AN ε-UNIFORM DEFECT-CORRECTION METHOD FOR A PARABOLIC CONVECTION-DIFFUSION PROBLEM

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The Dirichlet problem for a singularly perturbed parabolic equation with convective terms is considered on an interval. For the sufficiently smooth data, it is easy to construct a piecewise uniform mesh condensing in the boundary layer and a standard finite difference operator, which give an ε-uniformly convergent difference scheme with the accuracy $\mathcal{O}(N^{-1}\ln^2 N + K^{-1})$, where $N+1$ and $K+1$ is the number of nodes in the space and time meshes, respectively. Here ε-uniformly convergent schemes of high-order time-accuracy are constructed by a defect correction technique (by correction of the right-hand side). It allows us to achieve an arbitrary high order of accuracy $\mathcal{O}(N^{-1}\ln^2 N + K^{-n})$, $n \geq 2$. The efficiency of the new defect-correction scheme is confirmed by a series of numerical experiments.

1 Introduction

In this paper we study ε-uniform numerical methods, i.e. methods the accuracy of which is independent of the parameter ε, for time-dependent singular perturbation problems. For such problems ε-uniformly convergent difference schemes were introduced and analysed in Shishkin [1]. If the problem data are sufficiently smooth, the accuracy of these schemes is $\mathcal{O}(N^{-1}\ln^2 N + K^{-1})$, where N and K denote, respectively, the number of intervals in the space and time discretisation. The amount of computational work for this scheme is proportional to K. Therefore, it is natural to seek for a method that allows us to achieve the higher-order accuracy in time, without essentially increasing the amount of computational work. For reaction-diffusion equations, the improvement of time-accuracy by means of a defect correction technique, maintaining ε-uniform convergence, was studied in Hemker *et al* [2,3]. Particular attention was paid to the case of a Neumann problem in Hemker *et al* [4]. In the present paper we develop a new approach, also based on the defect correction method, but for the new class of problems, including convection-diffusion equations, that confirms the efficiency of this promising method for wide classes of singularly perturbed problems.

2 Problem Formulation

On the domain $G = (0,1) \times (0,T]$, with the boundary $S = \overline{G} \setminus G$, we consider the Dirichlet problem for the singularly perturbed parabolic equation with convective terms [a] :

$$L_{(1)}\, u(x,t) \equiv \left\{ \varepsilon\, a(x,t) \frac{\partial^2}{\partial x^2} + b(x,t)\, \frac{\partial}{\partial x} - c(x,t) - \right. \tag{1a}$$

$$\left. -p(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \quad (x,t) \in G, \quad \varepsilon \in (0,1]$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{1b}$$

For $S = S_0 \cup S^L$, we distinguish the lower base $S_0 = \{(x,t):\ x \in [0,1],\, t=0\}$ and the lateral boundary $S^L = S_0^L \cup S_1^L$, where S_0^L is the outflow part of the boundary, $S_d^L = \{(x,t):\ x = d,\ t \in (0,T]\}$, $d = 0,1$. As $\varepsilon \to 0$ in (1a), in the neighbourhood of the set S_0^L the solution exhibits a boundary layer, which is described by an ordinary differential equation (a regular layer).

In (1) $a(x,t), b(x,t), c(x,t), p(x,t), f(x,t)$, $(x,t) \in \overline{G}$ and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth and bounded functions which satisfy $a(x,t) \geq a_0 > 0$, $b(x,t) \geq b_0 > 0$, $p(x,t) \geq p_0 > 0$, $c(x,t) \geq 0$, $(x,t) \in \overline{G}$.

3 A classical finite difference scheme

To solve problem (1), we first consider a classical finite difference method. On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{2}$$

where $\overline{\omega}$ is a (in general) non-uniform mesh on $[0,1]$, $\overline{\omega}_0$ is a uniform mesh on the interval $[0,T]$. We define $\tau = T/K$, $h^i = x^{i+1} - x^i$, $h = \max_i h^i$, $h \leq MN^{-1}$, N and K are the numbers of intervals in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively. [b]

For problem (1) we use the difference scheme (cf. Samarskii [5])

$$\Lambda_{(3)} z(x,t) = f(x,t), \quad (x,t) \in G_h, \tag{3a}$$

$$z(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{3b}$$

Here $\quad G_h = G \cap \overline{G}_h, \quad S_h = S \cap \overline{G}_h,$

$$\Lambda_{(3)} z(x,t) \equiv \left\{ \varepsilon\, a(x,t) \delta_{\overline{x}\widehat{x}} + b(x,t)\delta_x - c(x,t) - p(x,t)\delta_{\overline{t}} \right\} z(x,t),$$

$$\delta_{\overline{x}\widehat{x}} z(x^i,t) = 2(h^{i-1} + h^i)^{-1} [\delta_x z(x^i,t) - \delta_{\overline{x}} z(x^i,t)],$$

[a] The notation $L_{(a)}$ denotes that this operator is first introduced in Eq. (a).

