2.5) \qquad U(t) = F(t) + \int_0^t F(t-y)\,U(dy) \qquad \text{for } t \geq 0.$$

Let the real-valued function $z(t)$, $t \geq 0$, be a Baire function *) and let $z$ be bounded on finite intervals. Consider the so-called *renewal equation*

$$(1.2.6) \qquad Z(t) = z(t) + \int_0^t Z(t-y)\,F(dy) \qquad \text{for } t \geq 0.$$

It is well-known that [15]

$$(1.2.7) \qquad Z(t) = z(t) + \int_0^t z(t-y)\,U(dy) \qquad \text{for } t \geq 0,$$

is the unique solution of the renewal equation (1.2.6) that is bounded on finite intervals. This can be easily verified by iterating (1.2.6), using (1.2.2) and using the fact that $F^{(n)}(t)$ tends to zero as $n \to \infty$ for each $t \geq 0$.

For any $t \geq 0$, let

$$\hat{U}(t) = \sum_{n=1}^{\infty} n\,F^{(n)}(t).$$

Using (1.2.2), we obtain $\hat{U}(t) = U(t) + \int_0^t \hat{U}(t-y)\,F(dy)$ and hence,

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*) A definition of a Baire function can be found in [15]. The class of the Baire functions coincides with the class of the Borel functions, where a real-valued function $g(x)$ of the real variable $x$ is said to be a Borel function if $\{x \mid g(x) \leq c\}$ is a Borel set for every real $c$.

by (1.2.7), we have (cf. [46])

$$\hat{U}(t) = U(t) + \int_0^t U(t-y)\, U(dy) \qquad \text{for } t \geq 0. \tag{1.2.8}$$

The simplest general result about the renewal function U(t) is the so-called *elementary renewal theorem* [39,46]

$$\lim_{t\to\infty} \frac{U(t)}{t} = \frac{1}{\mu} \qquad \text{for } t \geq 0. \tag{1.2.9}$$

It is well-known that if F is an exponential distribution, then [15]

$$U(t) = \frac{t}{\mu} \qquad \text{for } t \geq 0. \tag{1.2.10}$$

A very deep and important renewal theorem, with many applications in the theory of stochastic processes, is the key renewal theorem. Before we state this theorem, we give some definitions and a lemma.

A function z(t), $t \geq 0$, is said to be *directly Riemann integrable* if for fixed $h > 0$ the two series $\sum_1^\infty m_n(h)$ and $\sum_1^\infty M_n(h)$ converge absolutely and if

$$\lim_{h\downarrow 0} h \sum_{n=1}^{\infty} m_n(h) = \lim_{h\downarrow 0} h \sum_{n=1}^{\infty} M_n(h),$$

where we denote by $m_n(h)$ and $M_n(h)$, respectively, the largest and the smallest number such that $m_n(h) \leq z(t) \leq M_n(h)$ for $(n-1)h \leq t < nh$ (cf. [15]). We note that any directly Riemann integrable function is also Riemann integrable over $(0,\infty)$ in the ordinary sense. It is easily seen that a non-negative function z(t), $t \geq 0$, is directly Riemann integrable over $(0,\infty)$, if it is Riemann integrable over every finite interval (0,a) and if $\sum_1^\infty M_n(h) < \infty$ for some $h > 0$ (cf. [15]). From analysis it is well-known that a non-negative and non-decreasing function g(t), $t \geq 0$, is Riemann integrable over $(0,\infty)$ in the ordinary sense if and only if $\sum_0^\infty g(n) < \infty$. Hence we have the following useful criterion for directly Riemann integrability.

*Lemma 1.2.2.*

Let g(t), $t \geq 0$, be a monotone function which is Riemann integrable over $(0,\infty)$ in the ordinary sense. If the function z(t), $t \geq 0$, is Riemann integrable over every finite interval (0,a) and if $0 \leq z(t) \leq g(t)$ for all $t \geq 0$, then the function z(t) is directly Riemann integrable.

The distribution function $F(t)$ is said to be *arithmetic* if it is concentrated on a set of points $0,\lambda,2\lambda,\ldots$ for some $\lambda \geq 0$, i.e. $\sum_0^\infty P\{\underline{x}_1 = n\lambda\} = 1$.

We are now in a position to state the key renewal theorem. Its proof can be found in [15].

*Theorem 1.2.1. (Key Renewal Theorem).*

If $F(t)$ is non-arithmetic and if the Baire function $z(t)$, $t \geq 0$, is directly Riemann integrable, then

$$\lim_{t\to\infty} \int_0^t z(t-y)\, U(dy) = \frac{1}{\mu} \int_0^\infty z(t)dt.$$

*Remark 1.2.1.*

Let $F_0(t)$ be a probability distribution function concentrated on $[0,\infty)$. Let

$$V(t) = F_0(t) + \int_0^t U(t-x)\, F_0(dx), \qquad t \geq 0.$$

Note that $V(t) = U(t)$ if $F_0(t) = F(t)$, $t \geq 0$. We have (cf. [46])

$$\lim_{t\to\infty} \frac{V(t)}{t} = \frac{1}{\mu}\, .$$

Further, if $F(t)$ is non-arithmetic and if the Baire function $z(t)$ is directly Riemann integrable, then

$$\lim_{t\to\infty} \int_0^t z(t-y)\, V(dy) = \frac{1}{\mu} \int_0^\infty z(t)dt.$$

Hence the elementary renewal theorem and the key renewal theorem are also true for the so-called delayed renewal process.

For any $t \geq 0$, let

$$\underline{y}(t) = \underline{s}_{\underline{n}(t)+1} - t,$$

then $\underline{y}(t)$ is the length of the interval between $t$ and the epoch of the first renewal occurring after $t$. Using a standard argument from renewal theory, we have for each $t \geq 0$ that [15]

$$(1.2.11)\quad P\{\underline{y}(t) \leq y\} = F(t+y) - F(t) + \sum_{n=1}^{\infty} \int_0^t \{F(t+y-x) - F(t-x)\} F^{(n)}(dx) =$$

$$= F(t+y) - F(t) + \int_0^t \{F(t+y-x) - F(t-x)\}\, U(dx),\ y \geq 0.$$

It follows from (1.2.7) and (1.2.11) that if F is an exponential distribution, then for each $t \geq 0$ the random variable $\underline{y}(t)$ has the exponential distribution F.

An easy consequence of (1.2.11) and the key renewal theorem is that [15]

$$(1.2.12)\quad \lim_{t\to\infty} P\{\underline{y}(t) \leq y\} = \frac{1}{\mu} \int_0^y \{1 - F(x)\}dx, \qquad y \geq 0,$$

provided that F is non-arithmetic.

The following frequently used lemma is known as *Wald's equation* [13,15, 39].

*Lemma 1.2.3. (Wald's equation).*

Let $\{\underline{u}_n, n = 1,2,\ldots\}$ be a sequence of mutually independent, non-negative and identically distributed random variables having a finite expectation. Let $\underline{m}$ be a positive, integral-valued random variable with a finite expectation. If the event $\{\underline{m} = m\}$ is independent of $\underline{u}_{m+1}, \underline{u}_{m+2}, \ldots$, for every $m = 1,2,\ldots$, then $E(\underline{u}_1 + \ldots + \underline{u}_{\underline{m}}) = E\underline{u}_1 . E\underline{m}$.

Since the event $\{\underline{n}(t) + 1 = n\}$ occurs if and only if $\underline{s}_{n-1} \leq t$ and $\underline{s}_n \geq t$, we have for every $n \geq 1$ that the event $\{\underline{n}(t) + 1 = n\}$ is independent of $\underline{x}_{n+1}, \underline{x}_{n+2}, \ldots$ . Thus, by Wald's equation and (1.2.3),

$$(1.2.13)\quad E\underline{y}(t) = \mu(U(t) + 1) - t \qquad \text{for all } t \geq 0.$$

The following lemma is contained in a more general result about cumulative processes treated in [45,46]. Another proof of this lemma can be found in [39].

*Lemma 1.2.4.*

Let $\{(\underline{x}_n, \underline{y}_n), n = 1,2,\ldots\}$ be a sequence of mutually independent, non-negative and identically distributed two-dimensional random variables, where the $\underline{x}_k$ are the interarrival times of the above renewal process $\{\underline{s}_n\}$. Assume

$E\underline{y}_1 < \infty$. Then

$$\lim_{t\to\infty} \frac{1}{t} E\Big(\sum_{i=1}^{\underline{n}(t)} \underline{y}_i\Big) = \lim_{t\to\infty} \frac{1}{t} E\Big(\sum_{i=1}^{\underline{n}(t)+1} \underline{y}_i\Big) = \frac{E\underline{y}_1}{E\underline{x}_1} .$$

We note that the equality of the last two terms is a direct consequence of Wald's equation and the elementary renewal theorem.

In the remainder of this section we assume that the random variables $\underline{x}_n$ have a discrete probability distribution

$$f_j = P\{\underline{x}_n = j\}, \qquad \text{for } j = 0,1,\dots;\ n = 1,2,\dots,$$

where $f_0 < 1$. Let

$$(1.2.14)\quad f_0^{(0)} = 1, \qquad f_j^{(0)} = 0 \qquad \text{for } j = 1,2,\dots,$$

and

$$(1.2.15)\quad f_j^{(n)} = \sum_{k=0}^{j} f_{j-k}\, f_k^{(n-1)}, \qquad \text{for } j = 0,1,\dots;\ n = 1,2,\dots .$$

Note that $f_j^{(1)} = f_j$. Clearly $f_j^{(n)} = P\{\underline{s}_n = j\}$, $(j \geq 0;\ n \geq 0)$. Let

$$(1.2.16)\quad u_j = \sum_{n=1}^{\infty} f_j^{(n)} \qquad \text{for } j = 0,1,\dots .$$

By lemma 1.2.1 the renewal quantity $u_j$, $j \geq 0$, is finite. From (1.2.14), (1.2.15) and (1.2.16), it follows that

$$(1.2.17)\quad u_j = f_j + \sum_{k=0}^{j} f_{j-k}\, u_k, \qquad \text{for } j = 0,1,\dots .$$

We shall now consider the discrete renewal equation. Let $\{z_j, j \geq 0\}$ be a given sequence of finite numbers, and let the sequence $\{Z_j,\ j \geq 0\}$ be defined by

$$(1.2.18)\quad Z_j = z_j + \sum_{k=0}^{j} Z_{j-k}\, f_k \qquad \text{for } j = 0,1,\dots .$$

The discrete renewal equation (1.2.18) has the unique solution (cf. (1.2.7))

$$(1.2.19)\quad Z_j = z_j + \sum_{k=0}^{j} z_{j-k}\, u_k \qquad \text{for } j = 0,1,\dots .$$

The asymptotic behaviour of the solution $Z_j$ is described in the following discrete analogue of the key renewal theorem [14,15].

*Theorem 1.2.2.*

Suppose that $\sum_{j=0}^{\infty} |z_j| < \infty$. Then

(a) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} Z_k = \frac{1}{\mu} \sum_{j=0}^{\infty} z_j$.

(b) If the greatest common divisor of the indices n, where $f_n > 0$, is 1, then $\lim_{n\to\infty} Z_n = \frac{1}{\mu} \sum_{j=0}^{\infty} z_j$.

Finally, we consider the special case in which the random variables $\underline{x}_k$ have a geometric distribution,

$$f_0 = 0, \quad f_j = p(1-p)^{j-1} \qquad \text{for } j = 1,2,\ldots ,$$

where $0 < p \leq 1$. Note that $\mu = E\underline{x}_k = 1/p$. To determine the renewal quantity $u_j$ explicitly, let $\tilde{f}(s) = \sum_0^\infty f_j s^j$ and $\tilde{u}(s) = \sum_0^\infty u_j s^j$ for $|s| < 1$. From (1.2.17) it follows that $\tilde{u}(s) = \tilde{f}(s) + \tilde{u}(s)\,\tilde{f}(s)$ for $|s| < 1$. Since $\tilde{f}(s) = ps/(1-(1-p)s)$, we obtain $\tilde{u}(s) = ps/(1-s)$ for $|s| < 1$ and hence, by the uniqueness theorem for power-series, we have the well-known result

$$(1.2.20) \quad u_0 = 0, \quad u_j = p \qquad \text{for } j = 1,2,\ldots .$$

Finally, it follows from (1.2.11) and (1.2.20) that for each $k = 0,1,\ldots$ the random variable $\underline{y}(k)$ has the geometric probability distribution $\{p(1-p)^{j-1}\}$.

# CHAPTER II VARIOUS QUANTITIES FOR THE DYNAMIC (s,S) INVENTORY MODEL AND THE OPTIMALITY OF (s,S) POLICIES

This chapter deals with the periodic review, single item inventory model with an infinite planning horizon. In the sections 2.1 and 2.2 we consider the inventory model in which the demands in the successive periods are mutually independent, non-negative and identically distributed random variables having a discrete distribution. At the beginning of each period an order may be placed for any positive quantity of stock. An order placed at the beginning of period $t = 1,2,\dots$ is delivered at the beginning of period $t+\lambda$, where $\lambda$ is a *known* non-negative integer. When the demand during a period exceeds the inventory on hand, then the excess demand is *backlogged* until it is subsequently filled by a delivery. The following costs are involved. There is a fixed set-up cost for each order, a linear purchase cost, and a holding and shortage cost function. Finally, there is a fixed discount factor $\alpha$ with $0 < \alpha \leq 1$.

In section 2.3 we consider the same model except that the lead time of an order is zero and the demand in any period depends on the stock level at the beginning of the period.

We shall determine in section 2.1 a number of quantities for the inventory model in which the ordering policy followed is the familiar *(s,S) policy*, that is, if, at review, the stock on hand plus on order i is less than s, then S-i units are ordered; otherwise, no ordering is done. We give some preliminaries in subsection 2.1.1. In the subsections 2.1.2 and 2.1.3 both the known solution for the steady state behaviour of the stock level [29,30] and the known solution for the total expected discounted cost in the infinite period (s,S) model [2,48] are derived anew by using renewal theory in a systematic way. In addition, as a by-product we obtain the solution for the transient behaviour of the stock level and the solution for the total expected discounted cost in the finite period (s,S) model. Further, a sufficient condition is given under which the stock level has an ordinary limiting distribution. In subsection 2.1.4 the known solution for the average cost per period in the infinite period (s,S) model [2,27,29] is given. Further, we obtain in subsection 2.1.4 as a new result the asymptotic behaviour of the total expected cost in the n-period (s,S) model with $\alpha = 1$. The proofs and the results in section 2.1 can be adapted to the case in which the demand variables have an arbitrary distribution.

In section 2.2 we are concerned with the existence of optimal policies for the *infinite* period inventory model. In the *finite* period model the existence of an (s,S) policy minimizing the total expected cost was shown under different conditions by Scarf [41,43] and Veinott [49]. Scarf assumed that the one-period expected holding and shortage costs are *convex* and that the stock left over or backlogged demand remaining at the end of the final period has no salvage value. Veinott imposed on the one-period holding and shortage costs the weaker assumption that the negatives of these costs are *unimodal*, and further he assumed that stock left over or backlogged demand remaining at the end of the final period has a salvage value.

Under the assumption that the one-period holding and shortage costs are convex and using Scarf's results for the finite period model, Iglehart [26,27] has examined the *infinite* period model in which the demand distribution has a density. In [26] it was proved that an optimal (s,S) policy exists under the total discounted cost criterion ($\alpha < 1$) and in [27] the existence of an optimal (s,S) policy for the average cost criterion ($\alpha = 1$) has been shown (see also [48, pp. 530-531]). Under the assumption that the negatives of the one-period holding and shortage costs are unimodal, Johnson [28] has established the existence of optimal (s,S) policies for both the total discounted cost and the average cost criteria.Johnson gives a very elegant algorithm which generates only (s,S) policies and converges after a finite number of steps to an optimal (s,S) policy. However the proof of Johnson, based on Howard's policy improvement method, is typical for the discrete demand case and does not carry over to the case in which the demand distribution has a density.

A different and unified proof of the existence of optimal (s,S) policies for both the total discounted cost and the average cost criteria will be given in section 2.2 under the assumption that the negatives of the one-period holding and shortage costs are unimodal. This proof, which is inspired by Iglehart's optimality proof for the average cost criterion [27], carries over to the case in which the demand distribution has a density. Our approach, which treats the total discounted cost and the average cost criteria simultaneously, is based on results in Markovian decision theory and it does not use any optimality result for the finite model. To be more specific, our approach is a direct one which consists of analysing a certain function $a_\alpha(s,S)$ and verifying that some explicitly given function $v_\alpha^*(i)$ satisfies a functional equation that reduces to (1.1.1) of section 1.1 when $\alpha < 1$ and to (1.1.3) of section 1.1 when $\alpha = 1$. As a by-product we obtain upper and lower bounds on the optimal s and S. Further we find for the total discounted cost criterion a sufficient condition for the optimality of an (s,S) policy when s and S minimize the function $a_\alpha(s,S)$. These latter results sharpen similar results in [28,48]. It should be noted that optimality proofs for the infinite period model could also be given by using Veinott's results for the finite period model and modifying Iglehart's optimality proofs (cf. [49]). However this approach leads to separate proofs for the discounted cost and the average cost criteria, and,moreover, it does not yield a characterization of the optimal (s,S) policies for the discounted cost criterion.

In the final subsection of section 2.1 we generalize a uniqueness theorem for the optimal inventory equation given in [26].

In section 2.3 the proofs and the results of the sections 2.1 and 2.2 are adapted to the inventory model in which the lead time is zero and the demands depend on the stock level. This model was first treated by Johnson [28] who proved with the help of a finite algorithm the existence of optimal (s,S) policies in the infinite period model.

## 2.1. VARIOUS QUANTITIES FOR THE DYNAMIC (s,S) INVENTORY MODEL

### *2.1.1. Model and preliminaries*

We consider a single item dynamic (s,S) inventory model in which the demands $\underline{\xi}_1,\underline{\xi}_2,\ldots$ for a single item in periods 1,2,... are mutually independent , non-negative and identically distributed random variables with a discrete probability distribution $\phi(j) = P\{\underline{\xi}_t = j\}$, $(j = 0,1,\ldots;$ $t = 1,2,\ldots)$.*) It is assumed that

$$\phi(0) < 1,\ \mu = \sum_{j=0}^{\infty} j\phi(j) < \infty.$$

At the beginning of each period the stock on hand and on order is reviewed. If, at review, the stock on hand and on order $i < s$, then S-i units are ordered; otherwise, no ordering is done. The numbers s and S are arbitrary but fixed integers with $s \leq S$. An order placed at the beginning of period t is delivered at the beginning of period $t+\lambda$, where $\lambda$ is a known non-negative integer. The demand is assumed to take place at the end of each period. It is assumed that if the demand exceeds the supply, then the excess demand is backlogged.

There is specified a fixed discount factor $\alpha$, $0 < \alpha \leq 1$, so that a unit

*) Throughout this chapter the letters i,j,k,m,n and t will be reserved to denote integers.

cost incurred n periods in future has a present value of $\alpha^n$. Note that $\alpha = 1$ corresponds to the situation that the costs are not discounted.

The following costs are considered. The cost of ordering z units is $K\delta(z) + cz$, where $K \geq 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z > 0$. Assume that the ordering cost is incurred at the time of delivery of the order. By an appropriate discounting of the ordering cost we can always take care that this assumption is satisfied. For example, if the ordering cost is actually incurred at the time of ordering, then $K\delta(z) + cz$ should be replaced by $\alpha^{-\lambda}\{K\delta(z) + cz\}$. Let $g(j)$ be the holding and shortage cost in a period (to be charged at the beginning of the period) when j is the amount of stock on hand just after any additions to stock in that period. Finally, in the (s,S) inventory model with a finite planning horizon it is assumed that stock left over at the end of the final period can be salvaged with a return of d per unit. Similarly, any backlogged demand remaining at the end of the final period can be satisfied by a unit cost of d per unit. Usually $d = 0$ or $d = c$ in the literature.

Let

$$(2.1.1) \qquad \phi^{(0)}(0) = 1, \quad \phi^{(0)}(j) = 0 \qquad \text{for } j = 1,2,\ldots .$$

and

$$(2.1.2) \qquad \phi^{(t)}(j) = \sum_{k=0}^{j} \phi^{(t-1)}(k)\, \phi(j-k), \qquad j = 0,1,\ldots;\ t = 1,2,\ldots .$$

Note that $\phi^{(1)}(j) = \phi(j)$. Clearly, $\phi^{(t)}(j) = P\{\underline{\xi}_1 + \ldots + \underline{\xi}_t = j\}$, $(j \geq 0;\ t \geq 1)$.

Let

$$\Phi^{(t)}(j) = \sum_{k=0}^{j} \phi^{(t)}(k), \qquad j = 0,1,\ldots;\ t = 0,1,\ldots .$$

Define

$$m(j;\alpha) = \sum_{t=1}^{\infty} \alpha^t \phi^{(t)}(j) \text{ and } M(j;\alpha) = \sum_{t=1}^{\infty} \alpha^t \Phi^{(t)}(j), \qquad j = 0,1,\ldots .$$

Note that $M(j;\alpha) = m(0;\alpha)+\ldots+m(j,\alpha)$ for $j \geq 0$. For the case $\alpha < 1$ we have $M(j;\alpha) \leq \alpha/(1-\alpha)$ for all $j \geq 0$. The function $M(j;1)$ is also finite, since it is a renewal function (cf. lemma 1.2.1 in section 1.2). It is interesting to note that $1+M(S-s;1)$ can be interpreted as the expected number of periods between two successive orderings when an (s,S) policy is followed (cf. section 1.2).

Using (2.1.1) and (2.1.2), we have

$$m(j;\alpha) = \alpha\phi(j) + \alpha \sum_{k=0}^{j} \phi(j-k)\, m(k;\alpha), \qquad j \geq 0. \tag{2.1.3}$$

For any $j \geq 0$, let

$$\hat{m}(j;1) = \sum_{k=1}^{\infty} k\phi^{(k)}(j) \text{ and } \hat{M}(j;1) = \sum_{k=1}^{\infty} k\Phi^{(k)}(j). \tag{2.1.4}$$

It follows easily from (1.2.8) in section 1.2 that

$$\hat{m}(j;1) = m(j;1) + \sum_{k=0}^{j} m(j-k;1)\, m(k;1), \qquad j \geq 0.$$

For any $i \geq s$, let

$$\rho_i(k) = \begin{cases} 0 & k = 0, \\ \Phi^{(k-1)}(i-s) - \Phi^{(k)}(i-s), & k \geq 1. \end{cases} \tag{2.1.5}$$

Since

$$P\{\underline{\xi}_1 > j\} = 1 - \Phi^{(1)}(j) = \Phi^{(0)}(j) - \Phi^{(1)}(j), \qquad j \geq 0$$

and

$$P\{\underline{\xi}_1+\dots+\underline{\xi}_{k-1} \leq j, \underline{\xi}_1+\dots+\underline{\xi}_k > j\} = \Phi^{(k-1)}(j) - \Phi^{(k)}(j), \quad j \geq 0; \; k \geq 2,$$

we see that $\rho_i(k)$ can be interpreted as the probability that the cumulative demand will first exceed i-s during the $k^{th}$ period. Clearly we have for any $i \geq s$ that

$$\sum_{k=0}^{n} \rho_i(k) = 1 - \Phi^{(n)}(i-s), \qquad n \geq 1 \tag{2.1.6}$$

and

$$\sum_{k=0}^{n} k\rho_i(k) = 1 + \sum_{k=1}^{n-1} \Phi^{(k)}(i-s) - n\Phi^{(n)}(i-s), \qquad n \geq 1, \tag{2.1.7}$$

where we adopt the convention $\sum_a^b = 0$ if $a > b$. By lemma 1.2.1 in section 1.2, we have that $\phi^{(n)}(k)$ converges exponentially fast to zero as $n \to \infty$ for each $k \geq 0$, and hence

$$(2.1.8) \qquad \sum_{k=0}^{\infty} \rho_i(k) = 1, \ \sum_{k=0}^{\infty} k\rho_i(k) = 1 + M(i-s;1), \qquad i \geq s.$$

Hence we have for each $i \geq s$ that $\{\rho_i(k),\ k \geq 0\}$ constitutes a probability distribution with a finite, positive first moment.

Let

$$\rho(k;\alpha) = \alpha^k \rho_S(k), \qquad k \geq 0.$$

Note that $\rho(k;1) = \rho_S(k)$ for $k \geq 0$, and hence $\{\rho(k;1),\ k \geq 0\}$ constitutes a probability distribution with a finite, positive first moment. Define

$$(2.1.9) \qquad \rho^{(1)}(j;\alpha) = \rho(j;\alpha), \qquad j \geq 0$$

and

$$(2.1.10) \qquad \rho^{(t)}(j;\alpha) = \sum_{k=0}^{j} \rho^{(t-1)}(k;\alpha)\ \rho(j-k;\alpha), \qquad j \geq 0; t \geq 2.$$

Let

$$(2.1.11) \qquad r(j;\alpha) = \sum_{t=1}^{\infty} \rho^{(t)}(j;\alpha), \qquad j \geq 0.$$

The function $r(j;1)$ is finite (cf. lemma 1.2.1 in section 1.2), and hence the function $r(j;\alpha)$, $j \in I$, is finite for any $0 < \alpha \leq 1$. Note that $r(0;\alpha) = 0$, since $\rho(0;\alpha) = 0$.*) From (2.1.9) - (2.1.11) it follows that

$$(2.1.12) \qquad r(j;\alpha) = \rho(j;\alpha) + \sum_{k=0}^{j} \rho(j-k;\alpha)\ r(k;\alpha), \qquad k \geq 0.$$

From (2.1.5) we have

$$(2.1.13) \qquad \sum_{k=0}^{\infty} \alpha^k \rho_i(k) = \alpha - (1-\alpha)\ M(i-s;\alpha), \qquad i \geq s.$$

Using (2.1.10) and (2.1.13), it is easily verified by induction on t that

*) Note also that $\rho^{(t)}(j;\alpha) = \alpha^j \rho^{(t)}(j;1)$, and so $r(j;\alpha) = \alpha^j r(j;1)$. The quantity $r(j;1)$ can be interpreted as the probability that an order will be done in period j+1 when the initial stock is S and the (s,S) policy is used. I am indebted to dr. J. Wessels for these observations.

$$\sum_{j=0}^{\infty} \rho^{(t)}(j,\alpha) = \{\alpha - (1-\alpha)\, M(S-s;\alpha)\}^{t}, \qquad t \geq 1,$$

and hence

$$(2.1.14) \qquad \sum_{j=0}^{\infty} r(j;\alpha) = \frac{\{\alpha-(1-\alpha)M(S-s;\alpha)\}}{(1-\alpha)\{1+M(S-s;\alpha)\}} \qquad \text{for } \alpha < 1.$$

The following lemma is well-known from analysis.

*Lemma 2.1.1.*

Let $\{a_n, n \geq 0\}$ and $\{b_n, n \geq 0\}$ be two sequences such that $a_n \geq 0$ and $\sum a_n < \infty$. Suppose $b$ is a finite number. Let the sequence $\{c_n, n \geq 0\}$ be defined by $c_n = a_0 b_n + \dots + a_n b_0$, $n \geq 0$.

(a) If $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} b_k = b$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} c_k = b \sum_{j=0}^{\infty} a_j$.

(b) If $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} c_n = b \sum_{j=0}^{\infty} a_j$.

*Proof*

(a) Since the sequence $\{(b_0 + \dots + b_n)/n, n \geq 1\}$ has a finite limit $b$, this sequence is bounded by some number $N$. Define $b_{-n} = 0$ for $n \geq 1$. Then

$$\frac{1}{n} \sum_{k=0}^{n} c_k = \frac{1}{n} \sum_{j=0}^{k} \sum_{k=0}^{n} a_j b_{k-j} = \sum_{j=0}^{\infty} a_j \frac{1}{n} \sum_{k=0}^{n} b_{k-j}, \qquad n \geq 1.$$

Since for any $j \geq 0$ the sequence $\{(b_{-j} + \dots + b_{n-j})/n, n \geq 1\}$ is bounded by $N$ and has the finite limit $b$, an application of the Lebesgue dominated convergence theorem yields (a).
The assertion (b) is also a special case of the Lebesgue dominated convergence theorem.

In the sequel we shall use frequently the following well-known fact without special mention in each application: If the sequence $\{a_n, n \geq 0\}$ has a finite limit $a$, i.e. $a = \lim_{n\to\infty} a_n$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} a_k = a.$$

### 2.1.2. *The transient and the steady-state behaviour of the stock level*

In this section we shall determine the transient and the steady state behaviour of both the stock on hand plus on order and the stock on hand.

Denote by $\underline{x}_t$ and $\underline{a}_t$, respectively, the stock on hand plus on order *just before ordering* and the stock on hand plus on order *just after ordering* in period t( = 1,2,...). For any $t \geq 1$, we have

$$(2.1.15)\qquad \underline{a}_t = \begin{cases} \underline{x}_t & \text{if } \underline{x}_t \geq s, \\ S & \text{if } \underline{x}_t < s. \end{cases}$$

The stochastic processes $\{\underline{x}_t, t \geq 1\}$ and $\{\underline{a}_t, t \geq 1\}$ are Markov chains. We take the set

$$I = \{0, \pm 1, \pm 2, \ldots\},$$

i.e. the set of all integers, as state space for the Markov chains $\{\underline{x}_t, t \geq 1\}$ and $\{\underline{a}_t, t \geq 1\}$.

For any $n \geq 0$, let

$$p_{ij}^{(n)} = P\{\underline{a}_{n+1} = j \mid \underline{x}_1 = i\},\ r_{ij}^{(n)} = P\{\underline{x}_{n+1} = j \mid \underline{x}_1 = i\} \text{ for } i,j \in I.$$

By (2.1.15) and the relation

$$(2.1.16)\qquad \underline{x}_{t+1} = \underline{a}_t - \underline{\xi}_t, \qquad t \geq 1,$$

we see that the probability distribution of $\underline{x}_{n+1}$ can be obtained from the probability distribution of $\underline{a}_n$. We have

$$(2.1.17)\qquad r_{ij}^{(n)} = \sum_{k=s}^{\max(i,S)} p_{ik}^{(n-1)}\, \phi(k-j), \qquad i, j \in I;\ n \geq 1,$$

where we define $\phi(k) = 0$ for $k \leq -1$.

We shall now determine the probability distribution of $\underline{a}_{n+1}$. By (2.1.15), we have for any $n \geq 0$ that

$$(2.1.18)\qquad p_{ij}^{(n)} = \begin{cases} \phi^{(n)}(i-j), & j > S;\ i \in I, \\ 0 & j < s,\ i \in I, \end{cases}$$

and

$$(2.1.19)\qquad p_{ij}^{(n)} = p_{Sj}^{(n)} \qquad\qquad \text{for } i < s;\ j \in I,$$

where

$$\phi^{(n)}(k) \overset{\text{def}}{=} 0 \qquad\qquad \text{for } k \leq -1;\ n \geq 0.$$

We have already seen that for any $i \geq s$ the probability that the cumulative demand will first exceed $i - s$ during the $k^{th}$ period is $\rho_i(k)$. Hence, by using a standard argument from renewal theory, we have for any $n \geq 0$ that

$$(2.1.20)\qquad p_{ij}^{(n)} = \phi^{(n)}(i-j) + \sum_{k=0}^{n} p_{Sj}^{(n-k)}\, \rho_i(k), \quad s \leq j \leq \max(i,S);\ i \geq s.$$

In particular,

$$(2.1.21)\qquad p_{Sj}^{(n)} = \phi^{(n)}(S-j) + \sum_{k=0}^{n} p_{Sj}^{(n-k)}\, \rho(k;1), \quad s \leq j \leq S;\ n \geq 0.$$

For any $j$, $s \leq j \leq S$, the equation (2.1.21) is a discrete renewal equation. Hence (cf. section 1.2 and (2.1.11))

$$(2.1.22)\qquad p_{Sj}^{(n)} = \phi^{(n)}(S-j) + \sum_{k=0}^{n} \phi^{(n-k)}(S-j)\, r(k;1), \quad s \leq j \leq S;\ n \geq 0.$$

The relations (2.1.18) - (2.1.20) and (2.1.22) in conjunction yield the probability distribution of $\underline{a}_{n+1}$. The probability distribution of $\underline{x}_{n+1}$ is given by (2.1.17). It is easy to see that above analysis can be adapted to the case in which the independent, non-negative and identically distributed demand variables $\underline{\xi}_1, \underline{\xi}_2, \ldots$ have an arbitrary distribution. In a different but laborious way the distribution of $\underline{x}_n$ has been found in [20] for the case in which the demand distribution has a continuous density.

Next we shall determine the steady-state behaviour of the Markov chains $\{\underline{a}_t\}$ and $\{\underline{x}_t\}$.

*Theorem 2.1.1.*

$$\text{(a)}\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} p_{ij}^{(k)} = q_j \qquad\qquad \text{for all } i, j \in I,$$

where

$$q_j = \begin{cases} \{1+m(0;1)\}/\{1+M(S-s;1)\} & \text{for } j = S, \\ m(S-j;1)/\{1+M(S-s;1)\} & \text{for } s \le j < S, \\ 0 & \text{otherwise.} \end{cases}$$

If the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1, then $\lim_{n\to\infty} p_{ij}^{(n)} = q_j$ for all i, $j \in I$.

(b) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} r_{ij}^{(k)} = v_j \qquad \text{for all } i, j \in I,$$

where

$$v_j = \begin{cases} \{\phi(S-j) + \sum_{k=0}^{S-s} \phi(S-j-k)\, m(k;1)\}/\{1+M(S-s;1)\}, & j < s, \\ m(S-j;1)/\{1+M(S-s;1)\}, & s \le j \le S, \\ 0, & j > S. \end{cases}$$

If the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1, then $\lim_{n\to\infty} r_{ij}^{(n)} = v_j$ for all i, $j \in I$.

(c) The probability distribution $\{q_j, j \in I\}[\{v_j, j \in I\}]$ is the unique stationary probability distribution of the Markov chain $\{\underline{a}_t, t \ge 1\}[\{\underline{x}_t, t \ge 1\}]$.

*Proof*

(a) From (2.1.18) and the fact that $\phi^{(n)}(k)$ converges to zero as $n \to \infty$ for each $k \ge 0$ (cf. lemma 1.2.1 in section 1.2) it follows that if $j \notin [s,S]$, then assertion (a) holds for each $i \in I$. By (2.1.21), (2.1.8) and theorem 1.2.2 in section 1.2, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} p_{Sj}^{(k)} = \sum_{n=0}^{\infty} \phi^{(n)}(S-j) / \sum_{n=0}^{\infty} n\rho(n;1) = q_j, \qquad s \le j \le S,$$

where the sequence $\{p_{Sj}^{(n)}, n \ge 0\}$ has an ordinary limit for any $s \le j \le S$ if the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1. Hence, by (2.1.19), we have also proved assertion (a) for all $i < s$. From (2.1.20), lemma 2.1.1, (2.1.8) and the fact that $\phi^{(n)}(k)$ converges to zero as $n \to \infty$ for each $k \ge 0$ it follows that assertion (a) also holds for any $i \ge s$.

(b) From (2.1.17) and assertion (a) it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} r_{ij}^{(k)} = \sum_{k=s}^{S} q_k \phi(k-j) \qquad \text{for all } i,\ j \in I,$$

where the ordinary limit exists for all i and j, if the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1.
Since $\phi(k) = 0$ for $k \leq -1$, we have

$$\sum_{k=s}^{S} q_k \phi(k-j) = 0 \qquad \text{for } j > S.$$

By (2.1.3) and $\phi(k) = 0$ for $k \leq -1$, we have

$$\begin{aligned}\sum_{k=s}^{S} q_k \phi(k-j) &= \{\phi(S-j) + \sum_{k=s}^{S} m(S-k;1)\phi(k-j)\}/\{1+M(S-s;1)\} = \\ &= \{\phi(S-j) + \sum_{h=0}^{S-s} \phi(S-j-h)m(h;1)\}/\{1+M(S-s;1)\} = \\ &= \begin{cases} m(S-j;1)/\{1+M(S-s;1)\}, & s \leq j \leq S, \\ \{\phi(S-j) + \sum_{h=0}^{S-s} \phi(S-j-h)m(h;1)\}/\{1+M(S-s;1)\}, & j < s. \end{cases}\end{aligned}$$

This ends the proof of assertion (b).

(c) Clearly, both $\{q_j\}$ and $\{v_j\}$ are probability distributions. Moreover, both the Markov chain $\{\underline{a}_t\}$ and $\{\underline{x}_t\}$ have no two disjoint closed sets. From the theory of Markov chains [7] it follows now that both $\{\underline{a}_t\}$ and $\{\underline{x}_t\}$ have a unique stationary probability distribution. This ends the proof.

It is easy to verify that the results of theorem 2.1.1 can be adapted to the case in which the demand variables have an arbitrary distribution.

For the case in which the demand distribution has a density, in [29,30] the stationary distribution for the Markov process $\{\underline{x}_n\}$ has been determined by solving the "balance equations" for the stationary distribution. Using hard analysis, it is shown in [29] that if the demand density is uniformly continuous, then the Markov process $\{\underline{x}_t\}$ has an ordinary limiting distribution. In the derivation of the latter result the condition that the demand density be positive is inadvertently omitted. The approach we have followed, however, is simpler than that in [29,30] and has the additional advantage that it yields also the time dependent solution for the distribution of the stock level. Moreover, we find a sufficient condition under which the stock level has an ordinary limiting distribution. Finally, it is easy to give an example in which the stock level has no ordinary limiting distribution. Suppose $\phi(1) = 1$, $s = 1$ and $S = 2$. If $\underline{x}_1 = 0$, then $\underline{x}_n = 1$ for n even and $\underline{x}_n = 0$ for n odd.

