A BERRY-ESSEEN BOUND FOR L-STATISTICS WITH UNBOUNDED WEIGHT FUNCTIONS

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A Berry-Esseen bound of order $n^{-\frac{1}{2}}$ is established for linear combinations of order statistics. The theorem extends previous results for the case of bounded weights to a class of L-statistics with unbounded weight functions.

1. INTRODUCTION AND RESULT

Let $X_1, X_2, \ldots, X_n$ be independent random variables (r.v.) with common distribution function (df) F and let $X_{1:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics. Let J be a fixed real-valued weight function on (0,1). We consider L-statistics (or linear combinations of order statistics) of the form:

$$(1.1) \qquad T_n = \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{i/n} J(s)ds\, X_{i:n}.$$

Let

$$(1.2) \qquad F_n^*(x) = P(T_n^* \leq x) \qquad \text{for } -\infty < x < \infty$$

where

$$(1.3) \qquad T_n^* = (T_n - E(T_n))/\sigma(T_n).$$

In the past decade there has been considerable interest into the asymptotic distribution theory for L-statistics. It is well-known that T_n^* is asymptotically normally distributed under quite general conditions. A survey of such results was given by Serfling (1980). We also refer to a recent paper of Mason (1981), which contains the best result so far obtained in this area.

More recently attention has been paid to the problem of establishing Berry-Esseen bounds for L-statistics. We mention the work of Bjerve (1977), Helmers (1977,1981, 1982), Serfling (1980) and van Zwet (1983). These authors obtained Berry-Esseen bounds for L-statistics for the case of bounded weights. The purpose of this paper is to derive a Berry-Esseen bound for L-statistics with unbounded weight functions.

Let Φ denote the standard normal df and define F^{-1} by

$$F^{-1}(s) = \inf\{x : F(x) \geq s\} \text{ for } 0 < s < 1.$$

THEOREM 1. *Suppose there exists numbers* $\delta > 0$, $\varepsilon > 0$ *and* $K > 0$ *such that*

(I) *the function* J *satisfies a Lipschitz condition of order* 1 *on* $[\varepsilon, 1-\varepsilon]$, *whereas on neighbourhoods* $(0,\varepsilon)$ *and* $(1-\varepsilon,1)$ *of zero and one,* J *is twice differentiable with second derivative* J", *satisfying*

$$|J''(s)| \leq K[s(1-s)]^{-2} \tag{1.4}$$

(II) *the inverse* F^{-1} *satisfies*

$$|F^{-1}(s)| \leq K(s(1-s))^{-\frac{1}{4}+\delta} \quad \textit{for } 0 < s < 1 \tag{1.5}$$

and

$$|F^{-1}(s_1)-F^{-1}(s_2)| \leq K|s_1-s_2|[(s_1(1-s_1))^{-\frac{5}{4}+\delta}+(s_2(1-s_2))^{-\frac{5}{4}+\delta}] \tag{1.6}$$

for $0 < s_1,s_2 < \varepsilon$ *and* $1-\varepsilon < s_1,s_2 < 1$. *Then* $\sigma^2(J,F) > 0$ *where*

$$\sigma^2(J,F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))J(F(y))(F(\min(x,y))-F(x)F(y))dxdy \tag{1.7}$$

implies that

$$\sup_x |F_n^*(x) - \Phi(x)| = O(n^{-\frac{1}{2}}), \quad \textit{as } n \to \infty. \tag{1.8}$$

The theorem allows weight functions J tending to infinity in the neighbourhood of 0 and 1 at a logarithmic rate. An example is provided by the weight function Φ^{-1}, the normal quantile function. Then T_n is an asymptotically efficient L-estimator of normal scale.

Our method of proof resembles those of van Zwet (1977) and Does (1982) as these authors also combine smoothing techniques with appropriate conditioning arguments.

In section 2 we prove the theorem. The proofs of a number of lemmas are omitted, but these may be found in Helmers & Hušková (1984).

