CHARACTERIZING GRAPHIC MATROIDS BY A SYSTEM OF LINEAR EQUATIONS

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Abstract. Given a rank-r binary matroid we construct a system of $O(r^3)$ linear equations in $O(r^2)$ variables that has a solution over GF(2) if and only if the matroid is graphic.

1. Introduction

We prove the following result.

Theorem 1.1. *Let B be a basis in a binary matroid M. Then M is graphic if and only if the following system of linear equations admits a solution over $GF(2)$.*

- **(G1)** $\beta(a,b)+\beta(a,c)=0$, *for each* $(a,b,c)\in B^{(3)}$ *with* $C^*_b\cap C^*_c - C^*_a \neq \emptyset$.
- **(G2)** $\beta(a,b)+\beta(a,c)+\beta(b,a)+\beta(b,c)+\beta(c,a)+\beta(c,b)=1$, *for each* $(a,b,c)\in B^{(3)}$ *with* $C^*_a\cap C^*_b\cap C^*_c\neq\emptyset$.

Here $B^{(k)}$ denotes the set of all ordered k-tuples of distinct elements in B and C^*_e denotes the fundamental cocircuit of e with respect to B; that is, C^*_e is the complement of the hyperplane of M spanned by $B-\{e\}$. The variables and equations have a natural interpretation which is revealed in Section 2.

If M is a rank-r binary matroid with n elements, then the system (G1)-(G2) has $O(r^3)$ equations and $O(r^2)$ variables. The system can be easily determined in $O(nr^3)$-time and solved in $O(r^7)$-time. Mighton [3, 6] has a closely related characterization of graphic matroids that also gives an elementary algorithm. There are faster algorithms for testing graphicness, Bixby and Cunningham [1] have an $O(r^2n)$-time algorithm.

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2. Trees and paths

Let B be a basis of a binary matroid M. For each $f \in E(M) - B$, we define $P_f \subseteq B$ such that $P_f \cup \{f\}$ is the unique circuit contained in $B \cup \{f\}$; that is, $P_f \cup \{f\}$ is the fundamental circuit for f. Note that $e \in P_f$ if and only if $f \in C_e^*$ for each $e \in B$ and $f \in E(M) - B$. To avoid ambiguity, we will refer to the fundamental circuits and cocircuits of (M, B), as they rely on both M and B. Our linear system is motivated by the following well-known result; we include the proof for the sake of completeness.

Lemma 2.1. *If B is a basis of a binary matroid M, then M is graphic if and only if there is a tree T with $E(T) = B$ such that each of the sets $(P_f : f \in E(M) - B)$ is a path in T.*

Proof. Suppose that $M = M(G)$ for some graph G; we may assume that G is connected. Then B is a tree and each of the sets $(P_f : f \in E(G) - E(T))$ are paths in G.

Conversely, suppose that there is a tree T with $E(T) = B$ such that, for each $f \in E(G) - E(T)$, the set P_f is a path in T. Then there is a graph G such that the fundamental circuits of (M, B) coincide with the fundamental circuits of $(M(G), B)$. Since M and $M(G)$ are both binary, $M = M(G)$. □

Let $\vec{T}$ be an orientation of a tree T. For each $(a, b) \in E(T)^{(2)}$, we let $\beta_{\vec{T}}(a, b) = 1$ if the head of a is in the same component of $T - a$ as the edge b, otherwise we let $\beta_{\vec{T}}(a, b) = 0$. Note that, for $(a, b, c) \in E(T)^{(3)}$, the edge b lies between a and c in T if and only if $\beta_{\vec{T}}(b, a) + \beta_{\vec{T}}(b, c) = 1$. The following lemma characterizes paths in T by linear equations.

