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Strong Normalization and Perpetual Reductions in the Lambda Calculus

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Abstract. A lambda term M is called strongly normalizing if every reduction of M stops eventually (in the unique normal form), and weakly normalizing if some reduction of M ends in the normal form. Here we are interested in characterizing those reduction steps $M \to M'$ such that if M has an infinite reduction, M' has one too. A sufficient condition is that in the step $M \to M'$ a redex $R \equiv (\lambda x.\, A)\, B$ is contracted where x is free in A, i. e. R does not erase its "argument". (A corollary is the well-known fact that in the λI-calculus strong and weak normalization are equivalent.) An R whose contraction preserves the property of having an infinite reduction is called perpetual. In the present paper the perpetual redexes which do erase their argument are characterized.

Introduction. The relevance of λ-calculus to Computer Science, both practically and theoretically, is at present well established. The property of Strong Normalization of a λ-term is of obvious importance, because it allows one to attach to a λ-term a unique operational meaning. Also in the theory of Term Rewriting Systems (in which λ-calculus occurs as a prime example), much attention is devoted to this topic of termination (or normalization, as we call it) of reduction sequences.

In this paper we study the behaviour of a λ-term w.r.t. normalization in what seems to be a reversed way: if a λ-term M is not strongly normalizing, i.e. admits at least one infinite reduction, then we are interested in those reduction steps $M \to M'$ which preserve this property of having an infinite reduction. In fact we give a characterization of such steps. In this way we come to an understanding of what happens in a step $M \to M'$ which is critical in the sense that M has an infinite reduction but M' not (i.e. M' is strongly normalizing). Apart from the general insight into the normalization property which this approach via "perpetual" reductions yields, one may also think of systems in which non-termination rather than termination is desirable (such as operating systems).

In a technical respect, we have made an essential use of reduction strategies, a concept which seems to be of independent interest (see [3]). Reduction strategies occur in the Computer Science literature e.g. in definitions of operational semantics for data type specifications (see [5]).

We will now give a summary of the main definitions and results.

Let Λ be the set of λ-terms. A term $M \in \Lambda$ is called *strongly normalizing* if every reduction of M stops eventually (in the unique normal form). Let SN be the set of strongly normalizing λ-terms. Instead of $M \notin SN$ we will write ∞M, to indicate that there is an infinite reduction starting from M. Par abus de language we will call such a term M an *infinite* term.

We will study strong normalization by considering the question: supposing that one is interested to preserve the property ∞, which redexes can "safely" be contracted (i.e. without losing ∞) in any context? Let us call such redexes *perpetual*.

A partial answer to this question is obtained in [2]: the redex $(\lambda x.A)\,B$ is perpetual if it does not erase its argument, i.e. if $x \in FV(A)$. When applied to the λI-calculus this result yields at once two well-known facts:

(1) in the λI-calculus strong normalization is equivalent to having a normal form;
(2) a λI-term has a normal form iff all its subterms do.

In this paper we consider the redexes which do erase their argument, and we will arrive at a characterization of the perpetual redexes among them in terms of a certain quasi odering $\geqq_\infty$ on Λ, defined as follows:

$$A(\vec{x}) \geqq_\infty B(\vec{x}) \quad \text{iff} \quad \forall \vec{C} \in SN\ (\infty\, B(\vec{C}) \Rightarrow \infty\, A(\vec{C}))\,.$$

(So $A \geqq_\infty B$ iff every SN-substitution making B "explode" does the same for A.) To be precise, we will prove that the redex $(\lambda x.A)\,B$ where $x \notin FV(A)$, is perpetual iff $A \geqq_\infty B$. Together with the partial result in [2] for non-erasing redexes this yields a characterization of all perpetual redexes.

As in the proof of the result for non-erasing redexes in [2], our main tool will be the concept of a *perpetual reduction strategy*. An outline of the method employed in this paper will be given after introducing some terminology and preliminaries.

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1. Terminology. We will quickly introduce some basic concepts and notations.

Λ, the set of λ-terms, is defined inductively by (i) $x_i \in \Lambda$ ($i \in \mathbb{N}$); (ii) $M, N \in \Lambda \Rightarrow (MN) \in \Lambda$; (iii) $M \in \Lambda \Rightarrow (\lambda x.M) \in \Lambda$.

If in (iii) the proviso $x \in FV(M)$ is added, we get the *λI-terms*. Here $FV(M)$ is the set of free variables of M.

The usual *bracket convention* (association to the left) is employed. Writing $M(x_1, \ldots, x_n)$ means $FV(M) \subseteqq \{x_1, \ldots, x_n\}$; then $M(N_1, \ldots, N_n)$ is the result of simultaneous substitution of $N_1, \ldots, N_n$ for $x_1, \ldots, x_n$.

A term $R \equiv (\lambda x \cdot A)\,B$ is called a *redex*; $R' \equiv A[x := B]$, the result of substituting B for the free occurrences of x in A, is the *contractum* of R. A term not containing redexes is a *normal form*. In the sequel R, R' will exclusively be used for a redex and its contractum.

If $R = (\lambda x.A)\,B$ then $\text{Arg}(R)$, the *argument* of R, is B. If $x \in FV(A)$, R is called an *I-redex*. (But R need not be a λI-term.) If $x \notin FV(A)$, R is a *K-redex* and we will write $R = KAB$ (inspired by Combinatory Logic).

One step (β)-reduction is defined by $C[R] \to C[R']$ where R, R' are as above and $C[\ ]$ is a *context*. Contexts are λ-terms containing one hole $\square$; they can be inductively defined as follows: (i) $\square$ is a context, the trivial one; (ii) if $M \in \Lambda$ and N is a context then (MN) and (NM) are contexts; (iii) if M is a context then $(\lambda x \cdot M)$ is a context.

$C[M]$ is the result of substituting M for $\square$ in $C[\ ]$. The *subterm relation* $\subseteqq$ is defined by: $M \subseteqq N \Leftrightarrow \exists C[\ ]\ N \equiv C[M]$. ($\equiv$ denotes syntactical equality.)

Sometimes we will write $M \xrightarrow{R} N$ to indicate which redex R is contracted in the reduction step $M \to N$. (As everywhere in this paper we will tacitly assume that it is clear that we are speaking about *occurrences* of subterms, in casu R.) The *transitive* *reflexive* closure of $\to$ is denoted by $\twoheadrightarrow$. Reduction sequences $M_0 \to M_1 \to M_2 \to \dots \to M_n \to \dots$ will be denoted by $\mathcal{R}$, plus possibly subscripts. Although it is an abuse of notation, we will sometimes shorten $\mathcal{R} = M_0 \to \dots \to M_n$ to $\mathcal{R} = M_0 \twoheadrightarrow M_n$.

If M is not a normal form, the *leftmost redex* of M is that redex whose head-symbol λ is to the left of the head-λ of every other redex in M.

If $A, B \subseteq M$ we will write $A \lll B$ to denote that A and B are disjoint (i.e. incomparable w.r.t. $\subseteq$) and A is to the left of B.

