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Fundamentals of Partial Modal Logic

JAN JASPARS AND ELIAS THIJSSE

1 Introduction

This chapter gives a picture of the happy marriage of partial and modal logic. Rather than presenting a survey of the entire field[1], we focus on one type of partial modal logic. Here the models are the usual possible worlds structures (also known as Kripke models) where only the valuation function is partial. Validity can be characterized as relative truth. This paradigm is perhaps the most obvious way to combine the ideas of partiality and modality; most other approaches are variants of the one we are about to discuss. This style of partial modal logic is in our view perfectly suited for a whole range of applications. It should be noticed, however, that although only one way to do partial modal logic, our logic is still as flexible as normal modal logic, allowing for many different systems such as the "partial" counterparts of **K**, **S4** and **S5**.

Partiality and modality compared

The contributions of Langholm and Meyer and Van Der Hoek (this volume, Chapters 1 and 3, respectively), provide sufficient motivation for the two relevant phenomena: partiality and modality. We now highlight some of the resemblances and differences between these aspects.

Partiality and modality express two different dimensions of uncertainty with respect to a given piece of information. Partiality, the idea of not giving a (classical) truth value to every proposition, is at least different from modality in that it may not be reflected in the language: standard logical languages such as that of predicate calculus may be given a partial interpretation. So the basic inspiration for partial logic is *semantic*. On

[1]See the historical notes at the end for bibliographical information.

Partiality, Modality, and Nonmonotonicity
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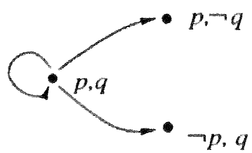

FIGURE 1 Partial model for $\Box(p \vee q)$

the other hand, modality, the idea of expressing the status of a proposition as *necessary* or *possible* (or some combination thereof), can and should be reflected in the logical language. Originally, modal logic was an entirely *syntactic* matter (cf. Lewis 1918). Through the work of Stig Kanger, Jaakko Hintikka and Saul Kripke in the late fifties, modal logic was finally given a set-theoretic semantics, using the idea of *possible worlds*. Since the language of partial logic has also been extended in order to deal with some of its peculiarities (see e.g. Langholm, Chapter 1), the difference between "syntactic" modality and "semantic" partiality is more or less cancelled, though it should be kept in mind that the basic motivation is different.

Due to the completeness results for elementary and modal logic, the distinction between syntax (deduction) and semantics (consequence) is, in some sense, a superficial one. More essential is the difference on the semantic level. Consider several cases of incomplete information:

Example 1.1 Assume you know p, but you don't know anything about q. Then a very simple model verifying this situation is the one in which only p is true: $\boxed{p}$. Indeed the partial model adequately represents the information contained in the non-modal formula p.

Example 1.2 Next suppose you know '$p \vee q$', without knowing p or knowing q. For example, you have been told that either Pat or Sue will come to your party, but not which of the two. A propositional model such as $\boxed{p}$ or $\boxed{q}$ now does not fit the situation, and in fact it can easily be shown that no such model exists: every propositional model that verifies $p \vee q$ either verifies p or verifies q, and therefore would imply too much knowledge. Here a Kripke model is called for, see figure 1.

These examples might indicate that partiality and modality are appropriate in distinct circumstances, but do not provide sufficient motivation for combining the two into one single system. Perhaps even the modal stance is the more general one?

Why combine partiality and modality?

It is true that the knowledge contained in example 1.1 could also be modeled in a possible world structure such as in Figure 2.

Yet it should be noted at this point that the two structures do not really

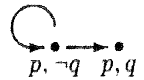

FIGURE 2 Partial Kripke model for p

model the same thing: the propositional structure $\boxed{p}$ models the plain *information* p, the above possible world structure the agent's *knowledge* $\Box p$. Apart from the gap between a simple proposition and the knowledge thereof, and differences of propositional theory (for example, whereas the latter model verifies $\neg p \vee q$, the former does not) the distinction is relatively harmless in the present case. But in general there is a clear difference between the two. This is not only clear from our second example, but particularly in cases where the two are compared (e.g. in $\Box p \Rightarrow p$), when some form of introspection is required (e.g. $\Box p \Rightarrow \Box\Box p$), and in cases involving several agents (e.g. $\Box_a \Box_b p$).

Still, this only proves the need for modal logic. Why partialize it? There are some very good reasons for doing so:

Naturalness We believe the partial approach to be much more intuitive, not only in its basic elements (partial worlds as mental reconstructions of the real world), but also in its behavior with respect to different phenomena. Partial Kripke-style semantics has been proposed for logic programming (see Przymusinski 1989 and Gelfond 1992), where a natural interpretation is provided for logic programs with explicit negation (Pearce and Wagner 1990) and disjunction (Minker 1982). See also, Busch's contribution in Chapter 2 of this volume, where both three-valued and intuitionistic logic are used as a basis for logic programming semantics.

Efficiency Classical possible world semantics leads to a combinatorial explosion: the less one knows, the bigger the model. For example, complete ignorance of n propositional variables leads to at least 2^n different worlds needed for a model that only represents what one knows. See Thijsse (1992) for a study of partial models that characterize a certain amount of knowledge. In addition, we refer to Levesque (1984) and Lakemeyer (1987) for tractable reasoning in other partial modal logics.

Flexibility Addition of (consistent) information can be accounted for in two ways: elimination of possible worlds and extension of propositional valuation. (see Jaspars 1991).

Adequacy Although classical and partial modal logic are similar to some extent, there are some interesting differences regarding the deductive systems. For a number of applications, such as natural language semantics (cf. Barwise 1981 and Muskens 1995) and awareness logic (see Thijsse, Chapter 8 of this volume), these differences make partial semantics preferable.

Partial and classical logic compared

Though quite similar to the modal system **K**, the partial modal logic **M** presented here is different in a number of ways.

Perhaps the most essential difference is that **M** has (almost) *no valid formulas*. This can be illustrated using the classical tautology $p \vee \neg p$: *tertium non datur* does not hold, since the formula is not true when p is undefined. As we will see, it can easily be shown that hardly any formula is always true, for the "empty" valuation, which leaves all propositional variables undefined, also produces a gap for every formula in some standard (modal) language.

Now where is the similarity between classical and partial logic? Notice that though (standard) formulas are not valid, *inference rules* may still apply. For example, the rule *ex contradictione sequitur quodlibet* (also known as 'ex falso') $\varphi \wedge \neg\varphi \Rightarrow \psi$ still holds: if the premise is true, so is the consequent (for this premise cannot be true). Indeed most classical rules transfer to partial (modal) logic. One noticeable exception is the rule of excluded middle: $\psi \not\Rightarrow \varphi \vee \neg\varphi$. Since the De Morgan property $\neg(\varphi \wedge \neg\varphi) \Leftrightarrow \varphi \vee \neg\varphi$ is valid, all this implies that contraposition does not hold.[2] Although this is a major departure from classical logic, we do not see any *a priori* reason why contraposition should be a necessary ingredient of a logical system.

The asymmetry in the inferential system, revealed by the absence of contraposition, has the effect that contraposition has to be explicitly stated in each case where it applies. For example, to formulate the (epistemic) logic of pre-ordered (reflexive + transitive) frames, the basic rule system **M** has to be extended with the rules:

(T) $\Box\varphi \Rightarrow \varphi,\ \varphi \Rightarrow \Diamond\varphi$
(4) $\Box\varphi \Rightarrow \Box\Box\varphi,\ \Diamond\Diamond\varphi \Rightarrow \Diamond\varphi$.

Such so-called frame completeness therefore usually involves pairs of dual rules. Exceptions are *ex falso*, of course, and self-dual rules such as $\Box\varphi \Rightarrow \Diamond\varphi$ (**D**) and $\Diamond\Box\varphi \Rightarrow \Box\Diamond\varphi$ (**G**), which appear as single rules. By turning to the weaker notion of model completeness, we will show in section 5 how to capture other single rules, such as $\Box\varphi \Rightarrow \varphi$.

For several applications, such as modelling explicit knowledge and belief, absence of contraposition is an advantage of partial modal logic, compared with classical modal logic. If $\Box\varphi$ is interpreted as 'knowing that φ', we would like to have the veridicality principle $\Box\varphi \Rightarrow \varphi$, saying that all

one (really) knows is true. Nevertheless $\varphi \Rightarrow \Diamond\varphi$ seems to be too strong in such an epistemic interpretation of modal operators. The latter rule says that anything which is true is considered possible by every agent. It would require full awareness of such agents to reason about the possibility of objective truths. Without this strong requirement, which clearly does not hold for conscious knowledge, $\varphi \Rightarrow \Diamond\varphi$ is no longer a tenable assumption. Before we return to this issue, we will define the language, semantics and inference system for partial modal logic.

2 Syntax and semantics

2.1 The modal language

As noticed in the introduction, the core language is just the usual one for modal logic. Let $\mathcal{P}$ be a (denumerable) set of propositional variables, and $\top, \neg, \wedge, \Box$ the logical constants. The language $\mathcal{L}$, the set of well-formed formulas over this vocabulary, is then defined recursively by:

- $\top \in \mathcal{L}$
- $\mathcal{P} \subseteq \mathcal{L}$
- if $\varphi \in \mathcal{L}$ then $\neg\varphi \in \mathcal{L}$ and $\Box\varphi \in \mathcal{L}$
- if $\varphi, \psi \in \mathcal{L}$ then $(\varphi \wedge \psi) \in \mathcal{L}$
- no other elements occur in $\mathcal{L}$ than those produced by the clauses above

Or, in concise BNF-style: (where $p \in \mathcal{P}$)

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \Box\varphi \mid (\varphi \wedge \varphi')$$

This is the basic language, which can be extended in two ways: (i) with symbols that serve as abbreviations and (ii) with new symbols. As for (i), the following definitions apply:

- $\bot = \neg\top$
- $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$
- $\Diamond\varphi = \neg\Box\neg\varphi$

As for (ii), the language can be extended with the exclusion negation $\sim$, resulting in $\mathcal{L}_{\sim}$. In addition, constants may be removed from the language; for example, removing $\top$ results in $\mathcal{L}_{-\top}$.

Finally, the following notation will be useful. The conjunction (disjunction) over all elements of a finite set Γ will be written as $\bigwedge\Gamma$ ($\bigvee\Gamma$, respectively); $\bigwedge\emptyset = \top$ and $\bigvee\emptyset = \bot$. If c is a unary logical constant (for example $\neg, \Box, \Diamond$) then $c\Gamma = \{c\varphi \mid \varphi \in \Gamma\}$ and $c^{-}\Gamma = \{\varphi \mid c\varphi \in \Gamma\}$.

2.2 Partial models and their properties

In partial modal semantics one often speaks of 'partial worlds' (or 'coherent situations') rather than of possible worlds; the valuation is supposed to semantics (cf. Barwise 1981 and Muskens 1995) and awareness logic (see Thijsse, Chapter 8 of this volume), these differences make partial semantics preferable.

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[2] See Blamey (1986). See also Langholm, Chapter 1 of this volume.

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2.2 Partial models and their properties

In partial modal semantics one often speaks of 'partial worlds' (or 'coherent situations') rather than of possible worlds; the valuation is supposed to manifest itself in the worlds. However, we usually refer to these indices as 'worlds' *simpliciter*. As in classical possible world semantics, a *partial Kripke model* M is a triple $\langle W, R, V\rangle$, where W is the set of worlds, $R \subseteq W \times W$ an *accessibility relation* between worlds, and the valuation V is now a *partial* function from $\mathcal{P} \times W$ into the set of classical truth values $\{0,1\}$. The graph $\langle W, R\rangle$ is called the *frame* of the model.

