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Invariant Manifolds for Volterra Integral Equations of Convolution Type

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In this paper we develop some elements of a qualitative theory for nonlinear Volterra integral equations of convolution type. Our starting point is a local semiflow associated with the equation and acting on a space of compactly supported forcing functions. Within that framework we discuss the variation-of-constants formula, the saddle point property, the center manifold and Hopf bifurcation. Some equations from population biology get special attention.

1. INTRODUCTION

In this paper we study the behaviour of solutions of nonlinear Volterra convolution-integral equations near a constant solution. The basic tool in our analysis will be a certain procedure to associate with the equation a local semiflow or dynamical system on a function space. Contrary to the usual situation in the theory of delay equations, the elements of this state space are forcing functions (and not initial functions). Once the construction of the semiflow is clarified, we will follow, as much as possible, the general lines of the qualitative theory of ordinary differential equations [20], functional differential equations [21] and parabolic partial differential equations [29]. At some points the specific situation requires special arguments (see, for instance, Theorem 4.4 and Section 8). Moreover, we try to give sharp results and easy proofs.

The paper consists of three parts. The first contains the definition of the semiflow, a discussion of the linear case and a derivation of the variation-of-constants formula.

The second part contains the construction of the stable and the unstable manifold corresponding to an equilibrium with no eigenvalues of the linearized equation on the imaginary axis (the saddle point property). As an application we discuss the existence and uniqueness of positive solutions, defined for all time, of certain equations arising in population dynamics and epidemiology.

The third part deals with the situation where some of the eigenvalues lie on the imaginary axis. We present the construction of a center manifold for an equation which, possibly, depends on some parameter(s). The result is then used to prove a Hopf bifurcation theorem. Subsequently, we derive an applicable formula for the direction of bifurcation and we show how this direction is related to the stability of the periodic solution. As a specific example we discuss the equation

$$x(t) = \gamma \left(1 - \int_0^1 x(t-\tau)\, d\tau\right) \int_0^1 B(\tau)\, x(t-\tau)\, d\tau$$

in some detail. This equation arises in mathematical epidemiology [17, 18].

In all parts some version of the variation-of-constants formula has a central position.

Finally, let us try to describe the place of our work within the existing literature. Local semiflows associated with Volterra integral equations and acting on forcing functions have been mainly studied from the point of view of topological dynamics. The emphasis then is on the analogy with nonautonomous ordinary differential equations (o.d.e.'s) and infinite delays are not excluded. See [37, 38].

On the other hand, there exists a wealthy qualitative theory of functional differential equations starting from a local semiflow acting on initial functions. See [21]. Part of this theory describes nonlinear autonomous equations in close analogy with autonomous o.d.e.'s. Moreover, it is widely accepted that an important special (and relatively easy) case is formed by equations with finite delay. Once the special case is fully analyzed and understood, one knows what to look for when generalizing to infinite delays and one can concentrate on the subtle technical difficulties involved.

In this paper we demonstrate that one can build a qualitative theory of (autonomous, finite delay) Volterra integral equations along the lines followed by Hale [21, 25, 26] and others, but within the framework of the semiflow acting on forcing functions introduced by Miller. This idea seems to be new. The main point consists of some reflection on the interpretation of "state" and "forcing." Once these notions have been clarified the technical difficulties are minimal. The resulting theory treats equations which would otherwise need the much more difficult theory of neutral functional differential equations and, in addition, the use of a first integral (see Hale [26]).

The ideas presented in this paper are also applicable to Volterra integro-differential equations. In the linear case this is related to the work [36] of Miller. It has been observed by Burns and Herdman [3] that, for linear equations with infinite delay, the two semigroup constructions are related to each other by duality. In [14] this point of view is elaborated in the context of the standard linear retarded functional differential equation with finite delay.

Our main objective is to show that results which are fairly standard for, e.g., retarded functional differential equations, can be proved quite easily for Volterra convolution equations once a certain approach is adopted. For this class of equations the results are either new, or the proofs are much simpler than known ones.

