All Schatten spaces endowed with the Schur product are Q-algebras

Jop Briët[a,1,*], Harry Buhrman[a,1], Troy Lee[b], Thomas Vidick[c]

[a] *CWI and University of Amsterdam. Science Park 123, 1098 SJ, Amsterdam, The Netherlands.*
[b] *Centre for Quantum Technologies, Block S15, 3 Science Drive 2 Singapore 117543*
[c] *Computer Science Division, 615 Soda Hall, University of California at Berkeley, Berkeley, CA 94720, USA*

Abstract

We prove that the Banach algebra formed by the space of compact operators on a Hilbert space endowed with the Schur product is a quotient of a uniform algebra (also known as a Q-algebra). Together with a similar result of Pérez-García for the trace class, this completes the answer to a long-standing question of Varopoulos.

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In this paper, we consider the commutative Banach algebras formed by p-Schatten spaces for $1 \leq p \leq \infty$ on the Hilbert space ℓ_2 endowed with the Schur product. In particular, we deal with the problem of determining if these algebras are quotients of a uniform algebra (Q-algebra).

The spectral theorem asserts that the space of compact operators on ℓ_2, which we denote by S_∞, consists of the operators A that admit a representation of the form

$$A = \sum_{i=1}^{\infty} \lambda_i \langle \cdot, e_i \rangle f_i,$$

where $(e_i)_i$ and $(f_i)_i$ are orthonormal bases for ℓ_2 and the sequence $(\lambda_i)_i \subset \mathbb{R}$ satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\lim_{i\to\infty} \lambda_i = 0$. The space S_∞ is endowed with the norm $\|A\| = \sup\{|\langle x, Ay\rangle| : \|x\|, \|y\| \leq 1\}$. For $1 \leq p < \infty$, the

*Corresponding author. Phone: +31 20 592 4051. Fax: +31 20 592 4199. Postal address: Science Park 123, 1098 SJ, Amsterdam, The Netherlands.

Email addresses: `j.briet@cwi.nl` (Jop Briët), `buhrman@cwi.nl` (Harry Buhrman), `troylee@gmail.com` (Troy Lee), `vidick@eecs.berkeley.edu` (Thomas Vidick)

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Schatten p-norm of A is given by $(\mathrm{Tr}|A|^p)^{1/p}$ where $|A| = (A^*A)^{1/2}$. The *p-Schatten space* $S_p \subseteq S_\infty$ is the subspace of compact operators that have finite Schatten p-norm. Common examples of these spaces are the trace class S_1 and the Hilbert-Schmidt operators S_2. The Schur product $*$ (also known as the Hadamard product) is a continuous and commutative multiplication for S_∞ defined as the entry-wise product when the elements of S_∞ are represented by matrices using the canonical basis for ℓ_2. Endowed with the Schur product, p-Schatten spaces form the commutative Banach algebras $(S_p, *)$. A commutative Banach algebra is said to be *uniform* if it is isometrically isomorphic to a closed subalgebra of $\mathcal{C}(K)$ the space of continuous functions on a closed Hausdorff topological vector space K.

Definition 1. *Let $\mathcal{X}$ be a commutative Banach algebra. Then $\mathcal{X}$ is a* Q-algebra *if there exists a uniform algebra $\mathcal{Y}$ and a closed ideal $\mathcal{I} \subseteq \mathcal{Y}$ such that $\mathcal{X}$ is isomorphic, as a Banach algebra, to the quotient algebra $\mathcal{Y}/\mathcal{I}$.*

The most interesting feature of Q-algebras, discovered by Cole (see [20]), is that they are isometrically isomorphic to a closed (commutative) subalgebra of $B(\mathcal{H})$, the algebra of bounded operators on a Hilbert space. In other words, Q-algebras are commutative operator algebras. In general, the converse is false [18], but Tonge [16] showed that it is true for every algebra generated by a set of commuting Hilbert-Schmidt operators when equipped with the regular matrix product.