2. PROOF

Let, for any $n \geq 1$, $(U_{1:n},\ldots,U_{n:n})$ denote the order statistics corresponding to a sample of size n from the uniform distribution on (0,1). For any integer $1 \leq m \leq [\varepsilon n]$, let $V = (V_{1:m-1},\ldots,V_{m-1:m-1})$, $Z = (Z_{1:n-2m},\ldots,Z_{n-2m:n-2m})$ and $W = (W_{1:m-1},\ldots,W_{m-1:m-1})$ be vectors of order statistics corresponding to samples of sizes m-1, n-2m, and m-1 from the uniform distribution on (0,1) and let V, Z and W, $U_{m:n}$, and $U_{n-m+1:n}$ be independent. Then the joint distribution of $(U_{1:n},\ldots,U_{n:n})$ is the same as that of

$$U_{m:n}\, V_{1:m-1},\ldots,U_{m:n}\, V_{m-1:m-1}, U_{m:n}, \tag{2.1}$$

$(U_{n-m+1:n}-U_{m:n})Z_{1:n-2m} + U_{m:n},\ldots,(U_{n-m+1:n}-U_{m:n})Z_{n-2m:n-2m} +$
$+ U_{m:n},\ U_{n-m+1:n},(1-U_{n-m+1:n})W_{1:m-1} + U_{n-m+1:n},\ldots,$
$(1-U_{n-m+1:n})W_{m-1:m-1} + U_{n-m+1:n}.$

Since the joint distribution of $X_{i:n}$, $i = 1,\ldots,n$ is the same as that of $F^{-1}(U_{i:n})$, $i = 1,\ldots,n$ it follows directly from (2.1) that the distribution of T_n (cf.(1.1)) can be identified with that of

$$T_{1n}(U_{m:n}) + \int_{\frac{m-1}{n}}^{m/n} J(s)ds\, F^{-1}(U_{m:n}) + T_{2n}(U_{m:n},U_{n-m+1:n}) + \int_{\frac{n-m}{n}}^{\frac{n-m+1}{n}} J(s)ds\, F^{-1}(U_{n-m+1:n}) + T_{3n}(U_{n-m+1:n}) \tag{2.2}$$

where

$$T_{1n}(U_{m:n}) = \sum_{i=1}^{m-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds\, F^{-1}(V_{i:m-1}U_{m:n}) \tag{2.3}$$

$$T_{2n}(U_{m:n},U_{n-m+1:n}) = \sum_{i=1}^{n-2m} \int_{\frac{i+m-1}{n}}^{\frac{i+m}{n}} J(s)ds.\ F^{-1}(Z_{i:n-2m} \cdot (U_{n-m+1:n}-U_{m:n})+U_{m:n}) \tag{2.4}$$

and

$$T_{3n}(U_{n-m+1:n}) = \sum_{i=1}^{m-1} \int_{\frac{i+n-m}{n}}^{\frac{i+n-m+1}{n}} J(s)ds\, F^{-1}(W_{i:m-1}(1-U_{n-m+1:n})+U_{n-m+1:n}). \tag{2.5}$$

Clearly, the r.v.'s $T_{1n}(U_{m:n})$,$T_{2n}(U_{m:n},U_{n-m+1:n})$ and $T_{3n}(U_{n-m+1:n})$ are conditionally independent, conditionally given $U_{m:n} = u$ and $U_{n-m+1:n} = v$ for any $0 < u < v < 1$. This fact will be crucial in what follows.