*Remark 2.1.1. A geometric demand distribution*

When the random variables $\underline{\xi}_1, \underline{\xi}_2, \ldots$ have a geometric distribution explicit expressions can be given for the probabilities $p_{ij}^{(n)}$, $r_{ij}^{(n)}$, $q_j$ and $v_j$. Suppose

$$\phi(0) = 0,\ \phi(j) = p(1-p)^{j-1} \qquad \text{for } j \geq 1,$$

where $0 < p \leq 1$. Then (cf. (1.2.20) in section 1.2)

$$m(0;1) = 0,\ m(j;1) = p \qquad \text{for } j \geq 1,$$

and hence, after some straightforward calculations, we obtain

$$q_j = \begin{cases} 1/\{1+p.(S-s)\} & \text{for } j = S, \\ p/\{1+p.(S-s)\} & \text{for } s \leq j < S, \end{cases}$$

and

$$v_j = \begin{cases} 0, & \text{for } j = S, \\ p/\{1+p.(S-s)\} & \text{for } s \leq j < S, \\ p(1-p)^{s-j-1}/\{1+p.(S-s)\} & \text{for } j < s. \end{cases}$$

We shall next determine the $p_{ij}^{(n)}$ and $r_{ij}^{(n)}$ explicitly. We shall first prove that

$$(2.1.23) \qquad \phi^{(k)}(m) - \phi^{(k+1)}(m) = \binom{m}{k} p^k (1-p)^{m-k}, \qquad k \geq 0; m \geq 0,$$

where we adopt the convention $\binom{m}{k} = 0$ for $k > m$. This relation can be proved by the following probabilistic argument. In a sequence of Bernoulli trials with the probability of success p we have that $\phi(j) = p(1-p)^{j-1}$ is the probability that the first success occurs at the $j^{th}$ trial. Hence $\Phi^{(k)}(m)$ is the probability that at least k successes occur in m Bernoulli trials. Consequently, $\Phi^{(k)}(m) - \Phi^{(k+1)}(m)$ is the probability that exactly k successes occur in m Bernoulli trials. This interpretation proves (2.1.23). In particular,

$$\rho(j;1) = \begin{cases} 0 & j = 0, \\ \binom{S-s}{j-1} p^{j-1}(1-p)^{S-s-j+1}, & j \geq 1. \end{cases}$$

By using the generating function approach we can now evaluate the renewal quantity $r(k;1)$ explicitly. Let

$$V(x) = \sum_{j=0}^{\infty} \rho(j;1)x^j, \quad R(x) = \sum_{j=0}^{\infty} r(j;1)x^j \qquad \text{for } |x| < 1.$$

By (2.1.12), we have $R(x) = V(x) + R(x)\,V(x)$, $|x| < 1$. Since $V(x) = x(1-p+px)^{S-s}$ for $|x| < 1$, we get

$$R(x) = V(x)/\{1-V(x)\} = \sum_{k=1}^{\infty} x^k(1-p+px)^{k(S-s)} =$$

$$= \sum_{k=1}^{\infty} x^k \sum_{m=1}^{\infty} \binom{k(S-s)}{m} (px)^m (1-p)^{k(S-s)-m}, \qquad |x| < 1.$$

By the uniqueness theorem for power-series, we obtain

$$r(j;1) = \sum_{k=1}^{j} \binom{k(S-s)}{j-k} \left(\frac{p}{1-p}\right)^{j-k} (1-p)^{k(S-s)}, \qquad j \geq 0.$$

Further the probabilities $\phi^{(n)}(j)$ can be given explicitly, since $\underline{\xi}_1+\ldots+\underline{\xi}_n$ has a negative binomial distribution. We have

$$\phi^{(n)}(j) = \binom{j-1}{n-1} p^n(1-p)^{j-n}, \qquad j \geq 0;\ n \geq 1.$$

Since $r(j;1)$ and $\phi^{(n)}(j)$ are explicitly found, we have also found an explicit expression for the probabilities $p_{ij}^{(n)}$ and $r_{ij}^{(n)}$ (cf. the formulas (2.1.17) - (2.1.22))

*Remark 2.1.2. The distribution of the stock on hand*

Denote by $\underline{y}_t$ and $\underline{b}_t$, respectively, the stock on hand *just before* any additions to stock and the stock on hand *just after* any additions to stock in period $t \geq 1$. If the lead time $\lambda$ is a positive integer, then the random variables $\underline{y}_1, \ldots, \underline{y}_\lambda$, $\underline{b}_1, \ldots, \underline{b}_\lambda$ can only be defined when the situation just before ordering in period 1 is specified. We assume that initially there are no outstanding orders.

If $\lambda = 0$, then $\underline{y}_t = \underline{x}_t$ and $\underline{b}_t = \underline{a}_t$ for $t \geq 1$. Consider the case in which $\lambda \geq 1$. Clearly

$$P\{\underline{y}_t = j | \underline{x}_1 = i\} = P\{\underline{b}_t = j | \underline{x}_1 = i\} = \phi^{(t-1)}(i-j), \quad i,j \in I; t = 1, \ldots, \lambda.$$

Since the lead time is fixed and excess demand is backlogged, we have

$$\underline{b}_t = \underline{a}_{t-\lambda} - \sum_{j=t-\lambda}^{t-1} \underline{\xi}_j, \; \underline{y}_t = \underline{b}_{t-1} - \underline{\xi}_{t-1}, \qquad t \geq \lambda + 1.$$

Hence

$$P\{\underline{b}_t = j | \underline{x}_1 = i\} = \sum_{k=s}^{\max(i,S)} p_{ik}^{(t-\lambda)} \phi^{(\lambda)}(k-j), \qquad i,j \in I; t \geq \lambda + 1$$

and

$$P\{\underline{y}_{\lambda+1} = j | \underline{x}_1 = i\} = \phi^{(\lambda)}(i-j), \; P\{\underline{y}_t = j | \underline{x}_1 = i\} =$$

$$= \sum_{k=s}^{\max(i,S)} p_{ik}^{(t-\lambda-1)} \phi^{(\lambda+1)}(k-j)$$

$$\text{for } i, j \in I; \; t \geq \lambda + 2.$$

By theorem 2.1.1, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} P\{\underline{b}_t = j | \underline{x}_1 = i\} = \sum_{k=s}^{S} q_k \, \phi^{(\lambda)}(k-j), \qquad i, j \in I,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} P\{\underline{y}_t = j | \underline{x}_1 = i\} = \sum_{k=s}^{S} q_k \, \phi^{(\lambda+1)}(k-j), \qquad i, j \in I,$$

where the ordinary limit exists for all $i, j \in I$ if the greatest common divisor of the indices $n$, where $\rho(n;1) > 0$, is 1.

### *2.1.3. The total discounted expected cost*

Assume that the function

$$L(k) = \sum_{j=0}^{\infty} g(k-j)\ \phi^{(\lambda)}(j), \qquad k \in I,$$

exists and is finite. Clearly $L(k)$ represents the expected holding and shortage cost in period $t+\lambda$ when $k$ is the stock on hand plus on order just after ordering in period $t$.

Let

$$G_\alpha(k) = L(k) + c(1-\alpha)k + \alpha c\mu, \qquad k \in I.$$

Note that $G_1(k) = L(k) + c\mu$.

We shall first consider the n-period (s,S) inventory model. In the n-period model ordering decisions are only made in the periods $1,\dots, n$ and only the expected discounted cost over the periods $\lambda+1,\dots, \lambda+n$ is taken into account. Remember that both stock left over and backlogged demand remaining at the end of period $\lambda+n$ have a salvage value in the n-period model. All costs are discounted to the beginning of period $\lambda+1$.

Define

$$g_0(i;\alpha) = 0 \qquad \text{for } i \in I$$

and

$$g_n(i;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E\{K\delta(\underline{a}_t-\underline{x}_t) + (\underline{a}_t-\underline{x}_t)c + L(\underline{a}_t) | \underline{x}_1 = i\} +$$
$$-\alpha^n dE\{\underline{x}_{n+1} - (\xi_{n+1}+\dots+\xi_{n+\lambda}) | \underline{x}_1 = i\}, \ i \in I;\ n \geq 1.$$

Note that from (2.1.15), (2.1.16) and the fact that $\mu = E\underline{\xi}_t < \infty$ it follows that the expectations exist and are finite. Clearly $g_n(i;\alpha)$ represents the expected discounted cost over the periods $\lambda+1,\dots,\lambda+n$ in the n-period (s,S) model when $\underline{x}_1 = i$.

We shall first write the formula for $g_n(i;\alpha)$ in a more convenient form. Using (2.1.16), we have (cf. [48])

$$g_n(i;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E\{K\delta(\underline{a}_t-\underline{x}_t) + (1-\alpha)c\ \underline{a}_t + \alpha c\underline{\xi}_t + L(\underline{a}_t) | \underline{x}_1 = i\} +$$
$$-\ ci + \alpha^n d\lambda\mu - \alpha^n(d-c)\ E(\underline{x}_{n+1} | \underline{x}_1 = i\}, \qquad i \in I;\ n \geq 1,$$

and hence

$$\text{(2.1.24)} \qquad g_n(i;\alpha) = g_n^*(i;\alpha) - ci + \alpha^n d\lambda\mu - \alpha^n(d-c)E(\underline{x}_{n+1}|\underline{x}_1 = i\},\ \ i \in I;\ n \geq 1,$$

where

$$g_n^*(i;\alpha) = \begin{cases} 0, & i \in I;\ n = 0, \\ \sum_{t=1}^{n} \alpha^{t-1} E\{K\delta(\underline{a}_t - \underline{x}_t) + G_\alpha(\underline{a}_t) | \underline{x}_1 = i\}, & i \in I;\ n \geq 1. \end{cases}$$

We shall next determine $g_n^*(i;\alpha)$. Clearly,

$$\text{(2.1.25)} \quad g_n^*(i;\alpha) = K + g_n^*(S;\alpha), \qquad i < s;\ n \geq 1.$$

The probability that the cumulative demand will first exceed i-s during the $k^{th}$ period is $\rho_i(k)$, $(i \geq s)$, and hence, by using a standard argument from renewal theory, we have

$$g_n^*(i;\alpha) = G_\alpha(i) + \sum_{k=1}^{n-1} \sum_{j=0}^{i-s} \alpha^k G_\alpha(i-j)\phi^{(k)}(j) + \sum_{k=1}^{n-1} \alpha^k \{K + g_{n-k}^*(S;\alpha)\}\rho_i(k)$$

for $i \geq s;\ n \geq 1$,

and hence

$$\text{(2.1.26)} \quad g_n^*(i;\alpha) = b_n(i;\alpha) + \sum_{k=0}^{n} g_{n-k}^*(S;\alpha)\ \alpha^k \rho_i(k), \qquad i \geq s;\ n \geq 0,$$

where

$$b_n(i;\alpha) = \begin{cases} 0 & i \geq s;\ n = 0, \\ G_\alpha(i) + \sum_{k=1}^{n-1} \sum_{j=0}^{i-s} G_\alpha(i-j)\alpha^k \phi^{(k)}(j) + \\ \qquad + K \sum_{k=1}^{n-1} \alpha^k \rho_i(k), & i \geq s;\ n \geq 1. \end{cases}$$

In particular,

$$\text{(2.1.27)} \quad g_n^*(S;\alpha) = b_n(S;\alpha) + \sum_{k=0}^{n} g_{n-k}^*(S;\alpha)\ \rho(k;\alpha), \qquad n \geq 0.$$

We are now in a position to prove the following lemma.

*Lemma 2.1.2.*

$$g_n(i;\alpha) = K+b_n(S;\alpha) + \sum_{k=0}^{n} b_{n-k}(S;\alpha)\ r(k;\alpha) - \alpha^n(d-c)(\sum_{j=s}^{S} jp_{Sj}^{(n-1)} - \mu)$$

$$- ci + \alpha^n d\lambda\mu, \qquad i < s;\ n \geq 1$$

and

$$g_n(i;\alpha) = b_n(i;\alpha) + \sum_{k=0}^{n} \{b_{n-k}(S;\alpha) + \sum_{j=0}^{n-k} b_{n-k-j}(S;\alpha)\ r(j;\alpha)\}\alpha^k \rho_i(k) +$$

$$- \alpha^n(d-c)(\sum_{j=s}^{\max(i,S)} jp_{ij}^{(n-1)} - \mu) - ci + \alpha^n d\lambda\mu, \quad i \geq s;\ n \geq 1.$$

*Proof*

Iterating (2.1.27) and using (2.1.10), yields for any $n \geq 0$

$$(2.1.28) \qquad g_n^*(S;\alpha) = b_n(S;\alpha) + \sum_{i=1}^{t} \sum_{k=0}^{n} b_{n-k}(S;\alpha)\ \rho^{(i)}(k;\alpha) + $$

$$+ \sum_{k=0}^{n} g_{n-k}^*(S;\alpha)\ \rho^{(t+1)}(k;\alpha), \qquad t \geq 1.$$

Since $\rho^{(t)}(k;\alpha) \leq \rho^{(t)}(k;1)$ and the probability $\rho^{(t)}(k;1)$ tends to zero as $t \to \infty$ for each $k \geq 0$, we obtain from (2.1.28)

$$(2.1.29) \qquad g_n^*(S;\alpha) = b_n(S;\alpha) + \sum_{k=0}^{n} b_{n-k}(S;\alpha)\ r(k;\alpha), \qquad n \geq 0.$$

The lemma follows now from (2.1.29), (2.1.26) - (2.1.24), and the fact that $E\ \underline{x}_{t+1} = E\ \underline{a}_t - \mu$, $(t \geq 1)$.

We shall next determine anew the known solution for the total expected discounted cost in the infinite period (s,S) inventory model with $\alpha < 1$.

*Theorem 2.1.2.*

Let $\alpha < 1$. Then

$$\lim_{n\to\infty} g_n(i;\alpha) = \frac{g_\alpha}{1-\alpha} - ci \qquad \text{for } i < s,$$

and

$$\lim_{n\to\infty} g_n(i;\alpha) = G_\alpha(i) + \sum_{j=0}^{i-s} G_\alpha(i-j)\ m(j;\alpha) +$$

$$+ \frac{g_\alpha}{1-\alpha}\{\alpha - (1-\alpha)\ M(i-s;\alpha)\} - ci \qquad \text{for } i \geq s,$$

where

$$g_\alpha = \frac{G_\alpha(S) + \sum_{k=0}^{S-s} G_\alpha(S-k)\, m(k;\alpha) + K}{1+M(S-s;\alpha)} .$$

*Proof*

Using (2.1.13), we have

$$\text{(2.1.30)}\quad \lim_{n\to\infty} b_n(i;\alpha) = G_\alpha(i) + \sum_{j=0}^{i-s} G_\alpha(i-j)\, m(j;\alpha) + K\{\alpha - (1-\alpha)\, M(i-s;\alpha)\}, \qquad i \geq s.$$

From (2.1.14), lemma 2.1.1(b), (2.1.29) and (2.1.30) we obtain

$$\text{(2.1.31)}\quad \lim_{n\to\infty} g_n^*(S;\alpha) = \lim_{n\to\infty} b_n(S;\alpha)/[(1-\alpha)\{1+M(S-s;\alpha)\}].$$

From (2.1.13), lemma 2.1.1(b), (2.1.26) and (2.1.31) it follows that

$$\text{(2.1.32)}\quad \lim_{n\to\infty} g_n^*(i;\alpha) = \{\alpha - (1-\alpha)\, M(i-s;\alpha)\} \lim_{n\to\infty} g_n^*(S;\alpha) + \lim_{n\to\infty} b_n(i;\alpha), \qquad i \geq s.$$

By (2.1.25), we have

$$\text{(2.1.33)}\quad \lim_{n\to\infty} g_n^*(i;\alpha) = K + \lim_{n\to\infty} g_n^*(S;\alpha), \qquad i < s.$$

From (2.1.24) it follows that

$$\lim_{n\to\infty} g_n(i;\alpha) = \lim_{n\to\infty} g_n^*(i;\alpha) - ci, \qquad i \in I.$$

The theorem now follows after some straightforward calculations from the formulas (2.1.30) - (2.1.33). This ends the proof.

For $\alpha < 1$, let

$$W(i;\alpha) = \lim_{n\to\infty} g_n(i;\alpha), \qquad i \in I.$$

In [48] the solution for $W(i;\alpha)$, $i \in I$, has been obtained by solving an equation for $W(i;\alpha)$ which is similar to the relation (2.1.26) for $g_n^*(i;\alpha)$. The total expected discounted cost $W(i;\alpha)$, $i \in I$, can also be determined by solving the following equation for $W(i;\alpha)$ (cf. [2,43]).

$$W(i;\alpha) = \begin{cases} K + (S-i)c + W(S;\alpha), & i < s, \\ G_\alpha(i) + \alpha \sum_{j=0}^{\infty} W(i-j;\alpha)\, \phi(j), & i \geq s. \end{cases}$$

This relation for $W(i;\alpha)$, $i \geq s$, can be easily converted in the standard form of the (defective) renewal equation.

*Remark 2.1.3.*

A direct consequence of theorem 2.1.2 is

$$\lim_{\alpha \uparrow 1} (1-\alpha)\, W(i;\alpha) = g_1 \qquad \text{for all } i \in I,$$

where

$$(2.1.34) \qquad g_1 = \frac{G_1(S) + \sum_{k=0}^{S-s} G_1(S-k)\, m(k;1) + K}{1+M(S-s;1)} \; .$$

*2.1.4. The average expected cost and the asymptotic behaviour of the total expected cost*

It is well-known that the average expected cost per period for the infinite period (s,S) model is $g_1$ for each initial stock [2,27,29].
We shall prove this result in the following theorem.*)

*Theorem 2.1.3.*

$$\lim_{n \to \infty} \frac{g_n(i;1)}{n} = \frac{G_1(S) + \sum_{k=0}^{S-s} G_1(S-k)\, m(k;1) + K}{1+M(S-s;1)} \qquad \text{for all } i \in I.$$

*Proof*

By (2.1.24), we have

*) This theorem follows also directly from remark 2.1.3 and the Tauberian theorem of Hardy - Littlewood - Karamata (cf. [47], p. 226).

$$\frac{g_n(i;1)}{n} = K \sum_{j<s} \frac{1}{n} \sum_{t=0}^{n-1} r_{ij}^{(t)} + \sum_{j=s}^{\max(i,S)} G_1(j) \frac{1}{n} \sum_{t=0}^{n-1} p_{ij}^{(t)} - \frac{ci}{n} + \frac{d\lambda\mu}{n} +$$

$$- \frac{(d-c)}{n} E(\underline{x}_{n+1}|\underline{x}_1=i), \qquad i \in I;\ n \geq 1.$$

Since $\{\frac{1}{n} \sum_{t=0}^{n-1} r_{ij}^{(t)},\ j \leq \max(i,S)\}$ constitutes a probability distribution for any $i \in I$ and $n \geq 1$, we obtain by using theorem 2.1.1(b) that

$$\lim_{n\to\infty} \sum_{j<s} \frac{1}{n} \sum_{t=0}^{n-1} r_{ij}^{(t)} = \lim_{n\to\infty} \{1 - \sum_{j=s}^{\max(i;S)} \frac{1}{n} \sum_{t=0}^{n-1} r_{ij}^{(t)}\} =$$

$$= 1 - \sum_{j=s}^{S} v_j = \frac{1}{1+M(S-s;1)} \qquad \text{for all } i \in I.$$

From theorem 2.1.1 (a) it follows that

$$\lim_{n\to\infty} \sum_{j=s}^{\max(i,S)} G_1(j) \frac{1}{n} \sum_{t=0}^{n-1} p_{ij}^{(t)} = \sum_{j=s}^{S} G_1(j)\ q_j =$$

$$= \{G_1(S) + \sum_{k=0}^{S-s} G_1(S-k)\ m(k;1)\}/\{1+M(S-s;1)\} \quad \text{for all } i \in I.$$

Finally, it follows from (2.1.15) and (2.1.16) that the sequence $\{E(\underline{x}_{n+1}|\underline{x}_1=i),\ n \geq 0\}$ is bounded, and hence $\frac{1}{n} E(\underline{x}_{n+1}|\underline{x}_1=i\}$ tends to zero as $n \to \infty$ for any $i \in I$. This ends the proof.

We shall next determine the asymptotic behaviour of the sequence $\{g_n(i;1) - ng_1,\ n \geq 1\}$.

Define

$$h_n(i) = g_n^*(i;1) - ng_1, \qquad i \in I;\ n \geq 0.$$

By (2.1.24), we have

$$(2.1.35) \qquad g_n(i;1) = ng_1+h_n(i)-ci+d\lambda\mu-(d-c)\ E(\underline{x}_{n+1}|\ \underline{x}_1=i\}, \qquad i \in I;\ n \geq 1.$$

From (2.1.25)

$$(2.1.36) \qquad h_n(i) = K + h_n(S), \qquad i < s;\ n \geq 1.$$

By (2.1.26), we have

$$h_n(i) = b_n(i;1)+\sum_{k=0}^{n} h_{n-k}(S)\rho_i(k)+g_1\sum_{k=0}^{n}(n-k)\rho_i(k)-ng_1, \quad i \geq s; n \geq 0.$$

Using (2.1.6) and (2.1.7), we obtain after some straightforward calculations

$$(2.1.37) \qquad h_n(i) = c_n(i) + \sum_{k=0}^{n} h_{n-k}(S)\ \rho_i(k), \qquad i \geq s;\ n \geq 0,$$

where $c_0(i) = 0$ for $i \geq s$, and

$$(2.1.38) \qquad c_n(i) = G_1(i) + \sum_{k=1}^{n-1}\sum_{j=0}^{i-s} G_1(i-j)\phi^{(k)}(j) + K\{1-\Phi^{(n-1)}(i-s)\} +$$

$$- g_1\{1 + \sum_{k=1}^{n-1} \Phi^{(k)}(i-s)\} \qquad \text{for } i \geq s;\ n \geq 1.$$

In particular,

$$(2.1.39) \qquad h_n(S) = c_n(S) + \sum_{k=0}^{n} h_{n-k}(S)\ \rho(k;1), \qquad n \geq 0.$$

This equation is a discrete renewal equation, and hence (cf. section 1.2)

$$(2.1.40) \qquad h_n(S) = c_n(S) + \sum_{k=0}^{n} c_{n-k}(S)\ r(k;1), \qquad n \geq 0.$$

A direct consequence of (2.1.35) - (2.1.37), (2.1.40) and the fact $E(\underline{x}_{n+1}|\underline{x}_1=i) = E(\underline{a}_n|\underline{x}_1=i) - \mu$, $(n \geq 1)$, is the following lemma.

*Lemma 2.1.3.*

$$g_n(i;1) = ng_1 + c_n(S) + \sum_{k=0}^{n} c_{n-k}(S)\ r(k;1) + K - ci + d\lambda\mu +$$

$$- (d-c)\ (\sum_{j=s}^{S} jp_{Sj}^{(n-1)} - \mu), \qquad i < s;\ n \geq 1,$$

and

$$g_n(i;1) = ng_1 + c_n(i) + \sum_{k=0}^{n}\{c_{n-k}(S) + \sum_{j=0}^{n-k} c_{n-k-j}(S)r(j;1)\}\rho_i(k) +$$

$$+ d\lambda\mu - ci - (d-c)\ (\sum_{j=s}^{\max(i,S)} jp_{ij}^{(n-1)} - \mu), \qquad i \geq s;\ n \geq 1.$$

To determine the asymptotic behaviour of $g_n(i;1) - ng_1$, we note first that, by lemma 1.2.1 in section 1.2 and (2.1.38),

$$\lim_{n\to\infty} c_n(i) = v(i), \qquad i \geq s, \tag{2.1.41}$$

where

$$v(i) = G_1(i) + \sum_{j=0}^{i-s} G_1(i-j)m(j;1) + K - g_1\{1 + M(i-s;1)\}, \quad i \geq s. \tag{2.1.42}$$

Moreover, it follows from lemma 1.2.1 that $c_n(i)$ tends exponentially fast to $v(i)$ as $n \to \infty$ for each $i \geq s$. From the definition (2.1.34) of $g_1$ it follows that $v(S) = 0$, and hence

$$\sum_{n=0}^{\infty} |c_n(S)| < \infty. \tag{2.1.43}$$

We shall next determine $\sum c_n(S)$. From (2.1.38) we obtain after some calculations

$$\sum_{n=0}^{N} c_n(S) = d_{1N} - d_{2N} - d_{3N} + d_{4N}, \qquad N \geq 1,$$

where

$$d_{1N} = N\{G_1(S) + \sum_{j=0}^{S-s} G_1(S-j) \sum_{k=1}^{N} \phi^{(k)}(j) + K\},$$

$$d_{2N} = \sum_{j=0}^{S-s} G_1(S-j) \sum_{k=1}^{N} k\phi^{(k)}(j) + K \sum_{n=1}^{N} \phi^{(n-1)}(S-s),$$

and

$$d_{3N} = Ng_1\{1 + \sum_{k=1}^{N} \phi^{(k)}(S-s)\}, \quad d_{4N} = g_1 \sum_{k=1}^{N} k\phi^{(k)}(S-s).$$

Using the definition (2.1.34) of $g_1$ and using lemma 1.2.1 in section 1.2, it is readily verified that

$$\lim_{N\to\infty} d_{1N} - d_{3N} = 0,$$

and hence

$$\sum_{n=0}^{\infty} c_n(S) = -\sum_{j=0}^{S-s} G_1(S-j)\hat{m}(j;1) - K\{1+M(S-s;1)\} + g_1 \hat{M}(S-s;1) = \tag{2.1.44}$$

$$= \sum_{j=0}^{S-s} \{g_1 - G_1(S-j)\}\hat{m}(j;1) - K\{1+M(S-s;1)\}.$$

From theorem 1.2.2 in section 1.2, (2.1.8), (2.1.39), (2.1.43) and (2.1.44) it follows that

$$(2.1.45)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} h_k(S) = \frac{\sum_{n=0}^{\infty} c_n(S)}{\sum_{n=0}^{\infty} n\rho(n;1)} = \frac{\sum_{j=0}^{S-s} \{g_1 - G_1(S-j)\}\hat{m}(j;1)}{1+M(S-s;1)} - K,$$

where the ordinary limit exists if the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1.
Next, it follows from (2.1.8), lemma 2.1.1, (2.1.36), (2.1.37), (2.1.41) and (2.1.45) that

$$(2.1.46)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} h_k(i) = \begin{cases} K + \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} h_k(S), & i < s, \\ v(i) + \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} h_k(S), & i \geq s, \end{cases}$$

where the ordinary limit exists for all $i \in I$, if the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1.
Further, from (2.1.16) and theorem 2.1.1 (a) it follows that

$$(2.1.47)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} E(\underline{x}_{k+1} \mid \underline{x}_1 = i) = \sum_{j=s}^{S} jq_j - \mu, \qquad i \in I,$$

where the ordinary limit exists for all $i \in I$ if the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1.

By (2.1.35), (2.1.42) and (2.1.45) - (2.1.47), we have the following result.

*Theorem 2.1.4.*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{g_k(i;1) - kg_1\} = \frac{\sum_{j=0}^{S-s} \{g_1 - G_1(S-j)\}\hat{m}(j;1)}{1+M(S-s;1)} - ci + d\lambda\mu +$$

$$- (d-c)\left(\sum_{j=s}^{S} jq_j - \mu\right) \qquad \text{for } i < s,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{g_k(i;1) - kg_1\} = G_1(i) + \sum_{j=0}^{i-s} G_1(i-j)\, m(j;1) +$$

$$- g_1\{1+M(i-s;1) + \frac{\sum_{j=0}^{S-s} \{g_1 - G_1(S-j)\}\hat{m}(j;1)}{1+M(S-s;1)} +$$

$$- ci + d\lambda\mu - (d-c)(\sum_{j=s}^{S} jq_j - \mu) \qquad \text{for } i \geq s,$$

where the ordinary limit exists for each $i \in I$ if the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1.

We note that theorem 2.1.3 is a direct corollary of theorem 2.1.4.

*Remark 2.1.4.*

A direct consequence of (2.1.35) and (2.1.46) is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \{g_k(i;1) - g_k(j;1)\} = v'(i) - v'(j) \quad \text{for all } i, j \in I,$$

where

$$v'(i) = \begin{cases} - ci + d\lambda\mu, & i < s, \\ G_1(i) + \sum_{j=0}^{i-s} G_1(i-j)m(j;1) - g_1\{1+M(i-s;1)\} - ci + d\lambda\mu, & i \geq s. \end{cases}$$

The ordinary limit exists for any $i, j \in I$, if the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1. Clearly, the quantity $v'(i)-v'(j)$ is a measure for the difference in total expected cost for the infinite period (s,S) model for initial stocks i and j.

*Remark 2.1.5.*

If s = S the results of theorem 2.1.4 can be further specified. Let $s = S = \bar{x}$. Then

$$\rho(k;1) = \{\phi(0)\}^{k-1} - \{\phi(0)\}^k = \{\phi(0)\}^{k-1}\{1 - \phi(0)\}, \qquad k \geq 1,$$

and hence by $\phi(0) < 1$ we have that $\rho(1;1) > 0$. Thus the sequence $\{g_n(i;1) - ng_1, n \geq 0\}$ is convergent for each $i \in I$. Using the relations

$$m(0;1) = \frac{\phi(0)}{1-\phi(0)}\ ,\ \hat{m}(0;1) = \frac{\phi(0)}{\{1-\phi(0)\}^2}\ ,\ g_1 = G_1(\bar{x}) + K(1-\phi(0)),$$

it is straightforward to verify that

$$\lim_{n\to\infty}[g_n(i;1)-n\{G_1(\bar{x})+K(1-\phi(0))\}] = K\phi(0)-ci+d\lambda\mu-(d-c)(\bar{x}-\mu), \qquad i < \bar{x}$$

and

$$\lim_{n\to\infty}[g_n(i;1)-n\{G_1(\bar{x})+K(1-\phi(0))\}] = G_1(i) + \sum_{j=0}^{i-\bar{x}} G_1(i-j)m(j;1) +$$
$$-\{G_1(\bar{x})+K(1-\phi(0))\}\{1+M(i-\bar{x};1)\}+K\phi(0)-ci+d\lambda\mu-(d-c)(\bar{x}-\mu), \qquad i \geq \bar{x}.$$

Finally, we shall give an example in which $\{g_n(i;1) - ng_1\}$ is divergent for each $i \in I$. Moreover, this counterexample will be of interest with regard to a conjecture of Iglehart [27].

Let $s = 1$ and $S = 2$. Suppose $\phi(1) = 1$, i.e. the demand in each period is 1, $c = d = 0$ and $K = 1$. Let $\lambda = 0$ and let the holding and shortage cost function $g(j)$ be such that $g(1) = g(2) = 0$, $g(j) > 1$ for $j \neq 1,2$, $g(j)$ is increasing for $j \geq 2$, and $g(j)$ is decreasing for $j \leq 1$. Note that $G_1(k) = L(k) = g(k)$ in this example.
Clearly

$$g_n(0;1) = \frac{n+1}{2} \text{ if } n \text{ odd, and } g_n(0;1) = \frac{n}{2} \text{ if } n \text{ even.}$$

Further $g_1 = \frac{1}{2}$. Hence the sequence $\{g_n(0;1) - ng_1,\ n \geq 0\}$ is divergent. Since $g_n(i;1) = g(i) + \ldots + g(1) + g_{n-i}(0;1)$ for $i \geq 1$; $n \geq i$, and $g_n(i;1) = g_n(0;1)$ for $i \leq 0$, we see that the sequence $\{g_n(i;1) - ng_1\}$ is divergent for all $i \in I$.

For this example it is readily verified that both for the finite period model with the total expected cost as optimality criterion and for the infinite period model with the average expected cost as optimality criterion the (1,2) policy is optimal among the class of all possible policies.

In [27] the conjecture is offered that if the ordering cost is of the form $K\delta(z) + c.z$, if $d = 0$ and if the one period holding and shortage cost function $g(j)$ is convex, then the minimal total expected cost in the n-period model minus n times the minimal average expected cost in the infinite period model has a finite limit for each initial stock. Hence above counterexample shows that this conjecture does not hold generally if the demand is bounded.

Probably, above conjecture is true when the ordinary limit is replaced by the Cesàro limit.

## 2.2. THE EXISTENCE OF OPTIMAL (s,S) POLICIES IN THE INFINITE PERIOD MODEL

In this section the infinite period inventory model will be considered for both the total expected discounted cost and the average cost criteria. Under certain conditions a unified proof for the existence of optimal (s,S) policies will be given. Finally, we shall prove a uniqueness theorem for the optimal inventory equation.

### *2.2.1. Model and preliminaries*

We consider the dynamic, infinite period inventory model in which the demands $\underline{\xi}_1, \underline{\xi}_2, \ldots$ for a single item in periods $t = 1,2,\ldots$ are mutually independent, non-negative and identically distributed random variables with the discrete probability distribution $\phi(j) = P\{\underline{\xi}_t = j\}$, $(j \geq 0; t \geq 1)$. It is assumed that

$$\phi(0) < 1, \quad \mu = \sum_{j=1}^{\infty} j\phi(j) < \infty.$$

At the beginning of each period the stock on hand plus on order is reviewed. An order may then be placed for any positive, integral quantity of stock. An order placed at the beginning of period t is delivered at the beginning of period $t+\lambda$, where $\lambda$ is a known non-negative integer. The demand is assumed to take place at the end of each period. All unfilled demand is backlogged until it is satisfied by a subsequent delivery.

A fixed discount factor $\alpha$, $0 < \alpha \leq 1$, is specified so that a unit cost incurred n periods in future has a present value of $\alpha^n$.

The following costs are considered. The cost of ordering z units is $K\delta(z) + cz$, where $K \geq 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z > 0$. Assume that the ordering cost is incurred at the time of delivery of the order. Let $g(j)$ be the holding and shortage cost in a period when i is the amount of stock on hand at the beginning of that period just after any additions to stock.

Let $\phi^{(0)}(0) = 1$, $\phi^{(0)}(j) = 0$ for $j > 0$, let $\phi^{(n)}(j)$ be the n-fold convolution of $\phi(j)$ with itself (cf. subsection 2.1.1), and let

$$m(j;\alpha) = \sum_{n=1}^{\infty} \alpha^n \phi^{(n)}(j), \quad M(j;\alpha) = \sum_{n=1}^{\infty} \alpha^n \phi^{(n)}(j), \qquad j \geq 0,$$

where $\Phi^{(n)}(j) = \phi^{(n)}(0) + \ldots + \phi^{(n)}(j)$, $j \geq 0$. When $n = 1$, we often write $\Phi^{(1)}(j) = \Phi(j)$. Assume that the function

$$L(k) = \sum_{j=0}^{\infty} g(k-j)\ \phi^{(\lambda)}(j), \qquad k \in I,$$

exists and is finite. I denotes the set of all integers. Clearly, $L(k)$ represents the expected holding and shortage cost in period $t+\lambda$ when $k$ is the stock on hand plus on order just after ordering in period t. Define

$$G_\alpha(k) = L(k) + (1-\alpha)ck + \alpha c\mu, \qquad k \in I.$$

The following conditions are imposed on $G_\alpha(k)$:

(i) There exists a finite integer $S_0$ such that $G_\alpha(i) \leq G_\alpha(j)$ for $j \leq i \leq S_0$ and $G_\alpha(i) \geq G_\alpha(j)$ for $i \geq j \geq S_0$,

(ii) $\lim_{|k| \to \infty} G_\alpha(k) = \infty$.

In words, it is assumed that there exists an integer $S_0$ such that $G_\alpha(k)$ is non-increasing for $k \leq S_0$ and $G_\alpha(k)$ is non-decreasing for $k \geq S_0$, where $G_\alpha(k)$ tends to $\infty$ as $|k| \to \infty$.

We shall assume that $S_0$ is the smallest integer at which $G_\alpha(k)$ takes on its absolute minimum. Let $S^0$ be the largest integer at which $G_\alpha(k)$ takes on its absolute minimum. Usually $S_0 = S^0$. Note that $G_\alpha(k) = G_\alpha(S_0)$ for $S_0 \leq k \leq S^0$.

Let $s_1$ be the smallest integer for which

$$G_\alpha(s_1) \leq G_\alpha(S_0) + (1-\alpha\phi(0))K \tag{2.2.1}$$

and let $S_1$ be the largest integer for which

$$G_\alpha(S_1) \leq G_\alpha(S_0) + \alpha K. \tag{2.2.2}$$

Note that $s_1 \leq S_0 \leq S^0 \leq S_1$.

We shall now formulate the inventory model as a Markovian decision model. At the beginnings of the periods 1,2,... the inventory system is observed to be in one of a possible number of states. The state of the system is defined as the stock on hand plus on order just before ordering. We take the set I of all integers as the set of all possible states. At the beginning of each period an ordering decision must be made, where any ordering decision is based on the stock on hand plus on order. Every ordering decision can be represented by the stock on hand plus on order just after ordering. Therefore, we say that in state i decision $a \geq i$ is made when $a-i$ units are ordered. We impose the following mild restriction on the choice of an ordering decision. Finite integers u and U are given such that nothing is ordered in state i with $i \geq U$, at most $U-i$ units are ordered in state i with $i < U$, and at least $u-i$ units are ordered in state i with $i < u$. We take $u \leq s_1$ and $U \geq S_1$, but for the rest u and U may be chosen arbitrarily. Let $A(i)$ denote the set of feasible decisions in state i. Then

$$A(i) = \begin{cases} \{i\} & \text{for } i \geq U, \\ \{i, i+1, \ldots, U\} & \text{for } u \leq i < U, \\ \{u, u+1, \ldots, U\} & \text{for } i < u. \end{cases}$$

An order, placed at the beginning of period t, cannot influence the holding and shortage costs incurred between the beginnings of period t and $t+\lambda$. Further, we consider only expected costs in our optimality criteria. Therefore, we may assign to decision a in state i the direct (expected) costs $K\delta(a-i) + (a-i)c + L(a)$. It will now be clear that the inventory model can be regarded as a Markovian decision model with I as state space, $A(i)$ as the set of possible decisions in state $i \in I$,

$$q_{ij}(a) = \begin{cases} \phi(a-j), & j \leq a;\ a \in A(i);\ i \in I, \\ \\ 0, & j > a;\ a \in A(i);\ i \in I, \end{cases}$$

and

$$w_{ia} = K\delta(a-i) + (a-i)c + L(a), \qquad a \in A(i);\ i \in I.$$

Denote by C the class of all possible policies for the inventory system (cf. section 1.1). We suppress the dependence of C on u and U.

Given an initial state $i \in I$ and a policy R to be used, let

$\underline{x}_t$ = stock on hand plus on order *just before* ordering in period t,
$\underline{a}_t$ = stock on hand plus on order *just after* ordering in period t.

Clearly

$$(2.2.3) \qquad u \le \underline{a}_t \le \max(\underline{x}_1, U), \qquad t \ge 1$$

and

$$(2.2.4) \qquad \underline{x}_{t+1} = \underline{a}_t - \underline{\xi}_t, \qquad t \ge 1.$$

From (2.2.3), (2.2.4) and the fact that $\mu = E\underline{\xi}_1 < \infty$ it follows that for each $i \in I$ there exists a finite constant $B_i$ such that for all $R \in C$ and $t \ge 1$,

$$(2.2.5) \qquad E_R\{|K\delta(\underline{a}_t-\underline{x}_t) + (\underline{a}_t-\underline{x}_t)c + L(\underline{a}_t)|\,|\underline{x}_1 = i\} \le B_i.$$

For the case $\alpha < 1$ we take as optimality criterion

$$V_\alpha(i;R) = \sum_{t=1}^{\infty} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + (\underline{a}_t-\underline{x}_t)c + L(\underline{a}_t)|\underline{x}_1 = i\}.$$

Note that by (2.2.5) the function $V_\alpha(i;R)$ exists and is finite. The quantity $V_\alpha(i;R)$ represents the total expected discounted cost over the periods $\lambda+1, \lambda+2, \ldots$, all discounted to the beginning of period $\lambda+1$, when i is the state in period 1 and the policy R is followed. Note that the costs over the first $\lambda$ periods cannot be influenced by any policy.

For the case $\alpha = 1$ we take as optimality criterion

$$g(i;R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + (\underline{a}_t-\underline{x}_t)c + L(\underline{a}_t)|\underline{x}_1 = i\}.$$

The function $g(i;R)$ exists and is finite. When the limit exists $g(i;R)$ represents the average expected cost per period when the initial state is i and policy R is followed.