2.1. Definition (Descendants and underlining). (I) Let $M \to M'$ and $N \subseteq M$. The subterm(s) $N' \subseteq M'$ which can be "traced back" to N are the *descendants* of N in M'. N is also called the *ancestor* of N'). (Notation: $N \succ\!\!- N'$).

The concept of descendant was first formulated in [7], [8]. Since [7], [8] are not generally accessible we will state the definition here. So let $M \xrightarrow{R} M'$ be a reduction step where $R \equiv (\lambda x.\, A)\, B$ is the contracted redex, and let $R' \equiv A[x := B]$ be the contractum in M'. Suppose $N' \subseteq M'$. According to the relative positions of N' and R' we distinguish four cases and define the ancestor N of N'.

(1) $N' \cap R' = \emptyset$ (N', R' are disjoint). Then there is a corresponding subterm $N \subseteq M$ which is the ancestor of N'.

(2) $N' \supsetneqq R'$ (R' is a proper subterm of N'). Then there is a unique $N \subseteq M$ such that $N \xrightarrow{R} N'$; and for this N we define $N \succ\!\!- N'$.

(3) $N' \equiv C[x := B]$ for some C where $C \subseteq A$ but $C \not\equiv x$; then $C \succ\!\!- N'$.

(4) N' is a subterm of some "copy" of B in $A[x := B]$. Then the ancestor of N' is N' itself as a subterm of B in M.

The notion $\succ\!\!-$ extends in the obvious way from one step reductions to arbitrary finite reductions. The transitive reflexive closure of $\succ\!\!-$ is denoted by $\gg\!\!-$. So if $\mathcal{R} = M_0 \to \dots \to M_n$ ($n \geqq 0$), $N \subseteq M_0$ and $N' \subseteq M_n$, then $N \gg\!\!- N'$ means that N' is a descendant of N (via $\mathcal{R}$).

Remark. (i) The notion of descendent of $N \subseteq M$ can easily be visualized by tracing the brackets which surround N.

(ii) Note that in the step $M \xrightarrow{R} M'$ the contracted redex $R \equiv (\lambda x.\, A)\, B$ has no descendants in M'. Also the (occurrences of) X in A have no descendants in M'.

(iii) Note that every $N' \subseteq M'$ has a unique ancestor $N \subseteq M$. On the other hand, an $N \subseteq M$ can have k descendants for every $k \geqq 0$.

(II) Often it is useful to attach some extra information to a λ-term, by specifying some of its subterms. This specification can be made simply by *underlining* those subterms. We define the set $\underline{\Lambda}$ of underlined λ-terms as follows: (1) x_i and $\underline{x_i} \in \underline{\Lambda}$ for all $i \geqq 0$; (2) $M, N \in \underline{\Lambda} \Rightarrow (MN)$, $(\underline{MN})$, $(\lambda x.\, M)$, $(\underline{\lambda x.\, M}) \in \underline{\Lambda}$. Reduction extends in a simple way to $\underline{\Lambda}$, by requiring that descendants of underlined subterms (and only those) are again underlined. E. g. $\underline{\underline{(\lambda x.\, \underline{x}x)\, (N\underline{M})}} \to N\underline{M}(N\underline{M})$.

2.2. Definition. Consider the reduction $\mathcal{R} = M_0 \to M_1 \to \dots$ and some subterm $L_0 \subseteq M_0$. The descendants of L_0 in $\mathcal{R}$ form a *tree*; e.g. see Fig. 1.

Now we define a *line of descendants* (l.o.d.) to be a branch in that tree.

Note that the l.o.d. $\mathcal{L} = L_0 \succ\!\!- L_1 \succ\!\!- L_2 \succ\!\!- \dots$ is in general not a reduction sequence, since there may be substitutions from "outside" into the L_i.

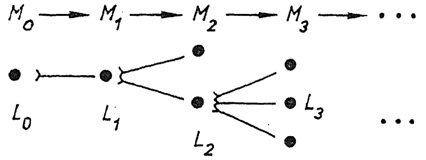


Fig. 1

2.3. Definition. Let $\mathcal{R} = M_0 \xrightarrow{R_0} M_1 \xrightarrow{R_1} M_2 \xrightarrow{R_2} M_3 \ldots$

$$\begin{array}{lcccc} & \cup\!\shortmid\!\shortmid & \cup\!\shortmid\!\shortmid & \cup\!\shortmid\!\shortmid & \cup\!\shortmid\!\shortmid \\ \text{and a l.o.d.} \quad \mathcal{L} = & L_0 >\!\!- & L_1 >\!\!- & L_2 >\!\!- & L_3 \ldots \end{array}$$

be given. Then $\mathcal{L}$ is called *passive* if for all i, $R_i \nsubseteq L_i$.

2.4. Definition. (i) Let $\mathcal{R}$ be as above. $\mathcal{R}$ is called *SN-substituting* if for all i, $\mathrm{Arg}(R_i) \in SN$.

(ii) Let $M \equiv M(x_1, \ldots, x_n)$ and $C_1, \ldots, C_n \in SN$. Then $M(C_1, \ldots, C_n)$ is called an *SN-substitution* of M.

2.5. Definition. Let $\mathcal{R}$ be as above where $R_i \equiv (\lambda x_i.\ A_i)\ B_i$ are the contracted redexes. $\mathcal{R}$ is called *simple* (w.r.t. substitution) if it is not the case that

$$\exists i\, \exists j > i\, \exists B_i' \subseteq M_j(B_i >\!\!- B_i' \ \&\ x_j \in FV(B_i'))\ .$$

(Here x_j is the bound variable of the redex R_j contracted in the step $M_j \to M_{j+1}$ of $\mathcal{R}$.)

Remark. This means that into a "substituted" subterm (i.e. a descendant B_i' of B_i) there is no substitution (of B_j) allowed; roughly said: there are no *double substitutions* allowed. An example where such a double substitution occurs, is

$$(\lambda x.\ ((\lambda y.\ yI)\ (xx)))\ \omega \to (\lambda x.\ xxI)\ \omega \to \omega\omega I\ .$$

Note that although this reduction is SN-substituting, the result of the double substitution is that the descendant $\omega\omega I$ of yI is not an SN-substitution of yI (here $\omega \equiv \lambda x.\ xx$). Note also that the reduction is not standard.

2.6. Definition. Let $\mathcal{R} = M_0 \xrightarrow{R_0} M_1 \xrightarrow{R_1} \ldots$ be a finite or infinite reduction sequence. In each M_n we will attach to some of the redex-λ's a marker * (meaning: this redex is henceforth forbidden to contract) as follows.

Basis. In M_0 all redex-λ's to the left of that of R_0 are marked.

Induction step. In M_{n+1} the following redex-λ's are marked:
(1) those that had already a mark in M_n,
(2) those to the left of the head-λ of R_{n+1}.
$\mathcal{R}$ is called a *standard reduction* if the restrictions imposed by the marks are not violated, i.e. if no marked redex is contracted.

