

Information Distance in Multiples

Paul M. B. Vitányi

Abstract—Information distance is a parameter-free similarity measure based on compression, used in pattern recognition, data mining, phylogeny, clustering and classification. The notion of information distance is extended from pairs to multiples (finite lists). We study maximal overlap, metricity, universality, minimal overlap, additivity and normalized information distance in multiples. We use the theoretical notion of Kolmogorov complexity which for practical purposes is approximated by the length of the compressed version of the file involved, using a real-world compression program.

Index Terms—Data mining, information distance, Kolmogorov complexity, multiples, pattern recognition, similarity.

I. INTRODUCTION

IN pattern recognition, learning and data mining one obtains information from objects containing information. This involves an objective definition of the information in a single object, the information to go from one object to another object in a pair of objects, the information to go from one object to any other object in a multiple of objects and the shared information between objects, [34].

The classical notion of Kolmogorov complexity [21] is an objective measure for the information in an a *single* object and information distance measures the information between a *pair* of objects [3]. This last notion has spawned research in the theoretical direction, among others [6], [30], [35], [37]–[39]. Research in the practical direction has focused on the *normalized* information distance, the similarity metric, which arises by normalizing the information distance in a proper manner and approximating the Kolmogorov complexity through real-world compressors [7]–[9], [26]. This normalized information distance is a parameter-free, feature-free, and alignment-free similarity measure that has had great impact in applications. A variant of this compression distance has been tested on all time sequence databases used in the last decade in the major data mining conferences (sigkdd, sigmod, icdm, icde, ssdb, vldb, pkdd, pakdd) [18]. The conclusion is that the method is competitive with all 51 other methods used and superior in heterogenous data clustering and anomaly detection. In [4] it was shown that the method is resistant to noise. This theory has found many applications in pattern recognition, phylogeny,

clustering and classification. For objects that are represented as computer files such applications range from weather forecasting, software, earthquake prediction, music, literature, ocr, bioinformatics, to internet [1], [2], [5], [8]–[10], [12], [19], [20], [22], [23], [25], [31]–[33], [40]. For objects that are only represented by name, or objects that are abstract like “red,” “Einstein,” “three,” the normalized information distance uses background information provided by Google, or any search engine that produces aggregate page counts. It discovers the “meaning” of words and phrases in the sense of producing a relative semantics. Applications run from ontology, semantics, tourism on the web, taxonomy, multilingual questions, to question-answer systems [13]–[15], [17], [36], [41]–[43]. For more references on either subject see the textbook [28] or Google Scholar for references to [8], [9], [26].

However, in many applications we are interested in shared information between many objects instead of just a pair of objects. For example, in customer reviews of gadgets, in blogs about public happenings, in newspaper articles about the same occurrence, we are interested in the most comprehensive one or the most specialized one. Thus, we want to extend the information distance measure from pairs to multiples.

A. Related Work

In [27] the notion is introduced of the information required to go from any object in a multiple of objects to any other object in the multiple. This is applied to extracting the essence from, for example, a finite list of internet news items, reviews of electronic cameras, tv’s and so on, in a way that works better than other methods. Let X denote a finite list of m finite binary strings defined by $X = (x_1, \dots, x_m)$, the constituting strings ordered length-increasing lexicographic. We use lists and not sets, since if X is a set we cannot express simply the distance from a string to itself or between strings that are all equal. Let U be the reference universal Turing machine, for convenience the prefix one as in Section II. We define the information distance in X by $E_{\max}(X) = \min\{|p| : U(x_i, p, j) = x_j \text{ for all } x_i, x_j \in X\}$. It is shown in [27, Theorem 2] that

$$E_{\max}(X) = \max_{x \in X} K(X|x) \quad (\text{I.1})$$

up to a logarithmic additive term. Define $E_{\min}(X) = \min_{x \in X} K(X|x)$. In [27, Theorem 3], they state that for every list $X = (x_1, \dots, x_m)$ we have

$$E_{\min}(X) \leq E_{\max}(X) \leq \min_{i:1 \leq i \leq m} \sum_{x_i, x_k \in X \& k \neq i} E_{\max}(x_i, x_k) \quad (\text{I.2})$$

up to a logarithmic additive term. This is not a corollary of (I.1) as stated in [27], but both inequalities follow from the definitions. The left-hand side (LHS) is interpreted as the program length of the “most comprehensive object that contains the most

Manuscript received May 20, 2009; revised November 15, 2010; accepted November 19, 2010. Date of current version March 16, 2011. This work was supported in part by the ESF QiT Programmme, the EU NoE PASCAL II and in part by the Netherlands BSIK/BRICKS project.