We shall next write the formulas for $V_\alpha(i;R)$ and $g(i;R)$ in a more convenient form. Using (2.2.4), we have (cf. [48])

$$\sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + (\underline{a}_t-\underline{x}_t)c + L(\underline{a}_t)|\underline{x}_1 = i\} =$$

$$= \sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + (1-\alpha)c\,\underline{a}_t + \alpha c\underline{\xi}_t + L(\underline{a}_t)|\underline{x}_1 = i\} - ci +$$

$$+ c\alpha^n E_R\{\underline{x}_{n+1}|\underline{x}_1 = i\} =$$

$$= \sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + G_\alpha(\underline{a}_t)|\underline{x}_1 = i\} - ci +$$

$$+ c\alpha^n E_R(\underline{x}_{n+1}|\underline{x}_1 = i), \qquad n \geq 1.$$

By (2.2.3) and (2.2.4), we have that $\{E_R(\underline{x}_{n+1}|\underline{x}_1 = i),\ n \geq 1\}$ is a bounded sequence for each $i \in I$. Thus

$$V_\alpha(i;R) = \sum_{t=1}^{\infty} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + G_\alpha(\underline{a}_t)|\underline{x}_1 = i\} - ci,$$

and

(2.2.6) $$g(i;R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + G_1(\underline{a}_t)|\underline{x}_1 = i\}.$$

Since the term $-ci$ is not affected by the choice of R, it is convenient to redefine $V_\alpha(i;R)$ by setting

(2.2.7) $$V_\alpha(i;R) = \sum_{t=1}^{\infty} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + G_\alpha(\underline{a}_t)|\underline{x}_1 = i\}, \quad i \in I;\ R \in C.$$

Note that if in the original model the unit purchase cost c is set equal to zero and $L(k)$ is replaced by $G_\alpha(k)$, then the formula (2.2.7) for the total expected discounted cost would also be obtained.

When $\alpha < 1$ a policy $R^* \in C$ is called *optimal* if

$$V_\alpha(i;R^*) \leq V_\alpha(i;R) \qquad \text{for all } i \in I \text{ and } R \in C.$$

When $\alpha = 1$ a policy $R^* \in C$ is called *optimal* if

$$g(i;R^*) \leq g(i;R) \qquad \text{for all } i \in I \text{ and } R \in C.$$

To prove the existence of an optimal (s,S) policy, we shall need the following two basic theorems.

*Theorem 2.2.1. (the discounted cost criterion)*

Let $\alpha < 1$. If for policy $R^* \in C$,

(2.2.8) $$V_\alpha(i;R^*) = \min_{k\in A(i)} \{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} V_\alpha(k-j;R^*)\phi(j)\},\ i \in I,$$

then the policy $R^*$ is optimal.

*Proof*

Fix some integer $i_0 \in I$. Let $U_0 = \max(i_0, U)$. Define the following Markovian decision process. The state space is $I' = \{i | i \leq U_0\}$, the set $A'(i)$ of feasible decisions in state $i \in I'$ is given by $A'(i) = \{a | \max(i,u) \leq a \leq U\}$ for $i \leq U$ and $A'(i) = \{i\}$ otherwise, the transition probabilities are given by $q'_{ij}(a) = \phi(a-j)$ for $i, j \in I'$; $a \in A'(i)$, where $\phi(k) = 0$ for $k < 0$, and the direct costs are given by $w'_{ia} = K\delta(a-i) + G_\alpha(a)$, $a \in A'(i)$; $i \in I'$. Denote by $C'$ the class of all possible policies for this decision process. Denote by $V'_\alpha(i;R)$ the total expected discounted cost for this new decision process. Since $A(i) = \{i\}$ for $i \geq U$ and $q_{ij}(a) = 0$ when $j > i \geq U$, there is a 1-1 correspondence between $C$ and $C'$; any policy of $C'$ can be seen as a restriction of a policy of $C$ to the states $i \leq U_0$. Clearly, $V'_\alpha(i;R) = V_\alpha(i;R)$ for all $i \leq U_0$ and $R \in C'$. Since (2.2.8) holds for all $i \leq U_0$, an application of theorem 1.1.1 (b) in section 1.1 shows that $V'_\alpha(i;R^*) \leq V'_\alpha(i;R)$ for all $i \leq U_0$ and $R \in C'$. Hence in particular we have $V_\alpha(i_0;R^*) \leq V_\alpha(i_0;R)$ for all $R \in C$. This ends the proof, since $i_0$ was chosen arbitrarily.

*Theorem 2.2.2. (the average cost criterion)*

Let $\alpha = 1$. Suppose there exists a set of finite numbers $\{g, v(i), i \in I\}$ such that

$$g+v(i) = \min_{k \in A(i)} \{K\delta(k-i) + G_1(k) + \sum_{j=0}^{\infty} v(k-j)\phi(j)\}, \qquad i \in I, \tag{2.2.9}$$

and

$$\lim_{n\to\infty} \frac{1}{n} E_R\,(v(\underline{x}_n) | \underline{x}_1 = i) = 0 \qquad \text{for all } i \in I \text{ and } R \in C. \tag{2.2.10}$$

Let $R^*$ be any stationary deterministic policy which, when in state $i$, prescribes a decision which minimizes the right-hand side of (2.2.9), then the policy $R^*$ is optimal. Further $g(i;R^*) = g$ for all $i \in I$, and the lim inf in (2.2.6) can be replaced by lim for policy $R^*$.

*Proof*

This theorem is a direct consequence of theorem 1.1.2 in section 1.1.

### 2.2.2. The function $a_\alpha(s,S)$

In this subsection we shall analyse for fixed $\alpha$, $0 < \alpha \leq 1$, the function $a_\alpha(s,S)$ which is defined as follows.

$$(2.2.11) \qquad a_\alpha(s,S) = \frac{G_\alpha(S) + \sum_{k=0}^{S-s} G_\alpha(S-k)m(k;\alpha) + K}{1+M(S-s;\alpha)}, \qquad s,\ S \in I \text{ and } s \leq S.$$

By the theorems 2.1.2 and 2.1.3, we have for an $(s,S)$ policy that

$$(2.2.12) \qquad V_\alpha(i;(s,S)) = \begin{cases} \dfrac{a_\alpha(s,S)}{1-\alpha}, & i < s, \\ G_\alpha(i) + \sum_{j=0}^{i-s} G_\alpha(i-j)m(j;\alpha) + \\ \qquad + \dfrac{a_\alpha(s,S)}{1-\alpha}\{\alpha - (1-\alpha)M(i-s;\alpha)\}, & i \geq s, \end{cases}$$

when $\alpha < 1$, and

$$(2.2.13) \qquad g(i;(s,S)) = a_1(s,S) \qquad \text{for all } i \in I.$$

*Lemma 2.2.1.*

Let $0 < \alpha \leq 1$. There exist finite integers $s^*$ and $S^*$ with $s^* \leq S^*$ such that $a_\alpha(s^*,S^*) \leq a_\alpha(s,S)$ for all $s$, $S \in I$, $s \leq S$.

*Proof*

To prove this lemma, we shall show that the minimization of $a_\alpha(s,S)$ can be restricted to a finite region. We first note that from (2.2.11) it follows, after some straightforward calculations, that for all $s$, $S \in I$ with $s \leq S$ holds

(2.2.14) $a_\alpha(s-1,S) - a_\alpha(s,S)$ is non-negative [positive] if and only if

$$m(S-s+1;\alpha)[G_\alpha(s-1)\{1 + M(S-s;\alpha)\} - (G_\alpha(S) + \\ + \sum_{j=0}^{S-s} G_\alpha(S-j)m(j;\alpha) + K)]$$

is non-negative [positive].

By the conditions imposed on $G_\alpha(k)$, we have for each fixed S' that an integer $s' \leq S'$ exists such that $G_\alpha(s-1) > G_\alpha(S'-k)$ for $k = 0,\ldots, S'-s$ and $s \leq s'$. Thus, by (2.2.14), for each $S' \in I$ there exists an integer $s' \leq S'$ such that

$$a_\alpha(s-1,S') - a_\alpha(s,S') \geq 0 \qquad \text{for all } s \leq s'. \tag{2.2.15}$$

Moreover, since $m(k;\alpha) > 0$ for infinitely many values of k, we have that in (2.2.15) the inequality sign holds for infinitely many values of s with $s \leq s'$.

Next we note that from (2.2.11) it follows that $a_\alpha(s,S) = G_\alpha(S) + K/\{1 + M(0;1)\} \leq G_\alpha(S) + K$ for all s and S, and hence in particular,

$$a_\alpha(S_0,S_0) \leq G_\alpha(S_0) + K, \tag{2.2.16}$$

where $S_0$ is defined on p. 45.Since $G_\alpha(k) \to \infty$ as $|k| \to \infty$, we can choose integers $r_1$ and $r_2$ with $r_1 \leq r_2$ such that

$$G_\alpha(k) > G_\alpha(S_0) + K \qquad \text{for both } k \leq r_1 \text{ and } k \geq r_2. \tag{2.2.17}$$

It follows trivially from (2.2.11) and (2.2.17) that

$$a_\alpha(s,S) > G_\alpha(S_0) + K \quad \text{for both } s \leq S \leq r_1 \text{ and } S \geq s \geq r_2. \tag{2.2.18}$$

Consider now the case in which $S > r_2 > s$. Let

$$R = \min(S-s,S-r_1)$$

Clearly, R depends on S and s. Using (2.2.18) and the fact that $G_\alpha(k) \geq G_\alpha(S_0)$ for all $k \in I$, we obtain

$$\begin{aligned} a_\alpha(s,S) \geq \{&G_\alpha(S)+(G_\alpha(S_0)+K)\sum_{j=0}^{S-r_2} m(j;\alpha)+G_\alpha(S_0)\sum_{j=S-r_2+1}^{R} m(j;\alpha) + \\ &+ (G_\alpha(S_0)+K)\sum_{j=R+1}^{S-s} m(j;\alpha)+K\}/\{1+M(S-s;\alpha)\} = \\ &= G_\alpha(S_0)+K+\{G_\alpha(S)-G_\alpha(S_0)-K\sum_{j=S-r_2+1}^{R} m(j;\alpha)\}/\{1+M(S-s;\alpha)\}. \end{aligned}$$

If $\alpha < 1$, then $M_\alpha(k) \leq \alpha/(1-\alpha)$ for all $k \geq 0$. Since $R \leq S-r_1$, we have that $M(R;1) \leq M(S-r_1;1)$. From formula (1.2.4) of section 1.2 we obtain $M(S-r_1;1) - M(S-r_2;1) \leq C$ for some constant C. Hence

$$a_\alpha(s,S) \geq G_\alpha(S_0) + K + \frac{G_\alpha(S) - G_\alpha(S_0) - \alpha K/(1-\alpha)}{1 + M(S-s;\alpha)} \qquad \text{if } \alpha < 1,$$

and

$$a_1(s,S) \geq G_1(S_0) + K + \frac{G_1(S) - G_1(S_0) - KC}{1 + M(S-s;1)} .$$

Thus, since $G_\alpha(k) \to \infty$ as $k \to \infty$, there exists an integer $M \geq r_2$ such that

$$(2.2.19) \qquad a_\alpha(s,S) > G_\alpha(S_0) + K \quad \text{for all } s < r_2 \leq M < S.$$

The lemma now follows from (2.2.15), (2.2.16), (2.2.18) and (2.2.19).

For any $\alpha$, $0 < \alpha \leq 1$, let

$$a_\alpha^* = \min_{\substack{s,S\epsilon I \\ s \leq S}} a_\alpha(s,S).$$

*Lemma 2.2.2.*

Let $0 < \alpha \leq 1$. Let $s^*$ and $S^*$ be any integers such that $s^* \leq S^*$ and $a_\alpha(s^*,S^*) = a_\alpha^*$. Then

(a) If $m(S^*-s^*+1;\alpha) > 0$, then $G_\alpha(s^*-1) \geq a_\alpha^*$.
(b) If $s^* = S^*$, then $G_\alpha(s^*) \leq a_\alpha^*$.
(c) If $s^* < S^*$ and if $m(S^*-s^*;\alpha) > 0$, then $G_\alpha(s^*) \leq a_\alpha^*$.
(d) If $\phi(1) > 0$, then $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$.
(e) If $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$, then $s_1 \leq s^* \leq S^0$ for $K = 0$, and $s_1 \leq s^* \leq S_0$ for $K > 0$.

*Proof*

(a) Since $a_\alpha(s,S)$ takes on its absolute minimum for $s = s^*$ and $S = S^*$, we have that $a_\alpha(s^*-1,S^*) - a_\alpha(s^*,S^*) \geq 0$. From (2.2.14) and $m(S^*-s^*+1;\alpha) > 0$ it follows

$$G_\alpha(s^*-1) \geq \frac{G_\alpha(S^*)+\sum_{j=0}^{S^*-s^*} G_\alpha(S^*-j)m(j;\alpha)+K}{1 + M(S^*-s^*;\alpha)} = a_\alpha(s^*,S^*) = a_\alpha^*.$$

(b) If $s^* = S^*$, then $a_\alpha^* = a_\alpha(s^*,s^*) = G_\alpha(s^*) + K/\{1 + M(0;\alpha)\} \geq G_\alpha(s^*)$.

(c) We have $a_\alpha(s^*, S^*) - a_\alpha(s^*+1,S^*) \leq 0$. From this and (2.2.11) it follows that $m(S^*-s^*;\alpha)[(1+M(S^*-s^*;\alpha))G_\alpha(s^*)-\{G_\alpha(S^*)+\Sigma_{j=0}^{S^*-s^*} G_\alpha(S^*-j)m(j;\alpha)+K\}] \leq 0$. Hence, since $m(S^*-s^*;\alpha) > 0$, we obtain $G_\alpha(s^*) \leq a_\alpha(s^*,S^*) = a_\alpha^*$.

(d) If $\phi(1) > 0$, then $\phi^{(k)}(k) \geq \{\phi(1)\}^k > 0$ for all $k \geq 1$, and hence $m(k;\alpha) > 0$ for all $k \geq 1$. From (a), (b) and (c) now follows (d).

(e) Since $1/\{1+M(0;\alpha)\} = 1 - \alpha\phi(0)$ and

$$G_\alpha(s^*) \leq a_\alpha^* \leq a_\alpha(S_0,S_0) = G_\alpha(S_0) + \frac{K}{1+M(0;\alpha)} ,$$

we have by definition (2.2.1) that $s_1 \leq s^*$. Next we distinguish between $K = 0$ and $K > 0$.

Consider first the case $K = 0$. Since $G_\alpha(k) \geq G_\alpha(S_0)$ for all $k \in I$, we have then $a_\alpha(s,S) \geq G_\alpha(S_0) = a_\alpha(S_0,S_0)$ for all $s$ and $S$. Hence

$$a_\alpha^* = a_\alpha(S_0,S_0) = G_\alpha(S_0) \qquad \text{if } K = 0. \tag{2.2.20}$$

From $a_\alpha^* \geq G_\alpha(s^*)$ and (2.2.20) it follows that $G_\alpha(s^*) \leq G_\alpha(S_0)$. Thus $G_\alpha(s^*) = G_\alpha(S_0)$, since $G_\alpha(k)$ takes on its absolute minimum at $S_0$. Hence, by the definition of $S^0$, we have that $s^* \leq S^0$.

Consider next the case $K > 0$. Assume to the contrary $s^* > S_0$. Since $G_\alpha(k)$ is non-decreasing on $[S_0,\infty)$, we have then

$$a_\alpha^* = a_\alpha(s^*,S^*) \geq G_\alpha(s^*-1) + \frac{K}{1+M(S^*-s^*;\alpha)} > G_\alpha(s^*-1).$$

This contradicts $G_\alpha(s^*-1) \geq a_\alpha^*$. Hence $s^* \leq S_0$. This ends the proof.

*Lemma 2.2.3.*

Let $0 < \alpha \leq 1$. There exist integers $s^*$ and $S^*$ with $s^* \leq S^*$ such that $a_\alpha(s^*,S^*) = a_\alpha^*$ and $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$. If $K = 0$, then $s^* = S_0$ and $S^* = S_0$ satisfy these conditions.

*Proof*

By lemma 2.2.1 there exist integers s' and S' such that $a_\alpha(s',S') = a_\alpha^*$. When $m(S'-s'+1;\alpha) = 0$, we have by the definition of $a_\alpha(s,S)$ that $a_\alpha(s'-1,S') = a_\alpha(s',S') = a_\alpha^*$. Further, $m(S'-s;\alpha) > 0$ for infinitely many values of s. It will now be clear that there exist integers s and S such that $a_\alpha(s,S) = a_\alpha^*$ and $m(S-s+1;\alpha) > 0$. By lemma 2.2.2 (a), we have now proved that the set of (s,S) policies

$$V = \{(s,S) \mid a_\alpha(s,S) = a_\alpha^* \leq G_\alpha(s-1)\}$$

is non-empty. Let $(s^*,S^*)$ be a policy in V such that $S^* - s^* \leq S-s$ for all $(s,S) \in V$. We shall prove that $a_\alpha^* \geq G_\alpha(s^*)$. When $s^* = S^*$ this follows from lemma 2.2.2 (b). Consider now the case $s^* < S^*$. Suppose to the contrary that $G_\alpha(s^*) > a_\alpha^*$. By lemma 2.2.2 (c), we have then $m(S^*-s^*;\alpha) = 0$. Next it follows from the definition of $a_\alpha(s,S)$ that $a_\alpha(s^*+1,S^*) = a_\alpha^*$. By $G_\alpha(s^*) > a_\alpha^*$, we have now found the contradiction $(s^*+1,S^*) \in V$. Thus $G_\alpha(s^*) \leq a_\alpha^*$.

Since $S_0$ is the smallest integer at which $G_\alpha(k)$ takes on its absolute minimum, we have $G_\alpha(S_0) < G_\alpha(S_0-1)$. Thus, by (2.2.20), we have for K = 0 that $a_\alpha^* = a_\alpha(S_0,S_0) = G_\alpha(S_0) < G_\alpha(S_0-1)$. This ends the proof of the lemma.

### *2.2.3. The optimality of an (s,S) policy*

In this subsection we shall give a unified proof of the existence of an optimal (s,S) policy; the cases $\alpha < 1$ and $\alpha = 1$ are treated simultaneously. As a by-product of the proof we obtain for $K > 0$ the important result that any (s,S) policy, such that $a_\alpha(s,S) = a_\alpha^*$ and $G_\alpha(s-1) \geq a_\alpha^* \geq G_\alpha(s)$, is optimal and has the property $s_1 \leq s \leq S_0 \leq S \leq S_1$, where $s_1$ and $S_1$ are defined by (2.2.1) and (2.2.2).

From now on $s^*$ and $S^*$ with $s^* \leq S^*$ are two fixed integers such that (cf. lemma 2.2.3)

$$a_\alpha(s^*,S^*) = a_\alpha^* \text{ and } G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*), \tag{2.2.21}$$

where we choose

$$s^* = S^* = S_0 \qquad \text{if } K = 0. \tag{2.2.22}$$

To give the existence proof, we shall define a function $v_\alpha^*(i)$, $i \in I$, which will be shown to satisfy a functional equation, which is closely related to the functional equations (2.2.8) and (2.2.9).
Let the function $v_\alpha^*(i)$, $i \in I$, be defined as follows.

$$(2.2.23)\qquad v_\alpha^*(i) = \begin{cases} 0 & \text{for } i < s^*, \\ G_\alpha(i) - a_\alpha^* + \alpha \sum_{j=0}^{i-s^*} v_\alpha^*(i-j)\phi(j) & \text{for } i \geq s^*. \end{cases}$$

Note that the finite function $v_\alpha^*(i)$, $i \in I$, is uniquely determined by the renewal equation (2.2.23).

*Remark 2.2.1.*

This remark is meant to motivate the definition of the function $v_\alpha^*(i)$. Consider first the case $\alpha = 1$. Suppose that $\{g, v(i)\}$ is a set of finite numbers satisfying (2.2.9) and suppose further that the right-hand side of (2.2.9) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$. Then $g = a_1(s^*,S^*) = a_1^*$ and (cf. theorem 2.2.2)

$$v(i) = \begin{cases} G_1(i) - g + \sum_{j=0}^{\infty} v(i-j)\phi(j), & i \geq s^*, \\ K + v(S^*), & i < s^*. \end{cases}$$

When c is a constant, the set of numbers $\{g, v(i)+c\}$ satisfies also (2.2.9). Normalizing $v(i)$ to be zero at $i = s^*-1$ explains now definition (2.2.23) for the case $\alpha = 1$. Consider next the case $\alpha < 1$. From

$$V_\alpha(i;(s^*,S^*)) = \begin{cases} G_\alpha(i) + \alpha \sum_{j=0}^{\infty} V_\alpha(i-j;(s^*,S^*))\phi(j), & i \geq s^*, \\ K + V_\alpha(S^*;(s^*,S^*)) = a_\alpha^*/(1-\alpha), & i < s^*, \end{cases}$$

(cf. (2.2.12) and see also theorem 2.2.1), it follows that

$$V_\alpha(i;(s^*,S^*)) - \frac{a_\alpha^*}{1-\alpha} = \begin{cases} G_\alpha(i) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} \{V_\alpha(i-j;(s^*,S^*)) - \frac{a_\alpha^*}{1-\alpha}\}\phi(j), & i \geq s^*, \\ 0, & i < s^*. \end{cases}$$

This suggests definition (2.2.23) for the case $\alpha < 1$.

Iterating (2.2.23) and using the convolution formula (2.1.2) in subsection 2.1.1, yields

$$v_\alpha^*(i) = G_\alpha(i) - a_\alpha^* + \sum_{t=1}^{n} \sum_{j=0}^{i-s^*} \{G_\alpha(i-j) - a_\alpha^*\}\alpha^t \phi^{(t)}(j) + \\ + \alpha^{n+1} \sum_{j=0}^{i-s^*} v_\alpha^*(i-j)\phi^{(n+1)}(j) \qquad \text{for all } n \geq 1;\ i \geq s^*.$$

Letting $n \to \infty$ and using the fact that $\phi^{(n)}(j) \to 0$ as $n \to \infty$ for each $j \geq 0$, we obtain

$$\text{(2.2.24)} \qquad v_\alpha^*(i) = \begin{cases} 0, & i < s^*, \\ G_\alpha(i) + \sum_{j=0}^{i-s^*} G_\alpha(i-j)m(j;\alpha) - a_\alpha^*\{1+M(i-s^*;\alpha)\}, & i \geq s^*. \end{cases}$$

It is interesting to note that for the case $\alpha < 1$ we have by (2.2.24) and (2.2.12) that

$$v_\alpha^*(i) - v_\alpha^*(j) = V_\alpha(i;(s^*,S^*)) - V_\alpha(j;(s^*,S^*)), \qquad i, j \in I.$$

For the case $\alpha = 1$, it follows from (2.2.24) and remark 2.1.4 in subsection 2.1.4 that

$$v_1^*(i) - v_1^*(j) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \{f_k(i) - f_k(j)\} \qquad \text{for all } i, j \in I,$$

where $f_k(i)$, $i \in I$, is the total expected cost over the periods $1,\dots, k$, when the initial state $\underline{x}_1 = i$ and the $(s^*,S^*)$ policy is followed $(c=d=0)$. These relations give the function $v_\alpha^*(j)$ a physical interpretation.

Let the function $J_\alpha(k)$ be defined by

$$\text{(2.2.25)} \qquad J_\alpha(k) = G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} v_\alpha^*(k-j)\phi(j) \qquad \text{for } k \in I.$$

By (2.2.23) and (2.2.25), we have

$$\text{(2.2.26)} \qquad J_\alpha(k) = G_\alpha(k) - a_\alpha^* \qquad \text{for } k < s^*,$$

(2.2.27) $J_\alpha(k) = v_\alpha^*(k)$ for $k \geq s^*$.

*Theorem 2.2.3.*

(a) $J_\alpha(k)$ is non-increasing on $(-\infty, s^*-1]$.
(b) $K + J_\alpha(S^*) = 0$, $J_\alpha(s^*-1) \geq 0$.
(c) $J_\alpha(k) \geq J_\alpha(S^*)$ for all $k \in I$.
(d) $J_\alpha(k) \leq 0$ for $s^* \leq k \leq S_0$.
(e) $K + J_\alpha(k) \geq J_\alpha(i)$ for $k \geq i \geq s^*$.
(f) $K + J_\alpha(k) > 0$ for $k > S_1$.
(g) $J_\alpha(k)$ is non-increasing on $[s^*, S_0]$.

*Proof*

(a) Since $G_\alpha(k)$ is non-increasing on $(-\infty, S_0]$ and since $s^* \leq S_0$ (cf. lemma 2.2.2 (e)), assertion (a) follows immediately from (2.2.26).

(b) From (2.2.27), (2.2.24), $a_\alpha^* = a_\alpha(s^*,S^*)$ and definition (2.2.11), it follows that

$$J_\alpha(S^*) = G_\alpha(S^*) + \sum_{j=0}^{S^*-s^*} G_\alpha(S^*-j)m(j;\alpha) - a_\alpha(s^*,S^*)\{1+M(S^*-s^*;\alpha)\} = -K.$$

By (2.2.21) and (2.2.26), we have $J_\alpha(s^*-1) = G_\alpha(s^*-1) - a_\alpha^* \geq 0$.

(c) Since $K \geq 0$, we have by (a) and (b) that

$$J_\alpha(k) \geq J_\alpha(s^*-1) \geq J_\alpha(S^*) \qquad \text{for } k < s^*.$$

Hence it remains to prove that $J_\alpha(k) \geq J_\alpha(S^*)$ for $k \geq s^*$. Suppose to the contrary that $J_\alpha(r) < J_\alpha(S^*)$ for some integer $r \geq s^*$. Then it follows from (2.2.24), (2.2.27) and (b) that

$$G_\alpha(r) + \sum_{j=0}^{r-s^*} G_\alpha(r-j)m(j;\alpha) - a_\alpha^*\{1+M(r-s^*;\alpha)\} + K < 0,$$

and hence, by definition (2.2.11), we obtain $a_\alpha(s^*,r) < a_\alpha^*$. This contradiction proves (c).

(d) Since $G_\alpha(k)$ is non-increasing on $[s^*,S_0]$, we have by (2.2.27), (2.2.24) and (2.2.21) that

$$J_\alpha(k) \leq \{G_\alpha(s^*) - a_\alpha^*\}\{1+M(k-s^*;\alpha)\} \leq 0 \qquad \text{for } s^* \leq k \leq S_0.$$

(e) By (b), (c) and (d), we have

$$K + J_\alpha(k) \geq K + J_\alpha(S^*) = 0 \geq J_\alpha(i) \qquad \text{for } k \in I \text{ and } s^* \leq i \leq S_0.$$

This relation proves (e) when $s^* \leq i \leq S_0$.

From (2.2.23) and (2.2.27) it follows that

$$(2.2.28) \qquad J_\alpha(k) = G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{k-s^*} J_\alpha(k-j)\phi(j) \qquad \text{for } k \geq s^*.$$

By (2.2.28), we have for $k \geq i \geq S_0$

$$J_\alpha(k) - J_\alpha(i) = G_\alpha(k) - G_\alpha(i) + \alpha \sum_{j=0}^{i-S_0}\{J_\alpha(k-j) - J_\alpha(i-j)\}\phi(j) + \\ + \alpha \sum_{j=i-S_0+1}^{k-s^*} J_\alpha(k-j)\phi(j) - \alpha \sum_{j=i-S_0+1}^{i-s^*} J_\alpha(i-j)\phi(j).$$

Since $G_\alpha(j)$ is non-decreasing on $[S_0, \infty)$, we have

$$G_\alpha(k) - G_\alpha(i) \geq 0 \geq \alpha K - K \qquad \text{for } k \geq i \geq S_0.$$

By (b) and (c), we have $J_\alpha(k) \geq -K$ for all $k \in I$. Further we have by (d) that $J_\alpha(k) \leq 0$ for $s^* \leq k \leq S_0$. Thus

$$J_\alpha(k) - J_\alpha(i) \geq \alpha K - K + \alpha \sum_{j=0}^{i-S_0}\{J_\alpha(k-j) - J_\alpha(i-j)\}\phi(j) + \\ - \alpha K\{\Phi(k-s^*) - \Phi(i-S_0)\} \qquad \text{for } k \geq i \geq S_0,$$

and hence

$$K + J_\alpha(k) - J_\alpha(i) \geq \alpha \sum_{j=0}^{i-S_0} \{K + J_\alpha(k-j) - J_\alpha(i-j)\}\phi(j) \text{ for } k \geq i \geq S_0.$$

Iterating this inequality and using the convolution formula (2.1.2) in subsection 2.1.1, yields for $k \geq i \geq S_0$,

$$K + J_\alpha(k) - J_\alpha(i) \geq \alpha^n \sum_{j=0}^{i-S_0}\{K + J_\alpha(k-j) - J_\alpha(i-j)\}\phi^{(n)}(j), \; n \geq 1.$$

Taking the limit as $n \to \infty$ and using the fact that $\phi^{(n)}(j) \to 0$ as $n \to \infty$ for each $j \geq 0$, we obtain

$$K + J_\alpha(k) - J_\alpha(i) \geq 0 \qquad\qquad \text{for } k \geq i \geq S_0,$$

from which (e) follows.

(f) By the definition (2.2.2) of $S_1$, we have that $G_\alpha(k) > G_\alpha(S_0) + \alpha K$ for $k > S_1$. From (2.2.16) we have $a_\alpha^* \leq a_\alpha(S_0,S_0) \leq G_\alpha(S_0) + K$. Further, we have by (b) and (c) that $J_\alpha(k) \geq -K$ for all $k \in I$. Hence, by (2.2.28), we obtain

$$K + J_\alpha(k) > K + G_\alpha(S_0) + \alpha K - G_\alpha(S_0) - K - \alpha K \sum_{j=0}^{k-s^*} \phi(j) \geq 0, \qquad k > S_1.$$

(g) By (2.2.28) and (d), we have for $s^* \leq i \leq k \leq S_0$ that

$$J_\alpha(i) - J_\alpha(k) \geq G_\alpha(i) - G_\alpha(k) + \alpha \sum_{j=0}^{i-s^*} \{J_\alpha(i-j) - J_\alpha(k-j)\}\phi(j).$$

Iterating this inequality, yields for $s^* \leq i \leq k \leq S_0$ that

$$J_\alpha(i) - J_\alpha(k) \geq G_\alpha(i) - G_\alpha(k) + \sum_{j=0}^{i-s^*} \{G_\alpha(i-j) - G_\alpha(k-j)\}m(j;\alpha).$$

The assertion (g) now follows from this inequality and the fact that $G_\alpha(j)$ is non-increasing on $[s^*,S_0]$ This ends the proof of the theorem.

*Theorem 2.2.4.*

(a) The set of numbers $\{a_\alpha^*, v_\alpha^*(i),\ i \in I\}$ satisfies the functional equation

$$\text{(2.2.29)} \qquad v_\alpha^*(i) = \min_{k \geq i}\{K\delta(k-i) + G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} v_\alpha^*(k-j)\phi(j)\},\ i \in I,$$

where the right-hand side of (2.2.29) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$.

(b) $\qquad s_1 \leq s^* \leq S_0 \leq S^* \leq S_1.$

*Proof*

(a) By (2.2.25), we have for each $i \in I$ that

$$K\delta(k-i) + G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} v_\alpha^*(k-j)\phi(j) = K\delta(k-i) + J_\alpha(k), \quad k \geq i.$$

Consider the function $K\delta(k-i) + J_\alpha(k)$ for $i$ fixed and $k \geq i$. Distinguish between $i < s^*$ and $i \geq s^*$.

(i) $i < s^*$. By theorem 2.2.3 (a), 2.2.3 (b) and 2.2.3 (c), we have

$$J_\alpha(i) \geq J_\alpha(s^*-1) \geq K + J_\alpha(S^*) = \min_{k>i} \{K + J_\alpha(k)\}.$$

Hence the right-hand side of (2.2.29) is minimized by $k = S^*$ for $i < s^*$. By theorem 2.2.3 (b) and (2.2.23), we have that $K + J_\alpha(S^*) = 0 = v_\alpha^*(i)$, $i < s^*$. This proves (a) for $i < s^*$.

(ii) $i \geq s^*$. By theorem 2.2.3 (e) and (2.2.27), we have

$$K + J_\alpha(k) \geq J_\alpha(i) = v_\alpha^*(i) \qquad \text{for } k \geq i \geq s^*,$$

from which (a) follows for $i \geq s^*$.

(b) By lemma 2.2.2 (e) and the choice $s^* = S_0$ when $K = 0$, we have $s_1 \leq s^* \leq S_0$. From theorem 2.2.3 (b) and 2.2.3 (f) it follows that $S^* \leq S_1$. To prove $S_0 \leq S^*$, assume to the contrary that $S_0 > S^*$. Since $S_0$ is the smallest integer at which $G_\alpha(k)$ takes on its absolute minimum, we have then $G_\alpha(S_0) < G_\alpha(S^*)$. From this inequality, (2.2.28), theorem 2.2.3 (d) and theorem 2.2.3 (g), it follows (note that $S_0 > S^* \geq s^*$)

$$\begin{aligned} J_\alpha(S_0) - J_\alpha(S^*) &= G_\alpha(S_0) - G_\alpha(S^*) + \\ &+ \sum_{j=0}^{S^*-s^*} \{J_\alpha(S_0-j) - J_\alpha(S^*-j)\}\ \phi(j) + \\ &+ \sum_{j=S^*-s^*+1}^{S_0-s^*} J(S_0-j)\phi(j) \leq G_\alpha(S_0) - G_\alpha(S^*) < 0. \end{aligned}$$

This contradicts theorem 2.2.3 (c). Thus $S_0 \leq S^*$. This completes the proof of the theorem.

We are now in a position to establish the optimality of the $(s^*,S^*)$ policy.

Consider first the case $\alpha < 1$. From (2.2.12), (2.2.21) and (2.2.24) it follows that

$$(2.2.30) \quad v_\alpha^*(i) = V_\alpha(i;(s^*,S^*)) - \frac{a_\alpha^*}{1-\alpha} \qquad \text{for all } i \in I.$$

Substituting (2.2.30) in (2.2.29), yields

$$(2.2.31) \quad V_\alpha(i;(s^*,S^*)) =$$

$$= \min_{k\geq i}\{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} V_\alpha(k-j;(s^*,S^*))\phi(j)\}, \qquad i \in I,$$

where the right-hand side of (2.2.31) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$. By theorem 2.2.4 (b), the restrictions u and U on the ordering level satisfy $u \leq s^*$ and $U \geq S^*$. Hence the function $V_\alpha(i;(s^*,S^*))$, $i \in I$, satisfies also (2.2.8). This proves the optimality of the $(s^*,S^*)$ policy.

Summarizing, we have proved the following theorem (see also the lemmas 2.2.2 and 2.2.3).

*Theorem 2.2.5. (the total discounted cost criterion)*

Let $\alpha < 1$, then

$$\min_{R\in C} V_\alpha(i;R) = V_\alpha(i;(s^*,S^*)) \qquad \text{for all } i \in I.$$

If $K = 0$, then the $(S_0,S_0)$ policy is optimal. If $K > 0$, then any $(s,S)$ policy such that $a_\alpha(s,S) = a_\alpha^*$ and $G_\alpha(s-1) \geq a_\alpha^* \geq G_\alpha(s)$, is optimal and has the property $s_1 \leq s \leq S_0 \leq S \leq S_1$. If $\phi(1) > 0$, then $a_\alpha(s,S) = a_\alpha^*$ implies $G_\alpha(s-1) \geq a_\alpha^* \geq G_\alpha(s)$.

Consider next the case $\alpha = 1$. By theorem 2.2.4, we have

$$(2.2.32) \quad a_1^* + v_1^*(i) = \min_{k\geq i}\{K\delta(k-i) + G_1(k) + \sum_{j=0}^{\infty} v_1^*(k-j)\phi(j)\}, \qquad i \in I,$$

where the right-hand side of (2.2.32) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$. Since $u \leq s^*$ and $U \geq S^*$, it follows that the set of numbers $\{a_1^*, v_1^*(i), i \in I\}$ satisfies (2.2.9), where the right-hand side of (2.2.9) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$. Finally, we check the condition (2.2.10). Since $v_1^*(j) = 0$ for $j < s^*$ and since $\underline{x}_t \leq \max(\underline{x}_1, U)$ for all $t \geq 1$, we have for each $i \in I$ and $R \in C$ that the sequence $\{E_R(v_1^*(\underline{x}_n) | \underline{x}_1 = i)\}$ is bounded. Hence the condition (2.2.10) is fulfilled. The optimality of the $(s^*,S^*)$ policy now **follows** from theorem 2.2.2.

Summarizing, we have proved the following theorem (see also the lemmas 2.2.2 and 2.2.3).

*Theorem 2.2.6. (the average cost criterion)*

Let $\alpha = 1$, then

$$\min_{R\in C} g(i;R) = a_1^* \qquad \text{for all } i \in I.$$

If $K = 0$, then the $(S_0,S_0)$ policy is optimal. If $K > 0$, then any $(s,S)$ policy such that $a_1(s,S) = a_1^*$ and $G_1(s-1) \geq a_1^* \geq G_1(s)$, is optimal and has the property that $s_1 \leq s \leq S_0 \leq S \leq S_1$. If $\phi(1) > 0$, then $a_1(s,S) = a_1^*$ implies $G_1(s-1) \geq a_1^* \geq G_1(s)$.

*Remark 2.2.2.*

In this remark we shall show that the condition i) imposed on $G_\alpha(k)$ (see p. 45) can be weakened slightly.*) Consider first the case $\alpha = 1$. Suppose the following conditions are imposed on $G_1(k)$:

(i) There exist finite integers $w$, $S_0$ and $W$ with $w \leq S_0 \leq W$ such that $G_1(k)$ is non-increasing on $[w,S_0]$, $G_1(k)$ is non-decreasing on $[S_0,W]$, $G_1(k) > G_1(S_0) + (1-\phi(0))K$ for $k < w$ and $G_1(k) > G_1(S_0) + K$ for $k > W$.

(ii) $G_1(k) \to \infty$ as $|k| \to \infty$.

Assume that $S_0$ is the smallest integer at which $G_1(k)$ takes on its absolute minimum. Let $s_1$ and $S_1$ be again defined by (2.2.1) and (2.2.2). Clearly, $s_1 \geq w$ and $S_1 \leq W$. Define a function $\hat{G}_1(k)$, $k \in I$, such that $\hat{G}_1(k) \leq G_1(k)$ for $k \in I$, $\hat{G}_1(k) = G_1(k)$ for $s_1-1 \leq k \leq S_1$, $\hat{G}_1(k)$ is non-increasing for $k \leq S_0$ and $\hat{G}_1(k)$ is non-decreasing for $k \geq S_0$.

Let $\hat{g}(i;R)$ and $\hat{a}_1(s,S)$ correspond to $\hat{G}_1(k)$. We have

$$\hat{g}(i;R) \leq g(i;R) \quad \text{and } \hat{a}_1(s,S) \leq a_1(s,S).$$

Since $\hat{G}_1(k)$ satisfies the conditions (i) and (ii) from subsection 2.2.1 (p. 45) we can choose by lemma 2.2.3 an $(s',S')$ policy such that

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*) The condition ii) can also be weakened; up to now this condition is only used to ensure the existence of $s_1$ and $S_1$ and to prove lemma 2.2.1.

