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Some remarks on Tits geometries

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ABSTRACT

A result of Tits' in his paper "A local approach to buildings" is somewhat strengthened: it is shown that each geometry of type D_n or E_6 is a building. A counterexample to the corresponding statement for E_7 is given. Moreover, a proof is given of the fact that any thick finite geometry of spherical type all of whose residues of type C_3 are buildings is itself a building.

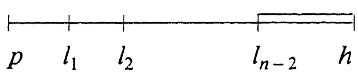
In [10] Tits proves the following

1. THEOREM. *Let G be a geometry of type $M=A_n$, $B_n(=C_n)$, D_n, E_6, E_7, E_8, or F_4. Then G is a building if and only if it has the following properties: if $M=C_n$, D_n or E_6, properties (O) and (LL), if $M=E_7$, properties (O), (LL) and (LH), if $M=E_8$ or F_4, properties (O), (LL), (LH) and (HH).*

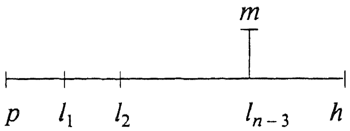
For notation and terminology, the reader is referred to [10]. We only recall the meaning of the properties (O), (LL), (LH), (HH). For this we need the following labelling of the diagrams:

A_n:

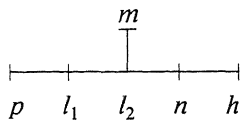
C_n:



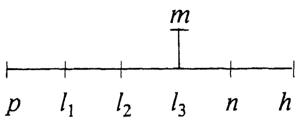
D_n:



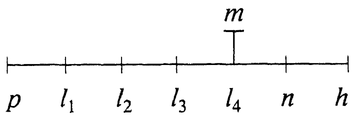
E_6:



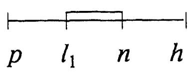
E_7:



E_8:



F_4:



The elements of type p, l_1 and h are called *points*, *lines* and *hyperlines* respectively.

(O) If two elements of type l_i (for some i) have the same shadow in the set of all points, they coincide.

(LL) If two lines are both incident to two distinct points, they coincide.

(LH) If a line and a hyperline are both incident to two distinct points, they are incident.

(HH) If two distinct hyperlines are both incident to two distinct points, the latter are incident to a line.

Here we show that any geometry of type D_n or E_6 is a building. For thick finite geometries of type D_n, this is stated in Timmesfeld [7], where the case $n=4$ is attributed to Th. Meixner; the proof in [loc. cit.], however, is valid in general. Also, Tits [11] has observed that thick finite geometries of type A_n, D_n, E_6 or E_7 are buildings on the basis of a case-by-case argument. Our final proposition extends this observation to geometries of arbitrary spherical type whose residues of type C_3 are buildings by use of a unified proof, valid in all cases.

2. COROLLARY. (a) *Every geometry of type A_n, D_n or E_6 is a building.*

(b) *A geometry of type E_7 is a building if and only if it satisfies* (LH).

(c) *A geometry of type E_8 is a building if and only if it satisfies* (LH) *and* (HH).

PROOF. By the above theorem, every geometry of type A_n is a building. In order to apply Tits' theorem to the other cases, we prove (O) and (LL) for D_n, E_6 and E_7, and – assuming (LH) – also for E_8.

For the sake of completeness we shall repeat the argument given by Timmesfeld [7] in case of D_n.

First, we introduce some terminology.

If $v_1, v_2, \ldots, v_t$ are elements of a geometry, we say that $v_1 - v_2 - \ldots - v_t$ is a chain whenever v_i and v_{i+1} are incident for each i $(1 \leqq i < t)$.

For any object X the set of points in Res(X) is called the *point shadow* of X, and the set of lines in Res(X) is called the *line shadow* of X.

We may assume $n > 3$.

Step 1. *If every geometry of type D_{n-1} is a building, then any geometry of type D_n satisfies (LL).*

Given lines L and L_1 both incident with two distinct points P and P_1, we can find a chain $L - M - H - L_1$, where M, H have types m, h respectively. (Such a chain exists, as may be seen, e.g., in Res(P) which is a building of type D_{n-1} by assumption.)