The author is with the National Research Center for Mathematics and Computer Science, The Netherlands (CWI), and also with the University of Amsterdam, CWI, Science Park 123, 1098XG Amsterdam, The Netherlands (e-mail: Paul.Vitanyi@cwi.nl).

Communicated by A. Krzyzak, Associate Editor for Pattern Recognition, Statistical Learning, and Inference.

Digital Object Identifier 10.1109/TIT.2011.2110130

information about all the others [all elements of X],” and the right-hand side (RHS) is interpreted as the program length of the “most specialized object that is similar to all the others.” The paper [27] develops the stated results and applications. It does not develop the theory in any detail. That is the purpose of the present paper.

B. Results

Information distance for multiples, that is, finite lists, appears both practically and theoretically promising. In nearly all cases below the results imply the corresponding ones for the pairwise information distance defined as follows. The information distance in [3] between strings x_1 and x_2 is $E_{\max}(x_1, x_2) = \max\{K(x_1|x_2), K(x_2|x_1)\}$. In the current paper $E_{\max}(X) = \max_{x: x \in X} K(X|x)$. These two definitions coincide for $|X| = 2$ since $K(x, y|x) = K(y|x)$ up to an additive constant term. We investigate the maximal overlap of information (Theorem 3.1) which for $|X| = 2$ specializes to Theorem 3.4 in [3], Corollary 3.2 shows (I.1) and Corollary 3.3 shows that the LHS of (I.2) can be taken to correspond to a single program embodying the “most comprehensive object that contains the most information about all the others” as stated but not argued or proved in [27]; metricity (Theorem 4.1) and universality (Theorem 5.2) which for $|XY| = 2$ (for metricity) and $|X| = 2$ (for universality) specialize to Theorem 4.2 in [3]; additivity (Theorem 6.1); minimum overlap of information (Theorem 7.1) which for $|X| = 2$ specializes to [29, Theorem 8.3.7] and the nonmetricity of normalized information distance for lists of more than two elements and certain proposals of the normalizing factor (Section VIII). In contrast, for lists of two elements we can normalize the information distance as in Lemma V.4 and Theorem V.7 of [26]. The definitions are of necessity new as are the proof ideas. Remarkably, the new notation and proofs for the general case are simpler than the mentioned existing proofs for the particular case of pairwise information distance.

II. PRELIMINARIES

Kolmogorov Complexity: This is the information in a single object [21]. The notion has been the subject of a plethora of papers. Informally, the Kolmogorov complexity of a finite binary string is the length of the shortest string from which the original can be losslessly reconstructed by an effective general-purpose computer such as a particular universal Turing machine. Hence it constitutes a lower bound on how far a lossless compression program can compress. For technical reasons we choose Turing machines with a separate read-only input tape, that is scanned from left to right without backing up, a separate work tape on which the computation takes place and a separate output tape. Upon halting, the initial segment p of the input that has been scanned is called the input “program” and the contents of the output tape is called the “output.” By construction, the set of halting programs is prefix free. We call U the reference universal prefix Turing machine. This leads to the definition of “prefix Kolmogorov complexity” which we shall designate simply as “Kolmogorov complexity.”

Formally, the *conditional Kolmogorov complexity* $K(x|y)$ is the length of the shortest input z such that the reference universal prefix Turing machine U on input z with auxiliary information

y outputs x . The *unconditional Kolmogorov complexity* $K(x)$ is defined by $K(x|\epsilon)$ where ϵ is the empty string (of length 0). In these definitions both x and y can consist of a nonempty finite lists of finite binary strings. For more details and theorems that are used in the present work see Appendix I.