[b] Here and in what follows we denote by M (or m) sufficiently large (or small) positive constants which do not depend on the values of the parameter ε or on the difference operators.

$$\delta_{\overline{x}} z(x^i,t) = (h^{i-1})^{-1}\left(z(x^i,t) - z(x^{i-1},t)\right),$$
$$\delta_x z(x^i,t) = (h^i)^{-1}\left(z(x^{i+1},t) - z(x^i,t)\right),$$
$$\delta_{\overline{t}} z(x^i,t) = \tau^{-1}\left(z(x^i,t) - z(x^i,t-\tau)\right),$$

$\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are the forward and backward differences, $\delta_{\overline{x}\widehat{x}} z(x,t)$ is the usual second difference of $z(x,t)$ on the non-uniform mesh.

The difference scheme (3), (2) is monotone (see Samarski [5]). Using the maximum principle and taking into account a-priori estimates of the derivatives, we find that the difference scheme (3), (2) converges for a fixed value of the parameter ε (see also Shishkin [6]):

$$| u(x,t) - z(x,t) | \leq M(\varepsilon^{-2}N^{-1} + \tau), \quad (x,t) \in \overline{G}_h. \tag{4}$$

Let $H^{(\alpha)}(\overline{G}) = H^{\alpha,\alpha/2}(\overline{G})$ is the Hölder space, where α is an arbitrary positive number (see Ladyzhenskaya [7]). We suppose that the functions $f(x,t)$ and $\varphi(x,t)$ satisfy compatibility conditions at the corner points, so that the solution of the boundary value problem is smooth for every fixed value of ε.

Assume that at the corner points $S_0 \cap \overline{S}^L$ the following conditions hold

$$\begin{aligned} &\frac{\partial^k}{\partial x^k}\varphi(x,t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, && k + 2k_0 \leq [\ \alpha\ ] + 2n, \\ &\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} f(x,t) = 0, && k + 2k_0 \leq [\ \alpha\ ] + 2n - 2, \end{aligned} \tag{5}$$

where $[\ \alpha\ ]$ is the integer part of a number α, $\alpha > 0$, $n \geq 0$ is an integer. We also suppose that $[\ \alpha\ ] + 2n \geq 2$.

Theorem 1 *Assume in Eq.* (1) *that* a, b, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 0$ *and let the condition* (5) *with* $n = 0$ *be fulfilled. Then, for a fixed value of the parameter* ε, *the solution of* (3), (2) *converges to the solution of* (1) *with an error bound given by* (4).

4 The ε-uniformly convergent scheme

Now we use the technique of the special meshes, condensed in the neighbourhood of the boundary layer. The way to construct the mesh for problem (1) is the same as in Shishkin [8] and in Hemker *et al* [2,3]. We take

$$\overline{G}_h = \overline{G}_h^{\,*} = \overline{\omega}^*(\sigma) \times \overline{\omega}_0\ , \tag{6}$$

where $\overline{\omega}_0$ is a uniform mesh with step-size $\tau = T/K$, i.e. $\overline{\omega}_0 = \overline{\omega}_{0(2)}$, and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a special *piecewise uniform* mesh depending on the parameter $\sigma = \sigma(\varepsilon, N) > 0$. We take $\sigma = \sigma_{(6)}(\varepsilon, N) = \min(\,1/2,\, m^{-1}\varepsilon \ln N\,)$, where m is any number from the interval $(0, m_0)$, and $m_0 = \min_{\overline{G}}[a^{-1}(x,t)b(x,t)]$. The mesh $\overline{\omega}^*(\sigma)$ is constructed as follows. We divide the interval $[0,1]$ in two parts $[0,\sigma]$ and $[\sigma,1]$, $\sigma \le 1/2$. In each part we use a uniform mesh, with $N/2$ subintervals in each interval $[0,\sigma]$ and $[\sigma,1]$. On such mesh $\overline{G}_h^*$ the scheme (3), (6) converges ε-uniformly, see Shishkin [1,6] :

$$|u(x,t) - z(x,t)| \le M(N^{-1}\ln^2 N + \tau), \quad (x,t) \in \overline{G}_h^*. \tag{7}$$

Theorem 2 *Let the conditions of Theorem* 1 *hold. Then the solution of* (3), (6) *converges ε-uniformly to the solution of* (1) *and the estimate* (7) *holds.*

5 Numerical results for scheme (3), (6)

We consider the model problem

$$L_{(8)}u(x,t) \equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right\}u(x,t) = f(x,t), \quad (x,t) \in G, \tag{8}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S, \quad T = 1,$$

where $f(x,t) = -4t^3$, $(x,t) \in \overline{G}$, $\varphi(x,t) = 0$, $(x,t) \in S$.