Notation

$\mathbb{R}^n$	n-dimensional Euclidean space
$\lvert \cdot \rvert$	norm in $\mathbb{R}^n$
$\mathbb{R}_+$	$\{x \in \mathbb{R} \mid x \geqslant 0\}$
$\mathbb{R}_-$	$\{x \in \mathbb{R} \mid x \leqslant 0\}$
X	Banach space of continuous functions $f(t)$ from $\mathbb{R}_+$ into $\mathbb{R}^n$ which vanish for t larger than some fixed positive number b
$\lVert \cdot \rVert$	supremum norm in X
$BC(Y; Z)$	Banach space of bounded continuous mappings of a Banach space Y into a Banach space Z
$\lvert\lvert\lvert \cdot \rvert\rvert\rvert$	supremum norm in $BC(Y; Z)$
$BC^{\eta}(\mathbb{R}_\pm, Z)$	Banach space of continuous mappings of $\mathbb{R}_\pm$ into the Banach space Z, such that the norm $\lvert\lvert\lvert \cdot \rvert\rvert\rvert^{\eta} < \infty$
$\lvert\lvert\lvert \cdot \rvert\rvert\rvert^{\eta}$	$\lvert\lvert\lvert F \rvert\rvert\rvert^{\eta} = \sup\{e^{-\eta s} \lVert F(s) \rVert_Z \mid s \in \mathbb{R}_\pm\}$ $\lvert\lvert\lvert F \rvert\rvert\rvert^{\eta} = \sup\{e^{-\eta \lvert s \rvert} \lVert F(s) \rVert_Z \mid s \in \mathbb{R}\}$
$NBV([0, b]; \mathbb{R}^n)$	Banach space of functions of bounded variation of the interval $[0, b]$ into $\mathbb{R}^n$, suitably normalized
$\mathrm{Var}_{[0,b]}$	total variation; norm of $NBV([0, b]; \mathbb{R}^n)$
$*$	$f * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$
α	bounded linear operator of X into $\mathbb{R}^n$ defined by $\alpha(f) = f(0)$
$f_s(\cdot)$	$f_s(t) = f(s+t)$
$\hat{\ }$	Laplace transform $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt$

PART I: VOLTERRA EQUATIONS, SEMIFLOWS AND THE VARIATION-OF-CONSTANTS FORMULA

2. Construction of the Semiflow

Let $B = B(\tau)$ be a given $n \times n$-matrix-valued L_2-function with support contained in $[0, b]$ for some b, $0 < b < \infty$. Let $g\colon \mathbb{R}^n \to \mathbb{R}^n$ be a given

C^k-smooth function ($k \geqslant 1$) with $g(0) = 0$. The translation invariant equation,

$$x(t) = \int_0^b B(\tau)\, g(x(t-\tau))\, d\tau, \tag{2.1}$$

admits the constant solution $x(t) \equiv 0$. We are interested in the behaviour of *all* solutions which are near to this constant solution in a sense to be specified.

Equation (2.1) is nonanticipative: the value of x at time t_0 does depend only on the values of $x(t)$ for $t < t_0$. So we can define an initial value problem by prescribing $x(t) = \phi(t)$, $t_0 - b \leqslant t \leqslant t_0$, and solving (2.1) for $t > t_0$. Note, however, that in general

$$x(t_0+) = \int_0^b B(\tau)\, g(\phi(t_0 - \tau))\, d\tau \neq \phi(t_0) = x(t_0-).$$

This observation clearly shows that a continuous initial function is possibly mapped onto a discontinuous function by translation along the solution. Consequently the definition of a semiflow via this mapping (as it is usual in the theory of functional differential equations, cf. Hale [21]) faces a serious difficulty. Although one can overcome this difficulty [26], we propose an alternative construction.

Behind the notion of a semiflow lies the idea to describe how certain objects (data), which single out a unique solution, evolve when time proceeds. It has been observed by Miller [37] and Miller and Sell [38] that in the theory of Volterra integral equations, forcing functions are such objects.