$\hat{a}_1(s',S') = \min \hat{a}_1(s,S)$ and $\hat{G}_1(s'-1) \geq \hat{a}_1(s',S') \geq \hat{G}_1(s')$, where we take $s' = S' = S_0$ for $K = 0$. By theorem 2.2.6, we have that $\hat{g}(i;R) \geq \hat{a}_1(s',S')$ for all $i \in I$ and $R \in C$ and further we have $s_1 \leq s' \leq S_0 \leq S' \leq S_1$. Since $\hat{G}_1(k) = G_1(k)$ on $[s_1,S_1]$, it follows that $\hat{a}_1(s',S') = a_1(s',S')$. Hence

(2.2.33) $\quad g(i;R) \geq a_1(s',S')$ for all $i$ and $R$, $\min\limits_{s,S} a_1(s,S) = \min\limits_{s,S} \hat{a}_1(s,S)$.

Thus we have proved that an optimal (s,S) policy exists under the weakened assumptions about $G_1(k)$. Moreover, we shall prove that theorem 2.2.6 remains true. We first note that the $(S_0,S_0)$ policy is optimal for $K = 0$. Let $K > 0$, and suppose the $(s',S')$ policy is such that $a_1(s',S') = \min a_1(s,S)$ and $G_1(s'-1) \geq a_1(s',S') \geq G_1(s')$. We have to show $s_1 \leq s' \leq S_0 \leq S' \leq S_1$. The proof that $s_1 \leq s'$ is the same as in the proof of lemma 2.2.2 (e). To prove that $s' \leq S_0$, we note that $S_0 < s' \leq S_1$ leads to the contradiction $a_1(s',S') > G_1(s'-1)$ and that $s' > S_1$ leads to the contradiction $a_1(s',S') > G_1(S_0) + K \geq a_1(S_0,S_0)$. Since $s_1 \leq s' \leq S_0$, we have by the properties of $\hat{G}_1(k)$ that $\hat{G}_1(s'-1) \geq a_1(s',S') \geq \hat{G}_1(s')$. Further, it follows from (2.2.33) that $a_1(s',S') = \min \hat{a}_1(s,S) = \hat{a}_1(s',S')$. By theorem 2.2.6 we now have that $s_1 \leq s' \leq S_0 \leq S' \leq S_1$. Finally, since the proofs and the results of lemma 2.2.1 and lemma 2.2.2 (a) - (d) remain true under the weakened assumptions about $G_1(k)$, we have that if $\phi(1) > 0$, then $a_1(s',S') = \min a_1(s,S)$ implies that $G_1(s'-1) \geq a_1(s',S') \geq G_1(s')$. This ends the proof.

Consider next the case $\alpha < 1$. Exploiting the fact that $V_\alpha(i;(s,S)) = a_\alpha(s,S)/(1-\alpha)$ for $i < s$, it can be proved in a similar way as for $\alpha = 1$ that theorem 2.2.5 remains valid when the condition that $G_\alpha(k)$ is non-increasing for $k < s_1$ is dropped. We note that in general the condition $G_\alpha(k)$ is non-decreasing for $k > S_1$ cannot be dropped, as can be easily seen by a counterexample with $\alpha$ close to zero.

*Remark 2.2.3.*

In this section we have laid emphasis on the *existence* of optimal (s,S) policies rather than their *computation*. For computational methods we refer to [28,48,50].

### *2.2.4. A uniqueness theorem for the optimal inventory equation*

In this subsection we shall prove that "the optimal inventory equation" has a unique solution which is finite and bounded from below.

Let $\alpha < 1$. Define

$$V_\alpha^*(i) = \min_{s_1 \le s \le S \le S_1} V_\alpha(i;(s,S)) \qquad \text{for } i \in I.$$

Note that by (2.2.12) the function $V_\alpha^*(i)$, $i \in I$, is finite and bounded from below. Let $s^*$ and $S^*$ be any integers such that $a_\alpha(s^*,S^*) = a_\alpha^*$ and $G_\alpha(s^*-1) \ge a_\alpha^* \ge G_\alpha(s^*)$, where we take $s^* = S^* = S_0$ if $K = 0$. By theorem 2.2.5, we have that $s_1 \le s^* \le S^* \le S_1$ and $V_\alpha^*(i) = V_\alpha(i;(s^*,S^*))$. From this and (2.2.31) it follows that

$$(2.2.34) \qquad V_\alpha^*(i) = \min_{k \ge i}\{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} V_\alpha^*(k-j)\phi(j)\}, \qquad i \in I,$$

where the right-hand side of (2.2.34) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \ge s^*$.

*Theorem 2.2.7.*

Let $\alpha < 1$. Let the function $u(i)$, $i \in I$, be a finite solution to the "optimal inventory equation"

$$(2.2.35) \qquad u(i) = \min_{k \ge i} \{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} u(k-j)\phi(j)\}, \qquad i \in I,$$

such that the function $u(i)$, $i \in I$, is bounded from below on $(-\infty, i_0]$ for some integer $i_0$. Then

$$u(i) = V_\alpha^*(i) \qquad \text{for all } i \in I.$$

*Proof*

For any $i \in I$, let $B(i)$ be the set of all integers $k \ge i$ for which the right-hand side of (2.2.35) is minimal. We shall first show that for each integer $L$ there exists an integer $N \ge L$ such that

$$(2.2.36) \qquad B(i) \cap \{k \mid k = N, N-1, \ldots\} \ne \emptyset \qquad \text{for all } i \le N.$$

We first note that if for each $i \le L$ there exists an integer $k \in B(i)$ with $k \le L$, then we choose $N = L$. Consider now the case where for some $i \le L$, say $i'$, there exists an integer $k \in B(i')$, say $k'$, such that $k' > L$. Since $k' > i'$, it follows from the definition of $B(i)$ that

$$G_\alpha(k) + \alpha \sum_{j=0}^{\infty} u(k-j)\phi(j) \ge G_\alpha(k') + \alpha \sum_{j=0}^{\infty} u(k'-j)\phi(j) \qquad \text{for } k \ge k'.$$

From this inequality it follows that for each $i \le k'$ there is an integer $k \in B(i)$ such that $k \le k'$. Thus we now choose $N = k'$. This proves (2.2.36).

Since the function $u(i)$ is bounded from below on $(-\infty, i_0]$ and is finite and since $G_\alpha(k) \to \infty$ as $k \to -\infty$, there exist integers $m_1$ and $m_2$ with $m_1 < m_2$ such that

$$G_\alpha(k)+\alpha \sum_{j=0}^{\infty} u(k-j)\phi(j) > K+G_\alpha(m_2)+\alpha \sum_{j=0}^{\infty} u(m_2-j)\phi(j) \qquad \text{for } k \le m_1.$$

This inequality shows that we can choose an integer $u' \le s_1$ such that $k \ge u'$ for all $k \in B(i)$ with $i < u'$. Let $M$ be an arbitrary integer with $M \ge S_1$. It follows from (2.2.36) that we can choose an integer $U' \ge M$ such that for each $i \le U'$ there exists an integer $k \in B(i)$ with $k \le U'$. From the choices of $u'$ and $U'$ it now follows that

$$\text{(2.2.37)} \qquad u(i) = \min_{k \in A'(i)} \{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} u(k-j)\phi(j)\} \qquad \text{for } i \le U',$$

where

$$A'(i) = \begin{cases} \{k \mid u' \le k \le U'\} & \text{for } i < u', \\ \{k \mid i \le k \le U'\} & \text{for } u' \le i \le U'. \end{cases}$$

The function $K\delta(k-i) + G_\alpha(k)$, where $k \in A'(i)$ and $i \le U'$, is bounded. Next it follows from theorem 1.1.1 (a) in section 1.1 that (2.2.37) has a unique bounded solution (see also the proof of theorem 2.2.1). From (2.2.34) and the relation $s_1 \le s^* \le S^* \le S_1$ it follows that the bounded function $V_\alpha^*(i)$, $i \le U'$, satisfies also (2.2.37). Thus

$$u(i) = V_\alpha^*(i) \qquad \text{for all } i \le U',$$

and consequently $u(i) = V_\alpha^*(i)$ for all $i \le M$. Since M was chosen arbitrarily, it follows that $u(i) = V_\alpha^*(i)$ for all $i \in I$. This completes the proof of the theorem.

This uniqueness theorem was proved in [26] for the case in which $G_\alpha(k)$ is convex and the demand has a density, where Scarf's results for the finite period model were used to establish the existence of a solution to the optimal inventory equation. The proof of theorem 2.2.7 is an adaptation of the uniqueness proof given in [26]. Finally, we note that in [4] a uniqueness theorem is given for the case $K = 0$ and no backlogging of excess demand.

## 2.3. THE DYNAMIC INVENTORY MODEL WITH DEMANDS DEPENDING ON THE STOCK LEVEL

In this section we shall consider a periodic review inventory model in which the demands depend on the stock level and the lead time of an order is zero. We give some preliminaries in subsection 2.3.1. In subsection 2.3.2 the results of section 2.1 will be generalized. The existence of optimal (s,S) policies for the infinite period model will be proved in subsection 2.3.3. Finally, the uniqueness theorem of subsection 2.2.4 will be generalized.

### *2.3.1. Model and preliminaries*

The stock level of a single item is reviewed at the beginnings of the periods 1,2,... and then an order may be placed for any positive, integral quantity of stock. We assume that the delivery of an order is *immediate*. Let $\phi(k,j)$ be the probability of demand j during any period for which the stock level is k at the beginning of that period just after any additions to stock. Excess demands are backlogged. Hence the stock level may take on negative values. We take the set I of all integers as set of all possible values for the stock level. It is assumed that

$$\phi(k,0) < 1 \quad \text{and} \quad \mu_k = \sum_{j=0}^{\infty} j\phi(k,j) < \infty \qquad \text{for all } k \in I.$$

The following costs are considered. The cost of ordering z units is $K\delta(z) + cz$, where $K \geq 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z \geq 1$. Let $g(k)$ be the holding and shortage cost in a period when k is the stock level just after any additions to stock. In the inventory model with a finite planning horizon, it is assumed that both stock left over and backlogged demand remaining at the end of the final period have a salvage value of d per unit.

Finally, there is a fixed discount factor $\alpha$ with $0 < \alpha \leq 1$. Let

$$(2.3.1) \qquad G_\alpha(k) = g(k) + (1-\alpha)ck + \alpha c\mu_k \qquad \text{for } k \in I.$$

We impose the following mild restrictions on the choice of an ordering decision. Finite integers u and U with $u \leq U$ are given, such that nothing is ordered, if, at review, the stock level i is larger than or equal to U, at most U-i units are ordered if the stock level i is less than U, and at least u-i units are ordered if the stock level i is less than u. The integers u and U may be chosen arbitrarily. Each ordering decision will be represented by the stock level just after ordering. Denote by A(i) the set of feasible decisions for stock level i. Then

$$A(i) = \{i\} \text{ for } i \geq U, \text{ and } A(i) = \{k \mid \max(i,u) \leq k \leq U\} \text{ for } i < U.$$

The above infinite period inventory model can be seen as a Markovian decision model in which the state space is given by the set I of all integers, the set of feasible decisions in state i is given by A(i), $q_{ij}(k) = \phi(k,k-j)$ for $i, j \in I$ and $k \in A(i)$, and $w_{ik} = K\delta(k-i)+(k-i)c+g(k)$ for $i \in I$ and $k \in A(i)$, where we put $\phi(i,j) = 0$ for $j < 0$.

Denote by C the class of all possible policies for above inventory model. Given an initial stock $i \in I$ and policy R to be used, let

$\underline{x}_t$ = the stock level just before ordering in period t, $t = 1,2,\ldots,$
$\underline{a}_t$ = the stock level just after ordering in period t, $t = 1,2,\ldots,$
$\underline{\xi}_t$ = the demand in period t, $t = 1,2,\ldots$ .

Clearly,

$$(2.3.2) \qquad u \leq \underline{a}_t \leq \max(\underline{x}_1, U) \qquad \text{for } t \geq 1,$$

and

$$(2.3.3) \qquad \underline{x}_{t+1} = \underline{a}_t - \underline{\xi}_t \qquad \text{for } t \geq 1.$$

By (2.3.2), we have for each $i \in I$ and $R \in C$ that $E_R(\underline{\xi}_t \mid \underline{x}_1 = i) \leq$ $\leq \max\{\mu_k \mid u \leq k \leq \max(i,U)\}$. Therefore, for each $i \in I$ there exists a finite number $B_i$ such that

$$(2.3.4) \qquad E_R\{|K\delta(\underline{a}_t-\underline{x}_t) + (\underline{a}_t-\underline{x}_t)c + g(\underline{a}_t)|\,|\underline{x}_1=i\} \leq B_i \qquad \text{for all } R \in C.$$

For any $i \in I$ and $R \in C$, let

$$(2.3.5) \qquad g_n(i;R;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + (\underline{a}_t-\underline{x}_t)c + g(\underline{a}_t)|\underline{x}_1=i\} + - \alpha^n d\, E_R(\underline{x}_{n+1}|\underline{x}_1=i), \qquad n \geq 1.$$

Clearly, $g_n(i;R;\alpha)$ represents the total expected discounted cost in the n-period model, when $\underline{x}_1 = i$ and policy R is used.

Using (2.3.3), we obtain

$$g_n(i;R;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t)+(1-\alpha)c\underline{a}_t+\alpha c\underline{\xi}_t+g(\underline{a}_t)|\underline{x}_1=i\} + - ci - \alpha^n(d-c)E_R(\underline{x}_{n+1}|\underline{x}_1=i\} \qquad \text{for } n \geq 1.$$

Since

$$E_R(\underline{\xi}_t|\underline{x}_1=i) = E_R(\mu_{\underline{a}_t}|\underline{x}_1=i) \qquad \text{for } t \geq 1,$$

we have by definition (2.3.1) that

$$(2.3.6) \qquad g_n(i;R;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + G_\alpha(\underline{a}_t)|\underline{x}_1=i\} + - ci - \alpha^n(d-c)E_R(\underline{x}_{n+1}|\underline{x}_1=i) \qquad \text{for } n \geq 1.$$

Since for each $i \in I$ and $R \in C$ the sequence $\{E_R(\underline{x}_{n+1}|\underline{x}_1=i)\}$ is bounded, we have for all $i \in I$ and $R \in C$ that

$$\lim_{n\to\infty} \alpha^n E_R(\underline{x}_{n+1}|\underline{x}_1=i) = 0 \text{ for } \alpha < 1, \; \lim_{n\to\infty} \frac{1}{n} E_R(\underline{x}_{n+1}|\underline{x}_1=i) = 0.$$

For any $i \in I$ and $R \in C$, let

$$(2.3.7) \qquad V_\alpha(i;R) = \sum_{t=1}^{\infty} \alpha^{t-1} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + G_\alpha(\underline{a}_t)|\underline{x}_1=i\} \qquad \text{for } \alpha < 1$$

and

$$(2.3.8) \qquad g(i;R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} E_R\{K\delta(\underline{a}_t-\underline{x}_t) + G_1(\underline{a}_t)|\underline{x}_1=i\}.$$

By (2.3.4), we have that $V_\alpha(i;R)$ and $g(i;R)$ exist and are finite. Clearly,

$$\lim_{n\to\infty} g_n(i;R;\alpha) = V_\alpha(i;R) - ci \text{ for } \alpha < 1,$$

and

$$\liminf_{n\to\infty} \frac{1}{n} g_n(i;R;1) = g(i;R).$$

The quantity $V_\alpha(i;R) - ci$ represents the total expected discounted cost over the periods 1,2,... , when the initial stock is i and policy R is followed. When the limit in (2.3.8) exists $g(i;R)$ represents the average expected cost per period in the infinite period model, when the initial stock is i and policy R is used.

In the infinite period model with $\alpha < 1$ a policy $R^* \in C$ is called *optimal* if $V_\alpha(i;R^*) \le V_\alpha(i;R)$ for all $i \in I$ and $R \in C$.

In the infinite period model with $\alpha = 1$ a policy $R^* \in C$ is called *optimal* if $g(i;R^*) \le g(i;R)$ for all $i \in I$ and $R \in C$.

To prove the existence of an optimal (s,S) policy in the infinite period model, we shall need in subsection 2.3.3 the following two basic theorems.

*Theorem 2.3.1. (the total discounted cost criterion)*

Let $\alpha < 1$. If for policy $R^* \in C$ holds

$$V_\alpha(i;R^*) = \min_{k\in A(i)} \{K\delta(k-i)+G_\alpha(k)+\alpha \sum_{j=0}^{\infty} V_\alpha(k-j;R^*)\phi(k,j)\}, \qquad i \in I,$$

then the policy $R^*$ is optimal.

*Proof*

The proof of this theorem is identical to the proof of theorem 2.2.1.

*Theorem 2.3.2. (the average cost criterion)*

Let $\alpha = 1$. Suppose there exists a set of finite numbers $\{g,v(i),i \in I\}$ such that

$$\text{(2.3.9)} \qquad g+v(i) = \min_{k \in A(i)} \{K\delta(k-i) + G_1(k) + \sum_{j=0}^{\infty} v(k-j)\phi(k,j)\}, \qquad i \in I,$$

and

$$\lim_{n\to\infty} \frac{1}{n} E_R(v(\underline{x}_n)|\underline{x}_1=i) = 0 \qquad \text{for all } i \in I \text{ and } R \in C.$$

Let $R^*$ be a stationary deterministic policy which, when in state i, prescribes an action which minimizes the right-hand side of (2.3.9), then policy $R^*$ is optimal. Further, $g(i;R^*) = g$ for all $i \in I$ and for policy $R^*$ the lim inf in (2.3.8) can be replaced by lim.

*Proof*

This theorem is a direct consequence of theorem 1.1.2 of section 1.1.

We shall now introduce a number of quantities connected with the demand process.

For any $i \in I$ and $j \geq 0$, let

$$\text{(2.3.10)} \qquad \phi^{(n)}(i,j) = \sum_{h=0}^{j} \phi^{(n-1)}(i,h)\phi(i-h,j-h), \qquad n \geq 1$$

and

$$\text{(2.3.11)} \qquad \Phi^{(n)}(i,j) = \sum_{k=0}^{j} \phi^{(n)}(i,k), \qquad n \geq 0,$$

where

$$\text{(2.3.12)} \qquad \phi^{(0)}(i,j) = \begin{cases} 1 & \text{for } j = 0, \\ 0 & \text{for } j \geq 1. \end{cases}$$

When n = 1 we often drop the superscript.

Note that under the conditions the initial stock is i and no orders are placed, $\phi^{(n)}(i,j)$ represents the probability that the total demand in the first n periods will be j.

For any $i \in I$ and $j \geq 0$, we have

$$\text{(2.3.13)} \qquad \phi^{(n+m)}(i,j) = \sum_{h=0}^{j} \phi^{(n)}(i,h)\phi^{(m)}(i-h,j-h), \qquad m, n \geq 0.$$

We prove (2.3.13) by induction on m, where we fix n. By (2.3.10) and (2.3.12), we have that (2.3.13) is true for m = 0,1. Assuming that (2.3.13) is true for the integer m, we have

$$\phi^{(n+m+1)}(i,j) = \sum_{h=0}^{j} \phi^{(n+m)}(i,h)\phi(i-h,j-h) =$$

$$= \sum_{h=0}^{j} \phi(i-h,j-h) \sum_{k=0}^{h} \phi^{(n)}(i,k)\phi^{(m)}(i-k,h-k) =$$

$$= \sum_{k=0}^{j} \phi^{(n)}(i,k) \sum_{h=k}^{j} \phi^{(m)}(i-k,h-k)\phi(i-h,j-h) =$$

$$= \sum_{k=0}^{j} \phi^{(n)}(i,k)\phi^{(m+1)}(i-k,j-k),$$

which proves (2.3.13) for m+1. This completes the induction proof.

By (2.3.11) - (2.3.13), we have for any $i \in I$ and $j \geq 0$ that

$$(2.3.14) \quad \Phi^{(n+m)}(i,j) = \sum_{h=0}^{j} \phi^{(n)}(i,h)\Phi^{(m)}(i-h,j-h), \qquad m, n \geq 0.$$

For any $i \in I$ and $j \geq 0$, let

$$(2.3.15) \quad m(i;j;\alpha) = \sum_{n=1}^{\infty} \alpha^n \phi^{(n)}(i,j),\ M(i;j;\alpha) = \sum_{n=1}^{\infty} \alpha^n \Phi^{(n)}(i,j).$$

Note that $M(i;j;\alpha) = m(i;0;\alpha) + \dots + m(i;j;\alpha)$, $j \geq 0$. Since $\Phi^{(n)}(i,j) \leq 1$ for all $i \in I$; $j, n \geq 0$, we have for all $i \in I$ and $j \geq 0$ that

$$M(i;j;\alpha) \leq \frac{\alpha}{1-\alpha} \qquad \text{for } \alpha < 1.$$

*Lemma 2.3.1.*

The function $M(i;j;1)$ is finite for all $i \in I$ and $j \geq 0$. Moreover, for any $i \in I$ and $j \geq 0$, $\Phi^{(n)}(i,j)$ converges exponentially fast to zero as $n \to \infty$ and $\Phi^{(1)}(i,j) + \dots + \Phi^{(n)}(i,j)$ converges exponentially fast to $M(i;j;1)$ as $n \to \infty$.

*Proof*

First let us prove that for any $i \in I$ and $j \geq 0$,

$$(2.3.16) \quad 0 \leq \Phi^{(n+1)}(i,j) \leq \Phi^{(n)}(i,j) \leq 1 \qquad \text{for } n \geq 0.$$

Clearly, (2.3.16) is true for $n = 0$, since $\Phi^{(0)}(i,j) = 1$. By (2.3.14), we have for any $i \in I$ and $j \geq 0$ that

$$\Phi^{(n)}(i,j) = \sum_{h=0}^{j} \phi(i,h)\Phi^{(n-1)}(i-h,j-h), \qquad n \geq 1, \tag{2.3.17}$$

from which (2.3.16) follows by induction on n.

We shall next prove that for each $i \in I$ and $j \geq 0$ there exists an integer $N \geq 1$ such that

$$\Phi^{(n)}(i,j) \leq \Phi^{(N)}(i,j) < 1 \qquad \text{for all } n \geq N. \tag{2.3.18}$$

Fix i and j. Suppose to the contrary that $\Phi^{(n)}(i,j) = 1$ for all $n \geq 0$. Then it follows from (2.3.17) and the assumption $\phi(i,0) < 1$ that an integer $h^*$ exists such that $1 \leq h^* \leq j$ and $\phi(i,h^*) > 0$. From this and the relations (2.3.16) and (2.3.17) it follows that $\Phi^{(n)}(i-h^*,j-h^*) = 1$ for all $n \geq 0$. Proceeding in this way, we see that there exists an integer $i^* < i$ such that $\Phi^{(n)}(i^*,0) = 1$ for all $n \geq 0$. This is a contradiction, since $\Phi^{(n)}(i^*,0) = \{\phi(i^*,0)\}^n < 1$ for $n \geq 1$. Thus $\Phi^{(n)}(i,j) < 1$ for some $n \geq 0$, and hence, by (2.3.16), we have proved (2.3.18).

We are now in a position to prove the lemma. Fix $i_0 \in I$ and fix $j_0 \geq 0$. From (2.3.18) follows the existence of a number $\delta$, $0 \leq \delta < 1$, and an integer $N \geq 1$ such that

$$\Phi^{(n)}(i,j) \leq \delta \qquad \text{for } i_0-j_0 \leq i \leq i_0;\ 0 \leq j \leq j_0;\ n \geq N.$$

For any $k \geq 0$, we have by (2.3.14) that

$$\begin{aligned}\Phi^{(kN+N)}(i_0,j_0) &= \sum_{h=0}^{j_0} \phi^{(N)}(i_0,h)\Phi^{(kN)}(i_0-h,j_0-h) \leq \\ &\leq \max_{0\leq h\leq j_0} \Phi^{(kN)}(i_0-h,j_0-h) \sum_{h=0}^{j_0} \phi^{(N)}(i_0,h) \\ &\leq \delta\Phi^{(kN)}(i_0',j_0')\end{aligned}$$

for some $i_0' \in [i_0-j_0,j_0]$ and $j_0' \in [0,j_0]$. From this we get

$$\Phi^{(kN+N)}(i_0,j_0) \leq \delta^{k+1}, \qquad k \geq 0. \tag{2.3.19}$$

It now follows from (2.3.16) and (2.3.19) that for any $i \in I$ and $j \geq 0$, $\phi^{(n)}(i,j)$ converges exponentially fast to zero as $n \to \infty$. This ends the proof of the lemma.

From (2.3.12), (2.3.13) and (2.3.15) it follows that

$$(2.3.20) \qquad m(i;j;\alpha) = \alpha\phi(i,j) + \alpha \sum_{h=0}^{j} \phi(i-h,j-h)m(i;h;\alpha) \quad \text{for } i \in I,\ j \geq 0,$$

and

$$(2.3.21) \qquad m(i;j;\alpha) = \alpha\phi(i,j) + \alpha \sum_{h=0}^{j} m(i-h;j-h;\alpha)\phi(i,h) \quad \text{for } i \in I;\ j \geq 0.$$

*Lemma 2.3.2.*

If $\sup_{k \geq k^*} \phi(k,0) < 1$ for some integer $k^*$, then for each $i_0 \in I$ there exists a constant $a_0$ such that $m(i;j;1) \leq a_0$ for all $0 \leq j \leq i-i_0$ and $i \geq i_0$.

*Proof*

Let

$$c_i = \frac{1}{1-\phi(i,0)} \qquad \text{for } i \in I.$$

From (2.3.21) it follows after some straightforward calculations that

$$(2.3.22) \qquad m(i;j;1) = \begin{cases} \phi(i,0)c_i & \text{for } i \in I;\ j = 0 \\ \phi'(i,j)c_{i-j} + \sum_{h=1}^{j-1} m(i-h;j-h;1)\phi'(i,h) & \text{for } i \in I;\ j \geq 1, \end{cases}$$

where

$$\phi'(i,j) = \frac{\phi(i,j)}{1-\phi(i,0)} \qquad \text{for } i \in I;\ j \geq 1.$$

Clearly, $\sum_{j=1}^{\infty} \phi'(i,j) = 1$ for all $i \in I$. Let $\delta = \sup_{k \geq k^*} \phi(k,0)$, then $c_k \leq 1/(1-\delta)$ for all $k \geq k^*$. Therefore, for each $i_0 \in I$ there exists a constant $a_0$ such that $c_i \leq a_0$ for all $i \geq i_0$. It now follows from (2.3.22) that

$$(2.3.23) \quad m(i;0;1) \leq a_0 \qquad \text{for all } i \geq i_0.$$

Assuming that $i \geq i_0$ is an integer such that

$$(2.3.24) \quad m(r;j;1) \leq a_0 \qquad \text{for all } 0 \leq j \leq r-i_0;\ i_0 \leq r \leq i,$$

we have by (2.3.22) that

$$m(i+1;j;1) \leq \phi'(i+1,j)a_0 + a_0 \sum_{h=1}^{j-1} \phi'(i+1,h) = a_0 \sum_{h=1}^{j} \phi'(i+1,h) \leq a_0,$$
$$1 \leq j \leq i+1-i_0.$$

From this relation and (2.3.23) it follows that (2.3.24) is also true when i is replaced by i+1, which ends the proof of the lemma.

We note that if $\delta = \sup_{k\epsilon I}\phi(k,0) < 1$, then it can be readily verified that $m(i;j;1) \leq 1/(1-\delta)$ for all $i \in I$ and $j \geq 0$.

For any $i \in I$ and $j \geq 0$, let

$$(2.3.25) \quad \hat{m}(i;j;1) = \sum_{n=1}^{\infty} n\phi^{(n)}(i,j).$$

The numbers $\hat{m}(i;j;1)$ satisfy

$$(2.3.26) \quad \hat{m}(i;j;1) = m(i;j;1) + \sum_{h=0}^{j} m(i-h;j-h;1)m(i;h;1), \quad i \in I;\ j \geq 0.$$

The proof of (2.3.26) proceeds as follows. From (2.3.10), (2.3.15) and (2.3.20) it follows easily that

$$\hat{m}(i;j;1) = m(i;j;1) + \sum_{h=0}^{j} \hat{m}(i;h;1)\phi(i-h,j-h), \qquad i \in I;\ j \geq 0.$$

From this and (2.3.13) with n = 1 it can be easily deduced by induction on r that

$$\hat{m}(i;j;1) = m(i;j;1) + \sum_{k=1}^{r} \sum_{h=0}^{j} m(i;h;1)\phi^{(k)}(i-h,j-h) + $$
$$+ \sum_{h=0}^{j} \hat{m}(i;h;1)\phi^{(r+1)}(i-h,j-h),$$
$$i \in I;\ j \geq 0;\ r \geq 0.$$

Taking the limit as $r \to \infty$ and using lemma 2.3.1, we obtain (2.3.26).

We close this subsection with the following lemma [28].

*Lemma 2.3.3.*

If $a_j$ and $b_j$, $j = 0,\ldots, N$, are non-negative real numbers such that

$$\sum_{j=0}^{N} a_j = \sum_{j=0}^{N} b_j, \quad \sum_{j=0}^{H} a_j \geq \sum_{j=0}^{H} b_j \quad \text{for } H = 0,\ldots, N-1, \tag{2.3.27}$$

and if $f$ and $g$ are functions on the integers $0,1,\ldots, N$ such that $f(j) \leq g(k)$ whenever $j \leq k$, then

$$\sum_{j=0}^{N} a_j\, f(j) \leq \sum_{j=0}^{N} b_j\, g(j).$$

The condition $f(j) \leq g(k)$ can be weakened to $f(j) \leq g(k)$ for any pair $(j,k)$ with $j \leq k$ and $a_j b_k > 0$.

*Proof*

Since $a_0 + \ldots + a_N = b_0 + \ldots + b_N$, it is no restriction to assume that the function $g$ is non-negative. Define the function $h(i)$, $i = 0,\ldots, N$, by

$$h(i) = \min \{g(i),\ldots, g(N)\} \qquad \text{for } i = 0,\ldots, N.$$

Then the function $h$ is non-negative and non-decreasing. Further we have that $f(i) \leq h(i) \leq g(i)$ for $i = 0,\ldots, N$. Since $h$ is non-negative and non-decreasing, the function $h$ can be expressed in the form (see also [12, p.123])

$$h(i) = \sum_{k=0}^{N} c_k\, h_k(i) \qquad \text{for } i = 0,\ldots, N,$$

where $c_i \geq 0$ for $i = 0,\ldots, N$, and

$$h_k(i) = \begin{cases} 0, & i < k, \\ 1, & i \geq k. \end{cases}$$

Then,

$$\sum_{j=0}^{N} (a_j-b_j)h(j) = \sum_{j=0}^{N} (a_j-b_j) \sum_{k=0}^{N} c_k h_k(j) = \sum_{k=0}^{N} c_k \sum_{j=k}^{N} (a_j-b_j) \leq 0.$$

Since $f \leq h \leq g$ and the numbers $a_i$ and $b_i$ are non-negative, we have

$$\sum_{j=0}^{N} a_j\, f(j) \leq \sum_{j=0}^{N} a_j\, h(j) \leq \sum_{j=0}^{N} b_j\, h(j) \leq \sum_{j=0}^{N} b_j\, g(j).$$

This ends the proof of the lemma.

*2.3.2. Some quantities for the dynamic (s,S) inventory model*

In this subsection we shall generalize the results found in section 2.1.

In this subsection s and S are fixed integers with $s, S \in I$ and $s \leq S$. For any $i \geq s$, let

$$(2.3.28) \qquad \rho_i(k) = \begin{cases} 0, & k = 0, \\ \Phi^{(k-1)}(i,i-s) - \Phi^{(k)}(i,i-s), & k \geq 1. \end{cases}$$

The quantity $\rho_i(k)$, $(k \geq 1)$, represents the probability that the cumulative demand will first exceed $i-s$ during the $k^{th}$ period, when the initial stock is i and no orders are placed.

From (2.3.28)

$$(2.3.29) \qquad \sum_{k=0}^{n} \rho_i(k) = 1 - \Phi^{(n)}(i,i-s), \qquad i \geq s;\ n \geq 1,$$

and

$$(2.3.30) \qquad \sum_{k=0}^{n} k\rho_i(k) = 1 + \sum_{k=1}^{n-1} \Phi^{(k)}(i,i-s) - n\Phi^{(n)}(i,i-s), \quad i \geq s;\ n \geq 1,$$

and hence, by lemma 2.3.1, we obtain

$$(2.3.31) \qquad \sum_{k=0}^{\infty} \rho_i(k) = 1, \ \sum_{k=0}^{\infty} k\rho_i(k) = 1 + M(i;i-s;1), \qquad i \geq s.$$

Hence we have for each $i \geq s$ that $\{\rho_i(k),\ k \geq 0\}$ constitutes a probability distribution with a finite first moment.

Let

$$\rho(k;\alpha) = \alpha^k\, \rho_S(k), \qquad k \geq 0.$$

Note that $\{\rho(k;1), k \geq 0\}$ is a probability distribution. Let

$$\rho^{(t)}(j;\alpha) = \begin{cases} \rho(j;\alpha), & j \geq 0;\ t = 1, \\ \sum_{k=0}^{j} \rho^{(t-1)}(k;\alpha)\rho(j-k;\alpha), & j \geq 0;\ t \geq 2, \end{cases}$$

and let

$$r(j;\alpha) = \sum_{t=1}^{\infty} \rho^{(t)}(j;\alpha), \qquad j \geq 0.$$

In the same way as in subsection 2.1.1 it can be shown that

$$\sum_{k=0}^{\infty} \alpha^k \rho_i(k) = \alpha - (1-\alpha)\ M(i;i-s;\alpha), \qquad i \geq s$$

and

$$\sum_{j=0}^{\infty} r(j;\alpha) = \frac{\alpha - (1-\alpha)\ M(S;S-s;\alpha)}{(1-\alpha)\{1+M(S;S-s;\alpha)\}} \qquad \text{for } \alpha < 1.$$

Consider now the $(s,S)$ policy. For any $n \geq 0$, let

$$p_{ij}^{(n)} = P_{(s,S)}\{\underline{a}_{n+1}=j\,|\,\underline{x}_1=i\},\quad r_{ij}^{(n)} = P_{(s,S)}\{\underline{x}_{n+1}=j\,|\,\underline{x}_1=i\}, \qquad i,\ j \in I.$$

By (2.3.3), we have

$$r_{ij}^{(n)} = \sum_{k=s}^{\max(i,S)} p_{ik}^{(n-1)}\phi(k,k-j), \qquad i,\ j \in I;\ n \geq 1.$$

For any $i \in I$ and $n \geq 0$, define $\phi^{(n)}(i,k) = 0$ for $k < 0$. Clearly,

$$\text{(2.3.32)} \qquad p_{ij}^{(n)} = \begin{cases} \phi^{(n)}(i,i-j), & i \geq s;\ j > S;\ n \geq 0, \\ 0, & i \in I;\ j < s;\ n \geq 0, \\ p_{Sj}^{(n)}, & i < s;\ j \in I;\ n \geq 0. \end{cases}$$

The quantity $\phi^{(n)}(i,j)$ represents the probability that the cumulative demand in the first n periods will be j, given that the initial stock is i and no orders will be placed in the periods $1,\ldots, n$. When the $(s,S)$ policy is used and $\underline{x}_1 = i \geq s$, the probability that the cumulative demand will first exceed $i-s$ during the $k^{th}$ period is $\rho_i(k)$. It will now be clear that for any $n \geq 0$

(2.3.33) $$p_{ij}^{(n)} = \phi^{(n)}(i,i-j) + \sum_{k=0}^{n} p_{Sj}^{(n-k)}\ \rho_i(k), \quad s \leq j \leq \max(i,S);\ i \geq s.$$

In particular,

$$p_{Sj}^{(n)} = \phi^{(n)}(S,S-j) + \sum_{k=0}^{n} p_{Sj}^{(n-k)}\ \rho(k;1), \qquad n \geq 0;\ s \leq j \leq S.$$

For each $j \in [s,S]$ this equation is a renewal equation, and hence (cf. section 1.2)

(2.3.34) $$p_{Sj}^{(n)} = \phi^{(n)}(S,S-j) + \sum_{k=0}^{n} \phi^{(n-k)}(S,S-j)\ r(k;1), \quad n \geq 0;\ s \leq j \leq S.$$

The relations (2.3.32) - (2.3.34) in conjunction yield the probability distribution of $\underline{a}_{n+1}$. We see that the derivation and the result are quite similar to those in subsection 2.1.2. The steady-state behaviour of the Markov chains $\{\underline{a}_t\}$ and $\{\underline{x}_t\}$ is given in the next theorem.

*Theorem 2.3.3.*

For all i, $j \in I$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} p_{ij}^{(k)} = q_j \text{ and } \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} r_{ij}^{(k)} = v_j,$$

where

$$q_j = \begin{cases} \{1 + m(S;0;1)\}/\{1 + M(S;S-s;1)\}, & j = S, \\ m(S;S-j;1)/\{1 + M(S;S-s;1)\}, & s \leq j < S, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$v_j = \begin{cases} \{\phi(S,S-j) + \sum_{k=0}^{S-s} \phi(S-k,S-j-k)m(S;k;1)\}/\{1+M(S;S-s;1)\}, & j < s, \\ m(S;S-j;1)/\{1+M(S;S-s;1)\}, & s \leq j \leq S, \\ 0 & j > S. \end{cases}$$

If the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1, then $\lim_{n\to\infty} p_{ij}^{(n)} = q_j$ and $\lim_{n\to\infty} r_{ij}^{(n)} = v_j$ for all i, $j \in I$.

Finally, the probability distribution $\{q_j, j \in I\}[\{v_j, j \in I\}]$

is the unique stationary probability distribution of the Markov chain $\{\underline{a}_t\}$ $[\{\underline{x}_t\}]$.

*Proof*

The proof is identical to the proof of theorem 2.1.1.

We shall next determine the solution for the total expected discounted cost in the (s,S) inventory model. To do this, let $g_0^*(i;\alpha) = 0$ for all $i \in I$, and let

$$g_n^*(i;\alpha) = \sum_{t=1}^{n} \alpha^{t-1} E_{(s,S)}\{K\delta(\underline{a}_t - \underline{x}_t) + G_\alpha(\underline{a}_t) | \underline{x}_1 = i\}, \qquad i \in I;\ n \geq 1.$$

Then , by (2.3.6), we have

$$g_n(i;(s,S);\alpha) = g_n^*(i;\alpha) - ci - \alpha^n(d-c)E(\underline{x}_{n+1}|\underline{x}_1 = i), \qquad i \in I;\ n \geq 1.$$

Clearly,

$$g_n^*(i;\alpha) = K + g_n^*(S;\alpha), \qquad i < s;\ n \geq 1.$$

From the interpretation of the probabilities $\rho_i(k)$ we have for $i \geq s$ and $n \geq 1$ that

$$g_n^*(i;\alpha) = G_\alpha(i) + \sum_{k=1}^{n-1} \sum_{j=0}^{i-s} \alpha^k G_\alpha(i-j)\phi^{(k)}(i,j) + \sum_{k=1}^{n-1} \alpha^k\{K + g_{n-k}^*(S;\alpha)\}\rho_i(k),$$

from which we get

$$(2.3.35) \qquad g_n^*(i;\alpha) = b_n(i;\alpha) + \sum_{k=0}^{n} g_{n-k}^*(S;\alpha)\alpha^k\rho_i(k), \qquad i \geq s;\ n \geq 0,$$

where $b_0(i;\alpha) = 0$ for $i \geq s$, and

$$b_n(i;\alpha) = G_\alpha(i) + \sum_{k=1}^{n-1} \sum_{j=0}^{i-s} \alpha^k G_\alpha(i-j)\phi^{(k)}(i,j) + K\sum_{k=1}^{n-1} \alpha^k \rho_i(k), \quad i \geq s; n \geq 1.$$

For $i = S$ the equation (2.3.35) is a renewal equation.