Davie [5] and Varopoulos [17] proved that the Banach algebra $(\ell_p, *)$ is a Q-algebra for all $1 \leq p \leq \infty$. Since the space of Hilbert-Schmidt operators is isometrically isomorphic to ℓ_2 it follows immediately that $(S_2, *)$ is also a Q-algebra. Varopoulos [19] asked the natural question if the same is true for all non-commutative analogues $(S_p, *)$.

Is it true that $(S_p, *)$ is a Q-algebra for all $1 \leq p \leq \infty$*?*

Recently, progress on this question was made by Le-Merdy [12] and Pérez-García [13], who proved that the property holds true for all $2 \leq p \leq 4$ and $1 \leq p \leq 2$, respectively. Mantero and Tonge [11] proved that $(S_\infty, *)$ fails to be a 1-summing algebra, which requires slightly stronger conditions than for being a Q-algebra. Nevertheless, in this paper we give a positive result for the high end of the spectrum.

Theorem 1. *The Banach algebra $(S_\infty, *)$ is a Q-algebra.*

A related result of Varopoulos himself [17] which characterizes the algebras $(S_p, *)$ for the intermediate values $1 < p < \infty$ via the complex interpolation method as intermediate algebras of the couple $\big((S_1, *), (S_\infty, *)\big)$, implies that the answer to his question is in fact completed.

Corollary 2. *For any $1 \leq p \leq \infty$, the Banach algebra $(S_p, *)$ is a Q-algebra.*

The proof of Theorem 1 relies on a simple characterization of Q-algebras due to Davie [5, Theorem 3.3]. We use a slight reformulation of it, as given in [7, Lemma 18.5 and Proposition 18.6]. Let $\mathbb{T}$ denote the closed unit disc in $\mathbb{C}$ and for Banach space X let $\mathcal{B}_X = \{A \in X : \|A\| \leq 1\}$ denote the unit ball in X. For positive integers n, N let $\{1, \ldots, n\}^N$ denote the N-fold Cartesian product of the set $\{1, \ldots, n\}$. For complex tensor $T : \{1, \ldots, n\}^N \to \mathbb{C}$, we abbreviate the coordinates $(i_1, \ldots, i_N) \in \{1, \ldots, n\}^N$ of T by I. We define the norm $\|T\|_\infty$ to be

$$\sup \left\{ \Big| \sum_{I \in \{1,\ldots,n\}^N} T[I] \chi_1(i_1) \cdots \chi_N(i_N) \Big| : \chi_1, \ldots, \chi_N : \{1, \ldots, n\} \to \mathbb{T} \right\}.$$

Theorem 3 (Davie). *Let $\mathcal{X} = (X, \cdot)$ be a commutative Banach algebra. Then $\mathcal{X}$ is a Q-algebra if and only if there exists a universal constant $K > 0$, such that for every choice of positive integers n, N, complex tensor $T : \{1, \ldots, n\}^N \to \mathbb{C}$, and X-valued sequences $A_1, \ldots, A_N : \{1, \ldots, n\} \to \mathcal{B}_X$, the inequality*

$$\Big\| \sum_{I \in \{1,\ldots,n\}^N} T[I] A_1(i_1) \cdots A_N(i_N) \Big\|_X \leq K^N \|T\|_\infty, \tag{1}$$

holds.

We prove that $(S_\infty, *)$ satisfies Davie's criterion using a multilinear generalization of the famous Grothendieck inequality, due to Blei [2] and Tonge [16] (see also [4]). The (complex) Grothendieck inequality [8, 10] states that there exists a universal constant K_G such that for every positive integer n, complex matrix $M \in \mathbb{C}^{n \times n}$ and complex vectors $x(1), \ldots, x(n), y(1), \ldots, y(n)$ in $\mathcal{B}_{\ell_2}$, the inequality

$$\left| \sum_{i,j=1}^{n} M_{ij} \langle x(i), y(j) \rangle \right| \leq K_G \|M\|_\infty,$$

holds. Currently the exact value of K_G is unknown, but it is known to be bounded as $1.3380 \lesssim K_G \lesssim 1.4049$. The lower and upper bounds on K_G were proved by Davie [6] and Haagerup [9], respectively.