Lists: A list is a multiple $X = (x_1, \dots, x_m)$ of $m < \infty$ finite binary strings in length-increasing lexicographic order. If X is a list, then some or all of its elements may be equal. Thus, a list is not a set but an ordered *bag* of elements. With some abuse of the common set-membership notation we write $x_i \in X$ for every i ($0 \leq i \leq m$) to mean that “ x_i is an element of list X .” The conditional prefix Kolmogorov complexity $K(X|x)$ of a list X given an element x is the length of a shortest program p for the reference universal Turing machine that with input x outputs the list X . The prefix Kolmogorov complexity $K(X)$ of a list X is defined by $K(X|\epsilon)$. One can also put lists in the conditional such as $K(x|X)$ or $K(X|Y)$. We will use the straightforward laws $K(\cdot|X, x) = K(\cdot|X)$ and $K(X|x) = K(X'|x)$ up to an additive constant term, for $x \in X$ and X' equals the list X with the element x deleted.

Information Distance: To obtain the *pairwise information distance* in [3] we take $X = \{x_1, x_2\}$ in (I.1). Then (I.1) is equivalent to $E_{\max}(x_1, x_2) = \max\{K(x_1|x_2), K(x_2|x_1)\}$.

III. MAXIMAL OVERLAP

We use the notation and terminology of Section I-A. Define $k_1 = E_{\min}(X)$, $k_2 = E_{\max}(X)$ and $l = k_2 - k_1$. We prove a *maximal overlap* theorem: the information needed to go from any x_i to any x_k in X can be divided in two parts: a single string of length k_1 and a string r of length l (possibly depending on x_i), everything up to an additive logarithmic term.

Theorem 3.1: A single program of length $k_1 + K(m, k_1, k_2) + \log m + O(1)$ bits concatenated with a string of l bits, possibly depending on i , suffice to find X from x_i for every $x_i \in X$. To find an arbitrary element $x_k \in X$ from x_i it suffices to concatenate at most another $\log m$ bits, possibly depending on i and k .

Proof: Enumerate the finite binary strings lexicographic length-increasing as s_1, s_2, \dots . Let $G = (V, E)$ be a graph defined as follows. Let A be the set of finite binary strings and B the set of vectors of strings in A defined by $v = (s_1, \dots, s_m)$ such that

$$\begin{aligned} \min_{j: 1 \leq j \leq m} \{K(s_1, \dots, s_m | s_j)\} &\leq k_1 \\ \max_{j: 1 \leq j \leq m} \{K(s_1, \dots, s_m | s_j)\} &\leq k_2. \end{aligned}$$

Given k_1 and k_2 the set B can be enumerated. Define $V = A \cup B$. Define E by length-increasing lexicographic enumerating $A \times B$ and put $(rs, v) \in E$ with $rs \in A$ and $v = (s_1, \dots, s_m) \in B$ if $s = s_j$ for some j ($1 \leq j \leq m$), where r is chosen as follows. It is the $\lceil i/2^{k_1} \rceil$ th string of length l where i is the number of times we have used $s \in A$. So the first 2^{k_1} times we choose an edge $(\cdot s, \cdot)$ we use 0^l , the next 2^{k_1} we use $0^{l-1}1$ and so on. In this way, $i \leq 2^{2k_1}$ so that $i/2^{k_1} \leq 2^l$. By adding r to s we take care that the degree of rs is at most 2^{k_1} and not at most 2^{2k_2} as it could be without the prefix r . The degree of a node $v \in B$ is trivially m .

In addition, we enumerate B length-increasing lexicographic and “color” everyone of the m edges incident with an enumerated vector $v \in B$ with the same binary string c of length $k_1 + \log m$. If $v = (s_1, \dots, s_m)$ and v is connected by edges to nodes $r_1 s_1, \dots, r_m s_m$, then choose c as the minimum color not yet appearing on any edge incident with any $r_j s_j$ ($1 \leq j \leq m$). Since the degree of every node $rs \in A$ is bounded by 2^{k_1} and hence the colors already used for edges incident on nodes $r_1 s_1, \dots, r_m s_m$ number at most $\sum_{1 \leq j \leq m} (2^{k_1} - 1) = m2^{k_1} - m$, a color is always available.