2.7. Lemma. *Standard reductions are simple (w.r.t. substitutions).*

Proof. Let $\mathcal{R} = M_0 \xrightarrow{R_0} M_1 \xrightarrow{R_1} \ldots$ where $R_i \equiv (\lambda x_i.\ A_i)\ B_i$ be a standard reduction. Consider a step

$$\begin{array}{l} M_i \equiv \text{-----}\, \lambda y_1 \text{---}\, \lambda y_2 \text{---}\, (\lambda x_i.\ A_i(x_i))\ B_i(y_1, y_2, \ldots)\ \text{-----} \\ \downarrow \\ M_{i+1} \equiv \text{----}\, \lambda^* y_1 \text{---}\, \lambda^* y_2 \text{----}\, A_i(B_i(y_1, y_2, \ldots))\ \text{-----} \end{array}$$

Obviously the free variables y_j in B_i (which are not free in M_i) must be bound by λ's before x_i. Hence by Definition 2.6 those y_j are „frozen" after this step. So $\mathscr{R}$ cannot make a substitution anymore into (a descendant of) B_i. □

2.8. Proposition. *Let $\mathscr{R}$ be a simple, SN-substituting reduction. Let $\mathscr{L}$: $M \twoheadrightarrow M'$ be a passive l.o.d. in $\mathscr{R}$. Then M' is an SN-substitution of M.*

Proof. Say $M \equiv M(x_1, \dots, x_n)$. Now it is obvious that the only thing happening in $\mathscr{L}$ is that some of the variables x_i in M are replaced by SN-terms C_i: e.g.

$$M(x_1, \dots, x_n) \rightarrowtail M(x_1, \dots, x_n) \rightarrowtail M(x_1, C_2, x_3, \dots, x_n) \rightarrowtail$$
$$M(C_1, C_2, x_3, \dots, x_n) \rightarrowtail \dots \rightarrowtail M' .$$

Since $\mathscr{R}$ is simple, there cannot be substitutions into the C_i and since $\mathscr{L}$ is passive there are no reductions inside the displayed terms. Hence $M' \equiv M(C_1, C_2, \dots, C_n)$ where some of the C_i may have remained x_i, and indeed all $C_i \in SN$. □

2.9. Definition (Reduction diagrams and projections). We will give a quick sketch of the definition of those concepts; precise definitions can be found in [1], [6].

If $L \leftarrow \mathscr{R}_2 \leftarrow M \rightarrow \mathscr{R}_1 \rightarrow N$ are two "divergent" reductions, it is by the well-known Church-Rosser theorem possible to find "converging" reductions $L \rightarrow \mathscr{R}_3 \rightarrow P \rightarrow \mathscr{R}_4 \rightarrow N$.

A stronger version of the CR-theorem asserts that this can be done in a canonical way, by adjoining "elementary diagrams" as suggested in Fig. 2. In this way the reduction diagram $\mathscr{D}(\mathscr{R}_1, \mathscr{R}_2)$ originates, and in [1], [6] it is proved that it "closes", i.e. the construction terminates and yields $\mathscr{R}_3$ and $\mathscr{R}_4$ as desired.

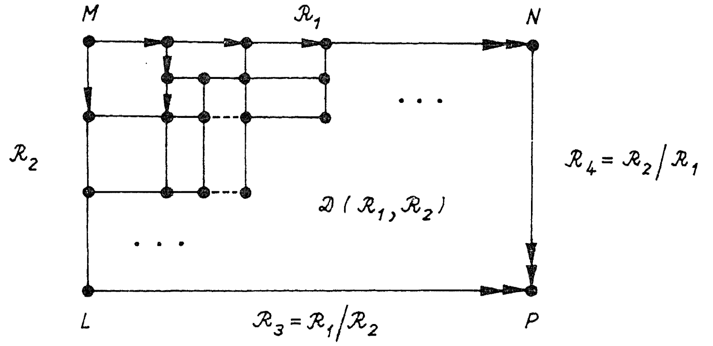
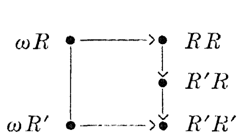
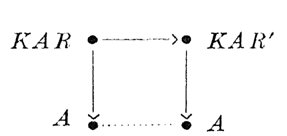


Fig. 2
(here —▶▶ means —»)

It is fairly evident how to define these elementary diagrams; we only give two examples:

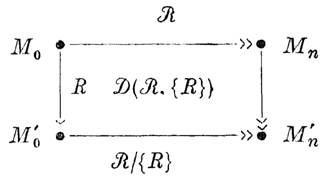




Notice that we have to introduce "empty steps" to keep things rectangular.

Now the in this way canonically found $\mathscr{R}_3$ is called the *projection* of $\mathscr{R}_1$ by $\mathscr{R}_2$, written: $\mathscr{R}_3 = \mathscr{R}_1/\mathscr{R}_2$. Similarly $\mathscr{R}_4 = \mathscr{R}_2/\mathscr{R}_1$.

Another notation: the reduction step $M_0 \xrightarrow{R} M_0'$ will be denoted by $\{R\}$. So by the above, $\mathcal{R}/\{R\}$, the projection of $\mathcal{R}$ by $\{R\}$, is the reduction displayed in the figure



(The reduction $\{R\}/\mathcal{R} = M_n \twoheadrightarrow M_n'$ is known as a "complete development" of the descendants of R in M_n.)

Remark. In one of the two examples of elementary diagrams above we saw that a reduction step may *vanish* when taking a projection of it; namely if $R \subseteq B$, then $\{R\}/C[KAB] \to C[A] = \emptyset$, the empty reduction.

3. Definition. (i) A map $F: \Lambda \to \Lambda$ is called a *one step reduction strategy* if $M \to F(M)$ unless M is in normal form, in which case $F(M) \equiv M$. All the strategies in this paper are one step strategies, so we will omit this qualification from now on. Reduction strategies were introduced in [2], see also [1].

Notation. $\mathcal{R}_{F,M}$ will be the reduction generated by repeated application of F:

$$\mathcal{R}_{F,M} = M \to F(M) \to F^2(M)\, F^3(M) \to \ldots$$

It is infinite or ends in the normal form of M.

(ii) A redex R is *perpetual* iff $\forall C[\;]\,(\infty\, C[R] \Rightarrow \infty\, C\,[R'])$, where R' is the contractum of R and $\infty\, M \Leftrightarrow M \notin SN$ as defined in 0.

(iii) A reduction strategy F is *perpetual* if

$$\forall M(\infty\, M \Rightarrow \infty\, F(M)) .$$

4. Definition. Let $A \equiv A(x_1, \ldots, x_n)$ and $B \equiv B(x_1, \ldots, x_n)$. Then $A \geqq_\infty B$ iff $\forall C_1 \ldots, C_n \in SN(\infty\, B(C_1, \ldots, C_n) \Rightarrow \infty\, A(C_1, \ldots, C_n))$. Obviously, $\geqq_\infty$ is a quasi ordering on Λ (i.e. reflexive and transitive).

5. Examples. Define $M >_\infty N$ iff $M \geqq_\infty N$ and $\neg\, M \leqq_\infty N$. Then it is not hard to prove that

(i) $xxx >_\infty xx$;

(ii) $xxI >_\infty xx$ $(I \equiv \lambda y.\, y)$;

(iii) xIx and xx are incomparable w.r.t. $\geqq_\infty$;

(iv) $A \supseteq B \Rightarrow A \geqq_\infty B$;

(v) $\infty\, A \Leftrightarrow \forall B\; A \geqq_\infty B$;

(vi) $B \in SN^\Sigma \Leftrightarrow \forall A\; A \geqq_\infty B$.