So consider the equation

$$x(t) = \int_0^t B(\tau)\, g(x(t-\tau))\, d\tau + f(t), \qquad t \geqslant 0, \tag{2.2}$$

which we shall frequently write in the form

$$x = B * g(x) + f.$$

For a given function f, say continuous, there exists a unique solution x defined for $0 \leqslant t < \omega(f)$. For each $s \in [0, \omega(f))$ we define $\Pi(f, s)$ by

$$\Pi(f, s)(t) = f(t+s) + \int_0^s B(t+s-\tau)\, g(x(\tau))\, d\tau. \tag{2.3}$$

This definition is suggested by the fact that $\Pi(f, s)$ is precisely the forcing function in the equation satisfied by the translated function $x_s(t) = x(t+s)$. Indeed,

$$x_s = B * g(x_s) + \Pi(f, s). \tag{2.4}$$

The uniqueness of solutions then implies the semigroup property

$$\Pi(\Pi(f, s_1), s_2) = \Pi(f, s_1 + s_2), \qquad s_1, s_2 \geqslant 0, s_1 + s_2 < \omega(f). \quad (2.5)$$

In this setting it becomes nontrivial to distinguish between an initial value problem for an autonomous equation and for a "truly" forced equation. In the linear case this distinction has been discussed in [13]. We repeat some of the arguments concerning autonomous problems here. We postpone a description of forced problems to Section 4.

If (2.2) is derived from (2.1) with x specified on $[-b, 0]$, i.e., $x(t) = \phi(t)$, $-b \leqslant t \leqslant 0$, then f is given by

$$f(t) = \int_t^b B(\tau)\, g(\phi(t-\tau))\, d\tau.$$

In particular, we observe that f vanishes for $t \geqslant b$. This motivates us to choose as underlying state space X, to which f belongs, a space of functions with support contained in the interval $[0, b]$. Note that this property is preserved under $\Pi(\cdot, s)$ (see (2.3) and recall that the support of B is contained in $[0, b]$).

The mapping $\phi \mapsto f$ need not to be surjective. But if f vanishes for $t \geqslant b$, (2.2) is precisely (2.1) for $t \geqslant b$. So, as far as the behaviour for $t \geqslant b$ is concerned, (2.1) and (2.2) are equivalent.

In order to avoid technical difficulties with the substitution operator generated by the nonlinear function g, we want the elements of X to be bounded functions and, finally, we need that translation is continuous. So we are led to define

$$X = \{f \in BC(\mathbb{R}_+; \mathbb{R}^n) \mid f(t) = 0, t \geqslant b\}$$

provided with the supremum-norm topology.

For a definition of the concept "local-semiflow" we refer to Miller [37, p. 153] or Miller and Sell [38, p. 4]. The following theorem is a corollary of Theorem 1.1 in [38] (in the notation of [38] we take the compatible pair (G_2, A_2), see Theorem II.3, and we observe that the mapping (1.5) changes neither g nor a; we remark that an elementary proof is possible if one assumes that g is globally Lipschitz). In particular the theorem expresses that Π is continuous.

THEOREM 2.1. *The mapping Π defined by* (2.3) *is a local semiflow on* X.

For given $f \in X$, formula (2.3) defines $\Pi(f, s)$ in terms of the solution x of (2.2). Conversely, one can bring out x again by means of the bounded linear operator $\alpha: X \to \mathbb{R}^n$

$$\alpha(f) = f(0). \quad (2.6)$$

Indeed

$$\alpha(\Pi(f, s)) = f(s) + \int_0^s B(s-\tau)\, g(x(\tau))\, d\tau = x(s). \tag{2.7}$$

In subsequent sections we shall frequently suppress the dependence on the function f in the notation. So instead of Π we use X-valued functions $F(s) = \Pi(F(0), s)$. Nevertheless it is enlightening to keep Π and its definition in the back of one's mind.

In this paper we concentrate on equations with a nonlinear function "inside" the integral, cf. (2.1). This is less restrictive than it seems since many equations of a different form can be transformed to (2.1). For instance, the equation

$$x(t) = \left(1 - \int_0^1 B^1(\tau)\, x(t-\tau)\, d\tau\right) \int_0^1 B^2(\tau)\, x(t-\tau)\, d\tau$$

is equivalent to a two-system of the form (2.1) (see Section 12; this observation makes some of the work in [16] redundant).