Lemma 2.2. *Let $\vec{T}$ be an orientation of a tree T and let $P \subseteq E(T)$. Then P is a path in T if and only if*

- **(H1)** $\beta_{\vec{T}}(a, b) + \beta_{\vec{T}}(a, c) = 0$, *for each* $(b, c) \in P^{(2)}$ *and* $a \in E(T) - P$.
- **(H2)** $\beta_{\vec{T}}(a, b) + \beta_{\vec{T}}(a, c) + \beta_{\vec{T}}(b, a) + \beta_{\vec{T}}(b, c) + \beta_{\vec{T}}(c, a) + \beta_{\vec{T}}(c, b) = 1$, *for each* $(a, b, c) \in P^{(3)}$.

Proof. Note that P is a path if and only if

(I1) P induces a connected subgraph of T, and
(I2) there is a path of T containing P.

Now (I1) and (H1) are clearly equivalent and (I2) is equivalent to each triple in $P^{(3)}$ being contained in a path of T. Consider $(a, b, c) \in P^{(3)}$. If there is a path of T containing a, b and c, then exactly one of those edges lies between the other two. On the other hand, if a, b and c do not lie on a path, then none of the edges lies between the other two. Thus (I2) is equivalent to (H2). □

The next lemma determines when $\beta : B^{(2)} \to GF(2)$ encodes a tree.

Lemma 2.3. *Let B be a finite set and let $\beta : B^{(2)} \to GF(2)$. Then there exists an oriented tree $\vec{T}$ such that $E(\vec{T}) = B$ and $\beta = \beta_{\vec{T}}$ if and only if the following condition is satisfied:*

- **(T)** *for each $(a, b, c) \in B^{(3)}$, either $\beta(b, a) + \beta(b, c) = 0$ or $\beta(a, b) + \beta(a, c) = 0$.*

Proof. If an edge b lies between edges a and c in an oriented tree $\vec{T}$, then a does not lie between b and c. Thus $\beta_{\vec{T}}$ satisfies (T).

Conversely, suppose that $\beta : B^{(2)} \to GF(2)$ satisfies (T). We may assume that there exists $(a, b, c) \in B^3$ such that $\beta(a, b) + \beta(a, c) = 1$ since otherwise we can readily construct an oriented star $\vec{T}$ satisfying the result. Let β' denote the restriction of β to $(B - \{a\})^{(2)}$. Inductively we may assume that there is an oriented tree $\vec{T}_a$ such that $E(\vec{T}_a) = B - \{a\}$ and $\beta' = \beta_{\vec{T}_a}$. Let $B_0 = \{e \in B - \{a\} : \beta(a, e) = 0\}$ and let $B_1 = \{e \in B - \{a\} : \beta(a, e) = 1\}$. Since $\beta(a, b) + \beta(a, c) = 1$, the sets B_0 and B_1 are both nonempty. If B_0 and B_1 each form connected subgraphs of $\vec{T}_a$, then it is straightforward to get the desired tree $\vec{T}$. Adding one to each of the values $(\beta(a, e) : e \in B - \{a\})$ gives another function satisfying (T) and this change swaps the roles of B_0 and B_1; this change corresonds to the operation of reversing the orientation on an edge in a tree. So we may assume that there exist $(e, f) \in B_0^{(2)}$ and $d \in B_1$ such that d lies between e and f in $\vec{T}_a$. Note that $\beta(d, e) \neq \beta(d, f)$, so, by possibly switching e and f, we may assume that $\beta(d, a) = \beta(d, e)$. Now $\beta(d, a) + \beta(d, f) = 1$ and $\beta(a, d) + \beta(a, f) = 1$, contradicting (T). □

Lemmas 2.1, 2.2, and 2.3 immediately imply the following results.

Lemma 2.4. *If B be a basis of a graphic matroid M, then the linear system (G1)-(G2) admits a solution.*

Lemma 2.5. *If B be a basis of a binary matroid M and there is a solution to the system (G1)-(G2) that satisfies (T), then M is graphic.*

To complete the proof of Theorem 1.1 we need to prove that, when (G1)-(G2) has a solution, there is a solution satisfying (T). We will prove a stronger result that, when $M(G)$ is 3-connected, every solution of (G1)-(G2) also satisfies (T).

3. Connectivity

The following two results are self-evident.