Knowing m, k_1, k_2 one can reconstruct G and color its edges. Given an element x from the list X and knowing the appropriate string r of length l and the color c of the edge (rx, X) , we can find X . Hence a single program, say p , of length $k_1 + K(m, k_1, k_2) + \log m + O(1)$ bits suffices to find X from rx for any $x \in X$ and with $|r| = l$. An additional $\log m$ bits suffice to select any element of X . Taking these $\log m$ bits so that they encode the difference from i to $k \bmod m$ we can compute from every $x_i \in X$ to every $x_k \in X$ and vice versa with the same program p of length $k_1 + K(m, k_1, k_2) + \log m + O(1)$ concatenated with a string r of length l and a string of length $\log m$, both possibly depending on i and k . Since we know m, k_1, k_2 from the fixed program p , where they are encoded as a self-delimiting prefix of length $K(m, k_1, k_2)$ say, we can concatenate these strings without separation markers and reconstruct them. ■

Corollary 3.2: Since $k_1 + l = k_2$, the theorem implies (I.1), that is, [27, Theorem 2].

It is not *a priori* clear that $E_{\min}(X)$ in the LHS of (I.2) corresponds to a *single* program that represents the information overlap of every shortest program going from any x_i to the list X . This seems in fact assumed in [27] where $E_{\min}(X)$ is interpreted as the [Kolmogorov complexity of] “the most comprehensive object that contains the most information about all the others.” In fact, for every $x_i \in X$ we can choose a shortest program going from x_i to the list X so that these programs have pairwise no information overlap at all (Theorem 7.1). But here we have proved the following.

Corollary 3.3: The quantity $E_{\min}(X)$ corresponds to a *single* shortest program that represents the maximum overlap of information of all programs going from x_i to the list X for any $x_i \in X$.

IV. METRICITY

We consider nonempty finite lists of finite binary strings, each list ordered length-increasing lexicographic. Let \mathcal{X} be the set of such ordered nonempty finite lists of finite binary strings. A *distance function* d on \mathcal{X} is defined by $d: \mathcal{X} \rightarrow \mathcal{R}^+$ where \mathcal{R}^+ is the set of nonnegative real numbers. Define $W = UV$ if W is a list of the elements of the lists U and V and the elements of W are ordered length-increasing lexicographical. A distance function d is a *metric* if $X, Y, Z \neq \emptyset$ and

- 1) *Positive definiteness:* $d(X) = 0$ if all elements of X are equal and $d(X) > 0$ otherwise.
- 2) *Symmetry:* $d(X)$ is invariant under all permutations of X .
- 3) *Triangle inequality:* $d(XY) \leq d(XZ) + d(ZY)$.

Theorem 4.1: The information distance for lists, E_{\max} , is a metric where the (in)equalities hold up to a $O(\log K)$ additive term. Here K is the largest quantity involved in the metric (in)equalities.

Proof: It is clear that $E_{\max}(X)$ satisfies positive definiteness and symmetry up to an $O(\log K)$ additive term where $K = K(X)$. It remains to show the triangle inequality.

Claim 4.2: Let X, Y, Z be three nonempty finite lists of finite binary strings and $K = \max\{K(X), K(Y), K(Z)\}$. Then, $E_{\max}(XY) \leq E_{\max}(XZ) + E_{\max}(ZY)$ up to an $O(\log K)$ additive term.

Proof: By Theorem 3.1

$$\begin{aligned} E_{\max}(XY) &= \max_{x:x \in XY} K(XY|x) = K(XY|x_{XY}) \\ E_{\max}(XZ) &= \max_{x:x \in XZ} K(XZ|x) = K(XZ|x_{XZ}) \\ E_{\max}(ZY) &= \max_{x:x \in ZY} K(ZY|x) = K(ZY|x_{ZY}) \end{aligned}$$

equalities up to a $O(\log K)$ additive term. Here x_{XY}, x_{XZ}, x_{ZY} are the elements for which the maximum is reached for the respective E_{\max} 's.