Since a formula may be neither true nor false, the meaning of logical constants cannot be spelled out by *truth conditions* alone, and *falsity conditions* have to be added: apart from the restriction of coherence, truth and falsity are independent. Here the partial truth relation between worlds and formulas is denoted by $\models$, the falsity relation by $\dashv$.[3] The truth and falsity conditions for $\mathcal{L}$ are recursively defined by: (for arbitrary $M = \langle W, R, V\rangle$, $w \in W$, $p \in \mathcal{P}$, and $\varphi, \psi \in \mathcal{L}$)

$M, w \models \top$ $\qquad$ $M, w \not\dashv \top$
$M, w \models p \Leftrightarrow V(p, w) = 1$ $\qquad$ $M, w \dashv p \Leftrightarrow V(p, w) = 0$
$M, w \models \neg\varphi \Leftrightarrow M, w \dashv \varphi$ $\qquad$ $M, w \dashv \neg\varphi \Leftrightarrow M, w \models \varphi$
$M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi$ and $M, w \models \psi$
$\quad M, w \dashv \varphi \wedge \psi \Leftrightarrow M, w \dashv \varphi$ *or* $M, w \dashv \psi$
$M, w \models \Box\varphi \Leftrightarrow \forall v : wRv \rightarrow M, v \models \varphi$
$\quad M, w \dashv \Box\varphi \Leftrightarrow \exists v : wRv$ and $M, v \dashv \varphi$

From these Tarskian conditions the truth and falsity clauses for the other connectives immediately follow.

$M, w \not\models \bot$ $\qquad$ $M, w \dashv \bot$
$M, w \models \varphi \vee \psi \Leftrightarrow M, w \models \varphi$ *or* $M, w \models \psi$
$\quad M, w \dashv \varphi \vee \psi \Leftrightarrow M, w \dashv \varphi$ and $M, w \dashv \psi$
$M, w \models \Diamond\varphi \Leftrightarrow \exists v : wRv$ and $M, v \models \varphi$
$\quad M, w \dashv \Diamond\varphi \Leftrightarrow \forall v : wRv \Rightarrow M, v \dashv \varphi$

M *verifies* (or *satisfies, supports*) φ in w whenever $M, w \models \varphi$. M *falsifies* (or *rejects*) φ in w whenever $M, w \dashv \varphi$. When $M, w \not\models \varphi$ and $M, w \not\dashv \varphi$, φ is said to be *undefined* on M in w. If $\Gamma \subseteq \mathcal{L}$ then $M, w \models \Gamma$ means that for all $\gamma \in \Gamma$: $M, w \models \gamma$.
A number of observations may help to gain some insight into this matter.

Semantic properties

First, notice that the given form of partial semantics is a correct generalization of classical semantics, since a partial model is a genuine Kripke model when V is a total function.

Observation 2.1 (Classicality) *If $M = \langle W, R, V\rangle$ is bivalent, i.e. $V : \mathcal{P} \times W \longrightarrow \{0,1\}$, then $M, w \models \varphi \Leftrightarrow M, w \not\dashv \varphi$ for all $\varphi \in \mathcal{L}$.*

(The proof is by induction on the structure of φ.)

[3]Alternative notations for $\models$ and $\dashv$ are: $\models^{+}$ and $\models^{-}$, $\models_T$ and $\models_F$, and $\models$ and $\dashv$.

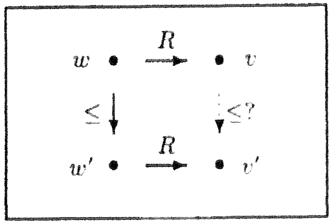

FIGURE 3 Propositional extension vs accessibility

One property for classical models is even transferred to the given partial semantics: the assignment of truth or falsity to a formula still takes place consistently.

Observation 2.2 (Coherence) *For no M, w, φ : $M, w \models \varphi$ and $M, w \dashv \varphi$.*

To avoid the impression that partial logic is not really that much different from classical logic, we note a simple but general case of undefinedness:

Observation 2.3 (Partiality) *There is a model M and a world w_0 such that for all formulas $\varphi \in \mathcal{L}_{-\top}$: $M, w_0 \not\models \varphi$ and $M, w_0 \not\dashv \varphi$.*

Proof. Consider a "no information" model with a single (self-accessible) empty world: let $W = \{w_0\}$, $w_0 R w_0$ and $V(p, w_0)$ be undefined for all $p \in \mathcal{P}$. The observation then follows by induction on the structure of φ. □

So there are no so-called *free* formulas (built up from the given constants, except for $\top$) which are valid. In other words, all tautologies essentially derive from $\top$.

A final important property of partial semantics is related to extension of information, i.e. further specification of truth values for propositional variables. So, we might introduce a relation of (propositional) extension $\leq$ by: $w_M \leq w'_{M'} \Leftrightarrow V'(p, w') = V(p, w)$ for $p \in \mathcal{P}$ such that $V(p, w)$ is defined. This definition, however, does not suit the modal language very well, for we would like this extension to hold for arbitrary modal formulas (so-called *persistence*.)

To see that $\leq$ does not produce the desired persistence, suppose for simplicity's sake that we have a model in which $w \models \Box p$, $w \leq w'$ and there is only one v such that wRv. So $v \models p$. Then for any v' which is R-accessible from w', $v' \models p$ should hold, but nothing forces $v \leq v'$: there is no compelling reason why the relations in the diagram of Figure 3 should commute.

One way to solve this is to strengthen the relation of extension. For example, we may define a *global* extension of one model to another based on the same frame. I.e., if $M = \langle W, R, V\rangle$, $M' = \langle W, R, V'\rangle$ and $w_M \leq w_{M'}$ for all $w \in W$, then by induction for all formulas φ:

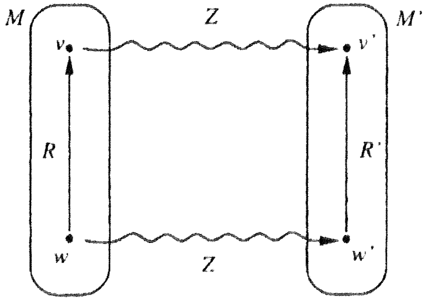

FIGURE 4 Bisimulation diagram

$$M, w \models \varphi \Rightarrow M', s \models \varphi \text{ and } M, w \dashv \varphi \Rightarrow M', s \dashv \varphi$$

A more flexible solution, also useful for *internal* persistence which fails in the above counter-example, is to move to the more subtle relation of bisimulation extension. This notion is in between the entirely local propositional extension and the fully global extension. The intermediate notion relates two worlds if their respective (n-step) alternatives are related by propositional extension. To be more precise, we reintroduce the notion of bisimulation.

Definition 2.1 A relation $Z \subseteq W \times W'$ is a *bisimulation* (or 'zigzag connection') between two frames $\langle W, R\rangle$ and $\langle W', R'\rangle$ iff

1. $(wZw'$ and $wRv) \Rightarrow \exists v' : (vZv'$ and $w'R'v')$ for all $w, v \in W, w' \in W'$
2. $(wZw'$ and $w'R'v') \Rightarrow \exists v : (vZv'$ and $wRv)$ for all $w \in W, w', v' \in W'$

This definition may be clarified by reformulating the conditions in relational algebra. If the relational composition $R_1 \bullet R_2$ denotes $\{\langle x, y\rangle \mid \exists z : xR_1z \text{ and } zR_2y\}$, and the reversal $Z^- = \{\langle y, x\rangle \mid \langle x, y\rangle \in Z\}$, then the above conditions amount to: $Z^- \bullet R \subseteq R' \bullet Z^-$ and $Z \bullet R' \subseteq R \bullet Z$. The back-and-forth nature of bisimulation can be seen as two ways to draw the arrows in the diagram of Figure 4.

Bisimulation or modal extension can now be structurally defined.

Definition 2.2 (Modal extension $\sqsubseteq$) Let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be partial Kripke models. Then $w_M \sqsubseteq w'_{M'}$ (i.e. w in M is *extended* to w' in M') iff there exists a bisimulation Z between $\langle W, R\rangle$ and $\langle W', R'\rangle$ such that wZw' and

$$\forall v \in W, v' \subset W' : vZv' \Rightarrow v_M \leq v'_{M'}.$$

When $M = M'$, $w_M \sqsubseteq w'_M$ is written as $w \sqsubseteq_M w'$.

Observation 2.4 *In general, $\sqsubseteq$ is a preorder, i.e. a reflexive and transitive relation, on the class of model-world pairs. This follows from three facts: the identity relation $xZy \Leftrightarrow x = y$ is a bisimulation on M, the composition of two bisimulations is a bisimulation, and $\leq$ is a preorder. Consequently, for a given model M, $\sqsubseteq_M$ is a preorder on W.*

Now for bisimulation extension we have the desired preservation property:

Observation 2.5 (Persistence) *If $w_M \sqsubseteq w'_{M'}$ then $M, w \models \varphi \Rightarrow M', w' \models \varphi$ for all $\varphi \in \mathcal{L}$.*[4]

(As usual in partial logic, the proof proceeds by simultaneous induction over the claim above together with $M, w \dashv \varphi \Rightarrow M', w' \dashv \varphi$.)

Without going very deeply into this matter, we notice that bisimulations between models, where Z is a frame-bisimulation such that wZw' implies $V(w) = V(w')$, yield an important definability result. Bisimulation invariance is considered to be *the* characteristic of the modal language, compared to the fragment of the first order language which contains translations of modal formulas, based on the standard truth and falsity conditions. The latter translations, and therefore the definability result, also have practical significance. For example, a similar formalization of the semantic meta-language has been used in one of the first studies of knowledge in AI, Moore (1980).

Logical consequence

So far, we have only defined the models and their properties, but not the notion of logical consequence. The notion of valid consequence that we consider particularly intuitive and useful for applications is what is called *strong consequence*, which amounts to relative verification (of one of the conclusions, given the truth of the premises). In general the class of relevant models matters. Let M be the class of partial Kripke models, and C be an arbitrary subclass of M.

Definition 2.3 (Strong consequence $\models$) $\Gamma \models_C \Delta$ iff for all $M \in C$ and w in M: if $M, w \models \Gamma$ then $M, w \models \delta$ for some $\delta \in \Delta$.

Here are a number of typical examples, some of which have already been mentioned in the introduction: (by default, $\models_M$ is written as $\models$)

- $\emptyset \models \top$ ($\models \top$ for short) and $\models \Box\top$, but $\not\models \Diamond\top$
- $\varphi, \neg\varphi \models \psi$, but $\varphi \not\models \psi, \neg\psi$
- $\Diamond(\varphi \wedge \neg\varphi) \models \psi$, but $\varphi \not\models \Box(\psi \vee \neg\psi)$
- $\Box\varphi \wedge \Box\psi \models \Box(\varphi \wedge \psi)$ and $\Diamond(\varphi \vee \psi) \models \Diamond\varphi \vee \Diamond\psi$
- $\Box\varphi \wedge \Diamond\psi \models \Diamond(\varphi \wedge \psi)$ and $\Box(\varphi \vee \psi) \models \Diamond\varphi \vee \Box\psi$

[4]The converse result can also be shown when the models are finite, cf. Thijsse (1992).

3 Derivation

We will use a sequential formulation for deduction in partial modal logic. The derivability relation $\vdash$ is a relation between two sets of formulas. $\Gamma \vdash \Delta$ denotes that Δ is derivable from Γ. Such an expression is called a *sequent*.[5]

3.1 Sequential rules for M

The basic logic corresponding to strong consequence on arbitrary partial Kripke models is called **M**. System **M** is triggered by the following sequential rules. The only structural rules are the START rule and two monotonicity rules, L-MON and R-MON. Furthermore the CUT rule is present.