By (2.3.3), we have

$$E(\underline{x}_{n+1}|\underline{x}_1 = i) = \sum_{j=s}^{\max(i,S)} (j-\mu_j)\, p_{ij}^{(n-1)}, \qquad i \in I;\ n \geq 1.$$

It will now be clear that lemma 2.1.2 is also true for the (s,S) model now under consideration, provided that we replace $g_n(i;\alpha)$ by $g_n(i;(s,S);\alpha)$, replace $\sum_j jp_{hj}^{(n-1)} - \mu$ by $\sum_j p_{hj}^{(n-1)}\{j-\mu_j\}$ for $h = S,i$, and put $\lambda = 0$.

*Theorem 2.3.4.*

Let $\alpha < 1$. Then

$$\lim_{n\to\infty} g_n(i;(s,S);\alpha) = \frac{a_\alpha(s,S)}{1-\alpha} - ci, \qquad i < s,$$

and

$$\lim_{n\to\infty} g_n(i;(s,S);\alpha) = G_\alpha(i) + \sum_{j=0}^{i-s} G_\alpha(i-j)m(i;j;\alpha) + \\ + \frac{a_\alpha(s,S)}{1-\alpha}\{\alpha-(1-\alpha)M(i;i-s;\alpha)\} - ci, \qquad i \geq s,$$

where

$$a_\alpha(s,S) = \frac{G_\alpha(S) + \sum_{k=0}^{S-s} G_\alpha(S-k)m(S;k;\alpha) + K}{1+M(S;S-s;\alpha)}.$$

*Proof*

The proof is identical to the proof of theorem 2.1.2.

Finally, we consider the total expected cost and the average expected cost in the (s,S) inventory model.

*Theorem 2.3.5.*

$$\lim_{n\to\infty} g_n(i;(s,S);1)/n = a_1(s,S) \qquad \text{for all } i \in I,$$

where

$$a_1(s,S) = \frac{G_1(S) + \sum_{k=0}^{S-s} G_1(S-k)m(S;k;1) + K}{1+M(S;S-s;1)}.$$

*Proof*

The proof is identical to the proof of theorem 2.1.3.

Let

$$h_n(i) = g_n^*(i;1) - na_1(s,S), \qquad i \in I;\ n \geq 0,$$

then $h_n(i) = K + h_n(S)$ for $i < s$; $n \geq 1$. It follows from (2.3.35), (2.3.30) and (2.3.29) that

$$(2.3.36) \qquad h_n(i) = c_n(i) + \sum_{k=0}^{n} h_{n-k}(S)\ \rho_i(k), \qquad i \geq s;\ n \geq 0,$$

where $c_0(i) = 0$ for $i \geq s$, and

$$c_n(i) = G_1(i) + \sum_{k=1}^{n-1}\sum_{j=0}^{i-s} G_1(i-j)\ \phi^{(k)}(i,j) + K\{1-\Phi^{(n-1)}(i,i-s)\} +$$

$$- a_1(s,S)\{1 + \sum_{k=1}^{n-1} \Phi^{(k)}(i,i-s)\}, \qquad i \geq s;\ n \geq 1.$$

For $i = S$ the equation (2.3.36) is a renewal equation.

It is now readily seen that lemma 2.1.3 is also true for the (s,S) model now under consideration, provided that we replace $g_n(i;1)$ and $g_1$, respectively, by $g_n(i;(s,S);1)$ and $a_1(s,S)$, replace $\sum_j jp_{hj}^{(n-1)} - \mu$ by $\sum_j p_{hj}^{(n-1)}\{j-\mu_j\}$ for $h = S,i$, and put $\lambda = 0$.
Using lemma 2.3.1, we obtain for any $i \geq s$ that

$$\lim_{n\to\infty} c_n(i) = G_1(i) + \sum_{j=0}^{i-s} G_1(i-j)m(i;j;1) + K - a_1(s,S)\{1+M(i;i-s;1)\},$$

where the convergence is exponentially fast. In particular, we have by the definition of $a_1(s,S)$ that $c_n(S)$ converges exponentially fast to zero as $n \to \infty$, and hence $\sum|c_n(S)| < \infty$. Moreover, similar to the proof of (2.1.44), we can show that

$$\sum_{n=0}^{\infty} c_n(S) = \sum_{j=0}^{S-s} \{a_1(s,S) - G_1(S-j)\}\hat{m}(S;j;1) - K\{1+M(S;S-s;1)\}.$$

It will now be clear that the following theorem can be proved in the same way as theorem 2.1.4.

*Theorem 2.3.6.*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \{g_k(i;(s,S);1)-ka_1(s,S)\} = \frac{\sum_{j=0}^{S-s} \{a_1(s,S)-G_1(S-j)\}\hat{m}(S;j;1)}{1+M(S;S-s;1)} - ci + $$
$$- (d-c) \sum_{j=s}^{S} (j-\mu_j)q_j, \qquad i < s,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \{g_k(i;(s,S);1)-ka_1(s,S)\} = G_1(i) + \sum_{j=0}^{i-s} G_1(i-j)m(i;j;1) +$$
$$- a_1(s,S)\{1+M(i;i-s;1)\} + \frac{\sum_{j=0}^{S-s} \{a_1(s,S)-G_1(S-j)\}\hat{m}(j;1)}{1+M(S;S-s;1)} +$$
$$- ci - (d-c) \sum_{j=s}^{S} (j-\mu_j)q_j, \qquad i \geq s,$$

where the ordinary limit exists for each $i \in I$ if the greatest common divisor of the indices n, where $\rho(n;1) > 0$, is 1.

### 2.3.3. *The optimality of (s,S) policies in the infinite period model*

In this subsection we shall establish the existence of optimal (s,S) policies in the infinite period model under certain conditions on the function $G_\alpha(k)$ and the probabilities $\phi(k,j)$. The total discounted cost and the average cost criteria are treated simultaneously. Finally, a uniqueness theorem for the optimal inventory equation is given.

The following conditions are imposed on the function $G_\alpha(k)$:

(i) There exists a finite integer $S_0$ such that $G_\alpha(i) \leq G_\alpha(j)$ for $j \leq i \leq S_0$ and $G_\alpha(i) \geq G_\alpha(j)$ for $i \geq j \geq S_0$.

(ii) $\lim_{|k|\to\infty} G_\alpha(k) = \infty$.

We assume that $S_0$ is the smallest integer at which $G_\alpha(k)$ takes on its absolute minimum. Let $S^0$ be the largest integer at which $G_\alpha(k)$ takes on its absolute minimum. Let $s_1$ be the smallest integer for which

$$G_\alpha(s_1) \leq G_\alpha(S_0) + (1-\alpha\phi(S_0,0))K$$

and let $S_1$ be the largest integer for which

$$G_\alpha(S_1) \le G_\alpha(S_0) + \alpha K.$$

Note that $s_1 \le S_0 \le S^0 \le S_1$.

We impose on the probabilities $\phi(i,j)$ the following condition

$$\Phi(i+1,j) \ge \Phi(i,j) \qquad \text{for all } i \ge S_0 \text{ and } j = 0,1,\dots \, . \tag{2.3.37}$$

That is, the distribution functions $\Phi(i,.)$, $i = S_0, S_0+1,\dots$, are assumed to be *stochastically increasing*. We note that in [28] a slightly different condition is imposed on the probabilities $\phi(i,j)$.

We now give some properties of the following function.

$$a_\alpha(s,S) = \frac{G_\alpha(S) + \sum_{k=0}^{S-s} G_\alpha(S-k)m(S;k;\alpha) + K}{1+M(S;S-s;\alpha)}, \qquad s,S \in I;\ s \le S.$$

We have by the theorems 2.3.4 and 2.3.5 that

$$V_\alpha(i;(s;S)) = \begin{cases} \dfrac{a_\alpha(s,S)}{1-\alpha} & \text{for } i < s, \\ G_\alpha(i) + \sum_{j=0}^{i-s} G_\alpha(i-j)m(i;j;\alpha) + \\ \qquad + \dfrac{a_\alpha(s,S)}{1-\alpha}\{\alpha-(1-\alpha)M(i;i-s;\alpha)\} & \text{for } i \ge s, \end{cases} \tag{2.3.38}$$

when $\alpha < 1$, and

$$g(i;(s,S)) = a_1(s,S) \qquad \text{for all } i \in I. \tag{2.3.39}$$

*Lemma 2.3.4.*

Let $0 < \alpha \le 1$. If $\sup_{k \ge k^*} \alpha\phi(k,0) < 1$ for some $k \ge k^*$ (note that this assumption is automatically satisfied if $\alpha < 1$), then there exist integers $s^*$ and $S^*$ such that $s^* \le S^*$ and $a_\alpha(s^*,S^*) \le a_\alpha(s,S)$ for all $s,S \in I$, $s \le S$.

*Proof*

The proof is identical to the proof of lemma 2.2.1, where we now have to use lemma 2.3.2 instead of formula (1.2.4) of section 1.2.

For any $\alpha$, $0 < \alpha \leq 1$, assume that

$$a_\alpha^* = \min_{\substack{s,S \in I \\ s \leq S}} a_\alpha(s,S)$$

exists.

*Lemma 2.3.5.*

Let $0 < \alpha \leq 1$. Let $s^*$ and $S^*$ be any integers such that $s^* \leq S^*$ and $a_\alpha(s^*,S^*) = a_\alpha^*$.

(a) If $m(S^*;S^*-s^*+1;\alpha) > 0$, then $G_\alpha(s^*-1) \geq a_\alpha^*$.
(b) If $s^* = S^*$, then $G_\alpha(s^*) \leq a_\alpha^*$.
(c) If $s^* < S^*$ and if $m(S^*;S^*-s^*;\alpha) > 0$, then $G_\alpha(s^*) \leq a_\alpha^*$.
(d) If $\phi(k,1) > 0$ for $s^*-1 \leq k \leq S^*$, then $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$.
(e) If $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$, then $s_1 \leq s^* \leq S^0$ when $K = 0$, and $s_1 \leq s^* \leq S_0$ when $K > 0$.

*Proof*

The proof is identical to the proof of lemma 2.2.2.

*Lemma 2.3.6.*

Let $0 < \alpha \leq 1$. There exist integers $s^*$ and $S^*$ with $s^* \leq S^*$ such that $a_\alpha(s^*,S^*) = a_\alpha^*$ and $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$. If $K = 0$, then $s^* = S_0$ and $S^* = S_0$ satisfy these conditions.

*Proof*

The proof is identical to the proof of lemma 2.2.3.

Let $0 < \alpha \leq 1$. From now on $s^*$ and $S^*$ with $s^* \leq S^*$ are two fixed integers such that

$$a_\alpha(s^*,S^*) = a_\alpha^* \quad \text{and} \quad G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*),$$

where we choose

$$s^* = S^* = S_0 \qquad \text{if} \qquad K = 0.$$

The function $v_\alpha^*(i)$, $i \in I$, is defined as follows (cf. subsection 2.2.3).

$$(2.3.40) \qquad v_\alpha^*(i) = \begin{cases} 0 & \text{for } i < s^*, \\ G_\alpha(i) - a_\alpha^* + \alpha \sum_{j=0}^{i-s^*} v_\alpha^*(i-j)\phi(i,j) & \text{for } i \geq s^*. \end{cases}$$

The function $v_\alpha^*(i)$ is uniquely determined by (2.3.40). Using (2.3.10), it can be easily verified by induction on n that

$$v_\alpha^*(i) = G_\alpha(i) - a_\alpha^* + \sum_{k=1}^{n} \sum_{j=0}^{i-s^*} \{G_\alpha(i-j) - a_\alpha^*\}\alpha^k \phi^{(k)}(i,j) +$$

$$+ \alpha^{n+1} \sum_{j=0}^{i-s^*} v_\alpha^*(i-j)\phi^{(n+1)}(i,j), \quad i \geq s^*;\ n \geq 0.$$

Taking the limit as $n \to \infty$ and using lemma 2.3.1, we obtain

$$(2.3.41) \qquad v_\alpha^*(i) = \begin{cases} 0, & i < s^*, \\ G_\alpha(i) + \sum_{j=0}^{i-s^*} G_\alpha(i-j)m(i;j;\alpha) - a_\alpha^*\{1+M(i;i-s^*;\alpha)\}, & i \geq s^*. \end{cases}$$

Let the function $J_\alpha(k)$ be defined by

$$(2.3.42) \qquad J_\alpha(k) = G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} v_\alpha^*(k-j)\phi(k,j) \qquad \text{for } k \in I.$$

From (2.3.40) and (2.3.42) it follows that

$$(2.3.43) \qquad J_\alpha(k) = G_\alpha(k) - a_\alpha^* \qquad \text{for } k < s^*,$$

and

$$(2.3.44) \qquad J_\alpha(k) = v_\alpha^*(k) \qquad \text{for } k \geq s^*.$$

*Theorem 2.3.7.*

(a) $J_\alpha(k)$ is non-increasing on $(-\infty, s^*-1]$.

(b) $K + J_\alpha(S^*) = 0$; $J_\alpha(s^*-1) \geq 0$.
(c) $J_\alpha(k) \geq J_\alpha(S^*)$ for all $k \in I$.
(d) $J_\alpha(k) \leq 0$ for $s^* \leq k \leq S_0$.
(e) $K + J_\alpha(k) \geq J_\alpha(i)$ for $k \geq i \geq s^*$.
(f) $K + J_\alpha(k) > 0$ for $k > S_1$.

*Proof*

The proofs of the parts (a) - (d) and (f) are identical to those of the parts (a) - (d) and (f) of theorem 2.2.3. However, the proof of (e) must be modified, since the demand probabilities depend on the stock level. The proof of (e) now proceeds as follows.

(e) From (b) - (d) it follows that

$$K + J_\alpha(k) \geq 0 \geq J_\alpha(i) \qquad \text{for } k \geq i \text{ and } s^* \leq i \leq S_0.$$

Assuming that $i-1$ is an integer such that $i-1 \geq S_0$ and $K + J_\alpha(k) \geq J_\alpha(r)$ for $k \geq r$ and $S_0 \leq r \leq i-1$, we shall demonstrate that $K + J_\alpha(k) \geq J_\alpha(i)$ for all $k \geq i$. When this induction step has been verified, (e) follows. Clearly, $K + J_\alpha(i) \geq J_\alpha(i)$. We have by (2.3.44) that $K + J_\alpha(k) \geq J_\alpha(i)$ for $k > i$ is equivalent to $K + v_\alpha^*(k) \geq v_\alpha^*(i)$ for $k > i$. Fix the integer $k$ with $k > i$. Since $G_\alpha(j)$ is non-decreasing on $[S_0,\infty)$, we have

$$G_\alpha(i) - G_\alpha(k) \leq 0 \leq K - \alpha K \sum_{j=0}^{k-s^*} \phi(k,j).$$

Using this inequality and the definition of $v_\alpha^*(j)$, it follows that

$$v_\alpha^*(i) - v_\alpha^*(k) \leq K\{1-\alpha\phi(i,0)\} + \alpha \sum_{j=1}^{i-s^*} v_\alpha^*(i-j)\phi(i,j) + \\ - \alpha \sum_{j=0}^{k-s^*} \{K+v_\alpha^*(k-j)\}\tilde{\phi}(k,j) + \alpha\phi(i,0)\{v_\alpha^*(i) - v_\alpha^*(k)\},$$

where

$$\tilde{\phi}(k,0) = \phi(k,0) - \phi(i,0) \text{ and } \tilde{\phi}(k,j) = \phi(k,j) \text{ for } 1 \leq j \leq k-s^*.$$

Using $K + v_\alpha^*(j) = K + J_\alpha^*(j) \geq 0$ for $j \geq s^*$ (cf. (b), (c) and (2.3.44)) and using $v_\alpha^*(s^*-1) = 0$, we obtain

$$\{v_\alpha^*(i)-v_\alpha^*(k)\}\{1-\alpha\phi(i,0)\} \le K\{1-\alpha\phi(i,0)\} + \alpha\sum_{j=1}^{i-s^*+1} v_\alpha^*(i-j)\tilde{\phi}(i,j) +$$
$$- \alpha\sum_{j=0}^{i-s^*+1} \{K+v_\alpha^*(k-j)\}\tilde{\phi}(k,j),$$

where

$$\tilde{\phi}(i,j) = \phi(i,j) \text{ for } 1 \le j \le i-s^*, \text{ and}$$

$$\tilde{\phi}(i,i-s^*+1) = \Phi(k,i-s^*+1) - \Phi(i,i-s^*).$$

Note that, by (2.3.37), the numbers $\tilde{\phi}(i,j)$ and $\tilde{\phi}(k,j)$ are non-negative. Let

$$a_j = \tilde{\phi}(i,i-s^*+1-j) \text{ for } 0 \le j \le i-s^*,\ a_{i-s^*+1} = 0, \text{ and}$$
$$b_j = \tilde{\phi}(k,i-s^*+1-j) \text{ for } 0 \le j \le i-s^*+1.$$

Furthermore, let

$$f(j) = v_\alpha^*(s^*-j+1) \text{ and } g(j) = K + v_\alpha^*(k-i+s^*-1+j),\ 0 \le j \le i-s^*+1.$$

Then

$$\sum_{j=0}^{i-s^*+1} a_j f(j) = \sum_{j=1}^{i-s^*+1} v_\alpha^*(i-j)\tilde{\phi}(i,j)\ ;$$

$$\sum_{j=0}^{i-s^*+1} b_j g(j) = \sum_{j=0}^{i-s^*+1} \{K+v_\alpha^*(k-j)\}\tilde{\phi}(k,j).$$

We shall now prove that lemma 2.3.3 can be applied with $N = i-s^*+1$ and $a_j, b_j, f(j)$ and $g(j)$ as defined above. Using assumption (2.3.37), it is straightforward to verify that condition (2.3.27) of lemma 2.3.3 is satisfied. Since $a_j = 0$ for $j = i-s^*+1$, it suffices to verify that $f(j) \le g(h)$ for all pairs $(j,h)$ with $0 \le j \le i-s^*$ and $j \le h \le i-s^*+1$. However, this follows directly from the relation $v_\alpha^*(s^*-1) = 0 \le K + v_\alpha^*(j)$ for $j \ge s^*$ (cf. (b), (c) and (2.3.44)) and from the induction hypothesis. Therefore $\sum_0^N a_j f(j) \le \sum_0^N b_j g(j)$, from which the inequality $v_\alpha^*(i) - v_\alpha^*(k) \le K$ follows. This ends the proof of the theorem.

We note that the condition (2.3.37) is used only in the proof of theorem 2.3.7 (e).

*Theorem 2.3.8.*

(a) The set of numbers $\{a_\alpha^*, v_\alpha^*(i), i \in I\}$ satisfies the functional equation

$$(2.3.45) \qquad v_\alpha^*(i) = \min_{k \geq i} \{K\delta(k-i) + G_\alpha(k) - a_\alpha^* + \alpha \sum_{j=0}^{\infty} v_\alpha^*(k-j)\phi(k,j)\}, \qquad j \in I,$$

where the right-hand side of (2.3.45) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$

(b) $\qquad s_1 \leq s^* \leq S_0 \quad$ and $\quad s^* \leq S^* \leq S_1$.

*Proof*

The proof is identical to the proof of theorem 2.2.4.

We are now in a position to establish the optimality of the $(s^*,S^*)$ policy.

Consider first the case $\alpha < 1$. From (2.3.38), $a_\alpha^* = a_\alpha(s^*,S^*)$ and (2.3.41) it follows that

$$(2.3.46) \qquad v_\alpha^*(i) = V_\alpha(i;(s^*,S^*)) - \frac{a_\alpha^*}{1-\alpha} \qquad \text{for all } i \in I.$$

Substituting (2.3.46) in (2.3.45), yields

$$(2.3.47) \qquad V_\alpha(i;(s^*,S^*)) = \min_{k \geq i} \{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} V_\alpha(k-j;(s^*,S^*))\phi(k,j)\}, \qquad i \in I,$$

where the right-hand side of (2.3.47) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$. Suppose now that the bounds $u$ and $U$ on the ordering level are chosen such that $u \leq s_1$ and $U \geq S_1$. Then, by theorem 2.3.1, the $(s^*,S^*)$ policy is optimal among the policies of the class C.

Summarizing, we have proved the following theorem (see also the lemmas 2.3.5 and 2.3.6).

*Theorem 2.3.9. (the total discounted cost criterion)*

Let $\alpha < 1$. Then

$$\min_{R \in C} V_\alpha(i;R) = V_\alpha(i;(s^*,S^*)) \qquad \text{for all } i \in I.$$

If $K = 0$, then the $(S_0,S_0)$ policy is optimal. If $K > 0$, then any $(s,S)$ policy such that $a_\alpha(s,S) = a_\alpha^*$ and $G_\alpha(s-1) \geq a_\alpha^* \geq G_\alpha(s)$, is optimal and has the property that $s_1 \leq s \leq S_0$ and $s \leq S \leq S_1$. If $\phi(k,1) > 0$ for $s-1 \leq k \leq S$, then $a_\alpha(s,S) = a_\alpha^*$ implies $G_\alpha(s-1) \geq a_\alpha^* \geq G_\alpha(s)$.

The following theorem dealing with the case $\alpha = 1$ can be proved in the same way as theorem 2.2.6.

*Theorem 2.3.10. (the average cost criterion)*

Let $\alpha = 1$. Then

$$\min_{R \in C} g(i;R) = a_1^* \qquad \text{for all } i \in I.$$

If $K = 0$, then the $(S_0,S_0)$ policy is optimal. If $K > 0$, then any $(s,S)$ policy such that $a_1(s,S) = a_1^*$ and $G_1(s-1) \geq a_1^* \geq G_1(s)$, is optimal and has the property that $s_1 \leq s \leq S_0$ and $s \leq S \leq S_1$. If $\phi(k,1) > 0$ for $s-1 \leq k \leq S$, then $a_1(s,S) = a_1^*$ implies $G_1(s-1) \geq a_1^* \geq G_1(s)$.

We close this subsection by generalizing the uniqueness theorem of subsection 2.2.4.

Let $\alpha < 1$. Define

$$V_\alpha^*(i) = \min_{s_1 \leq s \leq S \leq S_1} V_\alpha(i;(s,S)) \qquad \text{for } i \in I.$$

Note that by (2.3.38) the function $V_\alpha^*(i)$, $i \in I$, is finite and bounded from below. By theorem 2.3.9, we have that $V_\alpha^*(i) = V_\alpha(i;(s^*,S^*))$ for all $i \in I$. Hence, by (2.3.47), we obtain

$$(2.3.48) \qquad V_\alpha^*(i) = \min_{k \geq i} \{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} V_\alpha^*(k-j)\phi(k,j)\}, \qquad i \in I,$$

where the right-hand side of (2.3.48) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geq s^*$.

We have the following *uniqueness theorem*, which can be proved in the same way as theorem 2.2.7.

*Theorem 2.3.11.*

Let $\alpha < 1$. Let the function $u(i)$, $i \in I$, be a finite solution to the "optimal inventory equation"

$$u(i) = \min_{k \geq i} \{K\delta(k-i) + G_\alpha(k) + \alpha \sum_{j=0}^{\infty} u(k-j)\phi(k,j)\}, \qquad i \in I,$$

such that the function $u(i)$ is bounded from below on $(-\infty, i_0]$ for some integer $i_0$. Then, $u(i) = V_\alpha^*(i)$ for all $i \in I$.

*Remark 2.3.1.*

Let $s^*$ and $S^*$ be defined as on p. 84. Suppose $s^* < S_0$. It is easy to give an example showing that $S^*$ may be less than $S_0$. For example, let $\alpha = 1$, $c = 0$, $K = 3{,}75$, $g(k) = |k|$ for $k \leq 0$, $g(k) = k+3$ for $k \geq 1$, $\phi(i,1) = 1$ for $i \leq -1$, and $\phi(i,4) = 1$ for $i \geq 0$; then, $s_1 = -3$, $S_0 = S_1 = 0$, $a_1(s,0) = 3{,}75$ for $-3 \leq s \leq 0$, and the $(-3,-1)$ policy is optimal with $a_1(-3,-1) = 3{,}25$.

Suppose now that $\Phi(i+1,j) \geq \Phi(i,j)$ for all $i \geq s_1$ and $j \geq 0$, and that $\phi(i,0) = \phi(S_0,0)$ for $s_1 < i \leq S_0$. Now it can be proved that $J_\alpha(k)$ is non-increasing on $[s^*, S_0]$ and that $S^* \geq S_0$. The proof will only be sketched. It is straightforward to verify that $J_\alpha(s^*+1) \leq J_\alpha(s^*)$. Assuming that $J_\alpha(i)$ is non-increasing on $[s^*,k]$ for some $k$ with $s^* < k < S_0$, we can deduce the inequality

$$\{1-\alpha\phi(k,0)\}\{J_\alpha(k+1) - J_\alpha(k)\} \leq \alpha \sum_{j=1}^{k-s^*} J_\alpha(k+1-j)\phi(k+1,j) + -\alpha \sum_{j=1}^{k-s^*} J_\alpha(k+1-j)\phi(k,j) + \alpha J_\alpha(s^*)b_0 - \alpha J_\alpha(s^*)\, a_0,$$

where $a_0 = 1 - \Phi(k,k-s^*)$ and $b_0 = 1 - \Phi(k+1,k-s^*)$. From this and lemma 2.3.3 we can deduce that $J_\alpha(k+1) \leq J_\alpha(k)$. Next the relation $S^* \geq S_0$ can be established in a similar way as in the proof of theorem 2.2.4(b).

# CHAPTER III A PROBABILISTIC ANALYSIS OF AN (s,S) INVENTORY MODEL

In this chapter we deal with an (s,S) inventory model in which the demand epochs are generated by a *non-arithmetic* renewal process. The demands are independent, non-negative and identically distributed random variables and they are also independent of the renewal process. Excess demand is *backlogged* until it is subsequently filled by a delivery.

An order may be placed only at the demand epochs. The ordering policy followed is *an (s,S) policy*, that is, when the stock on hand plus on order falls below s, then this stock level is ordered up to S; otherwise, no ordering is done. The lead time of an order is a positive *constant* $\tau$.

The mathematical techniques of this chapter are based on renewal theory. In section 3.1 the renewal theoretic approach is prepared. Section 3.2 is devoted to the determination of the transient and the asymptotic behaviour of both the stock on hand plus on order and the stock on hand. In section 3.3 we are concerned with the determination of both the transient and the asymptotic behaviour of a number of random variables which are based on the number of outstanding orders. Further, we shall determine the distribution of the so-called busy period and the distributions of some related random variables. In section 3.4 we give certain averages which are measures of the merit of the (s,S) policy. Finally, we briefly consider the case in which the demand epochs are generated by an *arithmetic* renewal process.

Some of the results of this chapter extend related results in [16] and [31].

## 3.1. MODEL AND PRELIMINARIES

We consider the following inventory model with a single product and a single stocking point. Customers arrive at a stocking point at the epochs $\underline{s}_1, \underline{s}_2, \ldots$, where the interarrival times $\underline{s}_n - \underline{s}_{n-1}$ $(n \geq 1;\ \underline{s}_0 = 0)$ are independent and positive random variables with common probability distribution function $F(t) = P\{\underline{s}_n - \underline{s}_{n-1} \leq t\}$, $(n = 1,2,\ldots;\ t \geq 0)$. It is assumed that F is *non-arithmetic* (cf. section 1.2, p.15). Furthermore, we assume that

$$F(0) = 0 \quad \text{and} \quad \beta = \int_0^\infty tF(dt) < \infty.$$

Denote by $\underline{\xi}_n$ the demand size of the $n^{th}$ customer. The demand sizes $\underline{\xi}_1, \underline{\xi}_2, \ldots$ are independent, non-negative and integral-valued random variables with common probability distribution $\phi(j) = P\{\underline{\xi}_n = j\}$, $(j = 0,1,\ldots;$ $n = 1,2,\ldots)$*), and they are also independent of the arrival process $\{\underline{s}_n\}$. It is supposed that

$$\phi(0) < 1 \quad \text{and} \quad \mu = \sum_{j=0}^{\infty} j\phi(j) < \infty.$$

Excess demand is backlogged. Hence the stock on hand may take on negative values. A negative value of the stock on hand indicates the existence of a backlog. An order may be placed only at the epochs $\underline{s}_n$. The ordering policy followed is an (s,S) policy, that is, when the stock on hand plus on order i is less than s, then S-i units are ordered; otherwise, no ordering is done. The lead time of an order is a positive constant $\tau$. The numbers s and S are given integers with $s \leq S$.

We shall now give some preliminaries. First we introduce some notation. For any $n = 1,2,\ldots$ , let $F^{(n)}(t) = 0$ for $t < 0$ and let

$$F^{(n)}(t) = \int_0^t F(t-y)\, F^{(n-1)}(dy) \qquad \text{for } t \geq 0,$$

and

$$\phi^{(n)}(j) = \sum_{k=0}^{j} \phi(j-k)\, \phi^{(n-1)}(k) \qquad \text{for } j = 0,1,\ldots ,$$

where $F^{(0)}(t) = 1$ for $t \geq 0$, $F^{(0)}(t) = 0$ for $t < 0$, $\phi^{(0)}(0) = 1$, and $\phi^{(0)}(j) = 0$ for $j \geq 1$. Note that $F^{(n)}(t) = P\{\underline{s}_n \leq t\}$ and $\phi^{(n)}(j) = P\{\underline{\xi}_1 + \ldots + \underline{\xi}_n = j\}$.

For any $t \geq 0$ and $j = 0,1,\ldots$ , let

$$U(t) = \sum_{n=1}^{\infty} F^{(n)}(t), \quad m(j) = \sum_{n=1}^{\infty} \phi^{(n)}(j), \quad M(j) = \sum_{k=0}^{j} m(k).$$

The renewal functions U(t) and M(j) are finite (cf. lemma 1.2.1 of section 1.2). Further, we have

$$U(t) = F(t) + \int_0^t F(t-y)\, U(dy), \qquad t \geq 0$$

*) The proofs and the results of this chapter can be adapted to the case in which $\underline{\xi}_1, \underline{\xi}_2, \ldots$ are independent, non-negative and identically distributed random variables with an arbitrary distribution.

and

$$m(j) = \phi(j) + \sum_{k=0}^{j} \phi(j-k)\, m(k), \qquad k = 0,1,\ldots .$$

Let

$$\underline{n}(t) = \sup\, \{n \mid \underline{s}_n \leq t\}, \qquad t \geq 0,$$

and

$$\underline{m}_k = \sup\, \{n \mid \underline{\xi}_0 + \ldots + \underline{\xi}_n \leq k\}, \qquad k = 0,1,\ldots ,$$

where $\underline{\xi}_0 = 0$. Note that $\underline{n}(t)$ denotes the number of customers arriving in $[0,t]$ and that $\underline{m}_k$ denotes the number of customers before the cumulative demand exceeds k. We have (cf. formula (1.2.3) of section 1.2)

(3.1.1) $\quad E\underline{n}(t) = U(t)$ for $t \geq 0$, $\; E\underline{m}_k = M(k)$ for $k = 0,1,\ldots$ .

For any $t \geq 0$, let

$$\underline{a}(t) = \underline{\xi}_0 + \ldots + \underline{\xi}_{\underline{n}(t)}$$

The random variable $\underline{a}(t)$ denotes the cumulative demand in $[0,t]$. For any $t \geq 0$, let

$$a_k(t) = P\{\underline{a}(t) = k\}, \qquad k = 0,1,\ldots ;\; t \geq 0.$$

We have

$$a_k(t) = \sum_{n=0}^{\infty} P\{\underline{a}(t) = k \mid \underline{n}(t) = n\} P\{\underline{n}(t) = n\},\; k = 0,1,\ldots ;\; t \geq 0.$$

The event $\{\underline{n}(t) = n\}$ occurs if and only if $\underline{s}_n \leq t$ but $\underline{s}_{n+1} > t$, and hence $P\{\underline{n}(t) = n\} = F^{(n)}(t) - F^{(n+1)}(t)$, $(n \geq 0;\; t \geq 0)$. Further, the processes $\{\underline{s}_k\}$ and $\{\underline{\xi}_k\}$ are independent. Therefore,

(3.1.2) $$a_k(t) = \sum_{n=0}^{\infty} \phi^{(n)}(k)\, \{F^{(n)}(t) - F^{(n+1)}(t)\},\; k = 0,1,\ldots ;\; t \geq 0.$$

We note that $a_k(t)$ satisfies the following integral equations,

$$a_0(t) = 1 - F(t) + \int_0^t \phi(0)\, a_0(t-u)\, F(du), \qquad t \geq 0$$

and

$$a_k(t) = \sum_{j=0}^{k} \int_0^t \phi(j)\, a_{k-j}(t-u)\, F(du), \qquad k = 1,2,\ldots;\ t \geq 0.$$

We note that if

$$F(t) = 1 - e^{-\lambda t}, \qquad t \geq 0,$$

then for any $t \geq 0$ the probabilities $a_k(t)$, $k \geq 0$, satisfy (cf. [1])

$$a_0(t) = e^{-\lambda(1-\phi(0))t}$$

and

$$a_{k+1}(t) = \frac{\lambda t}{k+1} \sum_{j=0}^{k} (j+1)\phi(j+1)\, a_{k-j}(t), \qquad k = 0,1,\ldots .$$

This recurrence relation can be easily verified by applying Leibniz's theorem on the differentiation of products to the relation $g_t'(u) = \lambda t g_t(u)\psi'(u)$, $|u| \leq 1$, where $g_t(u)$ and $\psi(u)$ denote the generating functions of the probability distributions $\{a_k(t), k \geq 0\}$ and $\{\phi(k), k \geq 0\}$.

*Lemma 3.1.1.*

For each $k = 0,1,\ldots$ the function $a_k(t)$, $t \geq 0$, is a bounded Baire function which is directly Riemann integrable. Further,

$$\int_0^\infty a_k(t)\, dt = \beta\{\phi^{(0)}(k) + m(k)\}, \qquad k = 0,1,\ldots .$$

*Proof*

Since for any $k \geq 0$ the functions

$$\sum_{n=0}^{\infty} \phi^{(n)}(k)\, F^{(n)}(t) \quad \text{and} \quad \sum_{n=0}^{\infty} \phi^{(n)}(k)\, F^{(n+1)}(t), \qquad t \geq 0,$$

are bounded and are monotone and since a monotone function is a Baire function, it follows from (3.1.2) that $a_k(t)$, $t \geq 0$, is a Baire function. Moreover, since any bounded monotone function is Riemann integrable over finite intervals, it follows that the function $a_k(t)$, $t \geq 0$, is Riemann integrable over finite intervals. We have for each $k = 0,1,\dots$ that

$$0 \leq a_k(t) \leq g_k(t) \qquad \text{for } t \geq 0,$$

where

$$g_k(t) = \sum_{n=0}^{\infty} \phi^{(n)}(k)\{1-F^{(n+1)}(t)\}, \qquad t \geq 0.$$

For each $k \geq 0$ the function $g_k(t)$ is non-negative, non-increasing and uniformly bounded by $1+m(k)$. Using the fact that each term of the series $g_k(t)$ is a non-negative, non-increasing function and using well-known results for the Riemann and the Lebesgue integral, it follows that $g_k(t)$ is Riemann integrable over $(0,\infty)$ where

$$\int_0^\infty g_k(t)dt = \sum_{n=0}^{\infty} \phi^{(n)}(k) \int_0^\infty \{1-F^{(n+1)}(t)\}dt =$$

$$= \sum_{n=0}^{\infty} \phi^{(n)}(k)(n+1)\beta < \infty,$$

since $\phi^{(n)}(k)$ converges exponentially fast to zero as $n \to \infty$ for each $k \geq 0$ (cf. lemma 1.2.1 of section 1.2). It now follows from lemma 1.2.2 of section 1.2 that for each $k \geq 0$ the function $a_k(t)$ is directly Riemann integrable. Using (3.1.2), we obtain

$$\int_0^\infty a_k(t)dt = \sum_{n=0}^{\infty} \phi^{(n)}(k)\{(n+1)\beta - n\beta\} =$$

$$= \beta\{\phi^{(0)}(k) + m(k)\}, \qquad k \geq 0.$$

This ends the proof of the lemma.

For any $k = 1,2,\dots$ , let

$$\underline{t}_k = \underline{s}_{m_{k-1}+1}.$$

The random variable $\underline{t}_k$ represents the length of the time interval from $t = 0$ up to the epoch at which the cumulative demand exceeds $k-1$ for the first time. Since the event $\{\underline{t}_k \leq t\}$ occurs if and only if $\underline{a}(t) \geq k$, we have for any $k \geq 1$ that

$$P\{\underline{t}_k \leq t\} = 1 - \sum_{j=0}^{k-1} a_j(t), \qquad t \geq 0.$$

From Wald's equation and (3.1.1) it follows that

$$(3.1.3) \qquad E\underline{t}_k = E\underline{s}_1 \cdot E(\underline{m}_{k-1}+1) = \beta\{1+M(k-1)\}, \qquad k = 1,2,\ldots .$$

The following lemma will be needed.

*Lemma 3.1.2.*

Let H be a probability distribution and let $H^{(n)}$ be the n-fold convolution of H with itself. Let the sequence $\{a_n, n = 1,2,\ldots\}$ be such that $a_n \geq 0$, $(n \geq 1)$, and $\sum_1^\infty a_n = 1$. Let the probability distribution function G be defined by $G = \sum_{n=1}^\infty a_n H^{(n)}$. If H is non-arithmetic, then also G is non-arithmetic.

*Proof*

Let us first recall the following definition. A distribution function is said to be arithmetic if it is concentrated on a set of points of the form $0, \pm\lambda, \pm 2\lambda, \ldots$; a distribution function is said to be lattice if it is concentrated on a set of points $a, a\pm\lambda, a\pm 2\lambda, \ldots$ with a arbitrary.

We shall now prove the following assertion. If a distribution function H is non-arithmetic, then $H^{(n)}$ is non-arithmetic for all $n \geq 1$. To prove this, we distinguish between two cases. *Case 1.* H is both non-arithmetic and lattice. It readily follows from the above definition that a non-arithmetic distribution function B is lattice if and only if there exist numbers $a, b \neq 0$ such that a/b is irrational and a and b are points of increase of B. Thus there exist numbers $c, d \neq 0$ such that c/d is irrational and c and d are points of increase of H. Then the points nc and nd are points of increase of $H^{(n)}$ and nc/nd is irrational, which proves that $H^{(n)}$ is non-arithmetic. *Case 2.* H is non-lattice. From the theory of characteristic functions (cf. [15]) it follows that a probability distribution B is lattice if and only if $|b(t_0)| = 1$ for some $t_0 \neq 0$, where b denotes the characteristic function of B. Further, the characteristic function of $H^{(n)}$ is $h^n$, where h is the characteristic function of H. Hence $H^{(n)}$ is non-lattice, and so $H^{(n)}$

is non-arithmetic.