For vector $x \in \ell_2$, we will denote by x_ℓ, the number $\langle x, e_\ell \rangle$, where $e_1, e_2, \ldots$ are the canonical basis vectors for ℓ_2.

The multilinear extension of Grothendieck's inequality we use replaces the matrix M by a complex N-tensor T, and the inner product of pairs of unit vectors by the multilinear form (the *generalized inner product*) on N-tuples of vectors $x_1, \ldots, x_N \in \ell_2$ given by

$$\langle x_1, \ldots, x_N \rangle = \sum_{\ell=1}^{\infty} (x_1)_\ell \cdots (x_N)_\ell.$$

Theorem 4 (Tonge). *For all positive integers n, N, any complex tensor $T : \{1,\ldots,n\}^N \to \mathbb{C}$ and sequences $x_1,\ldots,x_N : \{1,\ldots,n\} \to \mathcal{B}_{\ell_2}$, the inequality*

$$\Big|\sum_{I\in\{1,\ldots,n\}^N} T[I]\big\langle x_1(i_1),\ldots,x_N(i_N)\big\rangle\Big| \le 2^{(N-2)/2}K_G\|T\|_\infty \tag{2}$$

holds.

This inequality was also used by Pérez-García [13] to prove that $(S_1,*)$ is a Q-algebra.

Proof of Theorem 1: We fix integers $n, N \in \mathbb{N}$, tensor $T : \{1,\ldots,n\}^N \to \mathbb{C}$ and operator-valued maps $A_1,\ldots,A_N : \{1,\ldots,n\} \to \mathcal{B}_{S_\infty}$. Define

$$M = \sum_{I\in\{1,\ldots,n\}^N} T[I]A_1(i_1) * \cdots * A_N(i_N).$$

By Theorem 3 (Davie's criterion) it suffices to show that the inequality

$$\|M\| \le K^N\|T\|_\infty, \tag{3}$$

holds for some constant K independent of n, N, T and $A_1,\ldots,A_N$.

We begin by making four small preliminary steps to show that without loss of generality we may assume that T is real valued and the A_i are finite-dimensional Hermitian matrices. Afterwards we will be able to apply Theorem 4 in order to prove Eq. (3). In the first step we show that without loss of generality, we may assume that the tensor T is real-valued. To this end, define the real-valued tensors T_R and T_C by $T_R[I] = \Re(T[I])$ and $T_C[I] = \Im(T[I])$ for every $I \in \{1,\ldots,n\}^N$. Define

$$\begin{aligned} M_R &= \sum_{I\in\{1,\ldots,n\}^N} T_R[I]A_1(i_1) * \cdots * A_N(i_N) \\ M_C &= \sum_{I\in\{1,\ldots,n\}^N} T_C[I]A_1(i_1) * \cdots * A_N(i_N) \end{aligned}$$

Since $M = M_R + iM_C$, we have $\|M\| \le 2\max\{\|M_R\|, \|M_C\|\}$. Proving Eq. (3) for real-valued tensors thus suffices.

In the second step we show that it suffices to consider the case where the operators $A_1(i_1),\ldots,A_N(i_N) \in \mathcal{B}_{S_\infty}$ are finite-dimensional matrices (in the canonical basis for ℓ_2). Recall that norm of M is given by

$$\|M\| = \sup\{|\langle u, Mv\rangle| : u, v \in \mathcal{B}_{\ell_2}\}.$$

For any $u \in \ell_2$ with $\|u\| \le 1$ and any $\varepsilon > 0$ there exists a $D \in \mathbb{N}$ such that the vector $u' = \sum_{\ell=1}^D u_\ell e_\ell$ has norm at least $1-\varepsilon$. Hence, for any $u, v \in \mathcal{B}_{\ell_2}$ and $\varepsilon > 0$ there exist $D \in \mathbb{N}$ and $u', v' \in \mathcal{B}_{\ell_2}$ supported only on $e_1,\ldots,e_D$ such that

$$|\langle u, Mv\rangle| \le |\langle u', Mv'\rangle| + \big(2\varepsilon(1-\varepsilon)+\varepsilon^2\big)|\langle u, Mv\rangle|.$$