Define, for $\frac{m}{n} \le s \le \frac{n-m}{n}$, the function ψ_n by

$$\psi_n(s) = \int_s^{\frac{n-m}{n}} J(y)dy - \frac{(\frac{n-m}{n} - s)}{\frac{n-2m}{n}} \int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y)dy \tag{2.6}$$

and note that $\psi_n(\frac{m}{n}) = \psi_n(\frac{n-m}{n}) = 0$. Let Γ_{n-2m} denote the empirical df based on $Z_1,\ldots,Z_{n-2m}$; i.e. $\Gamma_{n-2m}(s) = (n-2m)^{-1} \sum_{i=1}^{n-2m} I_{(0,s)}(Z_i)$ for $0 < s < 1$, where $Z_1,\ldots,Z_{n-2m}$ are independent uniform (0,1) r.v.'s corresponding to the order statistics $Z_{1:n-2m},\dots,Z_{n-2m:n-2m}$. Here and elsewhere $I_A(\cdot)$ denotes the indicator of a set A. For any r.v. X, with $0 < \sigma(X) < \infty$, we write $\tilde{X}$ for X-EX and X^* for $(X-EX)/\sigma(X)$.

Similarly as in Helmers (1981;1982) we can write

$$(2.7)\qquad T_{2n}(U_{m:n},U_{n-m+1:n}) = \\ = \int_0^1 \psi_n\left(\frac{m}{n} + \frac{n-2m}{n}\Gamma_{n-2m}(s)\right)d\,F^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s) + \\ + (n-2m)^{-1}\sum_{i=1}^{n-2m} F^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})Z_i)\cdot\int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y)dy.$$

To proceed we note that, as J is Lipschitz of order 1 on $[\varepsilon,1-\varepsilon]$ (cf. assumption (I)), we can approximate T_{2n} from above and below for sufficiently large n by r.v.'s T_{2n+} and T_{2n-} defined by

$$(2.8)\qquad T_{2n\pm}(U_{m:n},U_{n-m+1:n}) = \\ = \int_0^1 \{\psi_n\left(\frac{m}{n} + \frac{n-2m}{n}s\right) + \frac{n-2m}{n}(\Gamma_{n-2m}(s)-s)\psi_n'\left(\frac{m}{n} + \frac{n-2m}{n}s\right) \\ \pm 2^{-1}L\left(\frac{n-2m}{n}\right)^2(\Gamma_{n-2m}(s)-s)^2\, I_{[\varepsilon,1-\varepsilon]}(s) \\ + 2^{-1}\left(\frac{n-2m}{n}\right)^2(\Gamma_{n-2m}(s)-s)^2\psi_n''\left(\frac{m}{n} + \frac{n-2m}{n}s\right)I_{(0,\varepsilon)\cup(1-\varepsilon,1)}(s) \\ + 6^{-1}\left(\frac{n-2m}{n}\right)^3(\Gamma_{n-2m}(s)-s)^3\psi_n'''\left(\frac{m}{n} + \frac{n-2m}{n}(\lambda s+(1-\lambda)\Gamma_{n-2m}(s))\right)\cdot \\ \cdot I_{(0,\frac{\varepsilon}{2})\cup(1-\frac{\varepsilon}{2},1)}(s)\}d\,F^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s) \\ + (n-2m)^{-1}\sum_{i=1}^{n-2m} F^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})Z_i)\int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y)dy$$

where L is the Lipschitz constant and λ a random point in [0,1]; i.e.

$$(2.9)\qquad T_{2n-}(U_{m:n},U_{n-m+1:n}) \le T_{2n}(U_{m:n},U_{n-m+1:n}) \le T_{2n+}(U_{m:n},U_{n-m+1:n}).$$

Define (cf.(2.2))

$$(2.10)\qquad T_{n\pm} = T_n + T_{2n\pm}(U_{m:n},U_{n-m+1:n})-T_{2n}(U_{m:n},U_{n-m+1:n}).$$

In the following lemma we relate T_n^* with T_{n+}^* and T_{n-}^* (cf. Helmers(1981);(1982) for a similar approach).