$\Gamma \cap \Delta \neq \emptyset \Rightarrow \Gamma \vdash \Delta$ (START)	
$\dfrac{\Gamma \vdash \Delta \quad \Gamma \subseteq \Gamma'}{\Gamma' \vdash \Delta}$ (L-MON)	$\dfrac{\Gamma \vdash \Delta \quad \Delta \subseteq \Delta'}{\Gamma \vdash \Delta'}$ (R-MON)
$\dfrac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta}$ (CUT)[6]	

As usual, the introduction rules for the logical constants are not only separated into left and right rules, but due to partiality now also into TRUE and FALSE rules.

	$\Gamma \vdash \top, \Delta$ (R-TRUE $\top$)
$\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}$ (L-TRUE $\neg$)	
$\dfrac{\Gamma, \varphi \vdash \Delta}{\Gamma, \neg\neg\varphi \vdash \Delta}$ (L-FALSE $\neg$)	$\dfrac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \neg\neg\varphi, \Delta}$ (R-FALSE $\neg$)
$\dfrac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}$ (L-TRUE $\wedge$)	$\dfrac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}$ (R-TRUE $\wedge$)
$\dfrac{\Gamma, \neg\varphi \vdash \Delta \quad \Gamma, \neg\psi \vdash \Delta}{\Gamma, \neg(\varphi \wedge \psi) \vdash \Delta}$ (L-FALSE $\wedge$)	$\dfrac{\Gamma \vdash \neg\varphi, \neg\psi, \Delta}{\Gamma \vdash \neg(\varphi \wedge \psi), \Delta}$ (R-FALSE $\wedge$)
$\dfrac{\Gamma, \neg\varphi \vdash \neg\Delta}{\Box\Gamma, \neg\Box\varphi \vdash \neg\Box\Delta}$ (L-FALSE $\Box$)	$\dfrac{\Gamma \vdash \varphi, \neg\Delta}{\Box\Gamma \vdash \Box\varphi, \neg\Box\Delta}$ (R-TRUE $\Box$)

FALSE rules have a negation preceding the constant introduced. So every constant may have four sequential rules. Yet not all such rules are present: sometimes such absence hints at an essential departure of partial from classical logic: sometimes a (weaker) rule may be derivable, e.g. from the structural rules.

Definition 3.1 A set of formulas Δ is **M**-*derivable* from another set of formulas Γ whenever $\Gamma \vdash \Delta$, following a finite number of applications of (only) the rules above. The corresponding relation is denoted as $\Gamma \vdash_{\mathbf{M}} \Delta$.

Theorem 3.2 (Soundness of **M**) $\Gamma \vdash_{\mathbf{M}} \Delta \Rightarrow \Gamma \models_{\mathbf{M}} \Delta$

Proof. We have to show that the rules of **M** preserve valid consequence. To illustrate the procedure we check the R-TRUE $\Box$ rule. Let $\Gamma \models_{\mathbf{M}} \varphi, \neg\Delta$. This says that all worlds which support the premises (i.e. Γ) verify φ or falsify at least one of the elements of Δ. Suppose that $M = \langle W, R, V\rangle$ contains a world $w \in W$ which supports $\Box\Gamma$. So all $v \in W$ which are accessible from w support all elements of Γ. Because $\Gamma \models_{\mathbf{M}} \varphi, \neg\Delta$, we know that all accessible v must support φ, or there exists an accessible v which falsifies some element of Δ. In the former case we obtain $M, w \models \Box\varphi$, in the latter $M, w \models \neg\Box\delta$ for some $\delta \in \Delta$. Hence $\Box\Gamma \models_{\mathbf{M}} \Box\varphi, \neg\Box\Delta$. □

3.2 Some properties of M

The given sequent system clearly marks the difference with classical modal logic. This boils down to the absence of the R-TRUE $\neg$-rule

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}$$

Consequently, the law of excluded middle (*tertium non datur*) does not hold: $\not\vdash_{\mathbf{M}} \varphi \vee \neg\varphi$. Nevertheless there still are a lot of classically valid principles, such as De Morgan's laws, distributivity, associativity, commutativity, absorption and idempotence. Some important principles which will be used in this chapter are listed in Observation 3.1.

Observation 3.1

- $\Gamma \vdash_{\mathbf{M}} \Delta$ *and* Δ *finite* $\Rightarrow$ $\Gamma \vdash_{\mathbf{M}} \bigvee \Delta$
- $\Gamma \vdash_{\mathbf{M}} \Delta$ *and* Γ *finite* $\Rightarrow$ $\bigwedge \Gamma \vdash_{\mathbf{M}} \Delta$
- $\Gamma, \varphi \vdash_{\mathbf{M}} \Delta \Rightarrow \Box\Gamma, \Diamond\varphi \vdash_{\mathbf{M}} \Diamond\Delta$ (L-TRUE $\Diamond$)
- $\Gamma \vdash_{\mathbf{M}} \varphi, \Delta \Rightarrow \Box\Gamma \vdash_{\mathbf{M}} \Box\varphi, \Diamond\Delta$
- $\Diamond\bot \vdash_{\mathbf{M}} \emptyset$
- $\vdash_{\mathbf{M}} \Box\top$

The transformation of L-FALSE $\Box$ into L-TRUE $\Diamond$ will also be used frequently. Recall that some sequent rules are not included in the list of rules for **M** because they are redundant: for example (L-FALSE $\top$) $\Gamma, \neg\top \vdash \Delta$ is derivable (from R-TRUE $\top$ and L-TRUE $\neg$).

Another important property of **M** is its finiteness. This leads to what we will call the finiteness property throughout this chapter.

[5] To reduce the number of structural rules, the arguments of $\vdash$ are taken to be sets rather than sequences.

[6] Instead of $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \Delta$ we write Γ, φ and Γ, Δ respectively.

Observation 3.2 (Finiteness property) *If $\Gamma \vdash_M \Delta$ then there exist finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash_M \Delta'$.*

Since **M**-derivability is defined by making only a finite number of derivation steps, this can be proved by induction on the length of derivations.

3.3 Another sequential format

Instead of separating the introduction rules into TRUE and FALSE rules, we could also define two derivation relations, or use the four-place notation used by Langholm in Chapter 1 of this volume. In addition to true premises and true consequences, Langholm's *quadrants* contain two positions indicating falsity of other premises and consequences. So a sequent $\Gamma, \neg\Delta \vdash \Sigma, \neg\Theta$ (in our notation) is reformulated as

$$\begin{array}{c|c} \Gamma & \Sigma \\ \hline \Delta & \Theta \end{array}.$$

The obvious effect of the four-place notation is that the extra negations featuring in our FALSE-rules are eliminated in the quadrant notation. The price to pay is that in the latter notation one has to specify more positions, while departing from the traditional two-place notation.

An inference rule in the quadrant notation has the following format:

$$\begin{array}{c|c} \Gamma_1 & \Sigma_1 \\ \hline \Delta_1 & \Theta_1 \end{array} \text{ and } \dots \text{ and } \begin{array}{c|c} \Gamma_n & \Sigma_n \\ \hline \Delta_n & \Theta_n \end{array} \Rightarrow \begin{array}{c|c} \Gamma' & \Sigma' \\ \hline \Delta' & \Theta' \end{array}.$$

In our sequential format the rule is expressed as:

$$\frac{\Gamma_1, \neg\Delta_1 \vdash \Sigma_1, \neg\Theta_1 \quad \dots \quad \Gamma_n, \neg\Delta_n \vdash \Sigma_n, \neg\Theta_n}{\Gamma', \neg\Delta' \vdash \Sigma', \neg\Theta'}$$

For example, L-FALSE $\Box$ corresponds to the following four-place inference rule:

$$\begin{array}{c|c} \Gamma & \Sigma \\ \hline \Delta, \varphi & \Theta \end{array} \Rightarrow \begin{array}{c|c} \Box\Gamma & \Diamond\Sigma \\ \hline \Diamond\Delta, \Box\varphi & \Box\Theta \end{array}.$$

The L-TRUE $\neg$ can even be incorporated as a structural rule:

$$\begin{array}{c|c} \Gamma & \Sigma_1, \Sigma_2 \\ \hline \Delta & \Theta \end{array} \Rightarrow \begin{array}{c|c} \Gamma & \Sigma_2 \\ \hline \Delta, \Sigma_1 & \Theta \end{array}.$$

An advantage of this four-place variant is that the other introduction rules of system **M** show a nice symmetry. For example, L-TRUE $\wedge$ and R-FALSE $\wedge$ reappear as "north-west" and "south-east" rules:

$$\begin{array}{c|c} \Gamma, \varphi, \psi & \Sigma \\ \hline \Delta & \Theta \end{array} \Rightarrow \begin{array}{c|c} \Gamma, \varphi \wedge \psi & \Sigma \\ \hline \Delta & \Theta \end{array} \quad \text{and} \quad \begin{array}{c|c} \Gamma & \Sigma \\ \hline \Delta & \Theta, \varphi, \psi \end{array} \Rightarrow \begin{array}{c|c} \Gamma & \Sigma \\ \hline \Delta & \Theta, \varphi \wedge \psi \end{array}$$

It is a matter of taste which notation is preferable—we will stick to the explicit TRUE-FALSE distinction in sequential rules.

4 Saturation, completeness and decidability

In classical logic the notion of a *maximally consistent* set is an essential ingredient of Henkin-style completeness proofs. This is also the case for classical modal logic. Such a set of formulas is *consistent* with respect to the underlying logic, and it is *maximal* in the sense that it does not have consistent extensions. Maximally consistent sets enable us to get a grip on the semantic units, such as worlds in modal logic or interpretations and assignments in predicate logic. In other words, maximally consistent sets relate syntax and semantics. Completeness is then derived by making two steps.

One of these steps guarantees that a maximally consistent set verifies the formulas which are contained by it. This ensures that such a set does indeed behave like a world. This result is called the *truth lemma*.

The other step, which is normally made first, is called the *Lindenbaum Lemma*, and states that every consistent set can be extended to a maximally consistent set. The proof of this result proceeds by adding as much information to a given consistent set as possible, i.e. without losing consistency.

Then the general strategy of the Henkin completeness proof is as follows. In order to show that $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$, we prove that $\Gamma \nvdash \varphi \Rightarrow \Gamma \not\models \varphi$. So assume that φ is not derivable from Γ. Then, still in classical logic, $\Gamma \cup \{\neg\varphi\}$ is consistent. By the Lindenbaum Lemma we know that there exists a maximally consistent extension of $\Gamma \cup \{\neg\varphi\}$, say Γ^*. The truth lemma tells us that Γ^* verifies all elements of Γ but not φ. Therefore, Γ^*, interpreted as a world, shows that φ is not a valid consequence of Γ.

Although the general structure of the Henkin completeness proof for partial logic is the same, there are a number of differences due to the absence of *tertium non datur*. For example, if $\Gamma \nvdash \varphi$ then $\Gamma \cup \{\neg\varphi\}$ may be inconsistent. To wit $p \nvdash q \vee \neg q$, but $\{p, \neg(q \vee \neg q)\}$ is inconsistent. Therefore we need a generalization of the Lindenbaum Lemma: if $\Gamma \nvdash \varphi$ then Γ can be extended to a canonical set Γ^* which does not include φ. Now what precisely are the canonical sets of partial logic?

4.1 Saturated sets

In partial logic maximal consistency is not the correct characteristic of the syntactic counterparts of worlds. In fact we have to be more liberal in accepting sets of formulas as valuations or worlds. It could be the case that a formula φ is neither verified nor falsified. So, if we want to imitate such a phenomenon by means of sets of formulas, then these canonical sets do not have to be maximally consistent. The following definition of *saturated sets* gives the correct syntactic analogue of partial worlds.

Definition 4.1 Let **S** be a sequential derivation system for a language $\mathcal{L}$.