3. The Semigroup Associated with a Linear Equation

For given $f \in X$ the equation

$$x = B * x + f \tag{3.1}$$

has a unique solution given explicitly by

$$x = f - R * f, \tag{3.2}$$

where R, the so-called *resolvent*, is the unique, matrix-valued, locally integrable solution of

$$R = B * R - B. \tag{3.3}$$

For future use we mention that $B * R = R * B$ and that $R \in L_2^{\mathrm{loc}}(\mathbb{R}_+)$. See [37].

In this linear case the mapping $\Pi(f, s)$ is defined for all $s \geqslant 0$ and can be written as

$$\Pi(f, s) = T(s) f, \tag{3.4}$$

where for each s, $T(s)$ is a linear mapping of X into itself. Using (3.2) we find the explicit representation

$$(T(s) f)(t) = f(t+s) + (B_t - B_t * R) * f(s). \tag{3.5}$$

So $T(s)$ is the sum of a shift-operator $U(s)$

$$(U(s)f)(t) = f(t+s) \tag{3.6}$$

and a ("smoothing") operator $V(s)$

$$(V(s)f)(t) = (B_t - B_t * R) * f(s). \tag{3.7}$$

On X, $U(s)$ is nilpotent for each $s > 0$ (and vanishes for $s \geqslant b$) and $V(s)$ is compact. Formulas (3.5)–(3.7) unambiguously define the action of $T(s)$, $U(s)$ and $V(s)$ on any locally integrable function. We shall use the same symbols to denote the extended operators. Clearly (3.7) implies that the support of $V(s)f$ is contained in $[0, b]$. The following result is useful.

LEMMA 3.1. *For each $s \in \mathbb{R}_+$, $V(s)$ is a continuous linear mapping of $L_2^{\text{loc}}(\mathbb{R}_+; \mathbb{R}^n)$ into X.*

Next, we summarize some properties of $T(s)$.

THEOREM 3.2. *$\{T(s)\}$ forms a strongly continuous semigroup of bounded linear operators on X with infinitesimal generator A described by*

$$\mathscr{D}(A) = \{f \in X \mid f \text{ is absolutely continuous and } f'(\cdot) + B(\cdot)\,\alpha(f) \text{ is continuous}\}$$

$$(Af)(t) = f'(t) + B(t)\,\alpha(f). \tag{3.8}$$

The (closed) operator A has compact resolvent and

$$\sigma(A) = P_\sigma(A) = \{\lambda \mid \det(I - \hat{B}(\lambda)) = 0\}. \tag{3.9}$$

We refer to [13] for a detailed proof in a slightly different setting.

We emphasize that the spectrum of the generator coincides exactly with the roots of the characteristic equation $\det(I - \hat{B}(\lambda)) = 0$ because we restrict the semigroup to a space of functions of compact support.

THEOREM 3.3. *One can decompose X as*

$$X = X_- \oplus X_0 \oplus X_+$$

(with corresponding projection operators P_-, P_0 and P_+) such that

(i) *$T(s)$ and A are completely reduced by (X_-, X_0, X_+),*

(ii) *the spectrum of the restriction of A to X_-, X_0, X_+ is precisely the subset of $P_\sigma(A)$ that belongs to, respectively, the left half plane, the imaginary axis and the right half plane,*

(iii) *X_0 and X_+ are finite dimensional and on these subspaces $T(s)$ can be naturally defined for $s < 0$ such that the group property is maintained,*

(iv) *for any $\varepsilon > 0$ there exists a constant $K > 0$ such that*

$$\begin{aligned} \|T(s)f\| &\leqslant Ke^{(\gamma_+ - \varepsilon)s}\,\|f\| && \textit{for} \quad s \leqslant 0 \textit{ and } f \in X_+, \\ \|T(s)f\| &\leqslant Ke^{\varepsilon|s|}\,\|f\| && \textit{for} \quad -\infty < s < \infty \textit{ and } f \in X_0, \\ \|T(s)f\| &\leqslant Ke^{(\gamma_- + \varepsilon)s}\,\|f\| && \textit{for} \quad s \geqslant 0 \textit{ and } f \in X_-, \end{aligned} \tag{3.10}$$

where

$$\gamma_+ := \inf\{\operatorname{Re}\lambda \mid \lambda \in P_\sigma(A), \operatorname{Re}\lambda > 0\},$$
$$\gamma_- := \sup\{\operatorname{Re}\lambda \mid \lambda \in P_\sigma(A), \operatorname{Re}\lambda < 0\}.$$