LEMMA 2.1. *If the assumptions of Theorem* 1 *are satisfied, then*

$$P(T_n^* \le x) \le P(T_{n-}^* \le x_{n+}) \tag{2.11}$$

and

$$P(T_n^* \le x) \ge P(T_{n+}^* \le x_{n-}) \tag{2.12}$$

for appropriate sequences x_{n+}, $n = 1,2,\ldots$ *and* x_{n-}, $n = 1,2,\ldots$ *satisfying*

$$x_{n\underline{+}} = x(1+O(n^{-\frac{1}{2}}))+O(n^{-\frac{1}{2}}) \tag{2.13}$$

uniformly in x.

PROOF. See Helmers & Hušková (1984). □

In view of Lemma 2.1 it obviously suffices to show

$$\sup_x |P(T_{n\underline{+}}^* \le x) - \Phi(x)| = O(n^{-\frac{1}{2}}) \tag{2.14}$$

instead of (1.8). To prove (2.14) we show that for some sufficiently small $\gamma > 0$

$$\int_{|t|\le n^{\gamma}} |t|^{-1} |\rho_{n\underline{+}}^*(t) - e^{-\frac{1}{2}t^2}| dt = O(n^{-\frac{1}{2}}) \tag{2.15}$$

and

$$\int_{n^{\gamma}<|t|\le n^{\frac{1}{2}}} |t|^{-1} |\rho_{n\underline{+}}^*(t)| dt = O(n^{-\frac{1}{2}}), \tag{2.16}$$

where $\rho_{n\underline{+}}^*$ denotes the characteristic function (ch.f) of $T_{n\underline{+}}^*$. An application of Esseen's smoothing lemma (see, e.g., Feller (1971), p.538) will then complete the proof of (2.14).

We first prove (2.15). To start with we note that (2.1)-(2.5) and the remark following (2.5) directly yields

$$\rho_{n\underline{+}}^*(t) = E[\phi_{T_{1n}}^*(U_{m:n})(t)\ \phi_{T_{2n\underline{+}}}^*(U_{m:n},U_{n-m+1:n})(t)\ \phi_{T_{3n}}^*(U_{n-m+1:n})(t)$$
$$\exp(it\sigma_{n\underline{+}}^{-1}(E(T_{n\underline{+}}|U_{m:n},U_{n-m+1:n}) - ET_{n\underline{+}}))] \tag{2.17}$$

where $\sigma_{n\underline{+}}^2 = \sigma^2(T_{n\underline{+}})$ and, for any r.v. X with $E|X| < \infty$, (2.18) $\qquad \phi_X^*(t) = E(\exp(it\sigma_{n\pm}^{-1}(X-E(X|U_{m:n},U_{n-m+1:n})))|U_{m:n},U_{n-m+1:n})$.

Note that the expression within square brackets in (2.17) is precisely equal to the conditional ch.f. of T_{n+}^*, where the conditioning is on $U_{m:n}$ and $U_{n-m+1:n}$. The expectation operator $\bar{E}$ in (2.17) refers to the expected value taken w.r.t. $(U_{m:n},U_{n-m+1:n})$.

We continue with the analysis of $\rho_{n+}^*(t)$. In the next lemma we derive asymptotic approximations for the first and third factor within square brackets in (2.17); i.e. for $\phi_{T_{1n}(u)}^*(t)$ and $\phi_{T_{3n}(v)}^*(t)$ for $0 < u < \varepsilon$ and $1-\varepsilon < v < 1$

LEMMA 2.2. *If the assumptions of Theorem* 1 *are satisfied, then for any real* t *and* $0 < u < \varepsilon$

$$(2.19) \qquad |\phi_{T_{1n}(u)}^*(t) - 1 + \tfrac{1}{2}t^2\,\sigma_{n\pm}^{-2}\,\sigma^2(T_{1n}(u))| = $$
$$= O(n^{-3/2}(\log n)^3|t|^3\,u^{-3/4+3\delta_m 3/2})$$

and

$$(2.20) \qquad \sigma^2(T_{1n}(u)) = O(n^{-2}(\log n)^2\,u^{-\frac{1}{2}+2\delta_m}).$$

The relations (2.19) *and* (2.20) *remain valid if we replace* $T_{1n}(u)$ *by* $T_{3n}(v)$ *and* u *by* $1-v$.