- A set of formulas $\Gamma \subseteq \mathcal{L}$ is said to be **S**-*saturated* iff for all $\Delta \subseteq \mathcal{L}$: $\Gamma \vdash_{\mathbf{S}} \Delta \Rightarrow \Delta \cap \Gamma \neq \emptyset$.
- A set of formulas $\Gamma \subseteq \mathcal{L}$ is said to be **S**-*consistent* iff $\Gamma \nvdash_{\mathbf{S}} \emptyset$

It turns out that in classical logic saturation boils down to maximal consistency, so 'saturation' generalizes 'maximal consistency'. Before giving a reformulation of the notion of 'saturated set', we note that consistency can be expressed in many different ways, for example:

Observation 4.1 *If a sequent system* **S** *contains* R-MON *and uses language* $\mathcal{L}$, *then* $\Gamma \subseteq \mathcal{L}$ *is consistent iff* $\Gamma \nvdash \Delta$ *for some* $\Delta \subseteq \mathcal{L}$. *If* **S** *extends* **M** *then* Γ *is consistent* $\Leftrightarrow \Gamma \nvdash \bot \Leftrightarrow \Gamma \nvdash \varphi \wedge \neg\varphi$.

An alternative formulation of saturated sets uses a combination of perhaps more familiar concepts.

- Γ is a *theory* (deductively closed) iff $\Gamma \vdash \varphi$ implies $\varphi \in \Gamma$ for all φ.
- Γ is *prime* (or 'disjunction-saturated') iff for all φ and ψ, if $\Gamma \vdash \varphi \vee \psi$ then $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$.

Now notice that the three smallest cardinalities for Δ in the definition of saturated sets lead to three characteristic properties. Let Γ be saturated. If $\#\Delta = 0$, i.e. Δ is empty, then $\emptyset \cap \Gamma = \emptyset$ shows $\Gamma \nvdash_{\mathbf{S}} \emptyset$, i.e. Γ is consistent. To see that any saturated Γ is a theory, let $\Gamma \vdash \varphi$, then $\Gamma \cap \{\varphi\} \neq \emptyset$, so $\varphi \in \Gamma$ (in this case $\#\Delta = 1$). Finally, if $\Gamma \vdash \varphi \vee \psi$ then $\Gamma \vdash \varphi, \psi$, thus $\Gamma \cap \{\varphi, \psi\} \neq \emptyset$, so $\varphi \in \Gamma$ or $\psi \in \Gamma$, i.e. $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$, and therefore Γ is prime (here $\#\Delta = 2$). To summarize, a saturated set is a consistent prime theory. The converse also holds, for if Γ is a consistent prime theory then $\Gamma \vdash \Delta$ implies (by consistency and the finiteness property) that there are $\delta_1, \ldots, \delta_n \in \Delta$ such that $\Gamma \vdash \delta_1 \vee (\delta_2 \vee (\cdots \vee \delta_n) \cdots)$. By induction $\Gamma \vdash \delta_i$ for some $i \leq n$, and so $\delta_i \in \Gamma$.

Observation 4.2 *For any sequent system* **S** *that has the finiteness property and in which the conclusions function as a disjunction, a set is saturated iff it is a consistent prime theory.*

Saturated sets are smaller than (or equal to) maximally consistent sets. Indeed the construction of a maximally consistent set out of a given consistent set, as in the proof of the Lindenbaum Lemma for classical logic, usually does not involve limitations on "building-materials". Maximality is just the final stage of piling up arbitrary formulas, while preserving consistency. In partial logic the construction of saturated sets is often more restricted. As we will see in the completeness proof for **M**, saturated sets can be built into another set. The generalization of the Lindenbaum Lemma which we are about to give, guarantees a successful construction whenever this upper bound is rich enough to intersect all sequences which are derivable from a given set. Here is the formal implementation of this idea.

Definition 4.2 $\Lambda \subseteq \mathcal{L}$ is called an **S**-*saturator* of a set $\Gamma \subseteq \mathcal{L}$ iff for all $\Delta \subseteq \mathcal{L}$: $\Gamma \vdash_{\mathbf{S}} \Delta \Rightarrow \Delta \cap \Lambda \neq \emptyset$.

Observation 4.3 *Note that whenever a set* Γ *has an* **S**-*saturator it must be* **S**-*consistent:* $\Gamma \nvdash_{\mathbf{S}} \emptyset$. *Also note that a set is contained in all its saturators.*

The reader should keep in mind that a saturator does not have to be consistent itself, and thus need not be saturated either. A saturator merely helps to generate a saturated expansion.

The next result formally expresses the Lindenbaum-like result which we were looking for.

Lemma 4.3 (Generalized Lindenbaum Lemma, GLL) *Let* **S** *be a sequential derivation system in which sequences are taken to be sets. Suppose* **S** *contains the structural rules* START, L-MON, R-MON *and* CUT *and satisfies the finiteness property. If* $\Lambda \subseteq \mathcal{L}$ *is an* **S**-*saturator of* $\Gamma \subseteq \mathcal{L}$, *then* Λ *contains an* **S**-*saturated set* Γ^* *such that* $\Gamma \subseteq \Gamma^*$.

Proof. Let $\{\varphi_i\}_{i\in\omega}$ be an enumeration of Λ such that every element of Λ occurs infinitely many times. We define a sequence $\{\Gamma_i\}_{i\in\omega}$ such that Λ is a saturator of each Γ_i. The limit of $\{\Gamma_i\}_{i\in\omega}$ is the desired **S**-saturated Γ^*.

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if for all finite } \Delta \subseteq \mathcal{L}: \\ & \quad \Gamma_n, \varphi_n \vdash_{\mathbf{S}} \Delta \Rightarrow \Delta \cap \Lambda \neq \emptyset \\ \Gamma_n & \text{otherwise.} \end{cases}$$

$$\Gamma^* = \bigcup_{n\in\omega} \Gamma_n$$

As an immediate consequence of this definition, notice that

(1) if $\Gamma_n \vdash_{\mathbf{S}} \Delta$ then $\Delta \cap \Lambda \neq \emptyset$ for all finite $\Delta \subseteq \mathcal{L}$.

By the finiteness property this amounts to Λ being a saturator of each Γ_n, and so, by Observation 4.3, $\Gamma_n \subseteq \Lambda$, and therefore $\Gamma^* \subseteq \Lambda$.

To establish the saturation of Γ^* we prove by induction on the cardinality of $\Delta \cap \Lambda$ that

(2) $\Gamma_k \vdash_{\mathbf{S}} \Delta$ and Δ is finite $\Rightarrow \Delta \cap \Gamma^* \neq \emptyset$, for all $k \in \omega$.

- If $\Delta \cap \Lambda = \emptyset$ then $\Gamma_k \nvdash_{\mathbf{S}} \Delta$, because of (1). So (2) holds trivially for the basic step.
- Suppose (2) holds for all finite Δ such that $\#(\Delta \cap \Lambda) = n$ (the induction hypothesis). Now let Δ be a finite set with $\#(\Delta \cap \Lambda) = n+1$ and $\{\varphi_{k_1}, \ldots, \varphi_{k_{n+1}}\}$ be an enumeration of $\Delta \cap \Lambda$ such that $k \leq k_i$ for all $i \in \{1, \ldots, n+1\}$. The enumeration exists because all these formulas appear infinitely many times in $\{\varphi_i\}_{i\in\omega}$

immediately have the desired result. So, suppose that $\varphi_{k_{n+1}} \notin \Gamma^*$. Then there exists a finite set $\Delta' \subseteq \mathcal{L}$ such that

$$\Gamma_{k_{n+1}}, \varphi_{k_{n+1}} \vdash_S \Delta' \text{ and } \Delta' \cap \Lambda = \emptyset.$$

The CUT-rule entails

$$\Gamma_{k_{n+1}}, \Gamma_k \vdash_S \Delta', \Delta - \{\varphi_{k_{n+1}}\}.$$

Because $\Gamma_k \subseteq \Gamma_{k_{n+1}}$ we derive

$$\Gamma_{k_{n+1}} \vdash_S \Delta', \Delta - \{\varphi_{k_{n+1}}\}.$$

The induction hypothesis can be applied, because

$$\#((\Delta' \cup \Delta - \{\varphi_{k_{n+1}}\}) \cap \Lambda) = \#\{\varphi_{k_1}, \ldots, \varphi_{k_n}\} = n.$$

Thus $(\Delta' \cup \Delta - \{\varphi_{k_{n+1}}\}) \cap \Gamma^* \neq \emptyset$. Because $\Gamma^* \subseteq \Lambda$, we obtain $(\Delta' \cup \Delta - \{\varphi_{k_{n+1}}\}) \cap \Gamma^* \subseteq \{\varphi_{k_1}, \ldots, \varphi_{k_n}\}$, and therefore there is a $\varphi_{k_i} \in \Gamma^*$ for some $i \in \{1, \ldots, n\}$. So $\Delta \cap \Gamma^* \neq \emptyset$.

Because of the finiteness property of the derivation system **S**, we may conclude that if $\Gamma^* \vdash_S \Sigma$ then there exist finite subsets $\Gamma' \subseteq \Gamma^*$ and $\Sigma' \subseteq \Sigma$ such that $\Gamma' \vdash_S \Sigma'$. Thus, according to the definition of Γ^*, there exists an $n \in \omega$ such that $\Gamma_n \vdash_S \Sigma'$, and because of (2) we conclude $\Sigma' \cap \Gamma^* \neq \emptyset$, and therefore $\Sigma \cap \Gamma^* \neq \emptyset$. □

GLL turns out to be extremely useful in proving completeness results on the basis of canonical Henkin models, such as for **M**. The upper bound method also applies to completeness proofs for extensions of **M**, e.g. constructive logic with extra non-persistent connectives.

An important corollary of GLL is the (generalized) *saturation lemma*, which is in fact equivalent to GLL in every sequential system that contains R-MON.

Corollary 4.4 (Saturation lemma) *If* **S** *is a derivational system as in GLL and* $\Gamma \not\vdash_S \Delta$, *then there exists an* **S**-*saturated superset* Γ^* *of* Γ *such that* $\Gamma^* \cap \Delta = \emptyset$.

Proof. Notice that $\Gamma \not\vdash \Delta$ iff $\Delta^c = \mathcal{L} - \Delta$ is a saturator of Γ. For Δ^c is not a saturator of $\Gamma \Leftrightarrow \exists\Sigma : \Gamma \vdash \Sigma$ and $\Sigma \cap \Delta^c = \emptyset \Leftrightarrow \exists\Sigma : \Gamma \vdash \Sigma$ and $\Sigma \subseteq \Delta \Leftrightarrow$ (R-MON) $\Gamma \vdash \Delta$.
So, if $\Gamma \not\vdash_S \Delta$ then GLL shows that there is a saturated Γ^* such that $\Gamma \subseteq \Gamma^* \subseteq \Delta^c$ and therefore $\Gamma^* \cap \Delta = \emptyset$. Conversely, assume the saturation lemma and Λ to be a saturator of Γ. Then by the above remark $\Gamma \not\vdash \Lambda^c$, so there is a saturated $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \cap \Lambda^c = \emptyset$, i.e. $\Gamma^* \subseteq \Lambda$. □

One useful case of the saturation lemma is when Δ is a singleton set. If a formula $\varphi \in \mathcal{L}$ is not **S**-derivable from a set of assumptions Γ, then there exists a saturated extension of Γ that does not contain φ. In the following section we will show how these lemmas help us to obtain completeness for the partial system **M**.

4.2 Completeness

In classical modal logics the maximally consistent worlds are all put into one Kripke model: the so-called *canonical* or *Henkin* model. This is essentially what we will do with saturated sets in the case of partial modal logic. Apart from the obvious deviation from the classical case for the definition of the canonical valuation function, which now has to define both truth and falsity of propositional variables, the definition of the canonical accessibility relation is also different. In classical modal logic Δ is canonically accessible from Γ if $\Box^-\Gamma \subseteq \Delta$, or, equivalently, if $\Diamond\Delta \subseteq \Gamma$. In partial logic these conditions are not equivalent, yet both are required to arrive at an accessible saturated set.