Table 1: Errors $E(N,K,\varepsilon)$ for the special method (3), (6)

$\varepsilon \setminus N$	16	32	64	128	256	512
1.0	2.178e−03	1.179e−03	6.046e−04	2.989e−04	1.410e−04	6.073e−05
2^{-1}	6.460e−03	3.555e−03	1.840e−03	9.126e−04	4.312e−04	1.859e−04
2^{-2}	1.533e−02	8.465e−03	4.402e−03	2.188e−03	1.035e−03	4.467e−04
2^{-3}	2.950e−02	1.639e−02	8.544e−03	4.257e−03	2.017e−03	8.708e−04
2^{-4}	4.819e−02	3.275e−02	2.238e−02	1.148e−02	5.510e−03	2.399e−03
2^{-5}	6.342e−02	3.601e−02	2.334e−02	1.454e−02	8.192e−03	4.061e−03
2^{-6}	7.341e−02	4.263e−02	2.409e−02	1.498e−02	8.460e−03	4.192e−03
2^{-7}	7.763e−02	4.651e−02	2.495e−02	1.521e−02	8.601e−03	4.269e−03
2^{-8}	7.939e−02	4.819e−02	2.618e−02	1.534e−02	8.669e−03	4.307e−03
2^{-9}	8.015e−02	4.893e−02	2.673e−02	1.540e−02	8.707e−03	4.326e−03
2^{-10}	8.050e−02	4.927e−02	2.699e−02	1.543e−02	8.728e−03	4.336e−03
2^{-18}	8.082e−02	4.984e−02	2.730e−02	1.547e−02	8.749e−03	4.345e−03
$\overline{E}(N)$	8.082e−02	4.984e−02	2.730e−02	1.547e−02	8.749e−03	4.345e−03

In this table the function $E(N,K,\varepsilon)$ is defined by (9), where $z(x,t)$ is the solution of (3), (6) with $m = 2^{-1}$, $\overline{G}_h = \overline{G}_h^*$, $N^* = K^* = 2048$. In the bottom line $\overline{E}(N)$ gives the maximum over each column.

We evaluated the error $E(N,K,\varepsilon)$, defined by

$$E(N,K,\varepsilon) = \max_{(x,t)\in\overline{G}_h} |\, z(x,t) - u^*(x,t)\,|. \tag{9}$$

Here $u^*(x,t)$ is the linear interpolation obtained from the numerical solution $z(x,t)$ on the adapted mesh $G_{h(6)}$. Note that no special interpolation is needed along the t-axis.

We solve the problem using scheme (3), (6) with $K = N$ for $N = 2^k$, $k = 3, \dots, 9$ and for various values of ε. From the analysis of the numerical results we conclude that, in accordance with (7), the order of convergence is $\mathcal{O}(N^{-1}\ln^2 N + K^{-1})$. For large N the order of convergence is one with respect to the space and time variables that corresponds to the theoretical results.

6 Improved time-accuracy

6.1 The difference schemes of the second order accuracy in time

Here we construct the scheme with an order of accuracy with respect to τ higher than in (7).

The idea is similar to the one published in Hemker *et al* [2,3]. The error of the solution consists of two contributions: one caused by the discretisation of the time derivative and the other by the space derivative (or briefly, the time and the space errors). For the difference scheme (3), (6) the time error is associated with the truncation error given by the relation

$$\frac{\partial u}{\partial t}(x,t) - \delta_{\bar t}\, u(x,t) = 2^{-1}\,\tau \frac{\partial^2 u}{\partial t^2}(x,t) - 6^{-1}\,\tau^2 \frac{\partial^3 u}{\partial t^3}(x,t-\vartheta), \quad \vartheta \in [0,\tau]. \quad (10)$$