To prove that G is non-arithmetic, assume to the contrary that G is arithmetic. Since $a_n > 0$ for some $n \geq 1$, it follows from the definition of G that $H^{(n)}$ is arithmetic for some $n \geq 1$, and hence, by the above assertion, H is arithmetic. This contradiction proves the lemma.

I am indebted to Professor Runnenburg for help in this proof.

*Lemma 3.1.3.*

For each k = 1,2,... the distribution of $\underline{t}_k$ is non-arithmetic.

*Proof*

Let $\Phi^{(n)}(j) = \phi^{(n)}(0) + \ldots + \phi^{(n)}(j)$, $(j, n = 0,1,\ldots)$. Since $\Phi^{(n-1)}(j) - \Phi^{(n)}(j)$ is the probability that the cumulative demand will first exceed j at epoch $\underline{s}_n$, we have for each k = 1,2,... that

$$P\{\underline{t}_k \leq t\} = \sum_{n=1}^{\infty} \{\Phi^{(n-1)}(k-1) - \Phi^{(n)}(k-1)\}F^{(n)}(t), \qquad t \geq 0.$$

The lemma now follows from this relation and lemma 3.1.2, since F is assumed to be non-arithmetic.

For any $t \geq 0$, let

(3.1.4) $$\underline{y}(t) = \underline{s}_{\underline{n}(t)+1} - t.$$

The random variable $\underline{y}(t)$ represents the length of the time interval between time t and the epoch of the first demand occurring after t. We have for any $t \geq 0$ that (cf. section 1.2)

(3.1.5) $$P\{\underline{y}(t) \leq y\} = F(t+y) - F(t) + \int_0^t \{F(t+y-x) - F(t-x)\}U(dx), \quad y \geq 0,$$

and, since F is non-arithmetic,

(3.1.6) $$\lim_{t\to\infty} P\{\underline{y}(t) \leq y\} = \frac{1}{\beta}\int_0^y \{1-F(x)\}dx, \qquad y \geq 0.$$

For any $t \geq 0$, let

$$b_k(t,\tau) = P\{\underline{a}(t+\tau) - \underline{a}(t) = k\}, \qquad k = 0,1,\ldots,$$

i.e. $b_k(t,\tau)$ is the probability that the cumulative demand in $(t,t+\tau]$ will be k. We have for each $t \geq 0$ that

$$(3.1.7) \qquad b_0(t,\tau) = P\{\underline{y}(t) > \tau\} + \int_0^\tau \phi(0)a_0(\tau-y)dP\{\underline{y}(t) \leq y\},$$

and

$$(3.1.8) \qquad b_k(t,\tau) = \sum_{j=0}^{k} \int_0^\tau \phi(j)a_{k-j}(\tau-y)dP\{\underline{y}(t) \leq y\}, \qquad k = 1,2,\ldots .$$

We note that if F is an exponential distribution, then for each $t \geq 0$ the random variable $\underline{y}(t)$ has also the exponential distribution F (cf. section 1.2), and so in that case $b_k(t,\tau) = a_k(\tau)$.

Using (3.1.6), we obtain

$$(3.1.9) \qquad \lim_{t\to\infty} b_k(t,\tau) = b_k(\tau), \qquad k = 0,1,\ldots,$$

where

$$(3.1.10) \qquad b_0(\tau) = \frac{1}{\beta}\int_\tau^\infty \{1-F(x)\}dx + \frac{1}{\beta}\int_0^\tau \phi(0)a_0(\tau-y)\{1-F(y)\}dy$$

and

$$(3.1.11) \qquad b_k(\tau) = \frac{1}{\beta}\sum_{j=0}^{k}\int_0^\tau \phi(j)a_{k-j}(\tau-y)\{1-F(y)\}dy, \qquad k = 1,2,\ldots .$$

The above relations involve an interchange of the order of limit and integration justified by the fact that for each $k \geq 0$ the function $a_k$ can be written as the difference of two bounded, monotone functions and the fact that $\lim_{t\to\infty}P\{\underline{y}(t) \leq y\}$ is a continuous distribution function.

## 3.2. THE TRANSIENT AND THE ASYMPTOTIC BEHAVIOUR OF THE STOCK LEVEL

In this section we shall determine the transient and the asymptotic behaviour of both the stock on hand plus on order and the stock on hand.

For any $t \geq 0$, denote by $\underline{z}(t)$ the stock on hand plus on order at time t. As usual we take $\underline{z}(t)$ as continuous from the *right*. The set of the integers i with $i \geq s$ is taken as state space for the $\{\underline{z}(t)\}$ process. We note that the $\{\underline{z}(t)\}$ process is a *semi-Markov process*.*)

For any $t \geq 0$, let

$$p_{ij}(t) = P\{\underline{z}(t) = j | \underline{z}(0) = i\} \qquad \text{for } i, j \geq s.$$

For any $i \geq s$, let

$$G_i(t) = P\{\underline{t}_{i-s+1} \leq t\}.$$

Using a standard argument from renewal theory, we have

$$(3.2.1) \qquad p_{ij}(t) = a_{i-j}(t) + \int_0^t p_{Sj}(t-u)\, G_i(du), \qquad t \geq 0;\ i, j \geq s,$$

where $a_k(t) = 0$ for $k < 0$; $t \geq 0$. In particular,

$$(3.2.2) \qquad p_{Sj}(t) = 0, \qquad t \geq 0,\ j > S$$

and

$$(3.2.3) \qquad p_{Sj}(t) = a_{S-j}(t) + \int_0^t p_{Sj}(t-u)\, G_S(du), \qquad t \geq 0;\ s \leq j \leq S.$$

Let $G_S^{(n)}(t)$ be the n-fold convolution of $G_S(t)$ with itself, and let the renewal function $V_S(t)$ be defined by

$$V_S(t) = \sum_{n=1}^{\infty} G_S^{(n)}(t).$$

For each j, $s \leq j \leq S$, the equation (3.2.3) is a renewal equation. Hence, since $a_k(t)$ is a bounded Baire function, we have (cf. (1.2.7) of section 1.2)

$$(3.2.4) \qquad p_{Sj}(t) = a_{S-j}(t) + \int_0^t a_{S-j}(t-y)\, V_S(dy), \qquad t \geq 0;\ s \leq j \leq S.$$

---

*) It is interesting to note that at the demand epochs the stock on hand plus on order behaves exactly as the stock on hand plus on order in the classical periodic review (s,S) inventory model. This fact together with a limit theorem from the theory of semi-Markov processes [35,39,45] can also be used to determine the limiting distribution of the $\{\underline{z}(t)\}$ process.

The formulas (3.2.1), (3.2.2) and (3.2.4) in conjunction yield the time dependent solution of the distribution of the stock on hand plus on order.

From the key renewal theorem (cf. section 1.2), lemma 3.1.1, (3.1.3) and (3.2.4) it follows that

$$(3.2.5)\qquad \lim_{t\to\infty} p_{Sj}(t) = \frac{1}{E\underline{t}_{S-s+1}} \int_0^\infty a_{S-j}(t)\,dt = \frac{\phi^{(0)}(S-j) + m(S-j)}{1+M(S-s)} \qquad \text{for } s \le j \le S.$$

From (3.2.1) and (3.2.5) it now follows that

$$(3.2.6)\qquad \lim_{t\to\infty} p_{ij}(t) = q_j \qquad \text{for all } i,\ j \ge s,$$

where

$$(3.2.7)\qquad q_j = \begin{cases} \dfrac{\phi^{(0)}(S-j) + m(S-j)}{1+M(S-s)} & \text{for } s \le j \le S \\ 0 & \text{for } j > S. \end{cases}$$

Denote by $\underline{x}(t)$ the stock on hand at time t. As usual we take $\underline{x}(t)$ as continuous from the right. The set I of all integers is taken as state space for the $\{\underline{x}(t)\}$ process.

For any $t \ge \tau$, let

$$r_{ij}(t) = P\{\underline{x}(t) = j \mid z(0) = i\}, \qquad i \ge s;\ j \in I.$$

Since anything on order at time t will have arrived by time $t+\tau$ and since anything ordered after time t will arrive after time $t+\tau$, we have for any $i \ge s$ and $t \ge 0$ that

$$(3.2.8)\qquad r_{ij}(t+\tau) = \begin{cases} \sum\limits_{k=s}^{\max(i,S)} p_{ik}(t)\, b_{k-j}(t,\tau), & j \le \max(i,S), \\ 0, & \text{otherwise,} \end{cases}$$

where $b_k(t,\tau) = 0$ for $k < 0$. By (3.2.8) the time dependent solution of the distribution of the stock on hand is determined.

From (3.1.9), (3.2.6) and (3.2.8) it follows that the limiting distribution of the stock on hand is given by

$$\text{(3.2.9)} \qquad \lim_{t\to\infty} r_{ij}(t) = r_j, \qquad i \geq s;\ j \in I,$$

where

$$\text{(3.2.10)} \qquad r_j = \begin{cases} \sum_{k=s}^{S} q_k\, b_{k-j}(\tau), & j \leq S, \\ 0, & j > S, \end{cases}$$

where $b_k(\tau) = 0$ for $k = -1,-2,\ldots$ .

Let us now consider the special case $\phi(1) = 1$. Then $m(k) = 1$ for $k = 1,2,\ldots$ , and hence

$$q_j = \frac{1}{S-s+1} \qquad \text{for } s \leq j \leq S.$$

Further, since

$$a_n(t) = F^{(n)}(t) - F^{(n+1)}(t), \qquad n = 0,1,\ldots\ ;\ t \geq 0,$$

it follows from (3.1.11) that

$$\sum_{j=k}^{\infty} b_j(\tau) = \frac{1}{\beta} \int_0^{\tau} F^{(k-1)}(\tau-y)\{1-F(y)\}dy, \qquad k = 1,2,\ldots ,$$

and hence

$$r_j = \begin{cases} 0, & j > S, \\ \frac{1}{S-s+1} - \frac{1}{(S-s+1)\beta} \int_0^{\tau} F^{(S-j)}(\tau-y)\{1-F(y)\}dy, & s \leq j \leq S, \\ \frac{1}{(S-s+1)\beta} \int_0^{\tau} \{F^{(s-j-1)}(\tau-y)-F^{(S-j)}(\tau-y)\}\{1-F(y)\}dy, & j < s. \end{cases}$$

These results for the case $\phi(1) = 1$ are also contained in [16]. In [16] a number of results for the distribution of the stock level is given for an (s,S) inventory model with $\phi(1) = 1$ and a random lead time. This model is also considered in [31]. An (s,S) inventory model with $\phi(1) = 1$ and a Poisson arrival process $\{\underline{s}_n\}$ has been treated in [18], where the limiting distribution of the stock on hand has been determined explicitly for both the case in which the lead time is fixed and the case in which the lead time is exponentially distributed. In [19] a number of results are found for an (s,S) inventory model in which the customers arrive according to a Poisson process, the demands are independent and non-negative random variables with a common distribution, and the lead time is exponentially distributed.

Let us next consider the case in which $\phi(j) = p(1-p)^{j-1}$ for $j \geq 1$, where $0 < p \leq 1$. Then $m(j) = p$ for $j \geq 1$ (cf. (1.2.20) of section 1.2), and hence

$$q_j = \begin{cases} 1/\{1 + (S-s)p\} & \text{for } j = S, \\ \\ p/\{1 + (S-s)p\} & \text{for } s \leq j < S. \end{cases}$$

## 3.3. THE NUMBER OF OUTSTANDING ORDERS AND RELATED RANDOM VARIABLES

Let us introduce the following random variables. Denote by $\underline{\nu}(t)$ the number of outstanding orders at time t. As usual let $\underline{\nu}(t)$ be continuous from the right. It should be noted that the number of outstanding orders can be identified with the number of busy servers in a queueing model with infinitely many servers in which the customers arrive according to the interarrival distribution $G_S(t)$ and the service time for each customer is the constant $\tau$.

Let the random variable $\underline{\theta}$ be defined as the length of the time interval during which there are continuously orders outstanding, that is, if $\underline{\nu}(t-) = 0$ and $\underline{\nu}(t) = 1$ for some time t, then $\underline{\theta} = \inf_{w > t} \{w-t \mid \underline{\nu}(w) = 0\}$. Let us call $\underline{\theta}$ a busy period. Denote by $\underline{\chi}$ the stock on hand just after the end of a busy period and let $\underline{m}$ be the number of orders delivered in a busy period.

For any $t \geq \tau$, let $\underline{\rho}(t)$ denote the first epoch belonging to $[t,\infty)$ at which no orders are outstanding, that is, $\underline{\rho}(t) = \inf_{u \geq t} \{u-t \mid \underline{\nu}(u) = 0\}$.

For any $k = 0,1,\ldots,$ let

$$\underline{\Upsilon}_k(t) = \begin{cases} \text{the time between t and the epoch of the } (\underline{\nu}(t)-k)^{th} & \\ \text{delivery occurring after time t} & \text{if } \underline{\nu}(t) > k, \\ 0 & \text{if } \underline{\nu}(t) \leq k, \end{cases}$$

and for any $r = 1,2,\ldots,$ let

$$\underline{\delta}_r(t) = \begin{cases} \text{the time between t and the epoch of the } r^{th} \\ \text{delivery occurring after time t} & \text{if } \underline{\nu}(t) \geq r, \\ 0 & \text{if } \underline{\nu}(t) < r. \end{cases}$$

In this section we shall determine the distributions of the above random variables. Moreover, we shall give the limiting distributions of $\underline{\nu}(t)$, $\underline{\rho}(t)$, $\underline{\gamma}_k(t)$ and $\underline{\delta}_r(t)$. Finally, in remark 3.3.5 we give the joint distribution of $(\underline{\gamma}_0(t),\ldots,\underline{\gamma}_N(t))$ and that of $(\underline{\delta}_1(t),\ldots,\underline{\delta}_L(t))$ and their asymptotic behaviour for $t \to \infty$.

Let us first introduce some notation. Let $\underline{\tau}_0 = 0$ and denote by $\underline{\tau}_n$ $(n = 1,2,\ldots)$ the $n^{th}$ ordering epoch. The random variables $\underline{\tau}_n - \underline{\tau}_{n-1}$ $(n = 1,2,\ldots)$ are mutually independent. Given that $\underline{z}(0) = i$, the random variable $\underline{\tau}_1$ has the same distribution as the random variable $\underline{t}_{i-s+1}$, and the random variables $\underline{\tau}_n - \underline{\tau}_{n-1}$ $(n = 2,3,\ldots)$ have the same distribution as $\underline{t}_{S-s+1}$. The process $\{\underline{\tau}_n\}$ is a renewal process when $\underline{z}(0) = S$, and it is a so-called delayed renewal process when $\underline{z}(0) = i \neq S$. For any integer $i \geq s$, let

$$G_i^{(n)}(t) = P\{\underline{\tau}_n \leq t \mid \underline{z}(0) = i\}, \qquad n = 0,1,\ldots .$$

We have $G_i^{(0)}(t) = 1$, $G_i^{(1)}(t) = G_i(t)$ for $t \geq 0$, and

$$G_i^{(n)}(t) = \int_0^t G_S^{(n-1)}(t-u)\, G_i(du), \qquad n = 2,3,\ldots;\ t \geq 0.$$

For any $i \geq s$, let

$$V_i(t) = \sum_{n=1}^{\infty} G_i^{(n)}(t).$$

Then,

$$(3.3.1) \qquad V_i(t) = G_i(t) + \int_0^t V_S(t-u)\, G_i(du), \qquad i \geq s;\ t \geq 0.$$

We shall now determine the distribution of $\underline{\nu}(t)$. Note that for the determination of the distribution of $\underline{\nu}(t)$ for $t \geq \tau$ it is sufficient to know the stock on hand plus on order at epoch 0, because the lead time is a constant $\tau$. Note also that the number of outstanding orders at time t is the same as the number of orders placed in $(t-\tau,t]$. For any $i \geq s$ and $t \geq \tau$, let

$$v_{ij}(t) = P\{\underline{\nu}(t) = j \mid \underline{z}(0) = i\}, \qquad j = 0,1,\ldots .$$

*Theorem 3.3.1.*

(a) For any $i \geq s$ and $t \geq \tau$,

$$v_{ij}(t) = \begin{cases} 1 - \int_{(t-\tau)+}^{t} \{1-G_S(t-u)\}\, V_i(du), & j = 0^{*)}, \\ \int_{(t-\tau)+}^{t} \{G_S^{(j-1)}(t-u) - 2G_S^{(j)}(t-u) + G_S^{(j+1)}(t-u)\} V_i(du), & (j = 1,2,\ldots). \end{cases}$$

(b) $E\{\underline{\nu}(t)|\underline{z}(0) = i\} = V_i(t) - V_i(t-\tau)$, $(i \geq s;\ t \geq \tau)$.

(c) For any $i \geq s$,

$$\lim_{t\to\infty} v_{ij}(t) = \begin{cases} 1 - \frac{1}{\beta\{1+M(S-s)\}} \int_0^{\tau} \{1-G_S(t)\}dt, & j = 0, \\ \frac{1}{\beta\{1+M(S-s)\}} \int_0^{\tau} \{G_S^{(j-1)}(t)-2G_S^{(j)}(t)+G_S^{(j+1)}(t)\}dt, & (j = 1,2,\ldots). \end{cases}$$

(d) For any $i \geq s$, $\lim_{t\to\infty} E\{\underline{\nu}(t)|\underline{z}(0) = i\} = \tau/\{\beta(1+M(S-s))\}$.

*Proof*

(a) Since the event $\{\underline{\nu}(t) \geq j\}$ occurs if and only if at least j orders are placed in the time interval $(t-\tau,t]$, we have for any $i \geq s$ and $t \geq \tau$ that

$$\begin{aligned} P\{\underline{\nu}(t) \geq j|\underline{z}(0) = i\} &= \sum_{n=1}^{\infty} P\{t-\tau < \underline{\tau}_n \leq \underline{\tau}_{n+j-1} \leq t < \underline{\tau}_{n+j}|\underline{z}(0) = i\} = \\ &= \sum_{n=1}^{\infty} \int_{(t-\tau)+}^{t} \{G_S^{(j-1)}(t-u)-G_S^{(j)}(t-u)\}G_i^{(n)}(du) = \\ &= \int_{(t-\tau)+}^{t} \{G_S^{(j-1)}(t-u)-G_S^{(j)}(t-u)\}V_i(du),\ j=1,2,\ldots, \end{aligned}$$

which proves assertion (a).

---

*) The interval of integration of $\int_{a+}^{b}$ is given by (a,b].

(b) Since for each non-negative and integral-valued random variable $\underline{a}$ holds $E\underline{a} = \sum_{j=1}^{\infty} P\{\underline{a} \geq j\}$, assertion (b) follows from the above relation for $P\{\underline{\nu}(t) \geq j | \underline{z}(0) = i\}$.

(c) Let $z_0(t) = 1 - G_S(t)$ for $0 \leq t < \tau$, and zero otherwise. For any $j = 1,2,\ldots$, let $z_j(t) = G_S^{(j-1)}(t) - 2G_S^{(j)}(t) + G_S^{(j+1)}(t)$ for $0 \leq t < \tau$, and zero otherwise. Since $G_S^{(n)}(t)$ is a distribution function for each $n \geq 0$, it follows from lemma 1.2.2 of section 1.2 that for each $j \geq 0$ the function $z_j(t)$, $t \geq 0$, is directly Riemann integrable. Further, the distribution function $G_S(t)$ is non-arithmetic (cf. lemma 3.1.3). Using the key renewal theorem (see theorem 1.2.1 and remark 1.2.1 of section 1.2) and using (3.1.3), the assertion (c) follows from (a).

(d) Since $V_i(t) - V_i(t-\tau) = \int_{(t-\tau)+}^{\tau} V_i(dx)$, assertion (d) is an easy consequence of (b) and the key renewal theorem.

Theorem 3.3.1 (a) can also be obtained from a result in [16], and theorem 3.3.1 (c) is also a direct consequence of similar results in [16,31]. In the latter reference some results are obtained for the queueing system $G/G/\infty$.

*Remark 3.3.1.*

It is readily verified that the limiting distribution of $\underline{\nu}(\underline{\tau}_n-)$(= the number of outstanding orders just before ordering) is given by the probability distribution $\{G_S^{(k)}(\tau) - G_S^{(k+1)}(\tau),\ k = 0,1,\ldots\}$ (see also [16]), and hence $V_S(\tau)$ can be interpreted also as the expected number of outstanding orders in the steady state just before ordering.

*Remark 3.3.2.*

Denote by $\underline{w}(t;\, t_0)$ the number of orders delivered in the time interval $(t,\, t+t_0]$, where $t_0$ is a fixed positive number and $t \geq \tau$. Note that the number of orders delivered in $(t,\, t+t_0]$ is the same as the number of orders placed in $(t-\tau,\, t+t_0-\tau]$. The transient and the asymptotic behaviour of $\underline{w}(t;t_0)$ can be determined in the same way as those of $\underline{\nu}(t)$.

We shall next determine the distributions of $\underline{\theta}$, $\underline{\chi}$ and $\underline{m}$. To do this, let us first introduce some notation. Let

$$(3.3.2) \qquad \tilde{G}_S(t) = \begin{cases} G_S(t) & \text{for } t \leq \tau, \\ G_S(\tau) & \text{for } t > \tau. \end{cases}$$

Denote by $\tilde{G}_S^{(n)}(t)$ the n-fold convolution of $\tilde{G}_S(t)$ with itself, i.e., $\tilde{G}_S^{(n)}(t) = 0$ for $t < 0$,

$$\tilde{G}_S^{(1)}(t) = \tilde{G}_S(t) \text{ and } \tilde{G}_S^{(n)}(t) = \int_0^t \tilde{G}_S(t-x)\ \tilde{G}_S^{(n-1)}(dx)$$
$$\text{for } t \geq 0;\ n = 2,3,\dots\ .$$

Let

$$\tilde{V}_S(t) = \sum_{n=1}^{\infty} \tilde{G}_S^{(n)}(t).$$

We note that $\tilde{V}_S(t) = \tilde{G}_S(t) + \int_0^t \tilde{G}_S(t-x)\ \tilde{V}_S(dx)$ for $t \geq 0$.

*Theorem 3.3.2.*

Let $G_S(\tau) < 1$. Then

(a) $P\{\underline{\theta} \leq t\} = 0$ for $t < \tau$, $P\{\underline{\theta} \leq t\} = \{1-G_S(\tau)\}\{1+\tilde{V}_S(t-\tau)\}$ for $t \geq \tau$, and $\lim_{t\to\infty} P\{\underline{\theta} \leq t\} = 1$.

(b) $E\underline{\theta} = \tau + \{\int_0^\tau t\ G_S(dt)\}/\{1-G_S(\tau)\}$.

(c) $P\{\underline{m} = m\} = \{1-G_S(\tau)\}\{G_S(\tau)\}^{m-1}$ for $m = 1,2,\dots$ .

(d) $P\{\underline{\chi} = j\} = a_{S-j}(\tau)/\{1-G_S(\tau)\}$ for $j = s,\dots, S$.

*Proof*

(a) Using the fact that the orders are delivered in the same order as they are placed and using the fact that just after ordering the stock on hand plus on order is equal to S, it is readily seen that $\underline{\theta} = \tau$ if $\underline{t}_{S-s+1} > \tau$, and $\underline{\theta} = u + \underline{\theta}'$ if $\underline{t}_{S-s+1} = u$ with $u \leq \tau$, where $\underline{\theta}'$ has the same distribution as $\underline{\theta}$. Therefore,

$$(3.3.3) \qquad P\{\underline{\theta} \leq t\} = \begin{cases} 1-G_S(\tau) + \int_0^\tau P\{\underline{\theta} \leq t-u\}\ G_S(du) & \text{for } t \geq \tau, \\ 0 & \text{for } 0 \leq t < \tau. \end{cases}$$

Let $h(t) = 1-G_S(\tau)$ for $t \geq \tau$, and $h(t) = 0$ otherwise. From (3.3.2) and (3.3.3) it now follows that

$$(3.3.4) \qquad P\{\underline{\theta} \leq t\} = h(t) + \int_0^t P\{\underline{\theta} \leq t-u\} \, \tilde{G}_S(du) \qquad \text{for } t \geq 0.$$

Iterating the (defective) renewal equation (3.3.4) and using the fact that $\tilde{G}_S^{(n)}(t) \to 0$ as $n \to \infty$ for each $t \geq 0$, we obtain

$$(3.3.5) \qquad P\{\underline{\theta} \leq t\} = h(t) + \int_0^t h(t-u) \, \tilde{V}_S(du) \qquad \text{for } t \geq 0,$$

which proves the first part of (a). The other part of (a) follows from

$$\tilde{V}_S(\infty) = G_S(\tau)/\{1-G_S(\tau)\}.$$

(b) Denote by $g(z)$ and $f(z)$ the Laplace-Stieltjes transforms of $\tilde{G}_S(t)$ and $P\{\underline{\theta} \leq t\}$, respectively. From (3.3.2) and (3.3.5) we obtain

$$g(z) = \int_0^\infty e^{-zt} \, \tilde{G}_S(dt) = \int_0^\tau e^{-zt} \, G_S(dt)$$

and

$$f(z) = e^{-z\tau} \{1-G_S(\tau)\} + e^{-z\tau} \{1-G_S(\tau)\} \sum_{n=1}^{\infty} (g(z))^n =$$

$$= e^{-z\tau} \{1-G_S(\tau)\}/\{1-g(z)\}.$$

Assertion (b) now follows from $E\underline{\theta} = -f'(0)$.

(c) Using the same argument as in (a), we have

$$P\{\underline{m} = 1\} = 1-G_S(\tau), \text{ and } P\{\underline{m} = m\} = G_S(\tau) \, P\{\underline{m} = m-1\}, \; m = 2,3,\ldots,$$

from which (c) follows.

(d) Using the same argument as in (a), we have

$$P\{\underline{\chi} = j\} = a_{S-j}(\tau) + G_S(\tau) \, P\{\underline{\chi} = j\} \qquad \text{for } j = s,\ldots,S,$$

which proves (d).

*Remark 3.3.3.*

The joint distribution of $(\underline{\theta},\underline{m},\underline{\chi})$ satisfies

$$P\{\underline{\theta} \le t,\ \underline{m} = m,\ \underline{\chi} = j\} = \begin{cases} a_{S-j}(\tau), & t \ge \tau; m = 1; s \le j \le S, \\ \int_0^\tau P\{\underline{\theta} \le t-u,\ \underline{m} = m-1,\ \underline{\chi} = j\}\ G_S(du), & (t \ge \tau; m \ge 2; s \le j \le S), \\ 0, & \text{otherwise.} \end{cases}$$

*Remark 3.3.4.*

Consider the case in which the arrival process $\{\underline{s}_n\}$ is a Poisson process. Denote by $\underline{\eta}$ the length of the time interval between the end of a busy period and the next epoch at which an order is placed. Then

$$P\{\underline{\eta} \le t\} = \sum_{j=s}^{S} P\{\underline{\chi} = j\}\ P\{\underline{t}_{j-s+1} \le t\}.$$

*Theorem 3.3.3.*

Let $G_S(\tau) < 1$. Then

(a) For any $i \ge s$ and $t \ge \tau$,

$$P\{\underline{\rho}(t) \le u|\underline{z}(0) = i\} = 1 - \int_{(t-\tau)+}^{t} P\{\underline{\theta} > t+u-x\}\ G_i(dx) + \\ - \int_0^{t-\tau} V_i(dx) \int_{(t-\tau-x)+}^{t-x} P\{\underline{\theta} > t+u-x-y\}\ G_S(dy), \qquad u \ge 0.$$

(b) For any $i \ge s$,

$$\lim_{t\to\infty} P\{\underline{\rho}(t) \le u|\underline{z}(0) = i\} = 1 - \frac{1}{\beta\{1+M(S-s)\}} \int_\tau^\infty dv \int_{(v-\tau)+}^{v} P\{\underline{\theta} > v+u-y\}\ G_S(dy)$$

for $u \ge 0$.

*Proof*

(a) Assertion (a) follows from

$$P\{\underline{\rho}(t) \le u|\underline{z}(0) = i\} = 1 - P\{\underline{\rho}(t) > u|\underline{z}(0) = i\} =$$

$$= 1 - \sum_{n=0}^{\infty} P\{\underline{\tau}_n \le t-\tau < \underline{\tau}_{n+1} \le t,\ \underline{\theta} > t+u-\underline{\tau}_{n+1}|\underline{z}(0) = i\}.$$

(b) Let the function $z(v)$ be defined by

$$z(v) = \begin{cases} \int_{(v-\tau)+}^{v} P\{\underline{\theta} > v+u-y\}\, G_S(dy) & \text{for } v \geq \tau, \\ 0 & \text{for } 0 \leq v < \tau. \end{cases}$$

It is easily seen that on the interval $[\tau,\infty)$ the function z can be written as the difference of two bounded, monotone functions. Hence z is Riemann integrable over every finite interval $(0,a)$. Moreover the Baire function z satisfies $0 \leq z(v) \leq 1 - G_S(v-\tau)$ for $v \geq \tau$. The monotone function $1 - G_S(v-\tau)$ is Riemann integrable over $(\tau,\infty)$ in the ordinary sense. Note that $\int_\tau^\infty \{1-G_S(v-\tau)\}dv = E\underline{t}_{S-s+1}$. It now follows from lemma 1.2.2 of section 1.2 that the function z is directly Riemann integrable. Further, $G_S(t)$ is non-arithmetic. Using the key renewal theorem and (3.1.3), we now obtain (b) from (a).

Next we shall determine the distribution of $\underline{\gamma}_k(t)$ and that of $\underline{\delta}_r(t)$.

*Theorem 3.3.4.*

(a) For each $k = 0,1,\ldots$, $i \geq s$ and $t \geq \tau$,

$$P\{\underline{\gamma}_k(t) \leq u|\underline{z}(0) = i\} = 1 - \int_{(t+u-\tau)+}^{t} \{G_S^{(k)}(t-x) - G_S^{(k+1)}(t-x)\}\, V_i(dx) \qquad \text{for } 0 \leq u < \tau.$$

(b) For each $k = 0,1,\ldots$ and $i \geq s$,

$$\lim_{t\to\infty} P\{\underline{\gamma}_k(t) \leq u|\underline{z}(0) = i\} = 1 - \frac{1}{\beta\{1+M(S-s)\}} \int_0^{\tau-u} \{G_S^{(k)}(t) - G_S^{(k+1)}(t)\}dt \qquad \text{for } 0 \leq u < \tau.$$

*Proof*

(a) Assertion (a) follows from

$$P\{\underline{\gamma}_k(t) \leq u|\underline{z}(0) = i\} = 1 - P\{\underline{\gamma}_k(t) > u|\underline{z}(0) = i\} =$$

$$= 1 - \sum_{n=1}^{\infty} P\{t+u-\tau < \underline{\tau}_n \leq \underline{\tau}_{n+k} \leq t < \underline{\tau}_{n+k+1}|\underline{z}(0) = i\} \qquad \text{for } 0 \leq u < \tau.$$

(b) This assertion follows from (a) and an application of the key renewal theorem. Using lemma 1.2.2 of section 1.2, it is easily verified that the key renewal theorem may be applied to determine the limiting distribution of $\underline{\gamma}_k(t)$. This ends the proof of the theorem.

The above results for the distribution of $\underline{\gamma}_0(t)$ extend related results in [31]. For the special case in which $G_S(t)$ is an exponential distribution the above results for the distribution of $\underline{\gamma}_k(t)$ are contained in results in [44].

*Theorem 3.3.5.*

(a) For each $r = 1,2,\ldots$, $i \geq s$ and $t \geq \tau$,

$$P\{\underline{\delta}_r(t) \leq u \mid \underline{z}(0) = i\} = \sum_{j=0}^{r-1} v_{ij}(t) + \int_{(t-\tau)+}^{t+u-\tau} \{G_S^{(r-1)}(t+u-\tau-x) - G_S^{(r)}(t+u-\tau-x)\}\, V_i(dx)$$

for $0 \leq u < \tau$.

(b) For each $r = 1,2,\ldots$ and $i \geq s$,

$$\lim_{t\to\infty} P\{\underline{\delta}_r(t) \leq u \mid \underline{z}(0) = i\} = 1 - \frac{1}{\beta\{1+M(S-s)\}} \int_u^{\tau} \{G_S^{(r-1)}(t) - G_S^{(r)}(t)\}dt$$

for $0 \leq u < \tau$.

*Proof*

(a) From $P\{\underline{\delta}_r(t) \leq u \mid \underline{z}(0) = i\} = P\{\underline{\nu}(t) \leq r-1 \mid \underline{z}(0) = i\} +$ $+ \sum_{n=1}^{\infty} P\{t-\tau < \underline{\tau}_n \leq \underline{\tau}_{n+r-1} \leq t+u-\tau < \underline{\tau}_{n+r} \mid \underline{z}(0) = i\}$ we obtain (a).

(b) It readily follows from (a) and the key renewal theorem that

$$\lim_{t\to\infty} P\{\underline{\delta}_r(t) \leq u \mid \underline{z}(0) = i\} = \sum_{j=0}^{r-1} (\lim_{t\to\infty} v_{ij}(t)) + \frac{1}{\beta\{1+M(S-s)\}} \int_0^u \{G_S^{(r-1)}(t) - G_S^{(r)}(t)\}dt.$$

Assertion (b) now follows from this relation and theorem 3.3.1 (c).

*Remark 3.3.5.*

In this remark we shall determine the joint distribution of $(\underline{\gamma}_0(t),\ldots,$

$\underline{\gamma}_N(t))$ and that of $(\underline{\delta}_1(t),\ldots,\underline{\delta}_L(t))$ and their asymptotic behaviour for $t \to \infty$.

The simultaneous distribution of $(\underline{\gamma}_0(t),\ldots,\underline{\gamma}_N(t))$ can be obtained from the recurrence relation

$$P\{\underline{\gamma}_k(t) \le u_k \text{ for } k = 0,\ldots, N|\underline{z}(0) = i\} =$$

$$= P\{\underline{\gamma}_k(t) \le u_k \text{ for } k = 0,\ldots, N-1|\underline{z}(0) = i\} +$$

$$- P\{\underline{\gamma}_k(t) \le u_k \text{ for } k = 0,\ldots, N-1,\ \underline{\gamma}_N(t) > u_N|\underline{z}(0) = i\} =$$

$$= P\{\underline{\gamma}_k(t) \le u_k \text{ for } k = 0,\ldots, N-1|\underline{z}(0) = i\} +$$

$$- \sum_{n=1}^{\infty} P\{t+u_N-\tau < \underline{\tau}_n \le \underline{\tau}_{n+h} \le t+u_{N-h}-\tau \text{ for } h = 1,\ldots, N,\ \underline{\tau}_{n+N+1} > t|\underline{z}(0)=i\}$$

$$\text{for } 0 \le u_N \le u_{N-1} \le \cdots \le u_0 \le \tau.$$

For ease of notation, let us consider the case $N = 1$. The general case can be handled in the same way. For any $i \ge s$ and $t \ge \tau$ we have

$$P\{\underline{\gamma}_k(t) \le u_k \text{ for } k = 0,1|\underline{z}(0) = i\} = P\{\underline{\gamma}_0(t) \le u_0|\underline{z}(0) = i\} +$$
$$- \int_{(t+u_1-\tau)+}^{t+u_0-\tau} \Big[\int_0^{t+u_0-\tau-x} \{1-G_S(t-x-y)\}\ G_S(dy)\Big]\ V_i(dx)$$

$$\text{for } 0 \le u_1 \le u_0 \le \tau.$$

To determine the limiting distribution of $(\underline{\gamma}_0(t),\ \underline{\gamma}_1(t))$, we note that for all fixed $u_0$ and $u_1$ the function

$$z(v) = \begin{cases} \displaystyle\int_0^{v+u_0-\tau} \{1-G_S(v-y)\}\ G_S(dy), & \tau-u_0 \le v < \tau-u_1, \\ 0, & \text{otherwise}, \end{cases}$$

is directly Riemann integrable, since on the interval $[\tau-u_0,\tau-u_1)$ the function $z(v)$ can be written as the difference of two bounded, monotone functions (cf. lemma 1.2.2 of section 1.2). Using the key renewal theorem, it now follows that for any $i \ge s$,

$$\lim_{t\to\infty} P\{\underline{\gamma}_k(t) \leq u_k \text{ for } k = 0,1 | \underline{z}(0) = i\} = \lim_{t\to\infty} P\{\underline{\gamma}_0(t) \leq u_0 | \underline{z}(0) = i\} +$$

$$- \frac{1}{\beta\{1+M(S-s)\}} \int_{\tau-u_0}^{\tau-u_1} [\int_0^{v+u_0-\tau} \{1-G_S(v-y)\}\, G_S(dy)]dv, \quad 0 \leq u_1 \leq u_0 \leq \tau.$$

Related results for the distribution of $(\underline{\gamma}_0(t),\ldots,\underline{\gamma}_N(t))$ can be found in [31].