Here $^\Sigma$ is defined for subsets $X \subseteq \Lambda$ as follows:

$$X^\Sigma = \{M(x_1, \ldots, x_n) \in X \mid \forall N_1, \ldots, N_n \in X \quad M(N_1, \ldots, N_n) \in X\} .$$

(Using well-known properties of substitution one can verify that $X^\Sigma \subseteq X$ and $X^{\Sigma\Sigma} = X$.)

(vii) If $A' \twoheadrightarrow A \geqq_\infty B \twoheadrightarrow B'$, then $A' \geqq_\infty B'$.

(viii) The property $A \geqq_\infty B$ is not invariant under SN-substitution; for consider $A \equiv y$, $B \equiv \lambda z.\, zyx$ and the substitution $[x := aa]$. Then $A[x := aa] \equiv y \equiv A'$ and $B[x := aa] = \lambda z.\, zy(aa) \equiv B'$, and now $B' \not\leqq_\infty A'$. This fact will cause us some trouble later on.

Outline of the proof. As in [2], we employ the following method of proving that some redex R is perpetual. We search for a reduction strategy S such that

(i) S is perpetual,

(ii) if $C[\ ]$ is an arbitrary context, $M \equiv C[R]$ and ∞M (so by (i) $\mathcal{R}_{S,M}$ is infinite), then the projection $\mathcal{R}_{S,M}/\{R\}$ is also infinite.

(So if such an S exists, it delivers an infinite reduction $\mathcal{R}_{S,M}/\{R\}$ of $C[R']$, hence R is perpetual.)

To this end we define firstly a strategy $F: \Lambda \to \Lambda$, which is perpetual, SN-substituting, and yields standard reduction sequences $\mathcal{R}_{F,M}$. However, as the example in Section 14 shows, property (ii) fails for F.

From F we define another strategy F^* (this is our desired S) which has the same properties as F we just mentioned, and moreover satisfies (ii). F^* operates on Λ-terms plus some extra information: an underlining of some K-redexes in M. Now F^* is designed to have the following property. If $M \equiv C[\underline{KAB}]$ is an infinite term such that $A \geqq_\infty B$, then $\mathcal{R}_{F^*,M}$ is an infinite reduction such that (if the descendants $\underline{KA_iB_i}$ of $\underline{KAB}$ are also underlined throughout $\mathcal{R}_{F^*,M}$) no reduction step in $\mathcal{R}_{F^*,M}$ is taking place inside a $B_i \subseteqq \underline{KA_iB_i}$. This guarantees that the projection $\mathcal{R}_{F^*,M}/\{KAB\}$ is also infinite, since no reduction steps of $\mathcal{R}_{F^*,M}$ vanish when we take the projection (recall the remark at the end of Section 3). Hence $\infty\, C[A]$. So KAB with $A \geqq_\infty B$ is a perpetual redex. The reverse implication is easily established.

6. Definition. Let F be a reduction strategy, defined by induction on the structure of λ-terms as follows.

(1) if $M \in \Lambda$ is in normal form, then $F(M) \equiv M$;
(2) otherwise, let $(\lambda x.\, A)\, B$ be the leftmost redex in $M \equiv C[(\lambda x.\, A)\, B]$:

  (i) if $\neg \infty B$, i. e. $B \in SN$, then $F(M) \equiv C[A[x := B]]$
  (ii) if ∞B then $F(M) \equiv C[(\lambda x.\, A)\, F(B)]$.

It is easily proved that an equivalent definition of F is as follows.

Let $M \in \Lambda$ and $R_0, R_1, R_2, \ldots$ be the "*special sequence of redexes*" in M defined by:

- R_0 is the leftmost redex of M,
- R_{n+1} is the leftmost redex of $\mathrm{Arg}(R_n)$, if $\mathrm{Arg}(R_n)$ is not in normal form; otherwise the sequence stops with R_n.

Now let R_k be the first redex in this special sequence such that $\mathrm{Arg}(R_k) \in SN$. Then F contracts the redex R_k in M. If the special sequence is empty, which can only be the case if M is in normal form, then $F(M) \equiv M$.

7. Theorem. *If R is an I-redex, then R is perpetual.*

Proof (see [2], 5.8). The proof there is an application of a certain perpetual strategy F_∞ (the remarkable thing about F_∞ is that it is a recursive perpetual strategy, see [2] or [1]. It is defined as follows: F_∞ scans the special sequence of redexes of M, and picks out the first I-redex of that sequence, if there is one, otherwise it picks out the last redex in the special sequence to contract). This F_∞ satisfies for I-redexes the requirements (i), (ii) mentioned in the "outline of the proof" above. $\square$

8. Lemma. *F is a perpetual strategy.*

Proof. By induction on the structure of λ-terms we prove:

$$\infty M \Rightarrow \infty F(M) . \qquad (1)$$

Se let ∞M and suppose (induction hypothesis) that (1) is proved for all proper subterms of M.

Let $M \equiv C[(\lambda x.\, A)\, B]$ where the leftmost redex is displayed.

Case 1. If ∞B, then $F(M) \equiv C[(\lambda x.\, A)\, F(B)]$ and by the induction hypothesis $\infty F(B)$, hence $\infty F(M)$.

Case 2. If $\neg \infty B$, then $F(M) \equiv C[A(x := B)]$.

Case 2.1. If $(\lambda x.\, A)\, B$ was an I-redex, then by Theorem 7 $\infty F(M)$.

Case 2.2. If not, then $F(M) \equiv C[A]$. Now take an infinite reduction sequence $\mathcal{R} = M \to M_1 \to M_2 \to M_3 \to \dots$

Case 2.2.1. No descendant of $(\lambda x.\, A)\, B \equiv KAB$ is ever contracted in $\mathcal{R}$. So $M_i \equiv C_i[KA_iB_i]$, for $i = 1, 2, \dots$ Now since $\neg \infty B$ and since KA_iB_i remains the leftmost redex of M_i, we have for some m: $B_m \equiv B_{m+1} \equiv \dots$ But then evidently the projection

$$\mathcal{R}/\{KAB\} = C[A] \underset{\equiv}{\to} C_1[A_1] \underset{\equiv}{\to} \dots \underset{\equiv}{\to} C_m[A_m] \to \dots \to C_n[A_n] \to \dots$$

is an infinite reduction (since some of the steps up to $C_m[A_m]$, but not afterwards, may be trivial).