A better approximation than (3a) can be obtained by defect correction

$$\Lambda_{(3)} z^c(x,t) = f(x,t) + 2^{-1} p(x,t)\tau \frac{\partial^2 u}{\partial t^2}(x,t), \quad (11)$$

with $x \in \overline{\omega}$ and $t \in \overline{\omega}_0$, where $\overline{\omega}$ and $\overline{\omega}_0$ are as in (2); τ is step-size of the mesh $\overline{\omega}_0$; $z^c(x,t)$ is the "corrected" solution. Here we use for the approximation of $(\partial/\partial t)\, u(x,t)$ the expression

$$\delta_{\bar t}\, u(x,t) + \tau \delta_{\bar t \bar t}\, u(x,t)/2,$$

where $\delta_{\bar t \bar t}\, u(x,t) \equiv \delta_{t \bar t}\, u(x, t-\tau)$; $\delta_{t \bar t}\, u(x,t)$ is the second central divided difference. Instead of $(\partial^2/\partial t^2)\, u(x,t)$ we use the difference time derivative $\delta_{\bar t \bar t}\, z(x,t)$, where $z(x,t)$, $(x,t) \in G_{h(6)}$ is the solution of the difference scheme (3), (6). Thus, the new solution $z^c(x,t)$ has an accuracy of $\mathcal{O}(\tau^2)$ with respect to the time variable.

We shall use the notation $\delta_{k\bar t} z(x,t)$ for the backward difference of order k:

$$\delta_{k\bar t}\, z(x,t) = \left(\delta_{k-1\,\bar t}\, z(x,t) - \delta_{k-1\,\bar t}\, z(x,t-\tau)\right)/\tau, \quad t \ge k\tau, \quad k \ge 1;$$
$$\delta_{0\bar t}\, z(x,t) = z(x,t), \quad (x,t) \in \overline{G}_h.$$

On the mesh $\overline{G}_h$, we first write the difference scheme (3) for $z^{(1)}(x,t)$

$$\begin{aligned} \Lambda_{(3)} z^{(1)}(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z^{(1)}(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{12}$$

$z^{(1)}(x,t)$ is the solution of the discrete problem (12), (6)

Then we construct the difference equations for $z^{(2)}(x,t)$:

$$\Lambda_{(3)} z^{(2)}(x,t) = f(x,t) + \begin{cases} p(x,t)2^{-1}\,\tau \frac{\partial^2}{\partial t^2} u(x,0), & t = \tau, \\ p(x,t)2^{-1}\,\tau \delta_{2\bar{t}}\, z^{(1)}(x,t), & t \geq 2\tau, \quad (x,t) \in G_h, \end{cases}$$

$$z^{(2)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{13}$$

We shall call $z^{(2)}(x,t)$ the solution of the difference scheme (13), (12), (6) (or shortly, (13), (6)).

We suppose that the coefficients $a(x,t)$, $b(x,t)$ do not depend on t

$$a(x,t) = a(x), \quad b(x,t) = b(x), \quad (x,t) \in \overline{G} \tag{14}$$

and we take a homogeneous initial condition:

$$\varphi(x,0) = 0, \quad x \in [0,1]. \tag{15}$$

Under the conditions (14), (15), the following estimate holds for $z^{(2)}(x,t)$

$$\left| u(x,t) - z^{(2)}(x,t) \right| \leq M \left[N^{-1} \ln^2 N + \tau^2 \right], \quad (x,t) \in \overline{G}_h. \tag{16}$$

Theorem 3 *Let conditions* (14), (15) *hold and assume in Eq.* (1) *that* a, b, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 1$ *and let condition* (5) *be satisfied for* $n = 1$. *Then for the solution of difference scheme* (13), (6) *the estimate* (16) *holds.*

6.2 The difference scheme of the third order accuracy in time

Now we construct a difference scheme with third order accuracy in τ. On the mesh $\overline{G}_h$ we consider the difference scheme

$$\Lambda_{(3)} z^{(3)}(x,t) = f(x,t) + \\ + \begin{cases} p(x,t)\left(C_{11}\tau \frac{\partial^2}{\partial t^2} u(x,0) + C_{12}\tau^2 \frac{\partial^3}{\partial t^3} u(x,0)\right), & t = \tau, \\ p(x,t)\left(C_{21}\tau \frac{\partial^2}{\partial t^2} u(x,0) + C_{22}\tau^2 \frac{\partial^3}{\partial t^3} u(x,0)\right), & t = 2\tau, \\ p(x,t)\left(C_{31}\tau \delta_{2\bar{t}} z^{(2)}(x,t) + C_{32}\tau^2 \delta_{3\bar{t}} z^{(1)}(x,t)\right), & t \geq 3\tau, \quad (x,t) \in G_h, \end{cases} \tag{17}$$

$$z^{(3)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h.$$

Here $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$ are the solutions of problems (12), (6) and (13), (6) respectively, the derivatives $(\partial^2/\partial t^2)u(x,0)$, $(\partial^3/\partial t^3)u(x,0)$ are obtained from Eq. (1a), the coefficients C_{ij} are determined as in Hemker *et al* [2,3] and equal to $C_{11} = C_{21} = C_{31} = 1/2, \quad C_{12} = C_{32} = 1/3, \quad C_{22} = 5/6$.