PROOF. See Helmers & Hušková (1984). □

We also need an asymptotic approximation for $\phi_{T_{2n\pm}(u,v)}^*(t)$ for $0<u<\varepsilon, 1-\varepsilon<v<1$. Note that r.v. $S_n(u,v)$ appearing in the following lemma corresponds to the leading term in the stochastic expansion (2.8), conditional on $U_{m:n} = u$ and $U_{n-m+1:n} = v$.

LEMMA 2.3. *If the assumptions of Theorem* 1 *are satisfied, then for any* $|t| \le n^{\frac{1}{2}}$ *and* $0 < u < \varepsilon$, $1-\varepsilon < v < 1$.

$$(2.21) \qquad |\phi_{T_{2n\pm}(u,v)}^*(t) - \exp(-\tfrac{1}{2}t^2\sigma_{n\pm}^{-2}\,\sigma^2(S_n(u,v)))|$$
$$= O(n^{-\frac{1}{2}}(t^2+|t|^3)\exp(-\tfrac{1}{4}\,t^2\sigma_{n\pm}^{-2}\sigma^2(S_n(u,v)))$$
$$+ n^{-1}t^2((F^{-1}(u))^2+(F^{-1}(v))^2)+n^{-\frac{1}{2}}m^{-\frac{1}{2}}|t|(|F^{-1}(u)| +|F^{-1}(v)|))$$

where

$$(2.22) \qquad S_n(u,v) = -\left(\frac{n-2m}{n}\right)\int_0^1 J\left(\frac{m}{n} + \frac{n-2m}{n}\,s\right)(\Gamma_{n-2m}(s)-s)\,dF^{-1}(u+(v-u)s).$$

PROOF. Taylor expanding the ch.f. of $T_{2n\pm}(u,v)$ yields for any t and $0 < u < v < 1$

$$(2.23)\qquad \phi^*_{T_{2n\pm}(u,v)}(t) = E[\exp\{it\sigma^{-1}_{n\pm} S_n(u,v)\}(1+it\sigma^{-1}_{n\pm}\tilde{Q}_n(u,v))]+ +\tfrac{1}{2}t^2\sigma^{-2}_{n\pm}\sigma^2(Q_n(u,v)+|t|\sigma^{-1}_{n\pm} E|R_n(u,v)|$$

Here S_n is defined in (2.22), whereas Q_n and R_n are the quadratic and third order terms in (2.8). Exploiting the von-Mises statistic structure of $Q_n(u,v)$ and employing a bound for large deviation probabilities for the empirical df, due to Lai (1975), p.827, for the estimation of $E|R_n(u,v)|$ we arrive at (2.21). For details of the proof see Helmers & Hušková (1984). □

To deal with the fourth factor within square brackets in (2.17) it will be convenient to have

LEMMA 2.4. *If the assumptions of Theorem* 1 *are satisfied, then*

$$(2.24)\qquad E|E(T_{n\pm}|U_{m:n},U_{n-m+1:n})-ET_{n\pm}|^3 I_{(0,\varepsilon)}(U_{m:n})I_{(1-\varepsilon,1)}(U_{n-m+1:n}) = O(n^{-3/2}(\tfrac{m}{n})^{3/4+3\delta}(\log n)^3).$$

PROOF. See Helmers & Hušková (1984). □

We are now in a position to complete the proof of (2.15). Take $m = [n^{1/3}]$. Application of an exponential bound for uniform order statistics (see, e.g., Lemma A2.1 of Albers, Bickel and van Zwet (1976)) yields

$$(2.25)\qquad \int_{|t|\le n^\gamma} |t|^{-1}|\rho^*_{n\pm}(t) - Ee^{itT^*_{n\pm}} I_{(0,\varepsilon)}(U_{m:n})I_{(1-\varepsilon,1)}(U_{n-m+1:n})|dt = O(n^{-\frac{1}{2}}).$$