Case 2.2.2. A descendant of KAB is contracted in $\mathcal{R}$. So

$$\mathcal{R} = C[KAB] \to C_1[KA_1B_1] \to \dots \to C_m[KA_mB_m] \to$$
$$\to C_m[A_m] \to C_{m+1}[A_{m+1}] \to \dots$$

As in case 2.2.1, the projection $\mathcal{R}/\{KAB\}$ is an infinite reduction (after deleting some empty steps) starting with $C[A] \equiv F(M)$. Hence $\infty F(M)$. $\square$

9. Remark. From the definition it is clear that F is SN-substituting, i.e. $\mathcal{R}_{F,M}$ is SN-substituting. We remark that there is no perpetual NF-substituting strategy (NF is the set of normal forms); for consider $M \equiv (\lambda x.\, xx)\big(\lambda z.\, KI(zz)\big)$, then ∞M, but every NF-substituting strategy F' yields the reduction

$$\mathcal{R}_{F',M} = M \to (\lambda x.\, xx)\,(\lambda z.\, I) \to (\lambda z.\, I)\,(\lambda z.\, I) \to I .$$

10. Definition. Let $\underline{\Lambda}_K \subseteq \underline{\Lambda}$ be the set of λ-terms in which only *K-redexes* may be underlined. To be precise:

(i) $x_i \in \underline{\Lambda}_K$ for all $i \in N$;

(ii) $M, N \in \underline{\Lambda}_K \Rightarrow (MN) \in \underline{\Lambda}_K$;

(ii)′ if $M \equiv \lambda x.\, A$ (where $x \notin FV(A)$) and $B \in \underline{\Lambda}_K$, then $\underline{(\lambda x.\, A)\, B}$ ($\equiv \underline{KAB}$) $\in \underline{\Lambda}_K$,

(iii) $M \in \underline{\Lambda}_K \Rightarrow \lambda x.\, M \in \underline{\Lambda}_K$.

11. Definition. From F we define another perpetual, SN-substituting strategy $F^*\colon \underline{\Lambda}_K \to \underline{\Lambda}_K$. Let $M \in \underline{\Lambda}_K$.

Case (i). If $M \equiv C[\underline{KAB}]$ where ∞A and $\underline{KAB}$ is the leftmost underlined K-redex such that ∞A, then $F^*(M) \equiv C[\underline{KF^*(A)\,B}]$.

Case (ii). Otherwise $F^*(M) \equiv F(M)$.

12. Remark. So what happens is that F^* "zeroes in" via a chain (w.r.t. $\subseteqq$) of infinite underlined A_i's on its final target A_n, in which no infinite underlined A's appear; see Fig. 3.

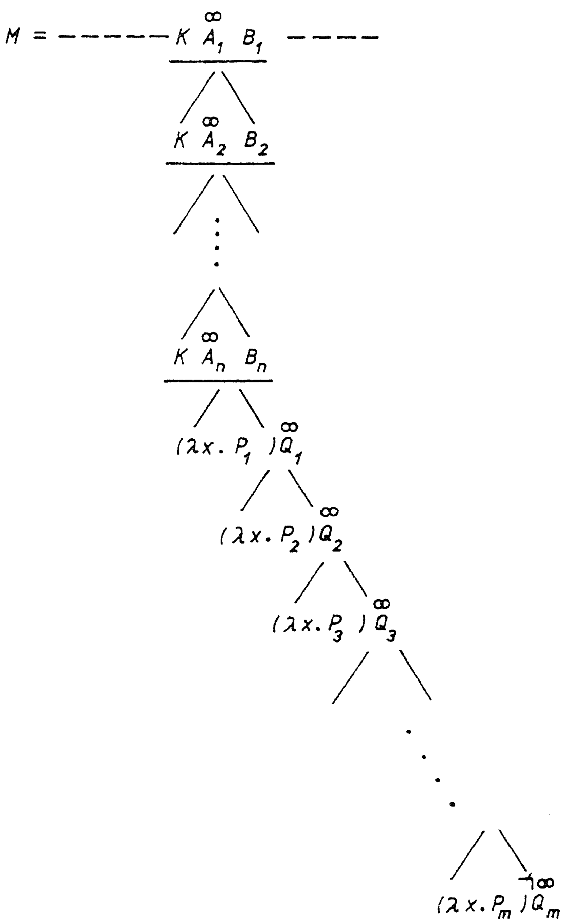


Fig. 3. Here the A_i, for $i = 1, \ldots, n-1$, are the "intermediate targets" of F^*; and A_n is the final target of F^*.

After F has found its final target A_n, it changes into F, the strategy which descends the chain of special redexes of A_n in search of the first one with SN-argument.

13. Notation and Remark. (i) Let $F^*[M]$ be the final target of F^* applied on M, and let $F^*\langle M\rangle$ be the redex which is selected by F^* as the one to contract $((\lambda x.\, P_m)\, Q_m$ above$)$.

(ii) Analogous: $F\langle M\rangle$ is the redex selected for contraction by F.

We note that

(1) $M \xrightarrow{F\langle M\rangle} F(M)$ and $M \xrightarrow{F^*\langle M\rangle} F^*(M)$,

(2) $F^*[M] \subsetneqq M \Rightarrow \infty\, F^*[M]$,

(3) $F^*\langle M\rangle \equiv F\langle F^*[M]\rangle$,

(4) $F^*[M] \equiv M$ if M does not contain a $\underline{KAB}$ such that $\infty\, A$.

14. Example. To illustrate the working of F^*, consider the following reductions shown in Fig. 4. Here $\omega := \lambda x.\, xx$ and $\Omega := \omega\omega$.

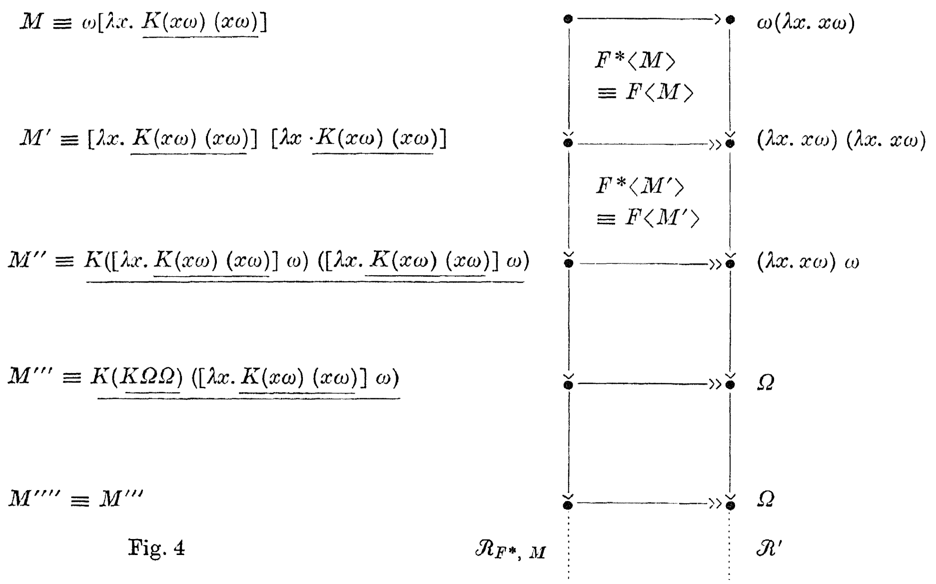


Fig. 4

Note that the projection $\mathcal{R}' = \mathcal{R}_{F^*, M}\{K(x\omega)\,(x\omega)\}$ is again infinite, which is what we wanted. Compare also $\mathcal{R}_{F,M}$ and note that $\mathcal{R}_{F,M}/\{K(x\omega)\,(x\omega)\}$ is finite (namely: $\omega\lambda x.\, x\omega \to (\lambda x.\, x\omega)\,(\lambda x.\, x\omega) \to (\lambda x.\, x\omega)\,\omega$).