It follows that for some $D \in \mathbb{N}$ and vectors $u', v' \in \mathcal{B}_{\ell_2}$ supported only on $e_1, \ldots, e_D$, we have

$$\|M\| \leq 2|\langle u', Mv' \rangle|. \tag{4}$$

Define for every $k = 1, \ldots, N$ and $i_k = 1, \ldots, n$ the D-by-D complex matrix $A'_k(i_k) = (\langle e_\ell, A_k(i_k) e_m \rangle)_{\ell,m=1}^D$. Note that $\|A'_k(i_k)\| \leq \|A_k(i_k)\| \leq 1$. Expanding the definition of M then gives

$$\langle u', Mv' \rangle = \Big\langle u', \sum_{I \in \{1,\ldots,n\}^N} T[I] A_1(i_1) * \cdots * A_N(i_N) v' \Big\rangle = \sum_{I \in \{1,\ldots,n\}^N} T[I] \langle u', A_1(i_1) * \cdots * A_N(i_N) v' \rangle = \sum_{I \in \{1,\ldots,n\}^N} T[I] \langle u', A'_1(i_1) * \cdots * A'_N(i_N) v' \rangle. \tag{5}$$

Define the complex number $\Theta = \langle u', Mv' \rangle$. Eq. (4) shows that to prove the theorem, it suffices to show that the inequality

$$|\Theta| \leq K^N \|T\|_\infty, \tag{6}$$

holds for some constant K, and Eq. (5) shows that we can write Θ using the matrix-valued maps $A'_1, \ldots, A'_N$.

In the third step we absorb the complex part of the number Θ into the matrix-valued map A'_1. Let us write Θ in polar coordinates as $|\Theta| e^{i\phi}$ for some $\phi \in [0, 2\pi]$. Define $A''_1(i_1) = e^{-i\phi} A'_1(i_1)$. Then by Eq. (5), we have

$$\sum_{I \in \{1,\ldots,n\}^N} T[I] \langle u', A''_1(i_1) * A'_2(i_2) * \cdots * A'_N(i_N) v' \rangle = |\Theta|. \tag{7}$$

In the fourth step we symmetrize the situation by making the matrices Hermitian. To this end, define the map $\rho : \mathbb{C}^{D \times D} \to \mathbb{C}^{2D \times 2D}$ by

$$\rho(A) = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}.$$

Define matrix-valued maps $B_1, \ldots, B_N : \{1, \ldots, n\} \to \mathbb{C}^{2D \times 2D}$ by

$$\begin{aligned} B_1(i_1) &= \rho\big(A''_1(i_1)\big) \\ B_2(i_2) &= \rho\big(A'_2(i_2)\big) \\ &\vdots \\ B_N(i_N) &= \rho\big(A'_N(i_N)\big). \end{aligned}$$

Note that $\|B_k(i_k)\| \leq 1$ for all $k = 1, \ldots, N$ and $i_k = 1, \ldots, n$, since the map ρ leaves the norm unchanged. Define the matrices

$$\begin{aligned} M' &= \sum_{I \in \{1,\ldots,n\}^N} T[I] A''_1(i_1) * A'_2(i_2) * \cdots * A'_N(i_N) \\ M'' &= \sum_{I \in \{1,\ldots,n\}^N} T[I] B_1(i_1) * B_2(i_2) * \cdots * B_N(i_N). \end{aligned}$$

Since the tensor T is real-valued we have $M'' = \rho(M')$.