Assume that $x_{XY} \in X$, the case $x_{XY} \in Y$ being symmetrical. Let z be some element of Z . Then

$$\begin{aligned} K(XY|x_{XY}) &\leq K(XYZ|x_{XY}) \\ &\leq K(XZ|x_{XY}) + K(Y|XZ, x_{XY}) \\ &\leq K(XZ|x_{XZ}) + K(Y|XZ, z) \\ &\leq K(XZ|x_{XZ}) + K(ZY|x_{ZY}). \end{aligned}$$

The first inequality follows from the general $K(u) \leq K(u, v)$, the second inequality by the obvious subadditive property of $K(\cdot)$, the third inequality since in the first term $x_{XY} \in XZ$ and the $\max_{x:x \in XZ} \{K(XZ|x)\}$ is reached for $x = x_{XZ}$ and in the second term both $x_{XY} \in X$ and for z take any element from Z and the fourth inequality follows by in the second term dropping X from the conditional and moving Z from the conditional to the main argument and observing that both $z \in ZY$ and the $\max_{x:x \in ZY} \{K(ZY|x)\}$ is reached for $x = x_{ZY}$. The theorem follows with (in)equalities up to an $O(\log K)$ additive term. ■

V. UNIVERSALITY

Let $X \in \mathcal{X}$. A priori we allow asymmetric distances. We would like to exclude degenerate distance measures such as $D(X) = 1$ for all X . For each d , we want only finitely many lists X such that $D(X) \leq d$. Exactly how fast we want the number of lists we admit to go to ∞ is not important; it is only a matter of scaling. For every distance D we require the following *density condition* for every $x \in \{0, 1\}^*$:

$$\sum_{X:x \in X \& D(X) > 0} 2^{-D(X)} \leq 1. \quad (\text{V.1})$$

Thus, for the density condition on D we consider only lists X with $|X| \geq 2$ and not all elements of X are equal. Moreover, we consider only distances that are computable in some broad sense.

Definition 5.1: An *admissible list distance* $D(X)$ is a total, possibly asymmetric, function from \mathcal{X} to the nonnegative real

numbers that is 0 if all elements of X are equal and greater than 0 otherwise (up to an additive $\log K$ additive term with $K = K(X)$), is upper semicomputable and satisfies the density requirement in (V.1).

Theorem 5.2: The list information distance $E_{\max}(X)$ is admissible and it is minimal in the sense that for every admissible list distance function $D(X)$ we have $E_{\max}(X) \leq D(X)$ up to an additive constant term.

Proof: It is straightforward that $E_{\max}(X)$ is a total real-valued function, is 0 only if all elements of X are equal and unequal 0 otherwise (up to an $O(\log K)$ additive term with $K = K(X)$) and is upper semicomputable. We verify the density requirement of (V.1). For every $x \in \{0, 1\}^*$, consider lists X of at least two elements not all equal and $x \in X$. Define functions $f_x(X) = 2^{-K(X|x)}$. Then, $f_x(X) \geq 2^{-E_{\max}(X)}$. It is easy to see that for every $x \in \{0, 1\}^*$

$$\begin{aligned} & \sum_{X:x \in X \& E_{\max}(X) > 0} 2^{-E_{\max}(X)} \\ & \leq \sum_{X:x \in X \& E_{\max}(X) > 0} f_x(X) \\ & = \sum_{X:x \in X \& E_{\max}(X) > 0} 2^{-K(X|x)} \leq \sum 2^{-l(p)} \end{aligned}$$

where the RHS sum is taken over all programs p for which the reference prefix machine U , given x , computes a finite list X of at least two elements not all equal and such that $x \in X$. This sum is the probability that U , given x , computes such a list X from a program p generated bit by bit uniformly at random. Therefore, the RHS sum is at most 1 and $E_{\max}(X)$ satisfies the density requirement (V.1).

We prove minimality. Fix any $x \in \{0, 1\}^*$. Since D is upper semicomputable, the function f defined by $f(X, x) = 2^{-D(X)}$ for X satisfying $x \in X$ and $D(X) > 0$ and 0 otherwise, is lower semicomputable. Since $\sum_{X:x \in X \& D(X) > 0} 2^{-D(X)} \leq 1$, we have $\sum_X f(X, x) \leq 1$ for every x . Note that given D we can compute f and hence $K(f) \leq K(D) + O(1)$. By the conditional version of Eq. (A.2) in the Appendix, that is [28, Theorem 4.3.2], we have $c_D \mathbf{m}(X|x \in X) \geq f(X, x)$ with $c_D = 2^{K(f)} = 2^{K(D)+O(1)}$, that is, c_D is a positive constant depending on D only. By the conditional version of Eq. (A.3) in the Appendix, that is [28, Theorem 4.3.4], we have for every $x \in X$ that $\log 1/\mathbf{m}(X|x) = K(X|x) + O(1)$. Hence, for every $x \in X$ we have $\log 1/f(X, x) \geq K(X|x) + \log 1/c_D + O(1)$. Altogether, for every admissible distance D and every $x \in \{0, 1\}^*$ and every list X satisfying $x \in X$, there is a constant c_D such that $D(X) \geq K(X|x) + \log 1/c_D + O(1)$. Hence, $D(X) \geq E_{\max}(X) + \log 1/c_D + O(1)$. ■