The simultaneous distribution of $(\underline{\delta}_1(t),\ldots,\underline{\delta}_L(t))$ can be obtained from the recurrence relation

$$P\{\underline{\delta}_k(t) \leq u_k \text{ for } k = 1,\ldots, L | \underline{z}(0) = i\} =$$

$$= P\{\underline{\delta}_k(t) \leq u_k \text{ for } k = 1,\ldots, L-1 | \underline{z}(0) = i\} +$$

$$- P\{\underline{\delta}_k(t) \leq u_k \text{ for } k = 1,\ldots, L-1,\ \underline{\delta}_L(t) > u_L | \underline{z}(0) = i\} =$$

$$= P\{\underline{\delta}_k(t) \leq u_k \text{ for } k = 1,\ldots, L-1 | \underline{z}(0) = i\} - \sum_{n=0}^{\infty} P\{\underline{\tau}_n \leq t-\tau < \underline{\tau}_{n+h} \leq$$

$$\leq t+u_h-\tau \text{ for } h = 1,\ldots, L-1,\ t+u_L-\tau < \underline{\tau}_{n+L} \leq t | \underline{z}(0) = i\}$$

$$\text{for } 0 \leq u_1 \leq u_2 \leq \cdots \leq u_L \leq \tau.$$

For ease of notation, let us consider the case $L = 2$. The general case can be handled in the same way. For any $i \geq s$ and $t \geq \tau$ we have

$$P\{\underline{\delta}_k(t) \leq u_k \text{ for } k = 1,2 | \underline{z}(0) = i\} = P\{\underline{\delta}_1(t) \leq u_1 | \underline{z}(0) = i\} +$$

$$- \int_{(t-\tau)+}^{t+u_1-\tau} \{G_S(t-x) - G_S(t+u_2-\tau-x)\}\, G_i(dx) +$$

$$- \int_0^{t-\tau} [\int_{(t-\tau-x)+}^{t+u_1-\tau-x} \{G_S(t-x-y) - G_S(t+u_2-\tau-x-y)\}\, G_S(dy)]\, V_i(dx)$$

$$\text{for } 0 \leq u_1 \leq u_2 \leq \tau.$$

To determine the limiting distribution of $(\underline{\delta}_1(t),\ \underline{\delta}_2(t))$, we note that for all fixed $u_1$ and $u_2$ the function

$$z(v) = \begin{cases} \int_{(v-\tau)+}^{v+u_1-\tau} \{G_S(v-y) - G_S(v+u_2-\tau-y)\}\, G_S(dy), & v \geq \tau, \\ \\ 0, & 0 \leq v < \tau, \end{cases}$$

is Riemann integrable over finite intervals, since on the interval $[\tau,\infty)$ the function $z(v)$ can be written as the difference of two bounded, monotone functions. Since $0 \leq z(v) \leq 1-G_S(v-\tau)$ for $v \geq \tau$ and since the monotone function $1-G_S(v-\tau)$ is Riemann integrable over $[\tau,\infty)$, it follows from lemma 1.2.2 of section 1.2 that the function $z(v)$ is directly Riemann integrable. Applying the key renewal theorem, we obtain for any $i \geq s$ that

$$\lim_{t\to\infty} P\{\underline{\delta}_k(t) \leq u_k \text{ for } k = 1,2 | \underline{z}(0) = i\} = \lim_{t\to\infty} P\{\underline{\delta}_1(t) \leq u_1 | \underline{z}(0) = i\} +$$

$$- \frac{1}{\beta\{1+M(S-s)\}} \int_\tau^\infty [\int_{(v-\tau)+}^{v+u_1-\tau} \{G_S(v-y) - G_S(v+u_2-\tau-y)\} \; G_S(dy)]dv$$

$$\text{for } 0 \leq u_1 \leq u_2 \leq \tau.$$

## 3.4. CERTAIN LONG-RUN AVERAGES FOR THE (s,S) POLICY

In this section we shall use the results of section 3.2 to determine certain long-run averages which may serve as a basis for comparing various (s,S) policies.

An important measure of the merit of the (s,S) policy is the average expected available stock, where the available stock at time t is defined as $\max(\underline{x}(t),0)$. For example, if, for each unit, the inventory cost is proportional to the time for which the unit remains in inventory, then the average expected inventory cost per unit time is equal to the proportionality constant times the average expected available stock. For each initial stock $\underline{z}(0) = i$, the average expected available stock is given by (cf. (3.2.10))

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \{\sum_{j=1}^{\max(i,S)} jr_{ij}(u+\tau)\}du = \sum_{j=1}^{S} jr_j.$$

Let us next determine the frequency of ordering. Define $\underline{k}(t) = \sup\{n | \underline{\tau}_n \leq t\}$, and let $K_i(t) = E\{\underline{k}(t) | \underline{z}(0) = i\}$. Then $K_i(t)$ is the expected number of orders in $[0,t]$ when the initial stock is i. By the elementary renewal theorem (cf. section 1.2) and (3.1.3), we have

$$\lim_{t\to\infty} \frac{K_i(t)}{t} = \frac{1}{E\underline{t}_{S-s+1}} = \frac{1}{\beta\{1+M(S-s)\}} \qquad \text{for } i \geq s.$$

Further, it follows from Wald's equation and (3.1.1) that the expected size of the $n^{th}$ order ($n \geq 2$) is equal to

$$(3.4.1)\qquad E\{\underline{\xi}_1 + \dots + \underline{\xi}_{\underline{m}_{S-s}+1}\} = \mu\{1+M(S-s)\}.$$

For example, if the ordering cost of k units is given by $K\delta(k) + ck$, where $\delta(0) = 0$, and $\delta(k) = 1$ for $k > 0$, then the average expected ordering cost per unit time is equal to $c\mu/\beta + K/\{\beta(1+M(S-s))\}$.

An important quantity is the expected fraction of time the system is out of stock. For example, one may ask to determine an (s,S) policy such that the expected fraction of time the system is out of stock is less than or equal to $\alpha$ for some $0 < \alpha < 1$. For each initial stock i, the expected fraction of time the system is out of stock is given by

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \{\sum_{j\leq 0} r_{ij}(u+\tau)\}du = 1 - \lim_{t\to\infty}\frac{1}{t}\int_0^t \{\sum_{j=1}^{\max(i,S)} r_{ij}(u+\tau)\}du =$$
$$= 1 - \sum_{j=1}^{S} r_j.$$

In the same way we obtain that for each initial stock the expected fraction of time that there are no orders outstanding is given by

$$\sum_{j=s}^{S} r_j.$$

Further, it may be of interest to know the average expected number of backorders on the books, where any unit backordered is counted as a backorder. For example, if, for each unit backordered, the backorder cost is proportional to the time for which the backorder exists, then the average expected backorder cost per unit time is equal to the proportionality constant times the average expected number of backorders on the books. Using (3.1.11) and $E\underline{a}(t) = \mu U(t)$ for $t \geq 0$, it is straightforward to verify that for any $i \geq s$ and $t \geq \tau$,

$$\sum_{j=-\infty}^{\max(i,S)} jr_{ij}(t+\tau) = \sum_{k=s}^{\max(i,S)} kp_{ik}(t) - \sum_{k=0}^{\infty} kb_k(t,\tau) =$$
$$= \sum_{k=s}^{\max(i,S)} kp_{ik}(t) - \mu\int_0^\tau \{1+U(\tau-y)\}\, dP\{\underline{y}(t) \leq y\}.$$

From renewal theory we have the identity (cf. [15])

$$\frac{1}{\beta}\int_0^t \{1+U(t-y)\}\{1-F(y)\}dy = \frac{t}{\beta} \qquad \text{for all } t \geq 0.$$

Using (3.1.6), it now follows that for any $i \geq s$,

$$(3.4.2)\qquad \lim_{t\to\infty} \sum_{j=-\infty}^{\max(i,S)} j r_{ij}(t) = \sum_{k=s}^{S} k q_k - \frac{\mu\tau}{\beta},$$

from which it follows that for each initial stock i the average expected number of backorders on the books is given by

$$\lim_{t\to\infty} \int_0^t \{\sum_{j\leq 0} j r_{ij}(u+\tau)\}du = \sum_{k=s}^{S} k q_k - \frac{\mu\tau}{\beta} - \sum_{k=1}^{S} k r_k.$$

Finally, we shall determine the average expected stock on order and the average expected number of outstanding orders. It follows from (3.4.2) that for each initial stock i the average expected stock on hand is given by

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \{\sum_{j=-\infty}^{\max(i,S)} j r_{ij}(u+\tau)\}du = \sum_{k=s}^{S} k q_k - \frac{\mu\tau}{\beta}.$$

For each initial stock i the average expected stock on hand plus on order is given by

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \{\sum_{j=s}^{\max(i,S)} j p_{ij}(u)\}du = \sum_{j=s}^{S} j q_j.$$

Thus for each initial stock the average expected stock on order is given by

$$(3.4.3)\qquad \frac{\mu\tau}{\beta}.$$

It is interesting to note that if F is an exponential distribution, then $E\underline{a}(\tau) = \mu\tau/\beta$, and hence the average expected stock on order is equal to the mean lead time demand.

Using the fact that the sizes of the outstanding orders are independent random variables which are also independent of the number of outstanding orders, it readily follows from (3.4.1) and (3.4.3) that the average expected number of outstanding orders is given by (see also theorem 3.3.1 (d)),

$$\tau/\{\beta(1+M(S-s))\}.$$

*Remark 3.4.1.*

We now consider briefly the case in which the distribution function F of the interarrival times is concentrated on the set of the positive integers. Moreover, we assume that $\tau$ is a positive integer.

Denote the probability distribution of the interarrival times $\underline{s}_n - \underline{s}_{n-1}$

by

$$f_j = P\{\underline{s}_n - \underline{s}_{n-1} = j\}, \qquad j = 0,1,\ldots;\ n = 1,2,\ldots\ ,$$

where $f_0 = 0$. The probability distribution of $\underline{s}_n$ is given by the n-fold convolution $f_j^{(n)}$ of $f_j$ with itself. Let

$$u(j) = \sum_{n=1}^{\infty} f_j^{(n)}, \qquad j = 0,1,\ldots\ .$$

For any $i \geq s$, let

$$g_i(j) = P\{\underline{t}_{i-s+1} = j\}, \qquad j = 1,2,\ldots\ .$$

Since $f_0 = 0$, it follows that for any $i \geq s$,

$$g_i(j) = \sum_{n=1}^{j} \{\Phi^{(n-1)}(i-s) - \Phi^{(n)}(i-s)\}\ f_j^{(n)}, \qquad j = 1,2,\ldots\ .$$

Since, in section 3.2, the derivation of the transient behaviour of the stock level does not use any fact about F, the results found in section 3.2 for the transient behaviour of the stock level carry over to the above case.

To determine the asymptotic behaviour of the stock level, we have to impose some conditions on the probabilities $g_S(j)$ and $f_j$. Using theorem 1.2.2 of section 1.2, it is readily seen that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}(k) = q_j \qquad \text{for all } i,\ j \geq s,$$

where the ordinary limit exists for all i and j if the greatest common divisor of the indices n for which $g_S(n) > 0$, is 1.

Moreover, it follows from theorem 1.2.2 of section 1.2 that the asymptotic behaviour of

$$P\{\underline{y}(k) = j\} = f_{k+j} + \sum_{h=0}^{k} f_{k+j-h}\ u(h), \quad j = 1,2,\ldots;\ k = 0,1,\ldots\ ,$$

is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} P\{\underline{y}(k) = j\} = \frac{1}{\beta} \sum_{h=0}^{\infty} f_{h+j} = \frac{1}{\beta}\{1-F(j-1)\}, \qquad j = 1,2,\ldots\ ,$$

where the ordinary limit exists for each j = 1,2,... if the greatest common divisor of the indices n for which $f_n > 0$, is 1. It now follows from (3.1.7) and (3.1.8) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n} b_k(m,\tau) = b_k(\tau), \qquad k = 0,1,\ldots , \tag{3.4.4}$$

where

$$b_0(\tau) = \frac{1}{\beta} \sum_{h=\tau+1}^{\infty} \{1-F(h-1)\} + \frac{1}{\beta} \sum_{h=1}^{\tau} \phi(0)\ a_0(\tau-h)\ \{1-F(h-1)\},$$

and

$$b_k(\tau) = \frac{1}{\beta} \sum_{j=0}^{k} \sum_{h=1}^{\tau} \phi(j)\ a_{k-j}(\tau-h)\ \{1-F(h-1)\}, \qquad k = 1,2,\ldots .$$

The ordinary limit in (3.4.4) exists for all $k = 0,1,\ldots$ if the greatest common divisor of the indices n for which $f_n > 0$, is 1. We note that if $\{f_j\}$ is a geometric distribution, then $b_k(m,\tau) = a_k(\tau)$ for all $k,m = 0,1,\ldots$.*) Using (3.2.8), it now follows that if g.c.d$\{n|f_n > 0\} = 1$ and if g.c.d$\{n|g_S(n) > 0\} = 1$, then

$$\lim_{n\to\infty} r_{ij}(n) = r_j \qquad \text{for all } i \geq s \text{ and } j \in I.$$

Finally, it is readily verified that the results of section 3.3 carry over to the above case if g.c.d$\{n|g_S(n) > 0\} = 1$, while the results of section 3.4 carry over to the above case if g.c.d$\{n|g_S(n) > 0\} = 1$ and if g.c.d$\{n|f_n > 0\} = 1$.

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*) As pointed out to me by Arie Hordijk the (s,S) model with an interarrival distribution $\{p(1-p)^{j-1}\}$ and a demand distribution $\{\phi(j)\}$ is equivalent to the (s,S) model with an interarrival distribution $\{f'_j\}$ and a demand distribution $\{\phi'(j)\}$, where $f'_1 = 1$, $\phi'(0) = 1-p+p\phi(0)$, and $\phi'(j) = p\phi(j)$ for $j \geq 1$.

# CHAPTER IV A FORMULA FOR THE LONG-RUN AVERAGE COST APPLIED TO AN (s,S) INVENTORY MODEL

In the first section of this chapter we shall derive a general formula for the average cost per unit time for a class of semi-Markovian decision processes. In section 4.2 this formula will be applied to an (s,S) inventory model.

Following De Leve [33], we shall analyse a decision process in the following way. A stochastic process, called the natural process, will be defined such that, roughly speaking, the decision process can be seen as a superposition of the natural process and decisions made in certain states of the natural process. The natural process can be thought of as a stochastic process that describes the evolution of the system when no decisions are made. Based upon this disintegration of the decision process De Leve [33] has given an original approach for the calculation of both the expected cost incurred between two successive decisions and the expected time between two successive decisions when the decision process has reached the "steady-state".

As part of an iteration method for a general class of Markovian decision models, De Leve [33] has given a formula for the long-run average cost per unit time which generalizes the classical one for decision processes with a regeneration point. However, in [33] the derivation of the former formula is obscured by a large number of assumptions imposed on the model considered there. For a model less general than in [33] we shall derive this formula anew under a more easily verifiable set of conditions.

In section 4.2 we shall apply the above formula to an (s,S) inventory model *) in which the times between the demand epochs have a geometric distribution, the demands have a discrete distribution, excess demand is backlogged, the lead time is fixed, and the ordering, holding and backorder costs consist of linear and fixed costs. The derivation of the average cost for this model carries over to the case in which the customers arrive according to a Poisson process.

## 4.1. THE AVERAGE COST PER UNIT TIME FOR A SEMI-MARKOVIAN DECISION PROCESS

Let $X$ be a countable set. The set $X$ will be called the state space.**) For any $i \in X$, let $F_i(t)$ be a probability distribution concentrated on $(0,\infty)$,

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*) The average cost for this model could also be obtained via the stationary distribution of the stock on hand plus on order.

**) The results of this section can be adapted to an arbitrary state space.

i.e. $F_i(0) = 0$. It is assumed that, for some $\delta > 0$,

$$(4.1.1) \qquad \inf_{i \in X} \{1-F_i(\delta)\} > 0.$$

For any i, $j \in X$, let $\pi(i,j,t)$, $t \geq 0$, be a non-negative Baire function in t such that

$$\sum_{j \in X} \pi(i,j,t) = 1 \qquad \text{for all } i \in X \text{ and } t \geq 0.$$

The *natural process* is defined as a stochastic process $\{(\underline{x}_n,\underline{\tau}_n),\ n = 0,1,\ldots\}$ that satisfies $\underline{\tau}_0 = 0$, $\underline{x}_0 = i_0$ for some $i_0 \in X$ and

$$P\{\underline{x}_k = i_k, \underline{\tau}_k \leq t_k \text{ for } k = 1,\ldots,n\} = \prod_{k=1}^{n} \int_0^{t_k} \pi(i_{k-1},i_k,t)\, F_{i_{k-1}}(dt)$$

$$\text{for all } i_k \in X,\ t_k \geq 0,\ (k = 1,\ldots,n), \text{ and } n \geq 1.$$

We shall say that a transition of the natural process has occurred at each of the epochs 0, $\underline{\tau}_1$, $\underline{\tau}_1+\underline{\tau}_2,\ldots$ . The random variable $\underline{x}_k$ denotes the state of the natural process at the $k^{th}$ transition and the random variable $\underline{\tau}_k$ denotes the length of the time interval between the $(k-1)^{th}$ and the $k^{th}$ transition. It follows from the above definition that if a transition of the natural process has just occurred into state i, then, "regardless of the history", the time until the next transition of the natural process has the distribution function $F_i$ and, under the condition that this time is t, the next transition will be into state j with probability $\pi(i,j,t)$. It should be noted that by assumption (4.1.1) the number of transitions in each finite time interval is finite with probability one.

For any $i,j \in X$, let $c(i,j,t)$, $t \geq 0$, be a non-negative Baire function in t. Assume that the following *cost structure* is imposed on the natural process. In the natural process no cost is incurred at epoch 0. Under the condition that $\underline{x}_{n-1} = i$, $\underline{\tau}_n = t$ and $\underline{x}_n = j$, a cost $c(i,j,t)$ is incurred at epoch $\underline{\tau}_1 + \ldots + \underline{\tau}_n$, $(n \geq 1)$.

The following assumption will be essential in our considerations:

*Assumption 1.* There is a non-empty set $A_0 \subset X$ *) such that

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*) We use the symbol $\subseteq$ to denote set inclusion and $\subset$ to denote proper set inclusion.

$$P\{\underline{x}_n \in A_0 \text{ for some } n \geq 0 \mid \underline{x}_0 = i\} = 1 \qquad \text{for all } i \in X.$$

We choose two non-empty sets

$$A_{01} \subseteq A_0 \text{ and } A_{02} \subseteq A_0$$

such that, for k = 1,2,

$$P\{\underline{x}_n \in A_{0k} \text{ for some } n \geq 0 \mid \underline{x}_0 = i\} = 1 \qquad \text{for all } i \in X.$$

For k = 1,2, let

$$\underline{\alpha}(k) = \inf\{n \geq 0 \mid \underline{x}_n \in A_{0k}\}.$$

For any $i \in X$, let

$$k_0(i) = E\{\sum_{j=1}^{\underline{\alpha}(1)} c(\underline{x}_{j-1}, \underline{x}_j, \underline{\tau}_j) \mid \underline{x}_0 = i\} \text{ and } t_0(i) = E\{\sum_{j=1}^{\underline{\alpha}(2)} \underline{\tau}_j \mid \underline{x}_0 = i\}.$$

It follows from the above definitions that $k_0(i) = 0$ for $i \in A_{01}$, and $t_0(i) = 0$ for $i \in A_{02}$. For initial state $\underline{x}_0 = i$ with $i \notin A_{01}$, we have that $k_0(i)$ is the expected cost incurred during the interval from 0 to the first epoch at which the natural process makes a transition into a state of $A_{01}$, including the cost incurred at the end of this interval. For initial state $\underline{x}_0 = i$ with $i \notin A_{02}$, we have that $t_0(i)$ is the expected time until the first transition of the natural process into a state of $A_{02}$. We shall see hereafter that the functions $k_0(i)$ and $t_0(i)$ will play a fundamental role in our considerations.

Let us next define a stochastic process called the decision process. Let $I$ be a given subset of $X$ such that

(4.1.2) $$A_0 \subseteq I \subset X.$$

Let $\psi: I \to I$ be a given function such that

(4.1.3) $$\psi(i) = i \qquad \text{for } i \notin I,$$

and

(4.1.4) $$\psi(i) \notin I \qquad \text{for } i \in I.$$

The *decision process* is defined as a stochastic process $\{(\underline{x}'_n, \underline{\tau}'_n),\ n = 0,1,\ldots\}$ that satisfies $\underline{\tau}'_0 = 0$, $\underline{x}'_0 = i_0$ for some $i_0 \in X$, and

$$P\{\underline{x}'_k = i_k,\ \underline{\tau}'_k \leq t_k \text{ for } k = 1,\ldots, n\} =$$

$$= \prod_{k=1}^{n} \int_0^{t_k} \pi(\psi(i_{k-1}),\ i_k, t)\ F_{\psi(i_{k-1})}(dt)$$

for all $i_k \in X, t_k \geq 0, (k = 1,\ldots,n)$, and $n \geq 1$.

Let us say that a transition of the decision process has occurred at each of the epochs $0,\ \underline{\tau}'_1,\ \underline{\tau}'_1+\underline{\tau}'_2,\ldots$ . The random variable $\underline{x}'_k$ denotes the state of the decision process at the $k^{th}$ transition and $\underline{\tau}'_k$ denotes the length of the time interval between the $(k-1)^{th}$ and the $k^{th}$ transition.

Let $d(i)$, $i \in X$, be a non-negative, *bounded* function such that

$$d(i) = 0 \text{ for } i \notin I.$$

We assume that a *cost structure* is imposed on the decision process in the following way. In the decision process the cost $d(i)$ incurred at epoch 0 when $\underline{x}'_0 = i$. Under the condition that $\underline{x}'_{n-1} = i$, $\underline{\tau}'_n = t$ and $\underline{x}'_n = j$, the cost $c(\psi(i),j,t)$ and the cost $d(j)$ are incurred in the decision process at epoch $\underline{\tau}'_1 + \ldots + \underline{\tau}'_n$, $(n \geq 1)$.

We see from the foregoing definitions that the decision process can be regarded as a superposition of the natural process and the "decision mechanism" $\psi$. Roughly speaking, when the initial state $i \notin I$ the decision process behaves exactly as the natural process up to the first epoch at which a transition occurs into a state of $I$ (say state $j$). Then the decision mechanism $\psi$ transfers the system to state $\psi(j)$ *) without loss of time but at the cost of an amount $d(j)$, and thereafter the decision process behaves again exactly as the natural process with initial state $\psi(j)$ up to the next epoch at which a transition occurs into a state of $I$, etc.

From now on we are concerned with the decision process. To study this process, we shall define an imbedded process $\{\underline{I}_n\}$. To do this, let $\underline{n}_1 < \underline{n}_2 < \ldots$ be the increasing sequence of positive indices n for which

*) The case in which the decision mechanism has a random effect can be treated in a similar way.

$\underline{x}'_n \in I$. Define $\underline{I}_0 = \underline{x}'_0$, and, for $n = 1,2,\ldots$, let $\underline{I}_n = \underline{x}'_{\underline{\tau}_n}$, that is, $\underline{I}_n$ ($n \geq 1$) denotes the state on the $n^{th}$ visit of the decision process to the set $I$. It is readily verified that the process $\{\underline{I}_n, n = 0,1,\ldots\}$ is a Markov chain with state space $X$. For any $n \geq 1$, let

$$p_{ij}^{(n)} = P\{\underline{I}_n = j \mid \underline{I}_0 = i\} \qquad \text{for } i,\ j \in X.$$

When $n = 1$ we often drop the superscript. Note that $p_{ij}^{(n)} = 0$ for $j \notin I$.

Before we impose a condition on the Markov chain $\{\underline{I}_n\}$, we prove the following elementary but useful theorem.

*Theorem 4.1.1.*

Let $(X,F)$ be a measurable space. Let the real-valued function $p(x,A)$, where $x \in X$ and $A \in F$, be such that $p(x,A)$ for fixed $x$ determines a probability measure in $A$, and $p(x,A)$ for fixed $A$ determines a x-function measurable with respect to $F$. Let $p^{(k)}(x,A)$ be defined recursively by $p^{(k)}(x,A) = \int_X p(\xi,A)\, p^{(k-1)}(x,d\xi)$, where $p^{(1)}(x,A) = p(x,A)$. Suppose there is a finite set $A^* \in F$, an integer $N \geq 1$ and a positive number $\rho$, such that $p^{(N)}(x,A^*) \geq \rho$ for all $x \in X$. Then the stochastic transition function $p(x,A)$ satisfies the *Doeblin condition.*

*Proof*

The Doeblin condition states (cf. [13]): There is a finite-valued measure $\phi$ of sets $A \in F$ with $\phi(X) > 0$, an integer $\nu \geq 1$ and a positive $\varepsilon$, such that for every $x \in X$,

$$p^{(\nu)}(x,A) \leq 1-\varepsilon \qquad \text{if } \phi(A) \leq \varepsilon.$$

To prove the Doeblin condition, suppose that $A^*$ consists of $M$ points. For any $A \in F$, let

$$\phi(A) = k\rho/M \qquad \text{if } A \cap A^* \text{ consists of } k \text{ points.}$$

Clearly, $\phi$ is a finite-valued measure on $(X,F)$ with $\phi(X) = \rho > 0$. Choose $\varepsilon$ such that $0 < \varepsilon < \rho/M$. If $\phi(A) \leq \varepsilon$, then $A \cap A^* = \emptyset$, and so in that case

$$p^{(N)}(x,A) \leq 1-p^{(N)}(x,A^*) \leq 1-\rho \leq 1-\varepsilon \qquad \text{for all } x \in X.$$

This ends the proof of the theorem. *)

We now impose the following condition on the Markov chain $\{\underline{I}_n\}$.

*Assumption 2.*

(i) There is a *finite* set $E \subseteq I$, an integer $\nu \geq 1$ and a positive number $\rho$, such that $\sum_{j \in E} p_{ij}^{(\nu)} \geq \rho$ for all $i \in I$.

(ii) The Markov chain $\{\underline{I}_n\}$ has no two disjoint closed sets of states. **)

This assumption appears to be often satisfied in practice.

It follows from assumption 2 (i) that $\sum_{j \in E} p_{ij}^{(\nu+1)} \geq \rho$ for all $i \in X$. Hence, by theorem 4.1.1, we have that the Markov chain $\{\underline{I}_n\}$ satisfies the Doeblin condition. Because of this property and assumption 2 (ii), it follows from the theory of Markov processes (cf. [13]) that a probability distribution $\{q_j, j \in X\}$ exists, such that

$$\lim_{n\to\infty} \sum_{j \in A} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(k)} = \sum_{j \in A} q_j \qquad \text{for all } A \subseteq X \text{ and } i,j \in X. \tag{4.1.5}$$

Moreover, the stationary probability distribution $\{q_j\}$ is the unique solution to (cf. [13])

$$q_j = \sum_{h \in X} p_{hj}\, q_h \qquad \text{for } j \in X, \quad \sum_{j \in X} q_j = 1. \tag{4.1.6}$$

Note that $q_j = 0$ for $j \notin I$.

We now introduce the following assumption.

*Assumption 3.*

The functions $k_0(i)$ and $t_0(i)$ are bounded on $X$.

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*) It is interesting to note that if X is denumerable and F is the class of all subsets of X, then the Doeblin condition is equivalent to the condition given in theorem 4.1.1.

**) A set C of states is said to be closed if $\sum_{j \in C} p_{ij} = 1$ for all $i \in C$.

We now define functions $k_1(i)$ and $t_1(i)$ which depend only on the natural process and the direct effect of the "single decision" $\psi(i)$ in state i. For any $i \in X$, let

$$k_1(i) = d(i) + k_0(\psi(i)) \text{ and } t_1(i) = t_0(\psi(i)).$$

Since the function d(i) is assumed to be bounded, it follows from assumption 3 that the functions $k_1(i)$ and $t_1(i)$ are bounded on $X$.

Given initial state $\underline{x}_0' = i \in X$, let us define the following random variables. Let $\underline{n} = \inf\{n | n \geq 1, \underline{x}_n' \in I\}$, and define

$$\underline{T}(i) = \sum_{j=1}^{\underline{n}} \underline{\tau}_j', \ \underline{K}(i) = d(i) + \sum_{j=1}^{\underline{n}} c(\underline{x}_{j-1}', \underline{x}_j', \underline{\tau}_j').$$

That is, $\underline{T}(i)$ represents the length of the time interval from 0 to the epoch at which the decision process makes a transition into a state of $I$ for the first time after t = 0, and $\underline{K}(i)$ represents the total cost incurred in this interval, where this interval is assumed to be left closed and right open with respect to the cost d(.) and left open and right closed with respect to the cost c(.,.,.).

Let

$$\lambda(i) = E\underline{T}(i) \text{ and } \chi(i) = E\underline{K}(i) \qquad \text{for } i \in X.$$

Since $I \supseteq A_{0k}$ for k = 1,2 and since the costs are non-negative, it follows that $0 \leq \chi(i) \leq k_1(i)$ and $0 \leq \lambda(i) \leq t_1(i)$ for all $i \in X$. Thus, since $k_1(i)$ and $t_1(i)$ are bounded, the functions $\chi(i)$ and $\lambda(i)$ are also bounded. Furthermore,

(4.1.7) $\qquad \lambda(i) > 0 \qquad$ for all $i \in X$.

The following well-known lemma will be needed.

*Lemma 4.1.1.*

Let $\mu$ and $\mu_n$, n = 1,2,... be finite measures on a measurable space $(X,F)$. Suppose $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for all $A \in F$. Then for any bounded measurable function f,

$$\lim_{n\to\infty} \int_X f(x)\ \mu_n(dx) = \int_X f(x)\ \mu(dx).$$

The proof of this lemma is standard. The lemma is trivially true when f is a simple function. For an arbitrary bounded measurable function the lemma is proved by using the fact that every bounded measurable function is the limit of a uniformly convergent sequence of simple functions.

*Theorem 4.1.2.*

For any $i \in X$,

$$\lim_{n\to\infty} \frac{E\{\sum_{h=0}^{n} \underline{K}(\underline{I}_h)|\underline{x}_0' = i\}}{E\{\sum_{h=0}^{n} \underline{T}(\underline{I}_h)|\underline{x}_0' = i\}} = \frac{\sum_{j\in I} \chi(j)q_j}{\sum_{j\in I} \lambda(j)q_j}.$$

*Proof*

It is easy to see that for any $i \in X$,

$$\frac{1}{n} E\{\sum_{h=0}^{n} \underline{K}(\underline{I}_h)|\underline{x}_0' = i\} = \frac{\chi(i)}{n} + \frac{1}{n}\sum_{h=1}^{n}\sum_{j\in I} \chi(j)\, p_{ij}^{(h)}, \qquad n \geq 1.$$

Using (4.1.5), the boundedness of the function $\chi(j)$ and lemma 4.1.1, it now follows that for any $i \in X$,

$$\lim_{n\to\infty} \frac{1}{n} E\{\sum_{h=0}^{n} \underline{K}(\underline{I}_h)|\underline{x}_0' = i\} = \lim_{n\to\infty} \sum_{j\in I} \chi(j)\, \frac{1}{n}\sum_{h=1}^{n} p_{ij}^{(h)} = \sum_{j\in I} \chi(j)q_j.$$

In the same way we obtain

$$\lim_{n\to\infty} \frac{1}{n} E\{\sum_{h=0}^{n} \underline{T}(\underline{I}_h)|\underline{x}_0' = i\} = \sum_{j\in I} \lambda(j)q_j \qquad \text{for all } i \in X.$$

This ends the proof of the theorem.

The left-hand member of the expression given in theorem 4.1.2 can be interpreted as the *long-run average expected cost per transition*. It will be proved that under certain conditions this quantity is equal to the *long-run average expected cost per unit time*.

The calculation of the functions $\chi(j)$ and $\lambda(j)$ may be difficult, whereas it is easy to determine the functions $k_1(j) - k_0(j)$ and $t_1(j) - t_0(j)$. The importance of this is demonstrated by the next theorem.

*Theorem 4.1.3.*

$$\sum_{j\in I} \chi(j)q_j = \sum_{j\in I} \{k_1(j) - k_0(j)\}q_j$$

and

$$\sum_{j\in I} \lambda(j)q_j = \sum_{j\in I} \{t_1(j) - t_0(j)\}q_j.$$

*Proof*

It readily follows from $I \supseteq A_{0k}$ $(k = 1,2)$ and the foregoing definitions that for any $j \in X$,

$$k_1(j) = \chi(j) + \sum_{h\in I} p_{jh}\, k_0(h) \text{ and } t_1(j) = \lambda(j) + \sum_{h\in I} p_{jh}\, t_0(h).$$

Using the boundedness of the functions $\chi(j)$ and $k_0(j)$ and using (4.1.6) together with the fact that $q_j = 0$ for $j \notin I$, we obtain

$$\sum_{j\in I} k_1(j)q_j = \sum_{j\in I} \chi(j)q_j + \sum_{j\in I} q_j \sum_{h\in I} p_{jh}\, k_0(h) =$$

$$= \sum_{j\in I} \chi(j)q_j + \sum_{h\in I} k_0(h) \sum_{j\in I} p_{jh}\, q_j =$$

$$= \sum_{j\in I} \chi(j)q_j + \sum_{h\in I} k_0(h)q_h.$$

This proves the first part of the theorem. The other part follows in the same way. This ends the proof of the theorem.

Before we introduce the last assumption needed to establish the equality of the average cost per transition and the average cost per unit time, we prove the following lemma.

*Lemma 4.1.2.*

Let a denumerable Markov chain be given with a state space J and a transition matrix $(q_{ij})$. Suppose this Markov chain satisfies the Doeblin condition and has no two disjoint closed sets. Then for each positive recurrent state r there exists a finite constant N, such that $f_{ir} = 1$ and $u_{ir} \leq N$ for all $i \in J$, where $f_{ir}$ is the probability that, starting from state i, the

Markov chain will ever reach state r, and $\mu_{ir}$ denotes the mean recurrence time from state i to state r.

*Proof*

Since the Markov chain with transition matrix $(q_{ij})$ satisfies the Doeblin condition and has no two disjoint closed sets, it follows from the theory of Markov chains (cf. [7] and [13]) that $J = L \cup L^c$ where the non-empty set L consists of positive recurrent states and $L^c$ is a transient set. Fix a state $r \in L$. Let $\tilde{q}_{ij} = q_{ij}$ for $i, j \in J$ with $i \neq r$, let $\tilde{q}_{rr} = 1$, and let $\tilde{q}_{rj} = 0$ for $j \in J$ with $j \neq r$.

We shall now prove that a Markov chain with transition matrix $(\tilde{q}_{ij})$ satisfies the Doeblin condition. Denote by $\tilde{q}_{ij}^{(k)}$ the k-step transition probabilities of this Markov chain. Let $q_{ij}^{(k)}$ be the k-step transition probabilities of the Markov chain with transition matrix $(q_{ij})$. Since the latter Markov chain satisfies the Doeblin condition, there is a finite-valued measure $\phi$ on the class of all subsets of J with $\phi(J) > 0$, an integer $\nu \geq 1$ and a positive number $\varepsilon > 0$, such that, for all $i \in J$,

$$\sum_{j \in A} q_{ij}^{(\nu)} \leq 1-\varepsilon \text{ if } \phi(A) \leq \varepsilon. \tag{4.1.8}$$

For any subset $A \subseteq J$, let $\tilde{\phi}(A) = \phi(A)$ if $r \notin A$, and let $\tilde{\phi}(A) = \phi(A) + 2\varepsilon$ if $r \in A$. It is easy to verify that $\tilde{\phi}$ is a finite-valued measure on the class of all subsets of J with $\tilde{\phi}(J) > 0$. If $\tilde{\phi}(A) \leq \varepsilon$, then $r \notin A$. Further, if $r \notin A$, then

$$\sum_{j \in A} \tilde{q}_{ij}^{(k)} \leq \sum_{j \in A} q_{ij}^{(k)} \qquad \text{for all } i \in J \text{ and } k = 1,2,\ldots, \tag{4.1.9}$$

because of the definition of the $\tilde{q}_{ij}$. It now follows from (4.1.8) and (4.1.9) that if $\tilde{\phi}(A) \leq \varepsilon$, then $\sum_{j \in A} \tilde{q}_{ij}^{(\nu)} \leq 1-\varepsilon$ for all $i \in J$. This proves that a Markov chain with transition matrix $(\tilde{q}_{ij})$ satisfies the Doeblin condition.

Since the state r is a positive recurrent state of the Markov chain with transition matrix $(q_{ij})$ and since this Markov chain has no two disjoint closed sets, it follows that the Markov chain with transition matrix $(\tilde{q}_{ij})$ has no two disjoint closed sets and has only transient states except the state r; this state is absorbing and hence aperiodic. Hence, by the Doeblin condition (cf. [13]), for all $i \in J$,

$$(4.1.10) \quad \lim_{k\to\infty} \tilde{a}_i(k) = 0, \text{ where } \tilde{a}_i(k) = \sum_{j\in J, j\neq r} \tilde{q}_{ij}^{(k)}.$$

Moreover, the convergence is exponentially fast and uniform in i. From the definition of the $\tilde{q}_{ij}$ it follows that $1 - f_{ir} = \lim_{k\to\infty} \tilde{a}_i(k) = 0$ for all $i \in J$. Further, since $E\underline{n} = \sum_0^\infty P\{\underline{n} > k\}$ for each non-negative, integral-valued random variable $\underline{n}$, we have $\mu_{ir} = 1 + \sum_{k=1}^\infty \tilde{a}_i(k)$ for all $i \in J$. This proves the lemma, since the convergence in (4.1.10) is uniform in i and exponentially fast.

We now introduce the last assumption.

*Assumption 4.* There is a state $r \in I$ such that

(i) $\qquad P\{\underline{x}_n' = r \text{ for some } n \geq 1 | \underline{x}_0' = i\} = 1 \qquad$ for all $i \in X$,

and

(ii) $\qquad E\underline{T}_r(i) < \infty$ and $E\underline{K}_r(i) < \infty \qquad$ for all $i \in X$,

where, for initial state $\underline{x}_0' = i$,

$$\underline{T}_r(i) = \sum_{j=1}^{\underline{\nu}} \underline{\tau}_j', \quad \underline{K}_r(i) = \sum_{j=0}^{\underline{\nu}-1} d(\underline{x}_j') + \sum_{j=1}^{\underline{\nu}} c(\underline{x}_{j-1}', \underline{x}_j', \underline{\tau}_j')$$

with $\underline{\nu} = \inf\{n | n \geq 1, \underline{x}_n' = r\}$. That is, for initial state $\underline{x}_0' = i$, $\underline{T}_r(i)$ denotes the length of the time interval from 0 to the epoch at which the decision process makes a transition into state r for the first time after $t = 0$, and $\underline{K}_r(i)$ denotes the total cost incurred in this interval, where this interval is assumed to be left closed and right open with respect to the cost $d(.)$ and left open en right closed with respect to the cost $c(.,.,.)$.

It follows from assumption 4 (i) that state r is a recurrent state of the Markov chain $\{\underline{I}_n\}$. Since this Markov chain is assumed to satisfy the Doeblin condition, it follows that state r is positive recurrent. Hence, by lemma 4.1.2, for any $i \in X$,

$$(4.1.11) \quad E\{\underline{N} | \underline{x}_0' = i\} < \infty \text{ where } \underline{N} = \inf\{n | n \geq 1, \underline{I}_n = r\}.$$

It should be noted that the result (4.1.11) can also be obtained without the use of assumption 2. The relation (4.1.11) can also be deduced from (4.1.1) and assumption 4 (see also lemma 1 in [38]).

We shall now prove that the average expected cost per transition is equal to the average expected cost per unit time.

Given initial state $\underline{x}_0' = i \in X$, denote by $\underline{w}_i(t)$ the total cost incurred in the decision process during the time interval $[0,t]$.

The following theorem is similar to a theorem given in [38,39].