Now $\{M, H\}$ is a flag, and Res($\{M, H\}$) is a projective space containing P and P_1 and hence also a line L_2 incident with P and P_1.

Moreover, Res(M) is a projective space containing P, P_1 and the lines L, L_2 incident to both points, whence $L = L_2$. Similarly for Res(H), we see that $L_1 = L_2$, so that $L = L_1$.

Step 2. *If a geometry of type D_n, E_6, E_7 or E_8 satisfies (LL) and if Res(P) satisfies (O) for each point P, then the geometry itself satisfies (O).*

Suppose M and M_1 are both objects of type l_i for some i with the same point shadow S. Let L be a line incident to M.

Take distinct points P and P_1 incident to L. Then P and P_1 belong to S, so there is a line L_1 incident to P, P_1 and M_1.

In view of (LL), we have $L = L_1$, so L belongs to the line shadow of M_1. Thus the line shadow of M_1 contains the line shadow of M, and by symmetry, the line shadows of M and M_1 coincide. Consequently, the point shadows of M and M_1 in Res(P) coincide. Since (O) holds for Res(P) by assumption, M and M_1 coincide.

Step 3. *Statement (a) holds.*

For the case D_n, this is immediate from the two previous steps by induction on n.

Thus, consider a geometry G of type E_6. In view of the theorem, Step 2 and statement (a), we need to show (LL).

Let L and L_1 be lines, both incident to the distinct points P and P_1. Since

Res(P) is a building of type D_5, we can find a hyperline H of G in Res(P) incident to both L and L_1. But Res(H) is a building of type D_5, too, so satisfies (LL). Since P and P_1 (being incident to L) are elements of Res(H), this yields $L = L_1$.

Step 4. *Every geometry of type E_7 satisfies* (O) *and* (LL). *In particular, statement* (b) *holds.*

In view of the two previous steps, we need only verify property (LL) for a geometry of type E_7.

Given lines L and L_1 both incident to two distinct points P and P_1, we can find a hyperline H in Res(P) which is incident to both L and L_1. (For Res(P) is a building of type E_6.)

Since P and P_1 are both elements of the building Res(H) of type D_6, it follows that $L = L_1$.

Step 5. *Let G be a geometry of type E_8. If G satisfies* (LL), *then* (O) *holds. If G satisfies* (LH), *then* (LL) *holds. In particular, statement* (c) *holds.*

The first part follows from Step 2, Step 4 and the theorem. Suppose G has property (LH). Let P and P_1 be two distinct points, both incident to the lines L and L_1. Take a hyperline H incident to L_1. Then P and P_1 are also incident to H, so applying (LH) to L and H, we find that L is incident to H. But Res(H) is a building due to (a), hence satisfies (LL). It follows that $L = L_1$.

This ends the proof of Step 5, and hence the proof of the corollary.

3. REMARKS. (i). Using a result of Buekenhout and Shult [1, 2] and an older (elementary) result of Tits [8] 7.3n, we can prove that a geometry of type D_n is a building without recourse to the above theorem.

(ii). For a geometry of type E_8 the properties (LH) and (HH) can be replaced by the following property:

For each pair of distinct hyperlines, both incident to two distinct points, there is a line incident to both points and both hyperlines.

4. EXAMPLE. We exhibit a quotient of the unique building G of type E_7 over the field $\mathbb{C}$ of complex numbers by a group of order 2, which is a geometry of type E_7 but not a building. This shows that Condition (LH) in the corollary is not superfluous. The example is an analogue of the ones given by Tits [10] for buildings of type C_n, and is given in terms of Ferrar's presentation [5] of E_7.

Let J_1 be the exceptional 27-dimensional Jordan algebra over the field $\mathbb{R}$ of the real numbers with positive definite trace form, and denote by J its complexification, by t complex conjugation with respect to the real form J_1, by $\langle .,. \rangle$ the standard bilinear form on J (positive definite on J_1), by $N(.)$ the standard cubic form, by $\langle .,.,. \rangle$ its linearization, and by * the cross product such that $\langle A * B, C \rangle = 6 \langle A, B, C \rangle$ for A, B, C in J.