VI. ADDITIVITY

Theorem 6.1: E_{\max} is not subadditive: neither $E_{\max}(X) + E_{\max}(Y) \leq E_{\max}(XY)$ nor $E_{\max}(X) + E_{\max}(Y) \geq E_{\max}(XY)$, the (in)equalities up to logarithmic additive terms, holds for all lists X, Y .

Proof: Below, all (in)equalities are taken up to logarithmic additive terms. Let x, y be strings of length n ,

$X = (\epsilon, x)$ and $Y = (\epsilon, y)$ with ϵ denoting the empty word. Then $E_{\max}(XY) = E_{\max}(\epsilon, \epsilon, x, y)$, $E_{\max}(X) = K(x)$ and $E_{\max}(Y) = K(y)$. If $x = y$ and $K(x) = n$, then $E_{\max}(XY) = E_{\max}(X) = E_{\max}(Y) = n$. Hence, $E_{\max}(XY) < E_{\max}(X) + E_{\max}(Y)$.

Let x, y be strings of length n such that $K(x|y) = K(y|x) = n$, $K(x), K(y) \geq n$, $X = (x, x)$ and $Y = (y, y)$. Then $E_{\max}(XY) = E_{\max}(x, x, y, y) = \max\{K(x|y), K(y|x)\} = n$, $E_{\max}(X) = 0$ and $E_{\max}(Y) = 0$. Hence, $E_{\max}(XY) > E_{\max}(X) + E_{\max}(Y)$. ■

Let $X = (x)$ and $Y = (y)$. Note that subadditivity holds for lists of singleton elements since $E_{\max}(x, y) = \max\{K(x|y), K(y|x)\} \leq K(x) + K(y)$, where the equality holds up to an additive $O(\log\{K(x|y), K(y|x)\})$ term and the inequality holds up to an additive constant term.

VII. MINIMAL OVERLAP

Let $X = (x_1, \dots, x_m)$ and p_i be a shortest program converting x_i to X ($1 \leq i \leq m$). Naively we expect that the shortest program that maps x_i to X contains the information about X that is lacking in x_i . However, this is too simple, because different short programs mapping x_i to X may have different properties.

For example, suppose $X = \{x, y\}$ and both elements are strings of length n with $K(x|y), K(y|x) \geq n$. Let p be a program that ignores the input and prints x . Let q be a program such that $y \oplus q = x$ (that is, $q = x \oplus y$), where \oplus denotes bitwise addition modulo 2. Then, the programs p and q have nothing in common.

Now let x and y be arbitrary strings of length at most n . Muchnik [29], see also the textbook [28, Theorem 8.3.7], shows that there exists a shortest program p that converts y to x (that is, $|p| = K(x|y)$ and $K(x|p, y) = O(\log n)$), such that p is simple with respect to x and therefore depends little on the origin y , that is, $K(p|x) = O(\log n)$. This is a fundamental coding property for individual strings that parallels related results about random variables known as the Slepian-Wolf and Csiszár-Körner-Martón theorems [11].

Theorem 7.1: Let $X = (x_1, \dots, x_m)$ be a list of binary strings of length at most n . For every $x_i \in X$ there exists a string p_i of length $K(X|x_i)$ such that $K(p_i|X) = O(\log mn)$ and $K(X|p_i, x_i) = O(\log mn)$.