Also we obtain with the aid of Theorem 1 of Mason (1981) that

$$(2.26)\qquad 0 < \lim_{n\to\infty} n\sigma^2_{n\pm} = \sigma^2(J,F) < \infty.$$

Using (2.17), (2.26) and the Lemma's 2.2, 2.3 and 2.4 we find after some elementary computations for all $|t| \le n^\gamma$ for some sufficiently small $\gamma > 0$

$$(2.27)\qquad |Ee^{itT^*_{n\pm}} I_{(0,\varepsilon)}(U_{m:n})I_{(1-\varepsilon,1)}(U_{n-m+1:n}) - e^{-\frac{1}{2}t^2}| \le |E[(1-\tfrac{1}{2}t^2\sigma^{-2}_{n\pm}\sigma^2(T_{1n}(U_{m:n})|U_{m:n}))(1-\tfrac{1}{2}t^2\sigma^{-2}_{n\pm} \sigma^2(T_{3n}(U_{n-m+1:n})|U_{n-m+1:n}))(\exp(-\tfrac{1}{2}t^2\sigma^{-2}_{n\pm}\sigma^2(S_n(U_{m:n},U_{n-m+1:n}))| U_{m:n},U_{n-m+1:n})(1+it(E(T^*_{n\pm}|U_{m:n},U_{n-m+1:n}) - \tfrac{1}{2}t^2 (E(T^*_{n\pm}|U_{m:n},U_{n-m+1:n}))^2)I_{(0,\varepsilon)}(U_{m:n})I_{(1-\varepsilon,1)}(U_{n-m+1:n})]$$

$$-e^{-\frac{1}{2}t^2}|+O(n^{-\frac{1}{2}}(t^2+|t|^3)\exp(-\frac{1}{5}t^2)) + O(n^{-\frac{1}{2}-\frac{1}{2}\delta}).$$

Combining now (2.25) through (2.27) we arrive after some calculations involving conditional moments (cf. Helmers & Hušková (1984)) at (2.15).

Next we prove (2.16). Take $m = [\frac{1}{4}\epsilon n]$. Using (2.17) once more we find for all $|t| \leq n^{\frac{1}{2}}$

$$|\rho^*_{n\pm}(t)| \leq E|\phi^*_{T_{2n\pm}(U_{m:n},U_{n-m+1:n})}(t)|. \tag{2.28}$$

Clearly $T_{2n\pm}(u,v)$ is the sum of a non-degenerate U-statistic of degree 2 with a kernel, which is bounded by $C(|F^{-1}(u)|+|F^{-1}(v)|)$ for some constant $C > 0$, and a remainder term satisfying $E|R_n(u,v)| = O(n^{-3/2}(|F^{-1}(u)|+|F^{-1}(v)|))$. Hence the argument given in Helmers and van Zwet (1982), p.504-505, cf. their relation (3.10) can essentially be repeated to find that for some sufficiently small $\gamma > 0$

$$\begin{aligned}
&\int_{n^\gamma<|t|\leq n^{\frac{1}{2}}} |t|^{-1}|\rho^*_{n\pm}(t)|dt \leq \\
&\leq \int_{n^\gamma<|t|\leq n^{\frac{1}{2}}} |t|^{-1}\, E|\phi^*_{T_{2n\pm}(U_{m:n},U_{n-m+1:n})}(t)|dt \\
&= O(n^{-\frac{1}{2}}E[|F^{-1}(U_{m:n})|^3+|F^{-1}(U_{n-m+1:n})|^3 + \\
&+ (F^{-1}(U_{m:n}))^2 + (F^{-1}(U_{n-m+1:n}))^2 + (|F^{-1}(U_{m:n})| + \\
&+ |F^{-1}(U_{n-m+1:n})|)]) \\
&= O(n^{-\frac{1}{2}})
\end{aligned} \tag{2.29}$$

which proves (2.16). This completes the proof of Theorem 1.

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