The ternary algebra M_1 is the 56-dimensional real vector space $\mathbb{R}+\mathbb{R}+ +J_1+J_1$, supplied with the alternating bilinear form $\{.,.\}$ given by

$$\{x_1,x_2\}=a_1b_2-a_2b_1+\langle A_1,B_2\rangle-\langle A_2,B_1\rangle$$

and the symmetric trilinear product $(x_1,x_2,x_3)\mapsto x_1x_2x_3$ obtained by linearizing the expression

$$\begin{aligned} xxx=6(&-a^2b+a\langle A,B\rangle-2N(b) ,\\ &b^2a-b\langle A,B\rangle+2N(a) ,\\ &(\ ab-\langle A,B\rangle)A-bB^*B+B^*(A^*A),\\ &(-ab+\langle A,B\rangle)B+aA^*A-A^*(B^*B)\ ), \end{aligned}$$

where $x=(a,b,A,B)$ and $x_i=(a_i,b_i,A_i,B_i)$ in M_1 for each i (thus a_i,b_i in $\mathbb{R}$ and A_i,B_i in J_1).

Then Aut M_1 is a real Lie group of type E_7. Denote by M the complexification of M_1 and retain the notation for the linear extensions to M of the bilinear form and the ternary multiplication on M_1.

Consider the subset S of M consisting of all members of M for which xxM is contained in $\mathbb{C}x$.

Instead of $\mathbb{C}x$, we shall also write $\langle x\rangle$. For x_1, x_2 in M with $\langle x_1\rangle$, $\langle x_2\rangle$ distinct elements of S, we call $\langle x_1\rangle$, $\langle x_2\rangle$ *adjacent* – notation $\langle x_1\rangle\sim\langle x_2\rangle$ – whenever Mx_1x_2 is contained in $\mathbb{C}x_1+\mathbb{C}x_2$. For details on M and S, the reader is referred to Faulkner [4] and Ferrar [5].

We are interested in the graph $(S,\sim)$. It has diameter 3, and two vertices $\langle x_1\rangle$ and $\langle x_2\rangle$ have distance at most 2 if and only if $\{x_1,x_2\}=0$. Moreover, $(S,\sim)$ is isomorphic to the graph $(S',\sim')$ obtained from the building G by letting S' be the set of points of G, and letting $P\sim' Q$ for distinct P, Q in S' stand for the existence of a line of G incident to both P and Q, i.e., for *collinearity* of P and Q.

It is known [3] that G can be uniquely reconstructed from $(S,\sim)$ up to isomorphism. Thus, any automorphism of $(S',\sim')$ extends uniquely to an automorphism of G.

From now on we shall identify $(S,\sim)$ and $(S',\sim')$.

Consider the semilinear transformation f of M given by

$$x^f=(-\bar{b},\bar{a},\ -B^t,A^t),\ \text{where}\ x=(a,b,A,B)\ \text{in}\ M.$$

The transformation f preserves S and induces an automorphism $f|S$ of $(S,\sim)$ of order two such that P has distance 3 to P^f in $(S,\sim)$ for any vertex P of S. It readily follows that the unique automorphism of G extending $f|S$ satisfies condition ($Q3$) of [10] and that the group F of order two which it generates acts freely on the set of all flags of G of corank 2.

Hence, by [loc. cit.], the quotient G/F is a geometry of type E_7, but not a building (e.g., since there are points in this quotient which are collinear to exactly two points of a line).

In order to deal with thick finite geometries we need the following lemma.