Proof: Muchnik's theorem as stated before gives a code p for x when y is known. There, we assumed that x and y have length at most n . The proof in [29] does not use any assumption about y . Hence we can extend the result to information distance in finite lists as follows. Suppose we encode the constituent list elements of X self-delimitingly in altogether $mn + O(\log mn)$ bits (now X takes the position of x and we consider strings of length at most $mn + O(\log mn)$). Substitute y by x_i for some i ($1 \leq i \leq m$). Then the theorem above follows straightforwardly from Muchnik's original theorem about two strings of length at most n . ■

The code p_i is not uniquely determined. For example, let $X = (x, y)$ and z be a string such that $|x| = |y| = |z| = n$, $K(y|z) = K(z|y) \geq n$ and $x = y \oplus z$. Then, both z and $y \oplus z$ can be used

for p with $K(p|X) = O(1)$ and $K(X|p, y) = O(1)$. But z and $y \oplus z$ have no mutual information at all.

Corollary 7.2: Let $X = (x_1, \dots, x_m)$. For every string x_i there is a program p_i such that $U(p_i, x_i) = X$ ($1 \leq i \leq m$), where $|p_i| = K(X|x_i)$ and $K(p_i) - K(p_i|p_j) = K(p_j) - K(p_j|p_i) = 0$ ($i \neq j$) and the last four equalities hold up to an additive $O(\log K(X))$ term.

VIII. NORMALIZED LIST INFORMATION DISTANCE

The quantitative difference in a certain feature between many objects can be considered as an admissible distance, provided it is upper semicomputable and satisfies the density condition (V.1). Theorem 5.2 shows that E_{\max} is universal in that among all admissible list distances in that always least. That is, it accounts for the dominant feature in which the elements of the given list are alike. Many admissible distances are absolute, but if we want to express similarity, then we are more interested in relative ones. For example, if two strings of 1 000 000 bits have information distance 1000 bits, then we are inclined to think that those strings are relatively similar. But if two strings of 1000 bits have information distance 1000 bits, then we find them very different.

Therefore, our objective is to normalize the universal information distance E_{\max} to obtain a universal similarity distance. It should give a similarity with distance 0 when the objects in a list are maximally similar (that is, they are equal) and distance 1 when they are maximally dissimilar. Naturally, we desire the normalized version of the universal list information distance metric to be also a metric.

For pairs of objects, say x, y , the normalized version e of E_{\max} defined by

$$e(x, y) = \frac{E_{\max}(x, y)}{\max\{K(x), K(y)\}} = \frac{\max\{K(x, y|x), K(x, y|y)\}}{\max\{K(x), K(y)\}} \tag{VIII.1}$$

takes values in $[0, 1]$ and is a metric. Several alternatives for the normalizing factor $1/\max\{K(x), K(y)\}$ do not work. Dividing by the length, either the sum or the maximum does not satisfy the triangle property. Dividing by $K(x, y)$ results in $e_1(x, y) = E_{\max}(x, y)/K(x, y) = \frac{1}{2}$ for $|x| = |y| = n$ and $K(x|y) = K(y|x) \geq n$ (and hence $K(x), K(y) \geq n$) and this is improper as $e_1(x, y)$ should be 1 in this case. We would like a proposal for a normalization factor for lists of more than two elements to reduce to that of (VIII.1) for lists restricted to two elements. This leads to the proposals below, which turn out to be improper.

As a counterexample to normalization take the following lists: $X = (x), Y = (y)$ and $Z = (y, y)$. With $|x| = |y| = n$ and the equalities below up to an $O(\log n)$ additive term we define: $K(x) = K(x|y) = K(x, y|y) = K(x, y, y|y) = n$, $K(y|x) = K(y, y|x) = K(x, y, y|x) = 0.9n$ and $K(y) = K(y, y) = 0.9n$. Using the symmetry of information (A.1) we have $K(x, y) = 1.9n$. Let U, V, W be lists. We show that for the proposals below the triangle property $e(UV) \leq e(UW) + e(WV)$ is violated.

- Consider the normalized list information distance

$$e(V) = \frac{K(V|x_V)}{K(V_{\max})}. \tag{VIII.2}$$

That is, we divide $E_{\max}(V)$ by $K(V_{\max})$. Here, $K(V|x_V) = E_{\max}(V)$, with $K(V_{\max}) = \max_i\{K(V_i)\}$ where the list V_i equals the list V with the i th element deleted ($1 \leq i \leq |V|$). Then, with equalities holding up to $O((\log n)/n)$ we have: $e(XY) = K(x, y|y)/K(x) = 1$, $e(XZ) = E_{\max}(XZ)/K((XZ)_{\max}) = K(x, y, y|y)/K(x, y) \approx \frac{1}{2}$ and $e(ZY) = E_{\max}(ZY)/K((ZY)_{\max}) = K(y, y, y|y)/K(y, y) = 0$. Hence the triangle inequality does not hold.