We remark that one can equally well decompose X according to a subdivision of $P_\sigma(A)$ relative to some line $\{\lambda \mid \operatorname{Re}\lambda = \text{const}\}$, and that in this case appropriate analogues of the estimates (3.10) are valid.

The appendix of [13] contains a detailed description of the elements of $\mathcal{N}(A - \lambda I)^k$ and $\mathcal{R}(A - \lambda I)^k$. Moreover, in [13] the adjoints of $T(s)$ and A are determined (though in an $L_1 - L_\infty$ context). It appears that the semigroup constructions on, respectively, initial functions and forcing functions are related to each other by duality. This point of view is developed more systematically in [14]. These results are useful if one wants actually to construct the operators P_+, P_0 and P_-.

The projection operators extend to L_1-functions with support in $[0, b]$ (and hence to the columns of B). Moreover, the range of the extended operators P_+ and P_0 is contained in X (even in $\mathcal{D}(A)$; see Section 10 for an example). There are at least two ways to see this. Either one can use an explicit representation of P_+ and P_0 in terms of solutions of the "adjoint" equation, or one can first discuss the semigroup, the generator and the spectrum in an L_1-setting (see [13]).

From (3.5), (3.3) and (3.2) we deduce that (cf. (2.7))

$$\alpha(T(s)f) = f(s) - R * f(s) = x(s). \tag{3.11}$$

Putting $f = B$ and using (3.3) once more we find

$$\alpha(T(s)\,B) = -R(s) \tag{3.12}$$

(we define the action of $T(s)$ on a matrix-valued function via the action on each separate column).

If $f \in X_-$, then $x = \alpha(T(\cdot)f)$ decays exponentially. However, this does not follow directly from the representation $x = f - R * f$. Therefore, motivated by (3.12), we introduce

$$R^-(s) = -\alpha(T(s)\,P_-B) \tag{3.13}$$

and we derive a representation in terms of this function. But first we state another auxiliary result which is the key point in the next section.

LEMMA 3.4

$$(V(s)f)(t)=\int_0^s (T(s-\tau)\,B)(t)f(\tau)\,d\tau.$$

We shall frequently suppress the variable t and write

$$V(s)f=\int_0^s T(s-\tau)\,Bf(\tau)\,d\tau. \tag{3.14}$$

It should be noted that this identity makes sense for any $f\in L_1^{\text{loc}}(\mathbb{R}_+;\mathbb{R}^n)$ and that $V(s)f\in X$ if $f\in L_2^{\text{loc}}(\mathbb{R}_+;\mathbb{R}^n)$ (Lemma 3.1).

LEMMA 3.5

(i) $R^-=B*R^--P_-B$,

(ii) $(T(s)\,P_-f)(t)=(P_-U(s)f)(t)+((P_-B)_t-B_t*R^-)*f(s)$,

(iii) *for* $s\geqslant b$, $\alpha(T(s)\,P_-f)=-R^-*f(s)$.

Proof. (i) From the definition of $T(s)$ (see (3.4) and (2.4)) we infer that R^- is the solution of the equation $Y=B*Y-P_-B$. Alternatively, one can define R^- as the solution of this equation and then derive (3.13).

(ii) By (i) and the definition of $T(s)$ we have

$$\begin{aligned}(T(s)\,P_-B)(t)&=-R_s^-(t)+B*R_s^-(t)=-R_s^-(t)+B*R^-(s+t)-B_t*R^-(s)\\&=(P_-B)_t(s)-B_t*R^-(s).\end{aligned}$$

Next, $T(s)=U(s)+V(s)$ and Lemma 3.4 imply that

$$T(s)\,P_-f=P_-U(s)f+\int_0^s T(s-\tau)\,P_-Bf(\tau)\,d\tau$$

and a combination of the identities proves the correctness of (ii). Note that we have changed the order of integration and application of the projection operator. For a discussion on this see the end of Section 4.