Definition 4.5 The **M**-*canonical model* is the triple $M_{\mathbf{M}} = \langle W_{\mathbf{M}}, R_{\mathbf{M}}, V_{\mathbf{M}} \rangle$ where

$W_{\mathbf{M}}$ is the set of all **M**-saturated sets

$$\Gamma R_{\mathbf{M}} \Delta \Leftrightarrow \Box^-\Gamma \subseteq \Delta \subseteq \Diamond^-\Gamma$$

$$V_{\mathbf{M}}(p, \Gamma) = 1 \Leftrightarrow p \in \Gamma \text{ and } V_{\mathbf{M}}(p, \Gamma) = 0 \Leftrightarrow \neg p \in \Gamma$$

The definition of $V_{\mathbf{M}}$ ensures the basic step for the inductive proof of the truth lemma, $R_{\mathbf{M}}$ that of the induction step for the intensional connective $\Box$. The definition of $R_{\mathbf{M}}$ explicitly states that a saturated set accessible from Γ is contained in the upper bound $\Diamond^-\Gamma$. The intuitive idea behind this upper bound is the requirement that an accessible world should never contain more information than what is determined as being possible by the original world. GLL will guide us whenever we look for particular accessible saturated sets in the truth lemma. The essence of proving completeness for intensional partial systems on the basis of GLL often boils down to finding a suitable saturator.

Since $\Delta \subseteq \Diamond^-\Gamma$ is not equivalent to $\Box^-\Gamma \subseteq \Delta$, we need both the upper and the lower bound for canonical accessibility. Nevertheless, if only one of these conditions holds, we can prove that there is a subset or superset of Δ which is accessible from Γ in the canonical model.

Lemma 4.6 *Let* $\Gamma \in W_{\mathbf{M}}$ *and* $\Delta \in W_{\mathbf{M}}$.

If $\Box^-\Gamma \subseteq \Delta$ then there exists a $\Delta' \subseteq \Delta$ such that $\Gamma R_{\mathbf{M}} \Delta'$.

If $\Delta \subseteq \Diamond^-\Gamma$ then there exists a $\Delta' \supseteq \Delta$ such that $\Gamma R_{\mathbf{M}} \Delta'$.

Proof. Let $\Gamma, \Delta \in W_{\mathbf{M}}$. For the first implication assume $\Box^-\Gamma \subseteq \Delta$. We will show that $\Delta \cap \Diamond^-\Gamma$ is an **M**-saturator of $\Box^-\Gamma$. Then because of GLL there is a Δ' such that $\Box^-\Gamma \subseteq \Delta' \subseteq \Delta \cap \Diamond^-\Gamma$. We will prove that for all finite $\Sigma \subseteq \mathcal{L}$

$$(3)\ \Box^-\Gamma \vdash_{\mathbf{M}} \Sigma \Rightarrow \Sigma \cap \Delta \cap \Diamond^-\Gamma \neq \emptyset$$

The finiteness property of $\mathbf{M}$ then guarantees that $\Delta \cap \Diamond^{-}\Gamma$ is an $\mathbf{M}$-saturator of $\Box^{-}\Gamma$.

Suppose $\Box^{-}\Gamma \vdash_{\mathbf{M}} \Sigma$ for finite Σ. Because $\Box^{-}\Gamma \subseteq \Delta$ and $\Delta \in W_{\mathbf{M}}$, we certainly have $\Sigma \cap \Delta \neq \emptyset$. We split Σ into a Δ-part, and a non-Δ part: $\Box^{-}\Gamma \vdash_{\mathbf{M}} \Sigma - \Delta, \Sigma \cap \Delta$, and define $\sigma := \bigvee(\Sigma - \Delta)$. Application of R-TRUE $\Box$ and L-MON yields:

$$\Gamma \vdash_{\mathbf{M}} \Box\sigma, \Diamond(\Delta \cap \Sigma).$$

Because $\Gamma \in W_{\mathbf{M}}$, we obtain $\Box\sigma \in \Gamma$ or $\Diamond(\Delta \cap \Sigma) \cap \Gamma \neq \emptyset$. The former disjunct contradicts $\Box^{-}\Gamma \subseteq \Delta$, because $\sigma \notin \Delta$ ($\Delta \in W_{\mathbf{M}}$). This means the latter disjunct should hold, which is just a reformulation of the desired conclusion in (3).

For the second implication assume that $\Delta \subseteq \Diamond^{-}\Gamma$. We will show that $\Diamond^{-}\Gamma$ is an $\mathbf{M}$-saturator of $\Box^{-}\Gamma \cup \Delta$, and so by GLL there is a Δ' such that $\Box^{-}\Gamma \cup \Delta \subseteq \Delta' \subseteq \Diamond^{-}\Gamma$.

Suppose $\Box^{-}\Gamma, \Delta \vdash_{\mathbf{M}} \Sigma$. By the finiteness property there exists a finite $\Delta' \subseteq \Delta$ such that $\Box^{-}\Gamma, \Delta' \vdash_{\mathbf{M}} \Sigma$. So, if $\delta = \bigwedge\Delta'$, $\mathbf{M}$ entails $\Box^{-}\Gamma, \delta \vdash_{\mathbf{M}} \Sigma$. L-TRUE $\Diamond$ and L-MON entail

$$\Gamma, \Diamond\delta \vdash_{\mathbf{M}} \Diamond\Sigma.$$

Furthermore $\delta \in \Delta$, because Δ is $\mathbf{M}$-saturated. Since $\Delta \subseteq \Diamond^{-}\Gamma$, we have $\Diamond\delta \in \Gamma$, and so $\Gamma \vdash_{\mathbf{M}} \Diamond\Sigma$, implying $\Sigma \cap \Diamond^{-}\Gamma \neq \emptyset$. □

Observation 4.4 *Note that the first result of the previous lemma also holds whenever Δ is an $\mathbf{M}$-saturator of $\Box^{-}\Gamma$ for $\Gamma \in W_{\mathbf{M}}$. The condition of the second result can analogously be weakened: let $\Gamma \in W_{\mathbf{M}}$ and $\Diamond^{-}\Gamma$ be a saturator of Δ. These results also hold for all inferential extensions $\mathbf{S}$ of $\mathbf{M}$, where the canonical model for $\mathbf{S}$, $M_{\mathbf{S}}$ is defined in the same way as for $\mathbf{M}$, but then restricted to the collection of $\mathbf{S}$-saturated sets.*

We now have the auxiliary results to give a relatively fast proof of the truth lemma for system $\mathbf{M}$.

Lemma 4.7 *For all $\varphi \in \mathcal{L}$ and all $\mathbf{M}$-saturated sets Γ:*

$$M_{\mathbf{M}}, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma \text{ and } M_{\mathbf{M}}, \Gamma \dashv \varphi \Leftrightarrow \neg\varphi \in \Gamma$$

Proof. The proof is by induction on the structure of φ. The basic step $p \in \mathcal{P}$ is immediate by the definition of $V_{\mathbf{M}}$. Since Γ is deductively closed, we know by R-TRUE $\top$ that $\top \in \Gamma$ and by Observation 4.1 that $\neg\top \notin \Gamma$, which implies that the truth lemma also holds for the basic step $\varphi = \top$.

The induction steps for the connectives $\neg$ and $\wedge$ are also straightforward. Only the case of $\Box$ is left. Assume the truth lemma to hold for φ and consider $\Box\varphi$. There are four different cases.

- First suppose $\Box\varphi \in \Gamma$. The definition of $R_{\mathbf{M}}$ tells us that all Δ such that $\Gamma R_{\mathbf{M}} \Delta$ contain φ. The induction hypothesis shows $M_{\mathbf{M}}, \Delta \models \varphi$ for any accessible set Δ, and therefore $M, \Gamma \models \Box\varphi$.
- Suppose $\Box\varphi \notin \Gamma$. Clearly $\Box^{-}\Gamma \nvdash_{\mathbf{M}} \varphi$, by R-TRUE $\Box$ and L-MON. The saturation lemma ensures the existence of an $\mathbf{M}$-saturated set Δ such that $\Box^{-}\Gamma \subseteq \Delta$ and $\varphi \notin \Delta$. Lemma 4.6 tells us that there exists an $\mathbf{M}$-saturated subset Δ' of Δ such that $\Gamma R_{\mathbf{M}} \Delta'$. Clearly $\varphi \notin \Delta'$. The induction hypothesis entails $M_{\mathbf{M}}, \Delta' \not\models \varphi$, hence $M_{\mathbf{M}}, \Gamma \not\models \Box\varphi$.
- Suppose $\neg\Box\varphi \notin \Gamma$. So for all Δ with $\Gamma R_{\mathbf{M}} \Delta$ that $\neg\varphi \notin \Delta$. By the induction hypothesis $M_{\mathbf{M}}, \Delta \not\dashv \varphi$ for all such Δ, and therefore $M_{\mathbf{M}}, \Gamma \not\dashv \Box\varphi$.
- Finally, let $\neg\Box\varphi \in \Gamma$, thus $\neg\varphi \in \Diamond^{-}\Gamma$. Suppose $\Box^{-}\Gamma, \neg\varphi \vdash_{\mathbf{M}} \Sigma$. By L-TRUE $\Diamond$ and L-MON we have $\Gamma \vdash_{\mathbf{M}} \Diamond\Sigma$. This implies that $\Sigma \cap \Diamond^{-}\Gamma \neq \emptyset$. So $\Diamond^{-}\Gamma$ is an $\mathbf{M}$-saturator of $\Box^{-}\Gamma \cup \{\neg\varphi\}$. Therefore GLL ensures the existence of an $\mathbf{M}$-saturated set Δ such that $\Box^{-}\Gamma \cup \{\neg\varphi\} \subseteq \Delta \subseteq \Diamond^{-}\Gamma$, thus $\Gamma R_{\mathbf{M}} \Delta$ and $\neg\varphi \in \Delta$. So by the induction hypothesis $M_{\mathbf{M}}, \Delta \dashv \varphi$, hence $M_{\mathbf{M}}, \Gamma \dashv \Box\varphi$.

□

Theorem 4.8 (Completeness M) *The system $\mathbf{M}$ is complete with respect to partial Kripke models, i.e. $\Gamma \models_{\mathbf{M}} \Delta \to \Gamma \vdash_{\mathbf{M}} \Delta$ for all $\Gamma, \Delta \subseteq \mathcal{L}$.*

Proof. Suppose $\Gamma \nvdash_{\mathbf{M}} \Delta$. The saturation lemma gives us an $\mathbf{M}$-saturated set Σ such that $\Gamma \subseteq \Sigma$ and $\Delta \cap \Sigma = \emptyset$. Because of Lemma 4.7 $M_{\mathbf{M}}, \Sigma \not\models \delta$ for all $\delta \in \Delta$, and $M_{\mathbf{M}}, \Sigma \models \gamma$ for all $\gamma \in \Gamma$, and therefore $\Gamma \not\models_{\mathbf{M}} \Delta$. □

4.3 Finite models

The decidability of a modal logic (i.e. of its consequence relation for finite sets) is mostly shown by proving the so-called *finite model property* (FMP) for this logic (see e.g. Hughes & Cresswell 1984). This means that every non-derivable finite sequence ('non-sequent', for short) has a finite countermodel, i.e. a model which shows the argument to be invalid. The combination of completeness and FMP establishes decidability.