Lemma 3.1. *Let B be a basis of a connected matroid M and let (X, Y) be a partition of B into nonempty sets. Then (X, Y) is a separation of M if and only if $P_x \subseteq X$ for each $x \in X - B$ and $P_y \subseteq Y$ for each $y \in Y - B$.*

Lemma 3.2. *Let B be a basis of a binary matroid M, and let (X, Y) be a partition of $E(M)$ with $|X|, |Y| \geq 2$. If $C^*_x \subseteq X$, for each $x \in X$, and there is a set $Z \subseteq X$ such that, for each $y \in Y$, either $C^*_y \cap X = \emptyset$ or $C^*_y \cap X = Z$, then (X, Y) is a 2-separation of M.*

The next lemma describes solutions to (G1).

Lemma 3.3. *Let B be a basis of a matroid M and let β be a solution to (G1). Then $\beta(b, a) = \beta(b, c)$ for each $(a, b, c) \in B^{(3)}$ where a and c are in the same component of $M \setminus C^*_b$.*

Proof. Suppose that the result fails and let N be the component of $M \setminus C^*_b$ containing a and c. Let $X = \{e \in E(N) : \beta(b, e) = \beta(b, a)\}$. By Lemma 3.1, there exists $f \in E(N) - B$ such that $P_f \cap X$ and $P_f - X$ are both nonempty. Let $a' \in P_f \cap X$ and $c' \in P_f - X$. Note that $b \notin P_f$, so, by (G1), $\beta(b, a') = \beta(b, c')$ — contradicting the definition of X. □

Let B be a basis of a matroid M. For $X \subseteq E(M)$, we let $M[B; X]$ denote $M/(B - X) \setminus (E(M) - (X - B))$. Note that $B \cap X$ is a basis of $M[B; X]$ and the fundamental cocircuits of $(M[B; X], B \cap X)$ are $(C_x \cap X : x \in B \cap X)$. Therefore, if β satisfies (G1)-(G2) for M, then the restriction of β to $X^{(2)}$ satisfies (G1)-(G2) for $M[B; X]$

We now reduce Theorem 1.1 to the 3-connected case.

Lemma 3.4. *Let B be a basis in a matroid M. If M is not graphic, then there exists $Z \subseteq E(M)$ such that $M[B; Z]$ is 3-connected and is not graphic.*

Proof. We may assume that M is not graphic and that, for each proper subset Z of $E(M)$, $M[B; Z]$ is graphic. Then M is connected. We may also assume that M is not 3-connected; let (X, Y) be a 2-separation in M. Note tht $r(X) + r(Y) = r(M) + 1$, so, up to symmetry, we may assume that $X \cap B$ is a basis of $M|X$. Thus $P_f \subseteq X$ for each $f \in X = B$. Then, by Lemma 3.1, there exists $y \in Y - B$ and $x \in X \cap B$ such that $x \in P_y$. By minimality, $M[B; X \cup \{y\}]$ and $M[B; Y \cup \{x\}]$ are both graphic. However, M is the 2-sum of $M[B; X \cup \{y\}]$ and $M[B; Y \cup \{x\}]$ and, hence, M is graphic. This contradiction completes the proof. □

4. The final step

Combining the following result with Lemmas 2.4, 3.4, and 2.5, completes the proof of Theorem 1.1.

Lemma 4.1. *Let B be a basis of a binary matroid M. If M is 3-connected, then every solution of (G1)-(G2) also satisfies (T).*

Proof. Let β be a solution to (G1)-(G2).