15. Proposition. *F^* is perpetual and SN-substituting.*

Proof. Immediately from the definition and because F has the same properties, by Lemma 8 and Remark 9. □

16. Proposition. *Let $M \in \Lambda$. Then $\mathcal{R}_{F,M}$ is a standard reduction.*

Proof. See Fig. 5. Consider M and say that R_n is the first redex in the special sequence of redexes of M with $\mathrm{Arg}(R_n) \in SN$. Then F in its search "jumps" from R_0 to R_n and contracts R_n. Mark every redex-λ to the left of that of R_n; so if $\mathcal{R}_{F,M}$ is to be standard. $R_0, R_1, \dots, R_{n-1}$ plus the redexes in $P_0, P_1, \dots, P_{n-1}$ are henceforth forbidden to contract.

Now it is easy to see that F respects those restrictions, since in all following steps F will also jump over all in this step marked λ's because the perpetuality of F conserves the property ∞ of (the descendants of) $Q_0, Q_1, \dots, Q_{n-1}$.

Similarly the marks originating in the following steps of $\mathcal{R}_{F,M}$ are respected by F. Hence F yields a standard reduction. □

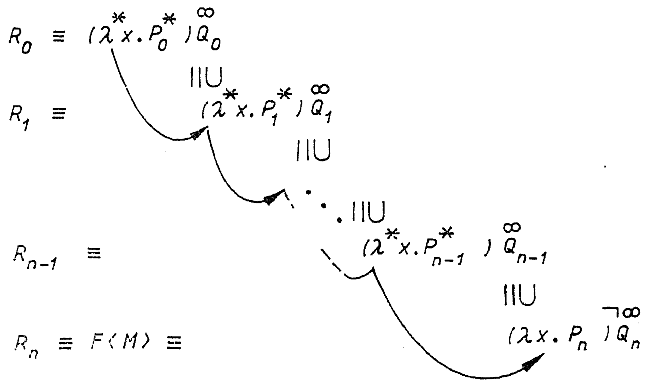


Fig. 5

17. Proposition. *Let $M \in \underline{\Lambda}_K$. Then $\mathcal{R}_{F^*, M}$ is a standard reduction.*

Proof. Let $\mathcal{R}_{F^*, M}$ be $M \equiv M_0 \xrightarrow{F^*\langle M_0\rangle} M_1 \xrightarrow{F^*\langle M_1\rangle} \ldots$. Suppose (induction hypothesis) that we have proved that $M_0 \to \ldots \to M_n$ is standard. See Fig. 6.

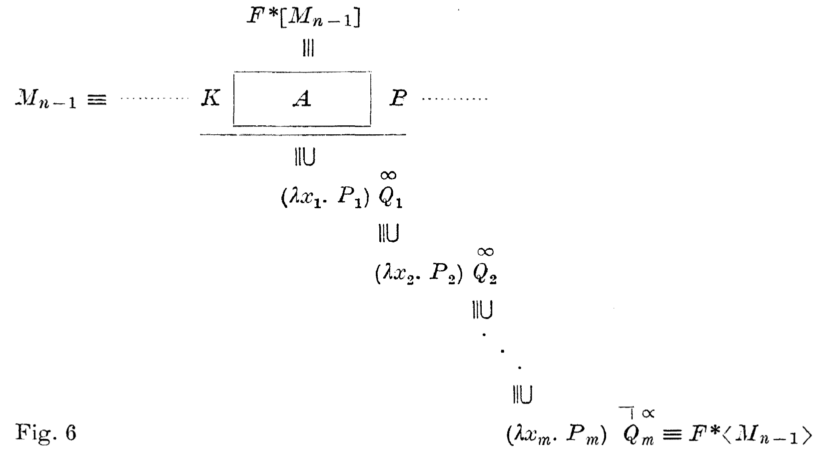


Fig. 6

Consider M_{n-1}, the special sequence of $F^*[M_{n-1}]$, and the contraction of $F^*\langle M_{n-1}\rangle$. There are two cases.

Case 1. In the step $M_{n-1} \to M_n$ a new $\underline{KFG}$ redex with ∞F appears. That is, there is a $\underline{KF'G'} \subseteqq M_{n-1}$ where $\neg\, \infty F'$ such that $\underline{KF'G'} \succ\!\!- \underline{KFG} \subseteqq M_n$ where ∞F.

This is only possible if there is a substitution into F', i.e. $F' \subseteqq P_m$ and hence $\underline{KF'G'} \subseteqq P_m$.

This implies that $F^*[M_n] \subseteqq$ the contractum of $F^*\langle M_{n-1}\rangle \equiv (\lambda x_m.\, P_m)\, Q_m$. Hence $F^*[M_n]$ and a fortiori $F^*\langle M_n\rangle$, are to the right of the descendants in M_n of $(\lambda x_1.\, P_1)$, … , $(\lambda x_{m-1}.\, P_{m-1})$. So indeed the standard requirement is fulfilled.

Case 2. Otherwise, it is evident that $F^*[M_n] \equiv$ the descendant of $F^*[M_{n-1}]$ (here we use the perpetuality of F).

Then Proposition 16 yields the result. □

18. Proposition. *Let* $M \in \Lambda$ *and let* $(\lambda x.\, A)\, B$ *be a redex in* M. *Then*

(i) $F\langle M\rangle \subseteq B \Rightarrow \infty B$ *and*

(ii) $F\langle M\rangle \not\subseteq A$.

Proof. Routine. □

19. Proposition. *Let* $M \in \underline{\Lambda}_K$ *and* $\underline{KAB} \subseteq M$. *Then*:

(i) $F^*\langle M\rangle \subseteq B \Rightarrow \infty B$;

(ii) $F^*\langle M\rangle \subseteq A \Rightarrow \infty A$.

Proof. (i) Suppose $M \supseteq \underline{KAB} \supseteq B \supseteq F^*\langle M\rangle$. If $M^0 \equiv F^*[M] \supseteq \underline{KAB}$, we are through by Proposition 18(i). So suppose $M^0 \not\supseteq \underline{KAB}$. Then, since $M^0 \supseteq F^*\langle M\rangle$, we must have $B \supseteq M^0 \supseteq F^*\langle M\rangle$. By the remark in Section 13, ∞M^0. Hence ∞B.

(ii) Suppose $M \supseteq \underline{KAB} \supseteq A \supseteq F^*\langle M\rangle$. There are 3 cases.

1. $M^0 \supseteq \underline{KAB}$ is not possible, by Proposition 18(ii).
2. If $A \supseteq M^0$ we are done since then $M^0 \subsetneqq M$, hence ∞M^0, hence ∞A.
3. The only remaining case $M^0 \equiv KA$ is impossible by definition of M^0. □

20. Proposition. *Let* $M \in \underline{\Lambda}_K$ *and let* $\underline{KAB} \subseteq M$, *say* $M \equiv C[\underline{KAB}]$. *Let* $M' = \,= C[KA'B]$ *where* $A \xrightarrow{F^*\langle M\rangle} A'$. *Then* $F^*\langle M'\rangle \subseteq A'$.