*Theorem 4.1.4.*

For any $i \in X$,

$$\lim_{t\to\infty} \frac{1}{t} E(\underline{w}_i(t)) = \lim_{n\to\infty} \frac{E\{\sum_{h=0}^{n} \underline{K}(\underline{I}_h) \mid \underline{x}_0'=i\}}{E\{\sum_{h=0}^{n} \underline{T}(\underline{I}_h) \mid \underline{x}_0'=i\}} .$$

*Proof*

Given initial state $\underline{x}_0' = i \in X$, we define the following random variables. Let $\underline{\nu}_0 = 0$, and let $\underline{\nu}_1 < \underline{\nu}_2 < \ldots$ be the increasing sequence of positive indices n for which $\underline{x}_n' = r$. Let $\underline{\gamma}_0 = 0$, and define $\underline{\gamma}_n = \sum_{k=\underline{\nu}_{n-1}+1}^{\underline{\nu}_n} \underline{\tau}_k'$, $(n = 1,2,\ldots)$. Denote by $\underline{\delta}_n$ $(n \geq 1)$ the total cost incurred in the decision process during the interval from epoch $\sum_{k=0}^{n-1} \underline{\gamma}_k$ to epoch $\sum_{k=0}^{n} \underline{\gamma}_k$, where we take this time interval left closed and right open with respect to the cost $d(.)$ and left open and right closed with respect to the cost $c(.,.,.)$. It is readily seen that $\underline{\gamma}_1, \underline{\gamma}_2, \ldots$ $[\underline{\delta}_1, \underline{\delta}_2, \ldots]$ are independent random variables. The random variable $\underline{\gamma}_1$ $[\underline{\delta}_1]$ has the same distribution as $\underline{T}_r(i)$ $[\underline{K}_r(i)]$, and the random variables $\underline{\gamma}_2, \underline{\gamma}_3, \ldots$ $[\underline{\delta}_2, \underline{\delta}_3, \ldots]$ have the same distribution as $\underline{T}_r(r)$ $[\underline{K}_r(r)]$. Note that, by assumption 4 and (4.1.7), $E\underline{\gamma}_2$ and $E\underline{\delta}_2$ are finite with $E\underline{\gamma}_2 > 0$. For any $t \geq 0$, let $\underline{n}(t) = \sup\{n \mid \underline{\gamma}_0 + \ldots + \underline{\gamma}_n \leq t\}$. Distinguish now two cases.

*Case 1.* $i = r$. Since the costs are non-negative, we have

$$\frac{1}{t} E\{\sum_{k=1}^{\underline{n}(t)} \underline{\delta}_k\} \leq \frac{1}{t} E(\underline{w}_r(t)) \leq \frac{1}{t} E\{\sum_{k=1}^{\underline{n}(t)+1} \underline{\delta}_k\} \qquad \text{for } t > 0,$$

where all expectations are understood to be conditioned on $\underline{x}_0' = r$. From this and lemma 1.2.4 of section 1.2 it follows that

$$(4.1.12) \qquad \lim_{t\to\infty} E(\underline{w}_r(t))/t = E\underline{\delta}_2/E\underline{\gamma}_2.$$

Let $\underline{N}_0 = 0$, and let $\underline{N}_1 < \underline{N}_2 < \dots$ be the increasing sequence of positive indices n for which $\underline{I}_n = r$. It is easy to see that $\{\underline{N}_k\}$ is a renewal process. Note that, by (4.1.11), $E\underline{N}_1$ is finite and positive. For any $n = 1,2,\dots$, let $\underline{m}_n = \sup\{k \mid \underline{N}_k \leq n\}$. From

$$\frac{1}{n} E(\sum_{k=1}^{\underline{m}_n} \underline{\delta}_k) \leq \frac{1}{n} E\{\sum_{h=0}^{n} \underline{K}(\underline{I}_h) \mid \underline{x}_0' = r\} \leq \frac{1}{n} E(\sum_{k=1}^{\underline{m}_n+1} \underline{\delta}_k), \qquad n \geq 1,$$

and

$$\frac{1}{n} E(\sum_{k=1}^{\underline{m}_n} \underline{\gamma}_k) \leq \frac{1}{n} E\{\sum_{h=0}^{n} \underline{T}(\underline{I}_h) \mid \underline{x}_0' = r\} \leq \frac{1}{n} E(\sum_{k=1}^{\underline{m}_n+1} \underline{\gamma}_k), \qquad n \geq 1,$$

we have, by lemma 1.2.4 of section 1.2,

$$\lim_{n\to\infty} \frac{1}{n} E\{\sum_{h=0}^{n} \underline{K}(\underline{I}_h) \mid \underline{x}_0' = r\} = E\underline{\delta}_2/E\underline{N}_1 \tag{4.1.13}$$

and

$$\lim_{n\to\infty} \frac{1}{n} E\{\sum_{h=0}^{n} \underline{T}(\underline{I}_h) \mid \underline{x}_0' = r\} = E\underline{\gamma}_2/E\underline{N}_1. \tag{4.1.14}$$

For $i = r$ the theorem now follows from (4.1.12) - (4.1.14).

*Case 2.* $i \neq r$. Let $v_r(t) = E(\underline{w}_r(t))$. Then, for any $t > 0$,

$$\frac{1}{t}\int_0^t v_r(t-u)G_i(du) \leq \frac{1}{t} E(\underline{w}_i(t)) \leq \frac{1}{t} E\underline{K}_r(i) + \frac{1}{t}\int_0^t v_r(t-u)G_i(du),$$

where $G_i$ is the probability distribution function of $\underline{T}_r(i)$. Taking the limit as $t \to \infty$ and using the bounded convergence theorem together with the fact that $v_r(t)/t$ has a finite limit, we obtain

$$\lim_{t\to\infty} \frac{1}{t} E(\underline{w}_i(t)) = \lim_{t\to\infty} \frac{1}{t} E(\underline{w}_r(t)), \tag{4.1.15}$$

from which the theorem follows, since the right-hand side of the expression given in theorem 4.1.4 does not depend on i (see theorem 4.1.2). This ends the proof.

Let us next prove that the average cost per unit time equals with probability one the average expected cost per unit time.

*Theorem 4.1.5.*

For any $i \in X$,

$$\lim_{t\to\infty} \frac{\underline{w}_i(t)}{t} = \lim_{t\to\infty} \frac{E(\underline{w}_i(t))}{t} \quad \text{with probability one.}$$

*Proof*

Following the notation used in the proof of theorem 4.1.4, we have for any $i \in X$ that

$$(4.1.16) \quad \frac{1}{t}\sum_{k=1}^{\underline{n}(t)} \underline{\delta}_k \le \frac{\underline{w}_i(t)}{t} \le \frac{1}{t}\sum_{k=1}^{\underline{n}(t)+1} \underline{\delta}_k \qquad \text{for } t > 0.$$

From renewal theory (cf. [39,45]) we have that $\lim_{t\to\infty} \underline{n}(t) = \infty$ with probability one and $\lim_{t\to\infty} \underline{n}(t)/t = 1/E\underline{\gamma}_2$ with probability one. It now readily follows from (4.1.16) and the strong law of large numbers for any $i \in X$,

$$(4.1.17) \quad \lim_{t\to\infty} \underline{w}_i(t)/t = E\underline{\delta}_2/E\underline{\gamma}_2 \text{ with probability one.}$$

The theorem now follows from (4.1.12), (4.1.15) and (4.1.17). This ends the proof.

A direct consequence of the theorems 4.1.2 - 4.1.5 is the next *main theorem.*

*Theorem 4.1.6.*

For any $i \in X$,

$$\lim_{t\to\infty} \frac{E\underline{w}_i(t)}{t} = \frac{\sum_{j\in I}\{k_1(j) - k_0(j)\}q_j}{\sum_{j\in I}\{t_1(j) - t_0(j)\}q_j}$$

and

$$\lim_{t\to\infty} \frac{\underline{w}_i(t)}{t} = \frac{\sum_{j\in I}\{k_1(j) - k_0(j)\}q_j}{\sum_{j\in I}\{t_1(j) - t_0(j)\}q_j} \quad \text{with probability one.}$$

*Remark 4.1.1.*

Using the fact that $\psi(i) \notin I$ for $i \in I$ (cf. (4.1.4)), it is easily verified that the functions $\chi(.)$ and $\lambda(.)$ do not depend on the values of the functions $\pi(i,.,.)$ and $c(i,.,.)$ for $i \in I$. Consequently (see the theorems 4.1.2 - 4.1.6), the average cost per unit time is independent of the values of the functions $\pi(i,.,.)$ and $c(i,.,.)$ for $i \in I$. This means that we may define the functions $\pi(i,.,.)$ and $c(i,.,.)$ for $i \in I$ in as convenient a manner as possible. This "flexibility" in the natural process may simplify the calculation of the functions $k_1(j) - k_0(j)$ and $t_1(j) - t_0(j)$.

*Remark 4.1.2.*

A close examination of the foregoing proofs and results shows that assumption 2 is only needed to establish the relation (4.1.5). However, using well-known results from Markov chain theory (cf. [7]), the relation (4.1.5) can also be deduced from assumption 4 and (4.1.1). We do not prove this. Thus the theorems 4.1.2 - 4.1.6 remain valid when we drop assumption 2, provided that we introduce assumption 4 at the place where assumption 2 was introduced. However, it seems that the foregoing proofs and results cannot be generalized to an arbitrary state space when assumption 2 is not imposed on the model.

*Remark 4.1.3.*

It is interesting to note that the quantity

$$g \overset{\text{def}}{=} [\sum_{j\in I} \{k_1(j) - k_0(j)\}q_j]/[\sum_{j\in I}\{t_1(j) - t_0(j)\}q_j$$

satisfies a countable set of linear equations. The following theorem can be proved (see also [11] and [33]).

*Theorem 4.1.7.*

Let $k_j = k_1(j) - k_0(j)$ and $t_j = t_1(j) - t_0(j)$ for $j \in X$.

(a) The countable set of linear equations in $(y, v_i, i \in X)$,

$$(4.1.18)\quad \begin{cases} v_i - yt_i - \sum_{j\in I} p_{ij}\, v_j = k_i, & i \in I, \\ v_i - \sum_{j\in I} p_{ij}\, v_j = 0\ , & i \notin I, \end{cases}$$

has a bounded solution.

(b) Let $(y, v_i)$ be a bounded solution of (4.1.18), then $y = g$.

(c) Let $h \in X$ be an arbitrary but fixed state, then (4.1.18) has under the condition $v_h = 0$ a unique bounded solution.

*Proof*

(a) For the proof of (a), we refer to [33, part I, pp. 42-47]. The proof given there exploits the fact that $\{\underline{I}_n\}$ satisfies the Doeblin condition.
(b) Let $(y, v_i)$ be a bounded solution of (4.1.18). Using the boundedness of the function $v_i$ and using (4.1.6) together with the fact that $q_j = 0$ for $j \notin I$, we obtain from (4.1.18) that

$$\sum_{i\in I} k_i q_i = \sum_{i\in I} \{v_i - y t_i - \sum_{j\in I} p_{ij} v_j\} q_i =$$

$$= \sum_{i\in I} v_i q_i - y \sum_{i\in I} t_i q_i - \sum_{j\in I} v_j q_j,$$

from which (b) follows.
(c) Let $(g, v_i)$ and $(g, v_i')$ be two bounded solutions of (4.1.18). Using the fact that $p_{ij} = 0$ for $j \notin I$, it follows from (4.1.18) that

$$v_i - v_i' = \sum_{j\in X} (v_j - v_j') p_{ij} \qquad \text{for all } i \in X.$$

Iterating this equality and using the boundedness of the function $v_i - v_i'$, we obtain for any $n \geq 1$ that

$$v_i - v_i' = \sum_{j\in X} (v_j - v_j') \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(k)} \qquad \text{for all } i \in X.$$

Taking the limit as $n \to \infty$ and using lemma 4.1.1, we obtain $v_i - v_i' = \sum_{j\in X} (v_j - v_j') q_j$ for all $i \in X$. Thus, for some constant c, $v_i = v_i' + c$ for all $i \in X$, from which (c) follows.

## 4.2. THE LONG-RUN AVERAGE COST FOR AN (s,S) INVENTORY MODEL

### *4.2.1. Model and preliminaries*

We consider the following inventory model with a single product and a single stocking point. Customers arrive at a stocking point at the epochs $\underline{s}_1, \underline{s}_2, \ldots$ where the interarrival times $\underline{s}_n - \underline{s}_{n-1}$ ($n = 1,2,\ldots;\ \underline{s}_0 = 0$) are independent and positive random variables with a common *geometric* probability distribution,

$$P\{\underline{s}_n-\underline{s}_{n-1} = t\} = p(1-p)^{t-1}, \qquad t = 1,2,\ldots;\ n = 1,2,\ldots ,$$

where $0 < p \leq 1$. Denote by $\underline{\xi}_n$ the demand size of the $n^{th}$ customer. The demand variables $\underline{\xi}_1, \underline{\xi}_2, \ldots$ are independent, non-negative and integral-valued random variables with a common probability distribution $\phi(j) = P\{\underline{\xi}_n = j\}$, $(j = 0,1,\ldots;\ n = 1,2,\ldots)$. It is assumed that

$$\phi(0) < 1 \text{ and } \mu = \sum_{j=1}^{\infty} j\phi(j) < \infty.$$

Moreover, the sequences $\{\underline{s}_n\}$ and $\{\underline{\xi}_n\}$ are assumed to be independent.

Excess demands are *backlogged.* An order may be placed only at the epochs $\underline{s}_n$. The ordering policy followed is an *(s,S) policy*, that is, when the stock on hand plus on order i is less than s, then S-i units are ordered; otherwise, no ordering is done. The numbers s and S are given integers with $S \geq s \geq 1$. The lead time of an order is a *fixed*, positive integer $\lambda$.

The costs involved are ordering costs, inventory costs and backorder costs. The costs are not discounted. The cost of ordering k units is $K\delta(k)$, where $K \geq 0$, $\delta(0) = 0$, and $\delta(k) = 1$ for $k \geq 1$. For any unit kept in stock inventory costs $c_{h1} + c_{h2}t$ are incurred when the unit is held in inventory for a time $t > 0$. For any unit backordered costs $c_{b1} + c_{b2}t$ are incurred when the backorder exists for a time $t > 0$. Note that, since the lead time is fixed and since $s \geq 1$, the backorder costs of any unit backordered are known in advance and, moreover, these costs will never exceed $c_{b1} + c_{b2}\lambda$.

We shall now give some preliminaries. For any $t = 0,1,\ldots,$ let

$$\underline{n}_t = \sup \{n | \underline{s}_n \leq t\}.$$

Note that $\underline{n}_0 = 0$. The random variable $\underline{n}_t$, $(t \geq 1)$, represents the number of customers arriving in [0,t]. Since the interarrival times $\underline{s}_n - \underline{s}_{n-1}$ have a geometric distribution, we have by a simple probabilistic argument (cf. the derivation of formula (2.1.23) of section 2.1),

$$P\{\underline{n}_t = j\} = \binom{t}{j} p^j (1-p)^{t-j}, \qquad j = 0,1,\ldots;\ t = 1,2,\ldots,$$

where we adopt the convention $\binom{a}{b} = 0$ if $b > a$. We review some well-known properties of the process $\{\underline{n}_t\}$. For any $t = 0,1,\ldots$, the random variable $\underline{s}_{\underline{n}_t+1} - t$ has also the geometric distribution $\{p(1-p)^{j-1}\}$(cf. section 1.2, p. 18). In words, the length of the time interval between t and the epoch of the first arrival occurring after t has the same geometric distribution as the interarrival times $\underline{s}_n - \underline{s}_{n-1}$. Moreover, the process $\{\underline{n}_t\}$ has stationary and independent increments.

Let $\phi^{(n)}(j)$ be the n-fold convolution of $\phi(j)$ with itself, i.e.,

$$\phi^{(n)}(j) = \sum_{k=0}^{j} \phi^{(n-1)}(k)\, \phi(j-k), \qquad j = 0,1,\ldots;\ n = 1,2,\ldots\ ,$$

where $\phi^{(0)}(0) = 1$ and $\phi^{(0)}(j) = 0$ for $j \geq 1$. Clearly, $\phi^{(n)}(j) = P\{\underline{\xi}_1 + \ldots + \underline{\xi}_n = j\}$. Let the renewal quantities m(j) and M(j) be defined by

$$m(j) = \sum_{n=1}^{\infty} \phi^{(n)}(j) \text{ and } M(j) = \sum_{k=0}^{j} m(k), \qquad j = 0,1,\ldots\ .$$

We have $m(j) = \phi(j) + \{\phi(j)\, m(0) + \ldots + \phi(0)\, m(j)\}$, $(j = 0,1,\ldots)$. Let

$$\underline{m}_k = \sup\, \{n \mid \underline{\xi}_0 + \ldots + \underline{\xi}_n \leq k\}, \qquad k = 0,1,\ldots\ ,$$

where $\underline{\xi}_0 = 0$. The random variable $\underline{m}_k$ represents the number of customers before the cumulative demand exceeds k. From renewal theory (cf. (1.2.3) of section 1.2),

$$(4.2.1) \qquad E\underline{m}_k = M(k), \qquad k = 0,1,\ldots\ .$$

For any $t = 0,1,\ldots$, let

$$\underline{a}(t) = \underline{\xi}_0 + \ldots + \underline{\xi}_{\underline{n}_t}.$$

The random variable $\underline{a}(t)$ represents the cumulative demand in the time interval $[0,t]$. For any $t = 0,1,\ldots$, let

$$(4.2.2) \qquad a_j(t) = P\{\underline{a}(t) = j\}, \qquad j = 0,1,\ldots\ .$$

We have $a_j(t) = \sum_{n=0}^{\infty} P\{\underline{a}(t) = j \mid \underline{n}_t = n\}\, P\{\underline{n}_t = n\}$, and hence

$$a_j(t) = \sum_{n=0}^{t} \phi^{(n)}(j) \binom{t}{n} p^n (1-p)^{t-n}, \qquad j = 0,1,\ldots;\ t = 0,1,\ldots\ .$$

Further we have

$$(4.2.3) \qquad E\underline{a}(t) = E\underline{\xi}_1 \cdot E\underline{n}_t = \mu t p, \qquad t = 0,1,\ldots .$$

For any $k = 1,2,\ldots$, let

$$(4.2.4) \qquad \underline{t}_k = \underline{s}_{\underline{m}_{k-1}+1} \text{ and } \underline{d}_k = \underline{\xi}_1 + \ldots + \underline{\xi}_{\underline{m}_{k-1}+1}.$$

In words, $\underline{t}_k$ is the length of the time interval from 0 to the epoch at which the cumulative demand exceeds k-1 for the first time and $\underline{d}_k$ is the cumulative demand in this time interval. Using (4.2.1) and Wald's equation, we obtain

$$(4.2.5) \qquad E\underline{t}_k = \frac{1}{p}\{1+M(k-1)\},\ E\underline{d}_k = \mu\{1+M(k-1)\}, \qquad k = 1,2,\ldots .$$

Since the event $\{\underline{t}_k \leq t\}$ occurs if and only if $\underline{a}(t) \geq k$, we have for any $k = 1,2,\ldots$ that

$$P\{\underline{t}_k \leq t\} = 1 - \sum_{j=0}^{k-1} a_j(t), \qquad t = 0,1,\ldots .$$

For any $k = 1,2,\ldots$, let

$$(4.2.6) \qquad d(j;k) = P\{\underline{d}_k = j\}, \qquad j = k,\ k+1,\ldots .$$

Using a standard argument from renewal theory, we have

$$(4.2.7) \qquad d(j;k) = \phi(j) + \sum_{n=1}^{\infty}\sum_{h=0}^{k-1} \phi^{(n)}(h)\ \phi(j-h) =$$

$$= \phi(j) + \sum_{h=0}^{k-1} \phi(j-h)\ m(h), \qquad j = k,\ k+1,\ldots;\ k = 1,2,\ldots .$$

Let the ι-function be defined by

$$\iota(x) = 1 \text{ for } x > 0, \text{ and } \iota(x) = 0 \text{ for } x \leq 0.$$

Then,

$$(4.2.8) \qquad E\iota(t-\underline{t}_k) = P\{\underline{t}_k < t\} = 1 - \sum_{j=0}^{k-1} a_j(t-1),\ k = 1,2,\ldots;\ t = 1,2,\ldots ,$$

and

$$(4.2.9) \qquad E\iota(\underline{t}_k - t) = P\{\underline{t}_k > t\} = \sum_{j=0}^{k-1} a_j(t), \qquad k = 1,2,\ldots;\ t = 0,1,\ldots .$$

Since the arrival process is "memoryless" and independent of the demands of the customers, we have by the theorem of total expectation that

$$(4.2.10) \qquad E\{(\underline{t}_k - t)\ \iota(\underline{t}_k - t)\} = \sum_{j=0}^{k-1} a_j(t)\ E\underline{t}_{k-j}, \quad k = 1,2,\ldots;\ t = 0,1,\ldots .$$

From this relation and the identity

$$E(\underline{t}_k - t) = E\{(\underline{t}_k - t)\ \iota(\underline{t}_k - t)\} + E\{(\underline{t}_k - t)\ \iota(t - \underline{t}_k)\},$$

it follows that for all $k = 1,2,\ldots$ and $t = 0,1,\ldots$ ,

$$(4.2.11) \qquad E\{(t - \underline{t}_k)\ \iota(t - \underline{t}_k)\} = \sum_{j=0}^{k-1} a_j(t)\ E\underline{t}_{k-j} + t - E\underline{t}_k .$$

*4.2.2. The long-run average cost for the (s,S) model*

We shall first define a natural process. Let the state space $X$ be defined by

$$X = \bigcup_{n=0}^{\lambda} X_n,$$

where

$$X_0 = \{i \mid i \text{ integer},\ i \leq S\},$$

and, for $n = 1,\ldots,\lambda$,

$$X_n = \{(i_0, i_1, \lambda_1, \ldots, i_n, \lambda_n) \mid i_0 \text{ integer};\ i_k \text{ and } \lambda_k \text{ positive integers for } k = 1,\ldots,n;$$

$$i_0 + i_1 + \ldots + i_n \leq S;\ 1 \leq \lambda_1 < \ldots < \lambda_n \leq \lambda;\ i_0 + i_1 + \ldots + i_n = S \text{ if } \lambda_n = \lambda\}.$$

We shall next define the distribution functions $F_x(t)$ and the transition probabilities $\pi(x,y,t)$ which determine the natural process. For any $x \in X$, we define $F_x(t)$ to be the distribution function of the geometric probability distribution $\{p(1-p)^{j-1}\}$. To define the transition probabilities $\pi(x,y,t)$, we fix a state $x \in X$ and a positive integer t. Let $\phi(j) = 0$ for $j < 0$. Distinguish between the following cases.

(a) $x \in X_0$. Then

$$\pi(x,y,t) = \begin{cases} \phi(x-y), & y \in X_0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $x = (i_0,i_1,\lambda_1,\ldots,i_n,\lambda_n) \in X_n$ with n fixed and $1 \leq n \leq \lambda$.

(b1) $t \geq \lambda_n$. Then

$$\pi(x,y,t) = \begin{cases} \phi(i_0+i_1+\ldots+i_n-y), & y \in X_0, \\ 0, & \text{otherwise.} \end{cases}$$

(b2) $\lambda_h \leq t < \lambda_{h+1}$ for some h with $1 \leq h \leq n-1$. Then

$$\pi(x,y,t) = \begin{cases} \phi(i_0+i_1+\ldots+i_h-k), & y = (k,i_{h+1},\lambda_{h+1}-t,\ldots,i_n,\lambda_n-t) \in X_{n-h}, \\ 0, & \text{otherwise.} \end{cases}$$

(b3) $t < \lambda_1$. Then

$$\pi(x,y,t) = \begin{cases} \phi(i_0-k), & y = (k,i_1,\lambda_1-t,\ldots,i_n,\lambda_n-t) \in X_n, \\ 0, & \text{otherwise.} \end{cases}$$

In words we can now describe the natural process as follows. The state space of the natural process is given by $X$. At the epochs $\underline{s}_0,\underline{s}_1,\underline{s}_2,\ldots$ the natural process is observed to be in one of the states of $X$. The state $i \in X_0$ corresponds to the situation in which the stock on hand is i and no orders are outstanding. The state $(i_0,i_1,\lambda_1,\ldots,i_n,\lambda_n)$ corresponds to the situation in which the stock on hand is $i_0$ and n orders are outstanding, where the $k^{th}$ order consists of $i_k$ units and will be delivered $\lambda_k$ units of time hence. At each demand epoch the state of the natural process describes the situation just after the demand has occurred. In the natural process no orders are placed, but orders already outstanding at epoch 0 will be delivered in the course of the natural process. The transitions of the natural process occur in the following way. If the present state is state i, then, under the condition that the next customer arrives t units of time hence and demands for j units, the next state of the natural process is state i-j. If the present state of the natural process is state $(i_0,i_1,\lambda_1,\dots,i_n,\lambda_n)$, then, under the condition that the next customer arrives t units of time hence and demands for j units, the next state of the natural process is, respectively, state $i_0+i_1+\dots+i_n-j$ if $t \geq \lambda_n$, state $(i_0+i_1+\dots+i_h-j,i_{h+1},\lambda_{h+1}-t,\dots,i_n,\lambda_n-t)$ if $\lambda_h \leq t < \lambda_{h+1}$ for some h with $1 \leq h \leq n-1$, and state $(i_0-j,i_1,\lambda_1-t,\dots,i_n,\lambda_n-t)$ if $t < \lambda_1$.

Let us now define the cost function $c(.,.,.)$. We shall not give a formal definition of the function $c(.,.,.)$ as done for the probabilities $\pi(.,.,.)$, since the expression for the function $c(.,.,.)$ is complicated. We shall confine ourselves to a description in words as to how the costs are incurred in the natural process. First we agree in which way the inventory costs are charged. To do this, we distinguish between the linear cost $c_{h2}$ and the fixed cost $c_{h1}$. For any unit kept in stock for some time $t > 0$ during the time interval $(\underline{s}_n,\underline{s}_{n+1}]$, inventory costs $c_{h2}t$ are charged at epoch $\underline{s}_{n+1}$. For any unit which comes from a delivery occurring in the time interval $(\underline{s}_n,\underline{s}_{n+1}]$ and which, moreover, is held in stock, inventory costs $c_{h1}$ are charged at epoch $\underline{s}_{n+1}$. Now consider the backorder costs. For any unit backordered at epoch $\underline{s}_{n+1}$, backorder costs $c_{b1}+c_{b2}t$ are charged at epoch $\underline{s}_{n+1}$ if this backorder is subsequently filled by a delivery in the natural process which arrives t units of time hence; otherwise, i.e., if this backorder is not satisfied by a delivery in the natural process, backorder costs $c_{b1}+c_{b2}\lambda$ are charged at epoch $\underline{s}_{n+1}$. By this description the function $c(.,.,.)$ is determined unequivocally.

We shall now show that assumption 1 of section 4.1 is satisfied. For any $x \in X$, let

$$e(x) = \begin{cases} i & \text{if } x = i, \\ i_0+i_1+\dots+i_n & \text{if } x = (i_0,i_1,\lambda_1,\dots,i_n,\lambda_n). \end{cases}$$

That is, e(x) represents the stock on hand plus on order for state x. It will now be clear that assumption 1 is satisfied for the choice

$$A_0 = \{x | e(x) < s\}.$$

Let us now define the set $I$ and the decision mechanism $\psi$ which determine the decision process. Let

$$I = A_0.$$

The "decision mechanism" $\psi(.)$ and the decision cost function $d(.)$ are defined by

$$\psi(x) = \begin{cases} (i, S-i, \lambda) & \text{for } x = i \in X_0 \cap I, \\ (i_0, i_1, \lambda_1, \ldots, i_n, \lambda_n, S-e(x), \lambda) & \\ \qquad \text{for } x = (i_0, i_1, \lambda_1, \ldots, i_n, \lambda_n) \in X_n \cap I, \end{cases}$$

and

$$d(x) = K \qquad \text{for } x \in I.$$

Note that $e(\psi(x)) = S$ for all $x \in I$. Note also that the function $d(.)$ is bounded. The above defined decision process agrees with the behaviour of the (s,S) inventory system.

We shall now prove that the Markov chain $\{\underline{I}_n\}$ satisfies the assumption 2 of section 4.1. To do this, we choose an integer r such that (cf. (4.2.6))

$$P\{\underline{d}_{S-s+1} = S-r\} > 0.$$

It will be obvious that such an integer r exists. Note that $r < s$. Let

$$\rho = P\{\underline{t}_{S-s+1} \geq \lambda,\ \underline{d}_{S-s+1} = S-r\}.$$

Then, since $e(\psi(x)) = S$ for all $x \in I$ and the lead time is fixed,

$$(4.2.12) \qquad P\{\underline{I}_1 = r | \underline{I}_0 = x\} = \rho > 0 \qquad \text{for all } x \in I,$$

from which it follows that assumption 2 is satisfied.

To determine the functions $k_1(x) - k_0(x)$ and $t_1(x) - t_0(x)$, we choose

$$A_{01} = \{x \mid e(x) \leq 0\} \qquad \text{and } A_{02} = A_0 .$$

Using the definition of the cost function $c(.,.,.)$, using $e(x) \leq S$ for all $x \in X$ and the fact that the interarrival times $\underline{s}_n - \underline{s}_{n-1}$ have an expectation $1/p$, it follows that for any $x \in X$,

$$0 \leq t_0(x) \leq \frac{S}{p} \; ; \; 0 \leq k_0(x) \leq c_{h1}S + c_{h2}\frac{S(S+1)}{2p} + (S+\mu)(c_{b1}+c_{b2}\lambda),$$

from which it follows that assumption 3 of section 4.1 is satisfied. Let us next check the assumption 4 of section 4.1. From (4.2.12) it follows that assumption 4 (i) is satisfied. To verify assumption 4 (ii), we first note that the times between successive orderings in the decision process are independent random variables with the same distribution as $\underline{t}_{S-s+1}$. Further, the costs incurred between successive orderings are bounded by a random variable which has the same distribution as $K + c_{h1}S + c_{h2}S \cdot \underline{t}_{S-s+1} + (c_{b1}+c_{b2}\lambda)\underline{d}_{S-s+1}$. Using (4.2.12) and Wald's equation, it now follows that assumption 4 (ii) is satisfied.

We shall now determine the functions $k_1(x) - k_0(x)$ and $t_1(x) - t_0(x)$. From the definition of $t_0(x)$ (see p. 120) it follows that

$$t_0(x) = \begin{cases} 0 & \text{for } x \in I, \\ E\underline{t}_{e(x)-s+1} & \text{for } x \in X \text{ with } e(x) \geq s, \end{cases}$$

from which we get

$$(4.2.13) \qquad t_1(x) - t_0(x) = E\underline{t}_{S-s+1} \qquad \text{for all } x \in I.$$

The determination of the function $k_1(x) - k_0(x)$ is somewhat less simple. However, we shall find that this function depends only on $e(x)$. From the definition of the cost function $c(x,y,t)$, the choice of $A_{01}$ and from the definitions of the functions $k_0(x)$ and $k_1(x)$ (see p. 120 and p.124), the following will be clear after some reflection. In $k_0(x)$ and $k_1(x)$ the same term appears for the expected inventory costs for the $e(x)$ units which represent the stock on hand plus on order in state x. Also, in $k_0(x)$ and $k_1(x)$ the same term appears for the expected backorder costs for the e(x) units which represent the stock on hand plus on order in state x. In $k_1(x)$ but not in $k_0(x)$ the term

$$(4.2.14)\qquad \sum_{k=\max(e(x),0)+1}^{S} E\{(c_{h1}+c_{h2}(\underline{t}_k-\lambda))\ \iota(\underline{t}_k-\lambda)\}$$

appears for the expected inventory costs for the S-e(x) units of the order placed in state x. Further, in $k_1(x)$ but not in $k_0(x)$ the term

$$(4.2.15)\qquad \sum_{k=\max(e(x),0)+1}^{S} E\{(c_{b1}+c_{b2}(\lambda-\underline{t}_k))\ \iota(\lambda-\underline{t}_k)\}$$

appears for the expected backorder costs for the S-e(x) units of the order placed in state x. If $e(x) \geq 1$, then the expected backorder costs

$$(4.2.16)\qquad (c_{b1}+c_{b2}\lambda)(E\underline{d}_{e(x)}-e(x))$$

contribute to $k_0(x)$ but not to $k_1(x)$. Finally, the expected backorder costs

$$(4.2.17)\qquad (c_{b1}+c_{b2}\lambda)(E\underline{d}_S-S)$$

contribute to $k_1(x)$ but not to $k_0(x)$. Summarizing, we have

$$k_1(x) - k_0(x) = \begin{cases} K + (4.2.14) + (4.2.15) + (4.2.17) - (4.2.16), & x \in I;\ e(x) \geq 1, \\ K + (4.2.14) + (4.2.15) + (4.2.17), & x \in I;\ e(x) \leq 0. \end{cases}$$

From this and the formulas (4.2.8) - (4.2.11) we obtain after some straightforward calculations that

$$(4.2.18)\qquad k_1(x) - k_0(x) = K + \sum_{k=i+1}^{S} \{(c_{h2}+c_{b2}) \sum_{j=0}^{k-1} a_j(\lambda)\ E\underline{t}_{k-j} + c_{h1} \sum_{j=0}^{k-1} a_j(\lambda) + c_{b2}(\lambda-E\underline{t}_k) + c_{b1}(1 - \sum_{j=0}^{k-1} a_j(\lambda-1))\} + (c_{b1}+c_{b2}\lambda)(E\underline{d}_S-E\underline{d}_i-S+i),\ x \in I;\ e(x) = i \geq 1,$$

and

$$(4.2.19)\qquad k_1(x) - k_0(x) = K + \sum_{k=1}^{S} \{(c_{h2}+c_{b2}) \sum_{j=0}^{k-1} a_j(\lambda)E\underline{t}_{k-j}+c_{h1} \sum_{j=0}^{k-1} a_j(\lambda) + + c_{b2}(\lambda-E\underline{t}_k) + c_{b1}(1 - \sum_{j=0}^{k-1} a_j(\lambda-1))\} + + (c_{b1}+c_{b2}\lambda)(E\underline{d}_S-S) \text{ for } x \in I \text{ with } e(x) \leq 0.$$

We see that, although the determination of each of the functions $k_0(x)$ and $k_1(x)$ is very difficult, the difference $k_1(x) - k_0(x)$ can be determined in a simple way.

Finally, we need the stationary probability distribution $\{q_x, x \in X\}$ of the Markov chain $\{\underline{I}_n\}$. Since the functions $k_1(x) - k_0(x)$ and $t_1(x) - t_0(x)$ depend only on $e(x)$, it suffices to determine $\sum_{x \in B(j)} q_x$ for $j = s-1, s-2, \ldots$ (cf. theorem 4.1.6), where

$$B(j) = \{x | e(x) = j\}, \qquad j = s-1, s-2, \ldots .$$

Note that $B(j) \subset I$ for each $j$. Since $e(\psi(x)) = S$ for all $x \in I$, we have for any integer $j < s$ (cf. (4.2.4) and (4.2.6)),

$$P\{\underline{I}_1 \in B(j) | \underline{I}_0 = x\} = P\{e(\underline{I}_1) = j | \underline{I}_0 = x\} = d(S-j; S-s+1), \quad x \in I.$$

Hence $P\{\underline{I}_1 \in B(j) | \underline{I}_0 = x\}$ is independent of $x$ for $x \in I$, and so

$$(4.2.20)\qquad \sum_{y \in B(j)} q_y = d(S-j; S-s+1), \qquad j = s-1, s-2, \ldots .$$

From theorem 4.1.6, (4.2.13) and (4.2.20) it now follows that the average cost per unit time for the (s,S) policy is equal to

$$(4.2.21)\qquad g(s,S) = \frac{1}{E\underline{t}_{S-s+1}} \left[ \sum_{j=-\infty}^{s-1} \{k_1(j) - k_0(j)\}\, d(S-j;\, S-s+1) \right],$$

where the function $k_1(j) - k_0(j)$ is given by (4.2.18) and (4.2.19).

The counterpart of this formula (with $c_{h1} = 0$) for the case in which the customers arrive according to a Poisson process can be found in [34].

We end this section with considering the special case $\phi(1) = 1$, that is, each customer demands for one unit. Then

$$a_k(\lambda) = \binom{\lambda}{k} p^k(1-p)^{\lambda-k},\ E\underline{t}_k = \frac{k}{p},\ E\underline{d}_k = k,\ (k = 1,2,\ldots), \text{ and}$$

$$d(S-s+1;S-s+1) = 1.$$

Let R = s-1 and Q = S-s+1. The right-hand side of (4.2.21) simplifies to

$$\frac{pK}{Q} + \frac{(c_{h2}+c_{b2})}{Q} \sum_{k=R+1}^{R+Q} \sum_{j=0}^{k-1} (k-j)\ a_k(\lambda) + c_{b2}\ \{p\lambda - R - \frac{(Q+1)}{2}\} +$$

$$+ c_{h1} \sum_{k=R+1}^{R+Q} \sum_{j=0}^{k-1} a_j(\lambda) + c_{b1} \sum_{k=R+1}^{R+Q} \{1 - \sum_{j=0}^{k-1} a_j(\lambda-1)\}.$$

The counterpart of this expression (with $c_{h1} = 0$) for the case in which the customers arrive according to a Poisson process can be found in [21,34].

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ERRATA tract 40

| page | line | | |
|---|---|---|---|
| 5 | 6 below | for than | read then |
| 10 | 2 below | for $V_\alpha(i;R_j)$ | read $V_\alpha(j;R_j)$ |
| | 1 below | for $V_\alpha(i)$ | read $V_\alpha(j)$ |
| 11 | 11 | for $x_2$ | read $\underline{x}_2$ |
| 14 | 8 below | for non-decreasing | read non-increasing |
| 16 | 10 below | for $\underline{s}_n \geq t$ | read $\underline{s}_n > t$ |
| 25 | 1 | for $\rho^{(t)}(j,\alpha)$ | read $\rho^{(t)}(j;\alpha)$ |
| 26 | 2 | for section | read subsection |
| 42 | 12, 14 | for $-ci+d\lambda u$ | read $-ci$ |
| 51 | 10 | for $a_\alpha(s,S)$ | read $a_\alpha(S,S)$ |
| | 11 | for all s and S | read all S |
| 52 | 1 | for $M_\alpha(k)$ | read $M(k;\alpha)$ |
| 62 | 8 below | for $s_1-1 \leq k \leq S_1$ | read $s_1 \leq k \leq S_1$ |
| 72 | 2 below | for $[i_0-j_0,j_0]$ | read $[i_0-j_0,i_0]$ |
| 82 | 6 | for $\hat{m}(j;1)$ | read $\hat{m}(S;j;1)$ |
| 96 | 20 | for non-arithmetic | read lattice |
| | 21 | for lattice | read non-arithmetic |
| 120 | 4 below | for $\psi : I \to I$ | read $\psi : X \to X$ |
| 131 | 10 | for numbers for | read numbers that for |
| 134 | 9 | add Finally, assume that either $p<1$ holds or $\phi(0)>0$. | |
| 139 | 10 below | for t | read $t>0$ |
| 140 | 2 below | add Note that $\rho>0$, since either $p<1$ holds or $\phi(0)>0$. | |

ADDENDA tract 40

page 52, line 5 below

*Lemma 2.2.2(f).* If $G_\alpha(s^*-1) \geq a_\alpha^* \geq G_\alpha(s^*)$ and if $K > 0$, then $s^* \leq s_1'$, where $s_1'$ is the smallest integer for which $G_\alpha(s_1') \leq G_\alpha(S_0) + (1-\alpha)K$.

*Proof.* If $\alpha = 1$, then $s_1' = S_0$ and so $s^* \leq s_1'$. If $\alpha < 1$, then the assertion follows from $G_\alpha(s^*-1) \geq a_\alpha^* > G_\alpha(S_0) + K(1 + \sum_1^\infty \alpha^n)^{-1}$ and $s^* \leq S_0$.

page 84, line 16

*Lemma 2.3.5(f).* The text is the same as that of lemma 2.2.2(f).