4.1.1. *Let $(a',b',c') \in B^{(3)}$ such that $\beta(b',a') + \beta(b',c') = 1$ and $\beta(a',b') + \beta(a',c') = 1$, and let $Z = C^*_{a'} \cap C^*_{b'}$. Then neither a' nor b' is in the same component of $M \setminus Z$ as c'.*

Proof of Claim. Let $Z' = (C^*_{a'} - \{a'\}) \cup (C^*_{b'} - \{b'\})$ and let N be the component of $M \setminus Z'$ containing c'. By Lemma 3.3, N contains neither a' nor b'. If the claim fails, then there exists $f \in Z' - Z$ such that $P_f \cap E(N) \neq \emptyset$. Up to symmetry, we may assume that $f \in C^*_{a'} - C^*_{b'}$. Then there is a component of $M \setminus C^*_{b'}$ containing $E(N) \cup \{a', f\}$, but this component contains a' and c' which contradicts Lemma 3.3. □

4.1.2. *Let $(a',b',c') \in B^{(3)}$ such that $\beta(b',a') + \beta(b',c') = 1$ and $\beta(a',b') + \beta(a',c') = 1$, and let $Z = C^*_{a'} \cap C^*_{b'}$. If $d \in B$ is in the same component of $M \setminus Z$ as c' and $C^*_d \cap Z \neq \emptyset$, then $\beta(b',a') + \beta(b',d) = 1$, $\beta(a',b') + \beta(a',d) = 1$, $\beta(d,a') + \beta(d,b') = 1$, and $Z \subseteq C^*_d$.*

Proof of Claim. By Lemma 3.3, $\beta(a',d) = \beta(a',c')$ and $\beta(b',d) = \beta(b',c')$. So $\beta(b',a') + \beta(b',d) = 1$ and $\beta(a',b') + \beta(a',d) = 1$. Note that $C^*_{a'} \cap C^*_{b'} \cap C^*_d \neq \emptyset$, so, by (G2), $\beta(d,a') + \beta(d,b') = 1$. Now, by 4.1.1, no two of a', b', d' are in the same component of $M \setminus Z$. Hence $Z \subseteq C^*_d$. □

Suppose that β does not satisfy (T) and let $(a,b,c) \in B^{(3)}$ such that $\beta(b,a) + \beta(b,c) = 1$ and $\beta(a,b) + \beta(a,c) = 1$. Let $Z = C^*_a \cap C^*_b$. By 4.1.2 and possibly changing our choice of c, we may assume that $C^*_c \cap Z \neq \emptyset$. Now, by 4.1.2, there is now symmetry among a, b, and c.

Let X_a and X_b be the ground sets of the components of $M \setminus Z$ that contain a and b respectively. By 4.1.2, for each $d \in (X_a \cup X_b) \cap B$, either $C^*_d - (X_a \cup X_b) = \emptyset$ or $C^*_d - (X_a \cup X_b) = Z$. Then, by Lemma 3.2, $(X_a \cup X_b, E(M) - (X_a \cup X_b))$ is a 2-separation of M, contradicting that M is 3-connected. □

5. Planar graphs

Our theorem was motivated by a result of Naji [4] who characterized the class of circle graphs by a system of linear equations over GF(2). Circle graphs are related to graphic matroids through the following two results: De Fraysseix [2] showed that the fundamental graph of a binary matroid M is a circle graph if and only if M is the cycle matroid of a planar graph. Whitney [7] proved that M is the cycle matroid of planar graph if and only if M is both graphic and cographic. By Whitney's theorem, any characterization for the class of graphic matroids immediately gives a characterization for the class of planar graphs; so we obtain the following corollary.

Corollary 5.1. *Let T be a spanning tree in a connected graph G. Then G is planar if and only if the following system of equations has a solution over $GF(2)$.*

- **(P1)** $\beta(a,b) + \beta(a,c) = 0$, *for each* $(a,b,c) \in (E(G) - E(T))^{(3)}$ *with* $P_b \cap P_c - P_a \neq \emptyset$.
- **(P2)** $\beta(a,b) + \beta(a,c) + \beta(b,a) + \beta(b,c) + \beta(c,a) + \beta(c,b) = 1$, *for each* $(a,b,c) \in (E(G) - E(T))^{(3)}$ *with* $P_a \cap P_b \cap P_c \neq \emptyset$.

It is not clear what the relationship is between our theorem and other characterizations of graphic matroids. Given that Theorem 1.1 is relatively easy to prove, it would be interesting if one could derive Mighton's Theorem [3] or Tutte's Theorem [5] from our theorem.

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