The completeness proof of $\mathbf{M}$ in the previous section not only helps to draw the last conclusion (decidability), it also presents a way to generate finite countermodels. We only have to modify the construction of the canonical model for $\mathbf{M}$ slightly. The (infinite) model $M_{\mathbf{M}}$ is essentially a countermodel for all non-sequents, whereas each finite countermodel will be constructed for just one non-sequent. The latter construction is possible since we only have to consider a certain class of *relevant* formulas. If this restricted class is (essentially) finite, this will result in a finite model. This selection, or *filtration* in technical terms, therefore depends on the set of relevant formulas, the *filtration set* in our terminology. The universe of the finite countermodel will consist of subsets of the filtration set which are saturated with respect to this filtration set.

Definition 4.9 Let $\mathbf{S}$ be a sequential derivation system for a language $\mathcal{L}$, and $\Phi \subseteq \mathcal{L}$.

- A set $\Gamma \subseteq \Phi$ is $\mathbf{S}$-saturated up to Φ iff for all $\Delta \subseteq \Phi$: $\Gamma \vdash_{\mathbf{S}} \Delta \Rightarrow \Delta \cap \Gamma \neq \emptyset$.
- $\Lambda \subseteq \mathcal{L}$ is an $\mathbf{S}$-saturator up to Φ of $\Gamma \subseteq \Phi$ iff for all $\Delta \subseteq \Phi$: $\Gamma \vdash_{\mathbf{S}} \Delta \Rightarrow \Delta \cap \Lambda \neq \emptyset$.

Observation 4.5

- *All* $\mathbf{S}$*-saturated sets up to* Φ *are* $\mathbf{S}$*-consistent.*
- *All* $\mathbf{S}$*-consistent sets are* $\mathbf{S}$*-saturated up to* $\emptyset$.
- *If* $\Phi \subseteq \Phi' \subseteq \mathcal{L}$ *and* $\Gamma \subseteq \Phi'$ *is* $\mathbf{S}$*-saturated up to* Φ', *then* $\Gamma \cap \Phi$ *is* $\mathbf{S}$*-saturated up to* Φ.

So, in particular, all saturated sets are saturated up to its supersets. We arrive at restricted forms of GLL, a saturation lemma and a completeness theorem.

Lemma 4.10 (Filtered GLL) *Let* $\mathbf{S}$ *be a sequent system as in GLL and let* $\Phi \subseteq \mathcal{L}$. *If* $\Lambda \subseteq \Phi$ *is an* $\mathbf{S}$*-saturator up to* Φ *of a set* $\Gamma \subseteq \Phi$, *then* Λ *contains an extension of* Γ *which is* $\mathbf{S}$*-saturated up to* Φ.

Proof. By the same construction as in GLL, where $\{\varphi_i\}_{i\in\omega}$ is an enumeration of Φ such that every member occurs infinitely many times. Let $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ if $\Gamma_n, \varphi \vdash_{\mathbf{S}} \Delta \Rightarrow \Lambda \cap \Delta \neq \emptyset$ for all finite $\Delta \subseteq \Phi$, and else $\Gamma_{n+1} = \Gamma_n$. Analogously to the proof of GLL we can show that the limit of this sequence $\{\Gamma_i\}_{i\in\omega}$ is $\mathbf{S}$-saturated up to Φ. □

Again we can rephrase this Lindenbaum Lemma as a saturation lemma for filtrations, which, modulo R-MON, is equivalent to the filtered GLL. We now explicitly state the relationship between a saturator and non-derivability already used implicitly in the proof of corollary 4.4.

Lemma 4.11 *If sequent system* $\mathbf{S}$ *contains rule* R-MON, *then* $\Gamma \not\vdash \Delta$ *iff* $\Phi - \Delta$ *is a saturator up to* Φ *of* Γ.

Proof. $\Phi - \Delta$ is not a saturator of $\Gamma \Leftrightarrow \exists \Sigma \subseteq \Phi : \Gamma \vdash \Sigma$ and $\Sigma \cap \Phi - \Delta = \emptyset \Leftrightarrow$ $\exists \Sigma \subseteq \Phi : \Gamma \vdash \Sigma$ and $\Sigma \subseteq \Delta \cup \Phi^c \Leftrightarrow \exists \Sigma \subseteq \Delta : \Gamma \vdash \Sigma \Leftrightarrow$ (R-MON) $\Gamma \vdash \Delta$.
□

Corollary 4.12 (Filtered saturation lemma)
Let $\mathbf{S}$ *be as in GLL. Assume* $\Phi \subseteq \mathcal{L}$ *and* $\Gamma, \Delta \subseteq \Phi$. *If* $\Gamma \not\vdash_{\mathbf{S}} \Delta$ *then there exists a set* Σ *which is* $\mathbf{S}$*-saturated up to* Φ *such that* $\Gamma \subseteq \Sigma$ *and* $\Sigma \cap \Delta = \emptyset$.

To use the techniques which were developed in the preceding sections we would like to be able to switch from filtered saturated sets to ordinary

saturated sets if necessary. The following lemma will justify this switch in the sense: every set which is $\mathbf{S}$-saturated up to Φ can be considered as the Φ-part of an $\mathbf{S}$-saturated set.

Lemma 4.13 *Let* $\mathbf{S}$ *be a sequential derivation system as in GLL, and* $\Phi \subseteq \mathcal{L}$. Γ *is* $\mathbf{S}$*-saturated up to* Φ *iff there exists an* $\mathbf{S}$*-saturated set* Γ^* *such that* $\Gamma^* \cap \Phi = \Gamma$.

Proof. If F is an $\mathbf{S}$-saturated set up to Φ, it is its own saturator up to Φ, so by Lemma 4.11 $\Gamma \not\vdash_{\mathbf{S}} \Phi - \Gamma$. The ordinary saturation lemma produces an $\mathbf{S}$-saturated set $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \cap (\Phi - \Gamma) = \emptyset$. In all, by a simple set-theoretic argument $\Gamma = \Gamma^* \cap \Phi$. The converse is a special case of Observation 4.5, taking $\Phi' = \mathcal{L}$. □

Observation 4.6 *By the last lemma, if* $\Phi \subseteq \Phi' \subseteq \mathcal{L}$ *and* Γ *is* $\mathbf{S}$*-saturated up to* Φ *then there exists a set* Γ' *which is* $\mathbf{S}$*-saturated up to* Φ' *and for which* $\Gamma' \cap \Phi = \Gamma$. *To get such a set, use* Γ^* *as in the proof above, and put* $\Gamma' = \Gamma^* \cap \Phi'$.

Theorem 4.14 (FMP M) $\mathbf{M}$ *has the finite model property.*

Proof. Let $\Gamma \not\vdash_{\mathbf{M}} \Delta$ for finite $\Gamma, \Delta \subseteq \mathcal{L}$. Call the set of all subformulas of Γ and Δ: $Sub(\Gamma, \Delta)$. Let the filtration set Φ be the set of all subformulas of elements of Γ and Δ, and their negations, i.e. $\Phi = Sub(\Gamma, \Delta) \cup \neg Sub(\Gamma, \Delta)$. We will show that there exists a finite model $M^{\Phi}_{\mathbf{M}} = \langle W^{\Phi}_{\mathbf{M}}, R^{\Phi}_{\mathbf{M}}, V^{\Phi}_{\mathbf{M}} \rangle \in \mathbf{M}$ which contains a world that verifies all formulas in Γ and does not verify any element of Δ. This can be established by taking $M^{\Phi}_{\mathbf{M}}$ to be the Φ-*filtration* of the $\mathbf{M}$-canonical model $M_{\mathbf{M}}$. Define $W^{\Phi}_{\mathbf{M}}$ to be the set of all sets that are $\mathbf{M}$-saturated up to Φ. Because Φ is finite $W^{\Phi}_{\mathbf{M}}$ is also finite. Furthermore, let $V^{\Phi}_{\mathbf{M}}(p, \Sigma) = 1$ iff $p \in \Sigma$ and $V^{\Phi}_{\mathbf{M}}(p, \Sigma) = 0$ iff $\neg p \in \Sigma$, and finally put

$$(i)\ \Sigma R^{\Phi}_{\mathbf{M}} \Theta \Leftrightarrow \Box^{-} \Sigma \subseteq \Theta \quad \text{and} \quad (ii)\ \neg\varphi \in \Theta \Rightarrow \neg\Box\varphi \in \Sigma \ \text{ for all } \varphi \in \Box^{-} \Phi$$

We will prove the truth lemma for this filtration:

$$M^{\Phi}_{\mathbf{M}}, \Sigma \models \varphi \Leftrightarrow \varphi \in \Sigma \text{ and } M^{\Phi}_{\mathbf{M}}, \Sigma \mathrel{=\!\!|} \varphi \Leftrightarrow \neg\varphi \in \Sigma \text{ for all } \varphi \in Sub(\Gamma, \Delta)$$

Apart from the restriction to $Sub(\Gamma, \Delta)$-elements, the inductive proof is similar to that of the preceding truth lemmas. The basic step for elements of $\mathcal{P}$ is immediate by the definition of $V^{\Phi}_{\mathbf{M}}$. If $\top \in Sub(\Gamma, \Delta)$ the result is obtained by $\Sigma \vdash_{\mathbf{S}} \top$ and $\Sigma \not\vdash_{\mathbf{S}} \neg\top$ for all elements Σ of $W^{\Phi}_{\mathbf{M}}$. For negation and conjunction the induction steps are also quite easy, by the induction hypothesis and the chosen filtration. The only step left is for the modal operator. Suppose $\Box\varphi \in Sub(\Gamma, \Delta)$. Again we consider four separate cases.

- If $\Box\varphi \in \Sigma$ then, by the definition of $R^{\Phi}_{\mathbf{M}}$, for all Θ which are accessible from Σ we infer that $\varphi \in \Theta$. The induction hypothesis gives $M^{\Phi}_{\mathbf{M}}, \Theta \models \varphi$ for all such Θ, and therefore $M^{\Phi}_{\mathbf{M}}, \Sigma \models \Box\varphi$.

- If $\Box\varphi \notin \Sigma$, take Σ' to be an **M**-saturated set such that $\Sigma = \Sigma' \cap \Phi$. By Lemma 4.13 there is such a set Σ'. Clearly $\Sigma' \not\vdash_{\mathbf{M}} \Box\varphi$ and therefore $\Box^-\Sigma' \not\vdash_{\mathbf{M}} \varphi$. As in the truth lemma proof for **M**, there must be an **M**-saturated set Θ' such that $\Box^-\Sigma' \subseteq \Theta' \subseteq \Diamond^-\Sigma'$ with $\varphi \notin \Theta'$. Take $\Theta = \Theta' \cap \Phi$. Then, by Lemma 4.13, $\Theta \in W^{\Phi}_{\mathbf{M}}$. Because $\Box^-\Sigma = \Box^-(\Sigma' \cap \Phi) = \Box^-\Sigma' \cap \Box^-\Phi \subseteq \Box^-\Sigma' \cap \Phi$, we have on the one hand $\Box^-\Sigma \subseteq \Theta' \cap \Phi = \Theta$. On the other hand, if $\neg\psi \in \Theta$ and $\Box\psi \in \Phi$, we obtain $\neg\Box\psi \in \Sigma'$ and $\neg\Box\psi \in \Phi$, and therefore $\neg\Box\psi \in \Sigma$. This implies that Θ is accessible from Σ in $M^{\Phi}_{\mathbf{M}}$: $\Sigma R^{\Phi}_{\mathbf{M}} \Theta$. Since $\varphi \notin \Theta$, the induction hypothesis shows $M^{\Phi}_{\mathbf{M}}, \Theta \not\models \varphi$, consequently $M^{\Phi}_{\mathbf{M}}, \Sigma \not\models \Box\varphi$.
- If $\neg\Box\varphi \notin \Sigma$ then we may conclude, by the definition of $R^{\Phi}_{\mathbf{M}}$ and the induction hypothesis, that $M^{\Phi}_{\mathbf{M}}, \Sigma \not\dashv \Box\varphi$.
- If $\neg\Box\varphi \in \Sigma$ then $\neg\Box\varphi$ must be a member of every **S**-saturated extension of Σ, and certainly for an **S**-saturated Σ' with $\Sigma' \cap \Phi = \Sigma$, which exists by Lemma 4.13. So, by the proof of the ordinary truth lemma of **M**, there exists an **M**-saturated set Θ' such that $\Box^-\Sigma' \cup \{\neg\varphi\} \subseteq \Theta' \subseteq \Diamond^-\Sigma'$. Again, take $\Theta = \Theta' \cap \Phi \in W^{\Phi}_{\mathbf{M}}$. By the same argument as in the case $\Box\varphi \notin \Sigma$, we have $\Sigma R^{\Phi}_{\mathbf{M}} \Theta$. Moreover, $\neg\varphi \in \Theta$, because $\neg\varphi \in \Phi$. Since $\varphi \in Sub(\Gamma, \Delta)$, we infer by induction that $M^{\Phi}_{\mathbf{M}}, \Theta \dashv \varphi$, and so $M^{\Phi}_{\mathbf{M}}, \Sigma \dashv \Box\varphi$.