Proof. Since $F^*\langle M\rangle \subseteq A$ and $F^*\langle M\rangle \subseteq F^*[M]$, clearly either $F^*[M] \subseteq A$ or $F^*[M] \supseteq A$. The latter case is impossible since $F^*\langle M\rangle \equiv F\langle F^*[M]\rangle$ cannot be a subterm of the function part of a redex (Proposition 18(ii)). So $F^*[M] \subseteq A$. Hence $\infty F^*[M]$, and by perpetuality of F, the descendant of $F^*[M]$ in M' is again infinite. Hence $\infty A'$. Therefore $F^*\langle M'\rangle \subseteq A'$. □

21. Proposition. *Let* $M \in \underline{\Lambda}_K$ *and* $\underline{KAB} \subseteq M$ *where* ∞A. *Then* $F^*\langle M\rangle \lll A$ *or* $F^*\langle M\rangle \subseteq A$.

Proof. An easy induction on the number of steps in which F zeroes in via the intermediate targets (see Section 12) on its final target $M^0 \equiv F^*[M]$, shows that $M^0 \lll A$ or $M^0 \subseteq A$.
Hence $F\langle M^0\rangle \equiv F^*\langle M\rangle \lll A$ or $\subseteq A$. □

22. Definition. Let $M \in \underline{\Lambda}_K$.

(i) A redex $\underline{KAB} \subseteq M$ such that $A \geqq_\infty B$ is a *p-redex*.

(ii) A redex $\underline{KAB} \subseteq M$ such that $\neg \infty B$ is a *q-redex*.

(iii) If every $\underline{KAB} \subseteq M$ is a p-redex, M is a *p-term*.

(iv) If every $\underline{KAB} \subseteq M$ is a p- or q-redex, M is a *pq-term*.

23. Proposition. *An SN-substitution of a p-redex is a p- or q-redex.*

Proof. Let $\underline{KAB}$ be a p-redex and let $\underline{KA'B'}$ be an SN-substitution. Suppose $\infty B'$. Then by definition of $\geqq_\infty$, $\infty A'$. Hence (Section 5(v)) $A' \geqq_\infty B'$, i.e. $\underline{KA'B'}$ is a *p-redex*. If $\neg \infty B'$, then $\underline{KA'B'}$ is a q-redex. □

24. Proposition. *Let* $M \in \underline{\Lambda}_K$ *be a p-term and let* $\mathcal{R}_{F^*,\,M}$ *be*

$$M \xrightarrow{R_0} F(M) \xrightarrow{R_1} F^2(M) \xrightarrow{R_2} \ldots$$

Then for all i, $F^{*i}(M)$ *is a pq-term.*

Proof. Consider the original p-redexes $\underline{KAB}$ in M. Evidently, it suffices to take an arbitrary such $\underline{KAB}$ and an arbitrary l.o.d. $\mathscr{L}$ through $\mathscr{R}$ starting with that $\underline{KAB}$:

$$\begin{array}{ccccccccccc} \mathscr{L} = & \underline{KAB} & >\!\!- & \underline{KA'B'} & >\!\!- & \underline{KA''B''} & >\!\!- \dots >\!\!- & \underline{KA^{(i)}B^{(i)}} & >\!\!- \dots \\ & \cap\!\| & & \cap\!\| & & \cap\!\| & & \cap\!\| & \\ \mathscr{R}_{F^*,M} = & M & \xrightarrow{R_0} & M_1 & \xrightarrow{R_1} & M_2 & \longrightarrow \dots \longrightarrow & M_i & \longrightarrow \dots \end{array}$$

and to prove that every $\underline{KA^{(i)}B^{(i)}}$ in $\mathscr{L}$ is a p- or q-redex.

Now say that j is the least natural number such that $R_j \subseteqq \underline{KA^{(j)}B^{(j)}}$. So the initial part of $\mathscr{L}$, $\underline{KAB} \gg\!\!- \underline{KA^{(j)}B^{(j)}}$ is a passive l.o.d. Since $\mathscr{R}_{F^*,M}$ is standard (Proposition 17), it is simple (Lemma 2.7). Since $\mathscr{R}_{F^*,M}$ is also SN-substituting, by Proposition 2.8 $\underline{KA^{(j)}B^{(j)}}$ is an SN-substitution of $\underline{KAB}$; now by Proposition 23 it is a p- or q-redex. Likewise the $\underline{KA^{(k)}B^{(k)}}$ for $k < j$ are p- or q-redexes. (The same argument in case there is no j as supposed.)

To treat the other part of $\mathscr{L}$, which is no longer passive, we distinguish the following cases.

Case 1. $R_j \equiv \underline{KA^{(j)}B^{(j)}}$. Then $\mathscr{L}$ stops in $M^{(j)}$ (since the contractum of a redex is not a descendant of that redex) and we are through.

Case 2. $R_j \subseteqq A^{(j)}$. By Proposition 19 (ii) we have: $R_{j+k} \subseteqq A^{(j+k)}$ for all k; and by Proposition 20, $\infty A^{(j+k)}$ for all k. Therefore by Section 5(v) all $\underline{KA^{(j+k)}B^{(j+k)}}$ are p-redexes.

Case 3. $R_j \subseteqq B^{(j)}$. Since $\mathscr{R}_{F^*,M}$ is standard, there will be no more substitutions into the $\underline{KA^{(j+k)}B^{(j+k)}}$ $(k \geqq 0)$ (because this would need a contraction of a redex whose head-λ is to the left of $B^{(j)}$, hence of R_j).

Now if $\underline{KA^{(j)}B^{(j)}}$ is a q-redex, i.e. $B^{(j)}$ is finite, then we are through: $B^{(j+k)}$ will remain finite for all k. And if it is a p-redex then we are also through, since by Section 5(vii) $A \geqq_\infty B \twoheadrightarrow B'$ implies $A \geqq_\infty B'$.

25. Lemma. *In the situation of Proposition* 24, $\mathscr{R}_{F^*,M}$ *contains no reduction steps inside an argument of an underlined K-redex (i.e. no* $R_i \subseteqq B \subseteqq \underline{KAB} \subseteqq M_i$, *for all* $i \geqq 0$ *and all* $\underline{KAB}$ in M_i).

Proof. By Proposition 24, all the M_i are pq-terms. Now suppose that there is an i and a $\underline{KAB} \subseteqq M_i$ such that $R_i \subseteqq B \subseteqq \underline{KAB}$. There are two cases:

1. $\underline{KAB}$ is a q-redex, i.e. B is finite. But this is impossible by Proposition 19.

2. $\underline{KAB}$ is a p-redex, i.e. $A \geqq_\infty B$. By Proposition 19, ∞B. Hence ∞A. But then by Proposition 21 it is impossible that R_i is to the right of A. — Contradiction. □

26. Corollary. $\infty C[KAB]$ & $A \geqq_\infty B \Rightarrow \infty C[A]$.

Proof. Underline KAB. The resulting underlined term $M_0 \equiv C[\underline{KAB}]$ is then a p-term. Since ∞M_0 and F^* is perpetual, the reduction $\mathscr{R}_{F^*,M}$ is infinite.