- Instead of dividing by $K(V_{\max})$ in (VIII.2) divide by $K(V')$ where V' equals V with x_V deleted. The same counterexample to the triangle inequality holds.
- Instead of dividing by $K(V_{\max})$ in (VIII.2) divide by $K(\{V_{\max}\})$ where $\{V_{\max}\}$ is the set of elements in V_{\max} . To equate the sets approximately with the corresponding lists, change Z to $\{y_1, y_2\}$ where y_i equals y but with the i th bit flipped ($i = 1, 2$). Again, the triangle inequality does not hold.
- Instead of dividing by $K(V')$ in (VIII.2) divide by $K(\{V'\})$ where $\{V'\}$ is the set of elements in V' . Change Z as in the previous item. Again, the triangle inequality does not hold.

APPENDIX

KOLMOGOROV COMPLEXITY THEORY

Theory and applications are given in the textbook [28]. Here we give some relations that are needed in the paper. The *information about x contained in y* is defined as $I(y : x) = K(x) - K(x|y)$. A deep and very useful, result due to Levin and Kolmogorov [44] and for the prefix version [16] called *symmetry of information* shows that

$$K(x, y) = K(x) + K(y|x) = K(y) + K(x|y) \tag{A.1}$$

with the equalities holding up to $O(\log K)$ additive precision. Here, $K = \max\{K(x), K(y)\}$. Hence, up to an additive logarithmic term $I(x : y) = I(y : x)$ and we call this the *mutual (algorithmic) information* between x and y .

The *universal a priori probability* of x is $Q_U(x) = \sum_{U(p)=x} 2^{-|p|}$. The following results are due to Levin [24].

There exists a lower semicomputable function $\mathbf{m} : \{0, 1\}^* \rightarrow [0, 1]$ with $\sum_x \mathbf{m}(x) \leq 1$, such that for every lower semicomputable function $P : \{0, 1\}^* \rightarrow [0, 1]$ with $\sum_x P(x) \leq 1$ we have

$$2^{K(P)} \mathbf{m}(x) \geq P(x) \tag{A.2}$$

for every x . Here $K(P)$ is the length of a shortest program for the reference universal prefix Turing machine to lower semicompute the function P . For every $x \in \{0, 1\}^*$

$$K(x) = -\log Q_U(x) = -\log \mathbf{m}(x) \tag{A.3}$$

with equality up to an additive constant independent of x . Thus, the Kolmogorov complexity of a string x coincides up to an ad-

ditive constant term with the logarithm of $1/Q_U(x)$ and also with the logarithm of $1/m(x)$. This result is called the ‘‘Coding Theorem’’ since it shows that the shortest upper semicomputable code is a Shannon-Fano code of the greatest lower semicomputable probability mass function.

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Paul M. B. Vitányi received the Ph.D. degree from the Free University of Amsterdam in 1978.

He is a CWI Fellow with the National Research Institute for Mathematics and Computer Science, Netherlands, CWI, and Professor of Computer Science with the University of Amsterdam. He has worked on cellular automata, computational complexity, distributed and parallel computing, machine learning and prediction, physics of computation, Kolmogorov complexity, information theory, quantum computing, publishing about 200 research papers and some books. Together with M. Li, they pioneered applications of Kolmogorov complexity and coauthored *An Introduction to Kolmogorov Complexity and its Applications* (New York: Springer-Verlag, 1993), (3rd ed. 2008) parts of which have been translated into Chinese, Russian, and Japanese. Web page: <http://www.cwi.nl/~paulv/>

Dr. Vitányi serves on the editorial boards of *Distributed Computing* (1987–2003), *Information Processing Letters*, *Theory of Computing Systems*, *Parallel Processing Letters*, *International Journal of Foundations of Computer Science*, *Entropy*, *Information*, *Journal of Computer and Systems Sciences* (guest editor), and elsewhere. He received a Knighthood (Ridder in de Orde van de Nederlandse Leeuw) in 2007.