This completes the truth lemma restricted to $Sub(\Gamma, \Delta)$ on the finite model $M^{\Phi}_{\mathbf{M}}$.

Because $\Gamma, \Delta \subseteq \Phi$ and $\Gamma \not\vdash_{\mathbf{M}} \Delta$, there exists an **M**-saturated set Σ up to Φ such that $\Gamma \subseteq \Sigma$ and $\Delta \cap \Sigma = \emptyset$, according to the filtered saturation lemma. Also because $\Gamma, \Delta \subseteq Sub(\Gamma, \Delta)$ we conclude, by the restricted truth lemma above, that $M^{\Phi}_{\mathbf{M}}, \Sigma \models \gamma$ for all $\gamma \in \Gamma$ and $M^{\Phi}_{\mathbf{M}}, \Sigma \not\models \delta$ for all $\delta \in \Delta$. □

By the completeness and the finite model property of **M** we conclude that

Corollary 4.15 $\vdash_{\mathbf{M}}$ *is a decidable relation on finite arguments.*

If, on the one hand, $\Gamma \models_{\mathbf{M}} \Delta$ for finite Γ and Δ, then, by the completeness theorem for **M**, $\Gamma \vdash_{\mathbf{M}} \Delta$, thus by enumerating proofs of possible sequents the valid inference will eventually turn up. If, on the other hand, $\Gamma \not\models_{\mathbf{M}} \Delta$ for finite Γ and Δ, then by the FMP there is a finite countermodel M which verifies all of Γ and nothing of Δ. Since the finite models for the language restricted to the propositional variables occurring in Γ and Δ (i.e. $\mathcal{P} \cap Sub(\Gamma, \Delta)$) can also be enumerated, the invalid argument will turn up as well.

We have proved a so-called strong finite model property. This means that we do not need completeness here because the above countermodel can never be larger than $2^{\#Sub(\Gamma,\Delta)+1}$. So we can give a strict upper bound for finding a countermodel for $\Gamma \not\vdash_{\mathbf{M}} \Delta$. We only need to check models with less worlds than this upper bound. The only valuations over these models are valuations which determine the propositional variables appearing in $Sub(\Gamma, \Delta)$.

5 Inferential extensions

In this section we will be concerned with a class of schematic extensions of **M** and with finding corresponding model-theoretic characterizations. In classical modal logic well-known schemes such as (**T**) $\Box\varphi \vdash \varphi$, (**4**) $\Box\varphi \vdash \Box\Box\varphi$, (**5**) $\Diamond\varphi \vdash \Box\Diamond\varphi$, (**B**) $\varphi \vdash \Box\Diamond\varphi$ and (**G**) $\Diamond\Box\varphi \vdash \Box\Diamond\varphi$ are characterized by well-defined classes of frames. For example, **T** corresponds to reflexive frames, and **B** to symmetric frames. These logics all fall within a wide and well-characterizable class of modal logics, where the added schematic extensions are of the form

$$\Diamond^k\Box^l\varphi \vdash \Box^m\Diamond^n\varphi$$

Such a schema is denoted as $\mathbf{G}^{k,l}_{m,n}$ and is called a (generalized) Geach rule. The indices refer to the corresponding number of $\Box$- and $\Diamond$-iterations. So for example

$$\mathbf{T} = \mathbf{G}^{0,1}_{0,0}, \mathbf{4} = \mathbf{G}^{0,1}_{2,0} \text{ and } \mathbf{5} = \mathbf{G}^{1,0}_{1,1}.$$

$\mathbf{G}^{k,l}_{m,n}$ can be characterized by the class of frames with an accessibility relation R such that

$$\forall x, y, z : xR^k y \text{ and } xR^m z \Rightarrow \exists w : yR^l w \text{ and } zR^n w$$

Here R^k is defined recursively by:

$$R^0 = \{\langle w, w\rangle \mid w \in W\}$$
$$R^{k+1} = R^k \bullet R$$

It can be proved that all these logics are *frame-complete*, that is complete with respect to the class of models which share the frame property. In other words, every non-sequent has a countermodel in the corresponding frame class.

In the sequel we will try to find similar characterizations for extensions of **M**. Partial modal logic will turn out to be quite different from classical modal logic in some respects. For example, in partial modal logic there is in general no *frame* completeness with respect to a single rule such as $\mathbf{G}^{k,l}_{m,n}$, but there is frame completeness with respect to

$$\mathbf{G}^{k,l}_{m,n} + \mathbf{G}^{m,n}_{k,l}.$$

Also, there is *model* completeness with respect to $\mathbf{G}^{k,l}_{m,n}$ (without its contrapositive). So we believe it is fair to say that, although the overall picture of completeness in partial modal logic is somewhat more complicated, it is also more subtle and interesting than in normal modal logic. Before we turn to Geach extensions, we will briefly view extensions of **M** from a more general point of view.

From now on, let **S** be a finite schematic extension of **M**. In particular, the extended system is assumed to satisfy the finiteness property. Notice that GLL and the saturation lemma still hold. Define the **S**-canonical model $M_{\mathbf{S}}$ like $M_{\mathbf{M}}$, but now with respect to **S**, of course. The universe $W_{\mathbf{S}}$ of the model consists of all **S**-saturated sets. Since every **S**-saturated set is an **M**-saturated set, $M_{\mathbf{S}}$ is a submodel of $M_{\mathbf{M}}$.[7] The accessibility relation $R_{\mathbf{S}}$ and valuation function $V_{\mathbf{S}}$ are thus, respectively, $R_{\mathbf{M}}$ and $V_{\mathbf{M}}$ restricted to $W_{\mathbf{S}}$. Finally we note that the counterpart of the comparison Lemma 4.6 holds for **S** as well. So we can prove the truth lemma for $M_{\mathbf{S}}$ and finally establish a very general and strong completeness theorem for **S**.

Theorem 5.1 (Completeness **S**) *A system* **S** *extending* **M** *is sound and complete with respect to* **S**-*models, in particular with respect to* $M_{\mathbf{S}}$: $\Gamma \vdash_{\mathbf{S}} \Delta \Leftrightarrow \Gamma \models_{M_{\mathbf{S}}} \Delta$ *for all* $\Gamma, \Delta \subseteq \mathcal{L}$.

Proof. For the soundness in the general proposition, we find that, by definition, **S**-models preserve the rules of **S**. The completeness half for the special case of the canonical model follows from the earlier argument, which shows that $M_{\mathbf{S}}$ is a countermodel to each non-sequent. To prove the missing parts, it suffices to show that $M_{\mathbf{S}}$ is an **S**-model. So, let $\Gamma \vdash \Delta$ and $M_{\mathbf{S}}, \Sigma \models \Gamma$, then by the truth lemma $\Gamma \subseteq \Sigma$, so (L-MON) $\Sigma \vdash \Delta$, and hence by **S**-saturation, there is a $\delta \in \Delta$ such that $\delta \in \Sigma$, and so again by the truth lemma $M_{\mathbf{S}}, \Sigma \models \delta$. □

Although this strong completeness theorem is perfectly general, it has the disadvantage that the resulting canonical model (or the class of **S**-models, for that matter) may be entirely chaotic. We would like to give characteristics of the models, preferably in terms of frames, or otherwise structural properties. This is what we will look for in the next few subsections. To make the exploration of such frame and structure properties possible, one more lemma is extremely useful. The lemma identifies $\sqsubseteq_{\mathbf{M}}$ on canonical models as set inclusion.

Lemma 5.2 *Let* **S** *be a schematic extension of* **M** *which satisfies the finiteness property, and* $M_{\mathbf{S}} = \langle W_{\mathbf{S}}, R_{\mathbf{S}}, V_{\mathbf{S}} \rangle$ *its canonical model as described above. If* Γ *and* Δ *are* **S**-*saturated sets, then*

$$\Gamma \subseteq \Delta \iff \Gamma \sqsubseteq_{M_{\mathbf{S}}} \Delta$$

Proof. We will prove, for the $\Rightarrow$-result, that the relation $\subseteq$ is a bisimulation over $M_{\mathbf{S}}$:

$$(\subseteq \bullet R_{\mathbf{S}}) \subseteq (R_{\mathbf{S}} \bullet \subseteq) \text{ and } (\supseteq \bullet R_{\mathbf{S}}) \subseteq (R_{\mathbf{S}} \bullet \supseteq).$$

In the first case we need to show that for all Γ, Γ', Δ' in $W_{\mathbf{S}}$ such that

[7]Technically speaking it is even a generated submodel of $M_{\mathbf{M}}$, generated by $W_{\mathbf{S}}$.

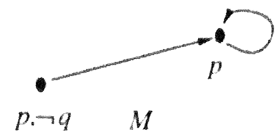

FIGURE 5 An instructive partial model

$\Gamma \subseteq \Gamma'$ and $\Gamma' R_{\mathbf{S}} \Delta'$, there exists a $\Delta \in W_{\mathbf{S}}$ such that $\Gamma R_{\mathbf{S}} \Delta$ and $\Delta \subseteq \Delta'$. Lemma 4.6 (for **S**) guarantees this existence, because

$$\Box^{-}\Gamma \subseteq \Box^{-}\Gamma' \subseteq \Delta'.$$

The second case follows from the second property in Lemma 4.6. If $\Gamma' \supseteq \Gamma$ and $\Gamma R_{\mathbf{S}} \Delta$ then we know

$$\Delta \subseteq \Diamond^{-}\Gamma \subseteq \Diamond^{-}\Gamma',$$

and therefore, according to Lemma 4.6, there exists $\Delta' \in W_{\mathbf{S}}$ such that $\Delta \subseteq \Delta'$ and $\Gamma' R_{\mathbf{S}} \Delta'$. The definition of $V_{\mathbf{S}}$ gives $\Gamma \sqsubseteq_{M_{\mathbf{S}}} \Sigma$.

The other direction of the lemma, $\Leftarrow$, is just a consequence of the persistence for $\sqsubseteq_{\mathbf{M}}$ and the truth lemma for **S**. □

5.1 A simple example

In the minimal system **M** of partial modal logic the law of contraposition is lacking: $\Gamma \models_{\mathbf{M}} \Delta \not\Rightarrow \neg\Delta \models_{\mathbf{M}} \neg\Gamma$. This creates an interesting perspective on its modal extensions. Adding a modal rule such as $\Box\varphi \vdash \varphi$ does not imply its contrapositive $\neg\varphi \vdash \neg\Box\varphi$. For example, consider the simple two world **M**-model of Figure 5.