Define the vector $w = (v' \oplus u')/\sqrt{2}$ and note that $\|w\| \leq 1$. We have

$$\begin{aligned}
\langle w, M''w\rangle &= \frac{1}{2}[(u')^*, (v')^*] \begin{bmatrix} 0 & M' \\ (M')^* & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \\
&= \Re\big(\langle u', M'v'\rangle\big) \\
&= \Re\left(\sum_{I\in\{1,\dots,n\}^N} T[I]\langle u', A_1''(i_1) * \cdots * A_N'(i_N)v'\rangle\right) \\
&= |\Theta|, \qquad (8)
\end{aligned}$$

where the last identity follows from Eq. (7), which shows that the term between brackets on the third line is the real number $|\Theta|$.

Next, we absorb the complex parts of the vector w into the matrix-valued map B_1. Using polar coordinates we can write

$$w = \sum_{\ell=1}^{2D} w_\ell e^{i\psi_\ell} e_\ell$$

for some moduli $w_\ell \in \mathbb{R}_+$ and arguments $\psi_\ell \in [0, 2\pi]$. Let $U \in \mathbb{C}^{D\times D}$ be the diagonal unitary matrix given by $U = \operatorname{diag}(e^{i\psi_1}, \dots, e^{i\psi_D})$. Define the non-negative real vector $w' = U^*w = \sum_{\ell=1}^{2D} w_\ell e_\ell$ and define the matrix-valued map B_1' by $B_1'(i_1) = U^*B_1(i_1)U$. Note that $\|B_1'(i_1)\| \leq \|B_1(i_1)\| \leq 1$.

Then, by Eq. (8) and by expanding the definition of M'' we have

$$\sum_{I\in\{1,\dots,n\}^N} T[I]\langle w', B_1'(i_1) * B_2(i_2) * \cdots B_N(i_N)w'\rangle = \langle w, M''w\rangle = |\Theta|. \qquad (9)$$

We can now make a connection to Theorem 4 using the following two claims.

Claim 5. *There exist real numbers $\mu_1, \dots, \mu_{2D} \geq 0$ such that*

$$0 \leq \sum_{\ell,m=1}^{2D} \mu_\ell \mu_m \min\{\ell, m\} \leq 1 \qquad (10)$$

and for $1_\ell = e_1 + \cdots + e_\ell$,

$$|\Theta| = \sum_{\ell,m=1}^{2D} \mu_\ell \mu_m \theta_{\ell,m}, \qquad (11)$$

where

$$\theta_{\ell,m} = \sum_{I\in\{1,\dots,n\}^N} T[I]\langle 1_\ell, B_1'(i_1) * B_2(i_2) * \cdots * B_N(i_N)1_m\rangle.$$

Proof: By relabeling the basis vectors $e_1, \ldots, e_{2D}$ appropriately, we may assume that the coefficients of the above vector w' satisfy $w_1 \geq w_2 \geq \cdots \geq w_{2D}$. Setting $\mu_\ell = (w_\ell - w_{\ell-1})$ for $\ell = 1, \ldots, 2D-1$ and $\mu_{2D} = w_{2D}$ gives

$$w' = \sum_{\ell=1}^{2D} \mu_\ell 1_\ell,$$

since $\langle w', e_k \rangle = \mu_k + \mu_{k+1} + \cdots + \mu_{2D} = w_k$. Eq. (10) follows as $0 \leq \langle w', w' \rangle \leq 1$ and $\langle 1_\ell, 1_m \rangle = \min\{\ell, m\}$, and Eq. (11) follows by expanding w' in Eq. (9). □

Claim 6. *For every $1 \leq \ell, m \leq 2D$, we have*

$$|\theta_{\ell,m}| \leq C_N \min\{\ell, m\} \|T\|_\infty, \tag{12}$$

where $C_N = 2^{(N-2)/2} K_G$.

Proof: Expanding the vectors 1_ℓ in the canonical basis gives

$$\Big\langle 1_\ell, B_1'(i_1) * B_2(i_2) * \cdots * B_N(i_N) 1_m \Big\rangle = \sum_{s=1}^{\ell} \sum_{t=1}^{m} \Big\langle e_s, B_1'(i_1) * B_2(i_2) * \cdots * B_N(i_N) e_t \Big\rangle. \tag{13}$$

Note that each term in the double sum on the right-hand side of Eq. (13) is simply the product of (s,t)-entries of the matrices $B_1'(i_1), B_2(i_2), \ldots, B_N(i_N)$.