We shall call $z^{(3)}(x,t)$ the solution of the difference scheme (17), (13), (12), (6) (or shortly, (17), (6)).

Assume the homogeneous initial condition

$$\varphi(x,0) = 0, \quad f(x,0) = 0, \quad x \in [0,1]. \tag{18}$$

Under conditions (14), (18) we come to the following estimate for the solution of difference scheme (17), (6)

$$\left| u(x,t) - z^{(3)}(x,t) \right| \le M \left[N^{-1} \ln^2 N + \tau^3 \right], \quad (x,t) \in \overline{G}_h. \tag{19}$$

Theorem 4 *Let conditions* (18) *hold and assume in Eq.* (1) *that* $a, b, c, p, f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 2$, *and let condition* (5) *be satisfied with* $n = 2$. *Then for the solution of scheme* (17), (6) *the estimate* (19) *is valid.*

The same technique can be used in the construction of difference schemes with an arbitrary high order of accuracy $\mathcal{O}(N^{-1}\ln^2 N + \tau^{n+1})$, $n > 2$.

7 On numerical experiments for the time-accurate schemes

We consider the following boundary value problem

$$\begin{gathered} L_{(8)}u(x,t) = 0, \quad 0 < x < 1, \quad 0 < t \le T, \\ u(0,t) = t^4, \quad 0 < t \le T, \quad u(x,t) = 0, \quad (x,t) \in S, \quad x > 0. \end{gathered} \tag{20}$$

It should be noted that the solution of this problem is singular.

It is very attractively to use the analytical solution of problem (20) for the computation of the errors in the approximate solution, as was in Hemker *et al* [2,3]. But here the suitable (for computation) representation of the solution $u(x,t)$ is not known. It is possible to use, as the exact solution, the solution of the grid problem on the mesh with a large number of nodes. But this method is not effective because the analysis of the order of accuracy for a defect-correction scheme requires a very dense mesh that leads to large computational expenses and, besides, to large round-off errors.

Here we use the method from Shishkin [9], different from the above-mentioned techniques. The solution of problem (20) is represented in the form of a sum

$$u(x,t) = V^{(1)}(x,t) + v(x,t), \quad (x,t) \in \overline{G}, \tag{21}$$

where $V^{(1)}(x,t)$ is the main singular part (two first terms) of the asymptotic expansion of the solution of problem (20), and $v(x,t)$ is the remainder term. The function $V^{(1)}(x,t)$ has a sufficiently simple analytical representation

$$V^{(1)}(x,t) = V_0(x,t) + V_1(x,t), \quad (x,t) \in \overline{G}, \qquad \text{where}$$

$$V_0(x,t) = t^4\Psi(x), \quad \Psi(x) = [1-\exp(-\varepsilon^{-1})]^{-1}[\exp(-\varepsilon^{-1}x) - \exp(-\varepsilon^{-1})],$$

$$V_1(x,t) = 4t^3[1-\exp(-\varepsilon^{-1})]^{-1}\left(-x\exp(-\varepsilon^{-1}x) - x\exp(-\varepsilon^{-1}) + \right.$$

$$\left. + 2\exp(-\varepsilon^{-1})[1-\Psi(x)]\right), \quad |V_0(x,t)| \le M, \quad |V_1(x,t)| \le M\varepsilon, \quad (x,t) \in \overline{G}.$$

The function $v(x,t)$ is the solution of the problem

$$L_{(8)}v(x,t) = 3t^{-1}V_1(x,t), \quad (x,t) \in G, \quad v(x,t) = 0, \quad (x,t) \in S. \tag{22}$$

For the function $v(x,t)$ the following estimate holds:

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} v(x,t)\right| \le M\varepsilon^2[1+\varepsilon^{-k}], \quad (x,t) \in \overline{G}, \quad k + 2k_0 \le 4, \quad k \le 3.$$

Then the function $v(x,t)$ and the product $\varepsilon^2(\partial^4/\partial x^4)v(x,t)$ are ε-uniformly bounded. Thus, we can consider $v(x,t)$ as the regular part of this solution.

We solve the grid problem, which approximates the boundary value problem (22) on a sufficiently dense mesh $\overline{G}_{h(6)}$, and there are no difficulties to find the function $v(x,t)$. Such a methodology allows us to perform the numerical experiments which illustrate the validity of the theoretical results.

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