All worlds in M that verify $\Box\varphi$ also verify φ. Therefore the model satisfies rule $\Box\varphi \vdash \varphi$, which we will call **T**□ henceforth. However, the model does not satisfy the rule $\varphi \vdash \Diamond\varphi = \mathbf{G}^{0,0}_{0,1}$, which we will call **T**◇ henceforth. So **T**◇ is logically independent of **T**□, when we compare them as extensions of **M**. Consequently $\Box\varphi \vdash \varphi$ defines a wider class than reflexive models only.

In order to find corresponding model-theoretic conditions for the extensions of **M** with rules such as **T**□, we need, apart from accessibility constraints, a structural comparison of the informational content of worlds in a partial Kripke model. In the example in figure 5, φ is true on all worlds which verify $\Box\varphi$ for arbitrary $\varphi \in \mathcal{L}$, because the left-hand world is a modal extension of the right-hand world. Informally, every world "sees a part" of itself. A corresponding constraint for $\varphi \vdash \Diamond\varphi$ singles out models in which every world extends at least one accessible world.

In fact the notion of modal extension $\sqsubseteq_{\mathbf{M}}$ enables us to formulate different forms of "pseudo-reflexivity", as were conceptually described before.

Definition 5.3 A partial Kripke model $M = \langle W, R, V \rangle$ is said to be *small-*

reflexive iff

$$\forall x\ \exists y : xRy \text{ and } y \sqsubseteq_{\mathbf{M}} x.$$

Such a model is called *big-reflexive* iff

$$\forall x\ \exists y : xRy \text{ and } x \sqsubseteq_{\mathbf{M}} y.$$

The class of small-reflexive models are denoted by T□ and the big-reflexive models are written as T◇.

Theorem 5.4 **T□** $= \mathbf{M} + \Box\varphi \vdash \varphi$ *is sound and complete with respect to* T□. **T◇** $= \mathbf{M} + \varphi \vdash \Diamond\varphi$ *is sound and complete with respect to* T◇.

Proof. Soundness is straightforward and thus left to the reader. For completeness, Theorem 5.1 tells us that it suffices to show that

$$M_{\mathbf{T}\Box} \in \mathsf{T}\Box \quad \text{and} \quad M_{\mathbf{T}\Diamond} \in \mathsf{T}\Diamond.$$

First we prove that $M_{\mathbf{T}\Box}$ is small-reflexive. This can be done by an application of Lemma 4.6. Let $\Gamma \in W_{\mathbf{T}\Box}$. By $\Box\varphi \vdash_{\mathbf{T}\Box} \varphi$ we conclude that $\Box^{-}\Gamma \subseteq \Gamma$, which means, according to Observation 4.4, that $R_{\mathbf{T}\Box}(\Gamma, \Delta)$ and $\Delta \subseteq \Gamma$ for some $\Delta \in W_{\mathbf{T}\Box}$. The latter conclusion also yields $\Delta \sqsubseteq_{\mathbf{T}\Box} \Gamma$ (Lemma 5.2), and so $M_{\mathbf{T}\Box}$ is small reflexive.

Big reflexivity for $M_{\mathbf{T}\Diamond}$ can be obtained in the same fashion, using the fact that $\Gamma \subseteq \Diamond^{-}\Gamma$ for all $\Gamma \in W_{\mathbf{T}\Diamond}$, and application of Lemma 4.6 and its generalization in Observation 4.4. □

5.2 Geach rules

A completeness result for the full class of partial Geach systems $\mathbf{M} + \mathbf{G}^{k,l}_{m,n}$, using $\sqsubseteq_{\mathbf{M}}$, is presented in Jaspars (1994). We will not go into the technicalities of the proof, but simply present the result in the following theorem.

Theorem 5.5 *The logic* $\mathbf{G}^{k,l}_{m,n}$ *is sound and complete with respect to the collection of models* $M = \langle W, R, V \rangle$ *which are 'semi-k, l, m, n confluent':*

$$\forall xyz \in W : xR^k y \text{ and } xR^m z \Rightarrow \exists vw \in W : v \sqsubseteq_{\mathbf{M}} w,\ yR^l v \text{ and } zR^n w.$$

Though quite general, the completeness theorem for Geach rules supplies sufficient information about the structure of the models. Insight into its applicability may be facilitated by the following examples. In Table 1, the left column lists the names of some Geach extensions of **M**, given in the middle column. The right column gives the corresponding constraints on the order within the models $M = \langle W, R, V \rangle$.

In agreement with our earlier terminology for pseudo-reflexivity, the four listed properties may be baptized small-symmetry, small-Euclidicity, small-transitivity and big-transitivity, respectively.

TABLE 1 Some conditions for geach extensions

MB□	$\varphi \vdash \Box\Diamond\varphi$	$\forall x, y : xRy \Rightarrow \exists z : yRz$ and $x \sqsubseteq_{\mathbf{M}} z$
M5□	$\Diamond\varphi \vdash \Box\Diamond\varphi$	$\forall x, y, z : xRy$ and $xRz \Rightarrow \exists w : zRw$ and $y \sqsubseteq_{\mathbf{M}} w$.
M4□	$\Box\varphi \vdash \Box\Box\varphi$	$\forall x, y, z : xRy$ and $yRz \Rightarrow \exists w : xRw$ and $w \sqsubseteq_{\mathbf{M}} z$
M4◇	$\Diamond\Diamond\varphi \vdash \Diamond\varphi$	$\forall x, y, z : xRy$ and $yRz \Rightarrow \exists w : xRw$ and $z \sqsubseteq_{\mathbf{M}} w$

5.3 Intermediate worlds .

There is an easy completeness proof for the system $\mathbf{T} = \mathbf{M} + \mathbf{T}\Box + \mathbf{T}\Diamond$ with respect to the class of partial models with a reflexive frame, for note that if Γ is a **T**-saturated set, then $\Box^{-}\Gamma \subseteq \Gamma$ by **T**□ and $\Gamma \subseteq \Diamond^{-}\Gamma$ by **T**◇, so $\Gamma R_{\mathbf{T}} \Gamma$. Notice that, in comparison with our present set-up, reflexivity is a stronger restriction than small- and big-reflexivity together. The possibility of restricting to the class of reflexive models can be understood by the following observation. It turns out that partial Kripke models are insensitive to adding and removing so-called *intermediate worlds*. Whenever a world w in a partial Kripke model M has two accessible worlds v and u, then every intermediate world x of v and u in M, meaning $v \sqsubseteq_{\mathbf{M}} x \sqsubseteq_{\mathbf{M}} u$, can be taken to be accessible from w as well, without changing truth and falsity in w.

Observation 5.1 *For* $M = \langle W, R, V \rangle \in \mathsf{M}$, *let* $w \in W$ *be such that for certain* $v, u \in W$: wRv *and* wRu. *Suppose there is an* $x \in M$ *such that* $v \sqsubseteq_{\mathbf{M}} x \sqsubseteq_{\mathbf{M}} u$, *and let* $M' = \langle W, R', V \rangle$ *where* $R' = R \cup \{\langle w, x \rangle\}$. *Then* $M, w \models \varphi \Leftrightarrow M', w \models \varphi$ *for all* $\varphi \in \mathcal{L}$.

(The proof is by induction on the structure of formulas: the □ and ¬□-steps follow from the persistence result for $\sqsubseteq_{\mathbf{M}}$.)

In a model which is both big- and small-reflexive, every world has access to a larger and a smaller world. That is, the original world is an intermediate world, and could therefore be assumed to be self-accessible as well without losing or gaining information. In this straightforward way we can transform every model which is big- and small-reflexive into a reflexive one.

6 Summary

We have illustrated the idea of partial modal logic (i.e. modal logic with a partial semantics) using one basic system, called **M** here, and its various extensions. Although the subject is not classical, the presentation was highly classical. After defining the language and its semantics, we gave a sequential inference system. Completeness of the sequent system was shown using a canonical Henkin model with saturated sets linking (inferential) syntax and semantics. Then decidability of the inference relation followed by means of the finite model property.

Finally, we showed how to extend the sequent system and restrict the class of admissible models in order to incorporate so-called Geach rules. Most applications related to belief and knowledge, for example, only use Geach rules, thus showing the significance of these extensions.

As was pointed out in the introduction, we restricted ourselves to just one type of partial modal logic, but the following notes suggest some other directions in this area.

7 Historical notes

The roots of partiality and the link to three-valued logic will have been made clear in the other "fundamental" contributions to this volume. Apart from the comprehensive introduction by Van der Hoek and Meyer (see Chapter 3, this volume), standard texts on modal logic are Hughes and Cresswell (1984), Chellas (1980) and Bull and Segerberg (1984), in order of increasing intricacy. Here we focus on partial modal logic.

An early combination of partial and modal logic is Segerberg (1967), where the connectives are characterized by 'weak Kleene' truth tables, and several non-standard interpretations are given to necessity. Schotch et al. (1978) present a system in which the underlying propositional logic is Łukasiewicz' three-valued logic $\mathbf{L}_3$, although necessity is given the text interpretation. Finally Morikawa (1989) proposes a family of modal logics based on arbitrary three-valued propositional logics and various non-standard interpretations for necessity (yet different from Segerberg's), one with only classical output values, another which is "weak" in the sense that $\Box\varphi$ is undefined when φ is undefined in some alternative.

The move to partialize the worlds was made independently by several authors, most explicitly by Humberstone (1981) and Barwise (1981). In fact even Hintikka's original formulation (in Hintikka 1962) of his 'model sets' had a partial element in it, which, however, was soon eliminated.

An account of completeness and definability for partial modal logic is given in Thijsse (1990) and Thijsse (1992, part I), where the relevant parameters are *valuation type* (partial and/or incoherent), *truth/falsity conditions*, *single* or *split* accessibility, *validity kind* ("always true" vs. "never false") and *rule concept* ('absolute', as in classical Necessitation, vs. 'relative', as in strong consequence). In these terms the system presented here may be characterized as relative verification on partial and coherent models with standard truth/falsity conditions and single accessibility.

Failure of contraposition, as in our system, is considered by Blamey (1986) to be a good reason for replacing strong ("positive") consequence ... barrelled ("full") consequence. See also Muskens (1995) and Langholm (Chapter 1). We obviously do not agree. To mention but one strange effect of the imposed symmetry: then (for coherent models) neither *ex falso* nor *tertium non datur* is valid, but the "unification" of these rules holds in the form $\varphi \wedge \neg\varphi \models \psi \vee \neg\psi$.

Our basic system $\mathbf{M}$ (without rules for $\top$) was called $\mathbf{M}^+$ in Thijsse (1992), where the derivation system resembles that of natural deduction, rather than the sequential format from Jaspars (1994) used here. The completeness proof for $\mathbf{M}^+$ contained repeated Lindenbaum constructions, which are avoided here by GLL and the saturation lemma. The latter was called 'generalized Lindenbaum Lemma' in Thijsse (1992), but is in fact a slight generalization of the saturation lemma in Troelstra and van Dalen (1988). An alternative is to use normal forms as in Jaspars (1993).

The concept of saturated sets can be traced back to Aczel (1968) and Thomason (1968), where they are used to give completeness proofs for intuitionistic predicate logic. Veltman (1985) uses saturated sets for a completeness proof of his data logic. Our definition of 'saturated' and our notion of S-saturator are from Jaspars (1994). Saturated sets may also be used in circumscribing knowledge based on a partial logic (see Van der Hoek, Jaspars and Thijsse 1994).

Other information orders are required for modal principles such as $\varphi \wedge \Diamond\neg\varphi \vdash \bot$, which is captured by the class of partial Kripke models in which every world only has access to worlds which are *coherent* with the initial world. Two worlds w and v in a model M are propositionally coherent if there exists no propositional variable p that has has different truth values in w and v. Such a pair is (modally) coherent if there exists a bisimulation Z on M such that wZv and all pairs in Z are propositionally coherent. Alternative ordering relations are studied in Jaspars (1994).

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