Suppose that $\ell \leq m$. Since the matrices $B_1'(i_1), B_2(i_2), \ldots, B_N(i_N)$ have norm at most 1, their rows belong to $\mathcal{B}_{\ell_2^m}$ (where ℓ_2^m is the set of length-m 2-summable sequences). Hence, the inner sum on the right-hand side of Eq. (13),

$$\sum_{t=1}^{m} \Big\langle e_s, B_1'(i_1) * B_2(i_2) \cdots * B_N(i_N) e_t \Big\rangle = \sum_{t=1}^{m} \langle e_s, B_1'(i_1) e_t \rangle \langle e_s, B_2(i_2) e_t) \rangle \cdots \langle e_s, B_N(i_N) e_t \rangle,$$

is the generalized inner product of a set of N vectors in $\mathcal{B}_{\ell_2^m}$. The result for the case $\ell \leq m$ now follows from the triangle inequality and Theorem 4, as

$$|\theta_{\ell,m}| = \left| \sum_{I \in \{1,\ldots,n\}^N} T[I] \Big\langle 1_\ell, B_1'(i_1) * B_2(i_2) * \cdots * B_N(i_N) 1_m \Big\rangle \right| \leq \sum_{s=1}^{\ell} \left| \sum_{I \in \{1,\ldots,n\}^N} T[I] \sum_{t=1}^{m} \langle e_s, B_1'(i_1) e_t \rangle \langle e_s, B_2(i_2) e_t) \rangle \cdots \langle e_s, B_N(i_N) e_t \rangle \right| \leq \ell 2^{(N-2)/2} K_G \|T\|_\infty.$$

The case $\ell \geq m$ is proved in the same manner. ♦

Putting Claim 5 and Claim 6 together gives

$$\begin{aligned}
|\Theta| &= \sum_{\ell,m=1}^{2D} \mu_\ell \mu_m \theta_{\ell,m} \\
&\leq \sum_{\ell,m=1}^{2D} \mu_\ell \mu_m |\theta_{\ell,m}| \\
&\leq C_N \|T\|_\infty \sum_{\ell,m=1}^{2D} \mu_\ell \mu_m \min\{\ell, m\} \\
&\leq C_N \|T\|_\infty.
\end{aligned}$$

We conclude that Eq. (6) (Davie's criterion) holds for $K \leq 4$. □

Corollary 2 now follows directly from the following two lemmas and the fact that both $(S_1, *)$ and $(S_\infty, *)$ are Q-algebras. Pietsch and Triebel [15] characterized the p-Schatten spaces for the intermediate values $1 < p < \infty$ via the complex interpolation method (see [1] for a detailed account).

Lemma 7 (Pietsch and Triebel). *For $0 \leq \theta \leq 1$, denote by $(S_\infty, S_1)_{[\theta]}$ the Banach space obtained via the complex interpolation method. Then, for $p = 1/\theta$, we have $(S_\infty, S_1)_{[\theta]} = S_p$.*

Varopoulos [17] proved that the property of being a Q-algebra is inherited under the complex interpolation method if it holds for both parent algebras.

Lemma 8 (Varopoulos). *Let $(\mathcal{X}_0, \mathcal{X}_1)$ be a compatible pair of complex Banach algebras. For $0 < \theta < 1$, denote by $(\mathcal{X}_0, \mathcal{X}_1)_{[\theta]}$ the Banach algebra obtained via the complex interpolation method. If $\mathcal{X}_0$ and $\mathcal{X}_1$ are Q-algebras, then $(\mathcal{X}_0, \mathcal{X}_1)_{[\theta]}$ is a Q-algebra.*

Remark 1. *Surprisingly, the main result of this paper came about in the context of quantum information theory [3], after a translation to an equivalent problem in this field was given by Pérez-García et al. [14].*

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