5. LEMMA. *Let s be a natural number. Suppose S is a finite regular connected graph with v points, valency $k \equiv 0$ (mod s) and diameter d such that for all i* $(1 \leq i \leq d)$ *and all x,y in S with mutual distance $d(x,y) = i$ we have*

$$\# \{z \in S \mid 1 = d(x,z) = i - d(y,z)\} \equiv 1 \ (mod\ s),$$

and

$$\# \{z \in S \mid 1 = d(x,z) = d(y,z) - i\} \equiv 0 \ (mod\ s).$$

Then

(i) *Each eigenvalue w of the adjacency matrix A of S distinct from k satisfies $w \equiv -1$ (mod p) for every maximal ideal p containing s of the ring of algebraic integers generated by the eigenvalues of A. Moreover, $v \equiv 1$ (mod s).*

(ii) *If g is an automorphism of S without fixed points such that each point in S is not adjacent to its image under g, then $s = 1$.*

PROOF. (i). By induction with respect to e, one can show that for every $e \geq 0$ the x,y entry of $(A+I)^e$ equals 0 if $d(x,y) > e$ and equals 1 (mod s) if $d(x,y) \leq e$. Thus, there is a matrix B with integral coefficients such that

$$(*) \qquad (A+I)^d = J + sB,$$

where J is the 'all 1' matrix.

Since S is regular of valency k, the matrices A, J, I and B commute and the 'all 1' vector j is a common eigenvector.

By (*), we get $(k+1)^d j = (v + si)j$ for some integer i, and hence that $(k+1)^d \equiv v$ (mod s). It follows that $v \equiv 1$ (mod s) as $k \equiv 0$ (mod s) by assumption. This proves the last statement of (i).

Since A and J are symmetric and S is connected, all other common eigenvectors are real and orthogonal to j (with respect to the standard inner product). Let w be an eigenvalue of A distinct from k corresponding to the common eigenvector b, say. Then $Jb = 0$, so by (*), we get

$$(w+1)^d b = (sm)b \text{ for some algebraic integer } m.$$

Consequently, $w + 1 \equiv 0$ (mod p) for each maximal ideal p containing s, whence (i).

(ii). Let M denote the permutation matrix of g on the points of S (considered as the basis of a real vector space). Suppose that p is a maximal ideal containing s. Now, A and M commute as g is an automorphism of S, so in view of (i) the matrix $(A+I)M$ has eigenvalues which are all 0 (mod p) except for the eigenvalue $(k+1)$ of multiplicity 1 with eigenvector j. As $k \equiv 0$ (mod p), we get trace $((A+I)M) \equiv 1$ (mod p). On the other hand, this trace equals 0 according to the assumption that no point is adjacent to or coincides with its image under g. This contradiction shows that s does not belong to any maximal ideal, whence $s = 1$ and we are done.

6. REMARK. Let M be one of the diagrams in Theorem 1. Consider a thick finite building of type M. It can easily be checked that the (collinearity) graph

S on the set of points in which two vertices x, y are adjacent if and only if there is a line incident to both x and y, satisfies the hypotheses of the above lemma, with s a power of the characteristic of the field over which the building is defined. In particular, $s>1$.

7. REMARK. We are indebted to F. Timmesfeld for pointing out to us the existence of a thick finite geometry of type C_3 associated with the alternating group on 7 letters, which is not covered by a building (in fact, the corresponding chamber system is 2-connected), see Kantor [6].

The involution exchanging opposite vertices of the octahedron (the thin building of type C_3) yields a quotient geometry of type C_3 which is not a building. In view of the existence of these geometries, the following result is to a certain extent best possible.

8. PROPOSITION. *Any thick finite geometry of spherical type all of whose residues of type C_3 are covered by buildings is a building.*

PROOF. For the types A_n, D_n and E_6 this is immediate from the corollary. However, we shall not make use of this.

Let G be a thick finite geometry of spherical type M. Without loss of generality, we may assume that M is connected.

If $n \leqq 2$, there is nothing to prove.

Suppose that $n \geqq 3$.

If $M=H_3$ or H_4, the statement is easily verified by use of the Feit-Higman Theorem.

Since each residue of type C_3 is covered by a building, Theorem 1 of [10] yields that $G=D/A$ (up to isomorphism), where D is a building of type M and A is a group of automorphisms of D acting freely on the set of all flags of corank 2 and satisfying ($Q1$) of [10]. Denote by $p: D \rightarrow G$ the canonical projection.