Now consider the projection $\mathscr{R}' = \mathscr{R}_{F^*,M}/\{\underline{KAB}\}$ shown in Fig. 7.

Claim. Each reduction $M'_i \twoheadrightarrow M'_{i+1}$ is in fact one step.

Proof. Consider in the figure showing $\mathscr{D}(\mathscr{R}_{F^*,M}, \{\underline{KAB}\})$ the subdiagram $\mathscr{D}(M_i \to M_{i+1}, \mathscr{R}_i)$. Here $\mathscr{R}_i = \{\underline{KAB}\}/M_0 \twoheadrightarrow M_i$ is a complete development of the underlined K-redexes $\underline{KA_jB_j}$.

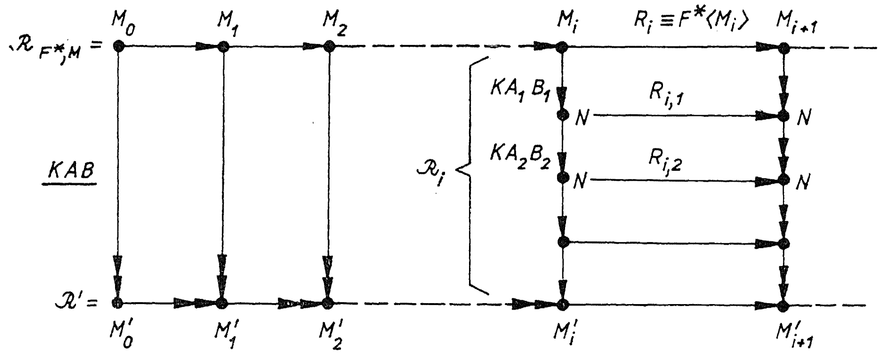


Fig. 7
(see Remark to Fig. 2)

Now by Lemma 25, $R_i \equiv F^*\langle M_i\rangle$ is not a subterm of the argument of an underlined K-redex. Hence $N_{i,1} \xrightarrow{R_{i,1}} N_{i+1,1}$ is indeed one step and $R_{i,1}$ is again not covered by the argument of an underlined K-redex, etc. This proves the claim.

So we proved that $\mathcal{R}'$ is infinite, and hence $\infty\, C[A]$. □

27. Proposition.

$$\infty(\lambda x_1, \dots x_n.\, A(x_1, \dots, x_n))\, C_1 \dots C_n \;\&\; C_1, \dots, C_n \in SN \Rightarrow \infty A(C_1, \dots, C_n).$$

Proof. Induction on n.

Basis. $n = 1$. Let $M \equiv (\lambda x.\, A(x))\, C$ where $C \in SN$ and suppose $\infty\, M$. Take an infinite reduction $\mathcal{R}$ of M.

Case 1. M's head-redex is never contracted in $\mathcal{R}$. Then clearly, since $C \in SN$, $\infty\, A(x)$. Hence also $\infty\, A(C)$.

Case 2. Otherwise $\mathcal{R} = M \twoheadrightarrow (\lambda x.\, A'(x)\, C' \to A'(C') \to \dots$ Hence $\infty\, A'(C')$, and because $A(x) \twoheadrightarrow A'(x)$, $C \twoheadrightarrow C'$ imply $A(C) \twoheadrightarrow A'(C')$, also $\infty\, A(C)$.

Induction step. The same argument. □

28. Theorem. *The redex KAB is perpetual $\Leftrightarrow A \geqq_\infty B$.*

Proof. ($\Leftarrow$) is Corollary 26.

($\Rightarrow$). Let $A \equiv A(\vec{x}) \equiv A(x_1, \dots, x_n)$ and $B \equiv B(\vec{x}) \equiv B(x_1, \dots, x_n)$ where $\{x_1, \dots, x_n\} = FV(A) \cup FV(B)$.
Suppose $\neg\, A(\vec{x}) \geqq_\infty B(\vec{x})$. Then for some $\vec{C} \in SN$, (i) $\infty\, B(\vec{C})$ but (ii) $\neg\, \infty\, A(\vec{C})$. Now consider the context $C[\;] = (\lambda\vec{x}.\, \Box)\, \vec{C}$.

Since $\infty\, KA(\vec{C})\, B(\vec{C})$ by (i), we have $\infty\, (\lambda\vec{x}.\, KA(\vec{x})\, B(\vec{x}))\, \vec{C}$, i.e. $\infty\, C[KAB]$. But $\neg\, \infty\, (\lambda\vec{x}.\, A(\vec{x}))\, \vec{C}$ by (ii) and Theorem 28, i.e. $\neg\, \infty\, C[A]$. Hence KAB is not perpetual, contradiction. □

29. Corollary. *The redex $(\lambda x.\, A)\, B$ is perpetual $\Leftrightarrow x \in FV(A)$ or $A \geqq_\infty B$.*

Proof. Corollary 29 and Theorem 7. □

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Резюме

λ-терм M называется строго нормализуемым, если каждая цепочка приведения обрывается, приводя к однозначно определенной нормальной форме, и слабо нормализуемым, если существует хотя бы одна обрывающаяся цепочка. В статье рассматриваются свойства таких шагов приведения $M \to M'$, для которых M' имеет бесконечную цепочку приведения, если таковую имеет M. Достаточным условием этого является то, что во время шага $M \to M'$ сокращению подвергается такой редекс (остаток) $R \equiv (\lambda x. A)\, B$, в котором x свободно входит в A, т. е. R не теряет своего ,,аргумента". (Следствием этого является тот хорошо известный факт, что в λI-исчислении строгая и слабая нормализуемости совпадают.) Редекс R, при сокращении которого сохраняется существование необрывающейся цепочки приведения, называется сохраняющим. В настоящей работе рассматриваются сохраняющие редексы, которые теряют свой аргумент.

Kurzfassung

Ein λ-Term heißt streng normalisierbar, falls jede Reduktionskette abbricht (in der eindeutigen Normalform), und schwach normalisierbar, falls eine Reduktionskette zur Normalform führt. Hier interessieren wir uns für die Charakterisierung solcher Reduktionsschritte $M \to M'$, für die M' eine unendliche Reduktionskette hat, falls M eine hat. Eine hinreichende Bedingung dafür ist, daß im Schritt $M \to M'$ ein solcher Redex $R \equiv (\lambda x. A)\, B$ kontraktiert wird, bei dem x in A frei vorkommt; d. h., R verliert sein „Argument" nicht. (Eine Folgerung ist der wohlbekannte Fakt, daß im λI-Kalkül strenge und schwache Normalisierbarkeit zusammenfallen.) Ein R mit der Eigenschaft, daß seine Kontraktion die Existenz unendlicher Reduktionsketten erhält, heißt vererbend. In der vorliegenden Arbeit werden diejenigen vererbenden Redexe charakterisiert, die ihr Argument verlieren.

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