Let X be an element of D. By induction on n, the residue of X^p is a building, so by [10] 6.1.8, the restriction of p to Res(X) is an isomorphism onto Res(X^p). Consequently, Res(X) is a finite building, and so is D by [9]. By ($Q1$), the restriction of p to Res(X) is induced by the quotient map with respect to the stabilizer of X in A. Thus, any automorphism in A fixing X fixes every element of Res(X), and, by connectedness of D, is the identity.

Suppose A contains a nontrivial automorphism a.

According to ($Q2'$) of [10] – a consequence of ($Q1$) – there is no element Y of D incident to both X and X^a. In particular, for each point P of D its image P^a is not collinear with P. This, however, contradicts (ii) of Lemma 5 in view of Remark 6. It follows that A is trivial, and that $G=D$ is a building.

9. REMARK. In the above proposition for geometries of type C_n we can weaken the 'thick' part of the hypothesis by only requiring that lines are incident to at least three points.

10. ACKNOWLEDGEMENT

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Appendix

On the distance between opposite vertices in buildings of spherical type

by J. Tits

In [LA], 1.4, example (b), I indicated how, starting from a compact Lie group of type B_n (and also from certain non compact groups), one can construct geometries of type B_n which are not buildings (they are not simply connected). In the above paper (§ 4), a similar construction is described in the case of E_7. The main purpose of this appendix is to show, by a uniform proof, that the same method applies to a compact simple Lie group G if and only if the connection index (order of the fundamental group = order of the centre of the universal covering) of G is $\leqq 2$; in other words, excluded are only the types A_n $(n \geqq 2)$, D_n, E_6. This shows that Corollary 2 (a) of the above paper is, in a certain sense, "best possible".

Let Φ be a reduced irreducible root system in a real vector space V (we suppose V spanned by Φ), let $A = \{a_1, \dots, a_l\}$ be a basis of Φ and let $d = \sum_{i=1}^{l} d_i a_i$ denote the highest root. The following lemma is well known.

LEMMA. *Let $i \in \{1, \ldots, l\}$, and let $a_{i_1}, \ldots, a_{i_m}$ denote the elements of A which are connected with a_i in the Dynkin diagram. Then, $d_i \geqq \frac{1}{2} \sum_{j=1}^{m} d_{i_j}$.*

Let $f: V \to \boldsymbol{R}$ be the coroot associated with a_i, that is, $f(v) = 2(a_i, v)/(a_i, a_i)$. The number $f(a_k)$ is equal to 2 if $k = i$, is a strictly negative integer if $k \in \{i_1, \ldots, i_m\}$, and is zero otherwise. Since $d + a_i$ is not a root, we have

$$0 \leqq f(d) = 2d_i + \sum_{j=1}^{m} f(a_{i_j}) d_{i_j} \leqq 2d_i - \sum_{j=1}^{m} d_{i_j},$$

hence the claim.

PROPOSITION. *For $i, j \in \{1, \ldots, l\}$, with $i \neq j$, set $\Phi(i, j) = \{\sum c_k a_k \in \Phi \mid c_i \neq 0, c_j \neq 0\}$ and let $V(i, j)$ denote the subspace of V spanned by $\Phi(i, j)$. Then, one has $V(i, j) \neq V$ if and only if $d_i = d_j = 1$.*

If $d_i = d_j = 1$, we have

$$V(i, j) \subset \boldsymbol{R} \cdot (a_i + a_j) + \sum_{k \neq i, j} \boldsymbol{R} \cdot a_k \neq V.$$

To prove the converse, we assume, without loss of generality, that $i = 1$, that $d_j > 1$ and that $(a_1, a_2, \ldots, a_j)$ is the unique chain joining a_1 and a_j on the Dynkin diagram (i.e., that a_k and a_{k+1} are connected in that diagram for $1 \leqq k \leqq j-1$). Set $b = a_1 + \ldots + a_j$ and let $b_0 = b, b_1, \ldots, b_m = d$ be a sequence of roots such that $b_r - b_{r-1} \in A$ for $1 \leqq r \leqq m$. Clearly, $V(i, j)$ contains all b_r $(0 \leqq r \leqq m)$, hence also the set $A' = \{b_r - b_{r-1} \mid 1 \leqq r \leqq m\}$. The smallest integer $j' \in [1, j]$ such that $d_k \geqq 2$ whenever $j' \leqq k \leqq j$ must be 1 or 2, otherwise the above lemma would imply $1 = d_{j'-1} > \frac{1}{2} d_{j'}$. Consequently, $a_2, a_3, \ldots, a_j$ belong to A' and so do of course $a_{j+1}, \ldots, a_l$. Therefore,

$$V(i, j) \supset \boldsymbol{R} \cdot b + \sum_{k \neq 1} \boldsymbol{R} \cdot a_k = V,$$

q.e.d.

Let $\sum$ denote the Coxeter complex associated with Φ, that is, "cut out" on the unit sphere of V (identified with its dual) by the kernels of the roots. Let Δ be a building of type $\sum$: here, "weak buildings" in the sense of [BN] are simply called "buildings", so that we allow Δ to be $= \sum$. Remember that two vertices of Δ are said to be in "generic position" if they belong to opposite chambers.

COROLLARY 1. *Two vertices p, q of Δ of nonopposite types i and j and in generic position are at distance at least two in the graph of vertices of Δ. They are at distance exactly two if and only if $d_i = d_j = 1$.*

By [BN], 3.9, it suffices to consider the case where $\Delta = \sum$. Let q' be the vertex of $\sum$ opposite to q and let j' be its type. We assume, without loss of generality, that p and q' are the vertices of types i and j' of the "fundamental chamber" corresponding to the basis A. Clearly, the kernels of all elements of $\Phi(i, j')$ (cf. the Proposition) separate p and q. Therefore, p and q are at distance at least 2 in (the graph of) Σ, and if they are at distance 2, any point connected with both of them must belong to the kernel of every element of $\Phi(i, j')$, which implies $d_i = d_j$ $(=d_{j'}) = 1$, by the Proposition. Conversely, if $d_i = d_j = 1$, the point $r \in \Sigma$ defined by

$$d(r) = a_k(r) = 0 \ (k \neq i, j') \text{ and } a_i(r) > 0$$

is connected with both p and q because it is separated from them by no kernel of root. Indeed, if a root $\sum c_k a_k$ is strictly positive on r, one has $c_i > c_{j'}$ (because $a_i(r) + a_{j'}(r) = d(r) = 0$), hence $c_i = 1$, $c_{j'} = 0$ or $c_i = 0$, $c_{j'} = -1$, and the root $\sum c_k a_k$ takes positive values in both p and q, hence the corollary.

COROLLARY 2. *The distance between a vertex of Δ of type i and any opposite vertex is $\geqq 3$; it is equal to 3 if and only if there exists $j \in \{1, \ldots, \hat{i}, \ldots, l\}$ such that $d_i = d_j = 1$.*

This is an immediate consequence of the previous corollary.

From Corollary 2 and [LA], 1.3 (cf. condition ($Q3$)), we deduce:

COROLLARY 3. *Suppose that $d_i = 1$ for at most one value of i (i.e. that Σ is of type C_n, F_4, E_7 or E_8) and let Γ be an automorphism group of Δ whose orbits consist of pairwise opposite vertices. Then Δ/Γ is a geometry of type M (in the sense of [LA]), where M is the Coxeter matrix of Σ.*

EXAMPLE. Let K be a field, L a separable quadratic extension, Γ the Galois group, G a simple algebraic group, defined and anisotropic over K and whose relative Weyl group over L is of type $M = C_n$, F_4, E_7 or E_8 (e.g. $K = \mathbf{R}$, $L = \mathbf{C}$ and G is a compact group of type B_n, C_n, F_4, E_7 or E_8), and Δ the building of G over L. The proof of 4.7 in [GR] shows that if P, P' are two parabolic subgroups of G defined over L and permuted by the nontrivial element of Γ, the unipotent radical $R_u(P \cap P')$ is defined over K. Since G is K-anisotropic, it follows that $R_u(P \cap P')$ is trivial; in other words, $P \cap P'$ is reductive, which means that P and P' are opposite. Consequently, Δ/Γ is a geometry of type M, by Corollary 3.

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