(iii) This is a consequence of (i), (ii) and the fact that $U(s)=0$ for $s\geqslant b$. ∎

The definition (3.13), the fact that the columns of $T(b)\,P_-B$ belong to X (Lemma 3.1), the semigroup property and the exponential estimate (3.10) for $T(s)\,P_-$ jointly imply that the function $\tau\to R^-(\tau)\,e^{-(\gamma_-+\varepsilon)\tau}$ is absolutely integrable on $\mathbb{R}_+$ for any $\varepsilon > 0$. It is this property which makes Lemma 3.5 useful.

As a side remark we mention that R^- admits an interpretation in terms of Laplace transforms. The function $R^+ := R - R^-$ corresponds, under inverse transformation, to the singularities of $-(I - \hat{B}(\lambda))^{-1}\,\hat{B}(\lambda)$ in the closed right half plane. Compare for instance with [32]. Alternatively, one can describe R^+ in terms of quantities related to the generalized eigenspaces of $A - \lambda I$ and $A^* - \lambda I$, with $\lambda \in P_\sigma(A)$ and $\operatorname{Re}\lambda \geqslant 0$.

4. The Variation-of-Constants Formula

In our approach a "forced" linear equation is an equation of the form

$$x = B * x + f + h,$$

where both f and h are given functions. Here f belongs to X and represents the "state" at time $t = 0$, while h describes the forcing (so there are no restrictions on the support of h). In this form the initial time is $t = 0$. More generally, we denote the initial time by σ and the initial state by $F(\sigma)$. Our aim is to derive a formula which expresses the state at time s, denoted by $F(s)$, in terms of $F(\sigma)$ and h.

THEOREM 4.1. *Let $h \in C(\mathbb{R}; \mathbb{R}^n)$, $\sigma \in \mathbb{R}$ and $F(\sigma) \in X$ be given. Let x: $[\sigma, \infty) \to \mathbb{R}^n$ denote the unique solution of*

$$x_\sigma = B * x_\sigma + F(\sigma) + h_\sigma. \tag{4.1}$$

For each $s \geqslant \sigma$ define $F(s)$ by

$$x_s = B * x_s + F(s) + h_s. \tag{4.2}$$

Then $F(s) \in X$ and

$$F(s) = T(s - \sigma)\,F(\sigma) + \int_\sigma^s T(s - \tau)\,Bh(\tau)\,d\tau. \tag{4.3}$$

Proof. Recalling the definition of $T(s)$ (see (2.4), (3.4)) we observe that

$$F(s) + h_s = T(s - \sigma)(F(\sigma) + h_\sigma) = T(s - \sigma)\,F(\sigma) + h_s + V(s - \sigma)\,h_\sigma.$$

So Lemma 3.1 implies that $F(s) \in X$. Moreover, by Lemma 3.4,

$$F(s) = T(s - \sigma)\,F(\sigma) + \int_0^{s-\sigma} T(s - \sigma - \tau)\,Bh_\sigma(\tau)\,d\tau$$

$$= T(s - \sigma)\,F(\sigma) + \int_\sigma^s T(s - \tau)\,Bh(\tau)\,d\tau. \quad ∎$$

One can as well use (4.3) to define $F(s)$ and subsequently bring out the solution of (4.1) via the operator α. More precisely, we put

$$x(s) = \alpha(F(s)) + h(s), \qquad s \geqslant \sigma, \tag{4.4}$$

and we shall show that x thus defined satisfies (4.1). Applying α to (4.3), with $s = \sigma + t$, and using (3.11) and (3.12), we see that

$$\alpha(F(\sigma + t)) = F(\sigma)(t) - (R * F(\sigma))(t) - R * h_\sigma(t).$$

On the other hand, by (4.4), $\alpha(F(\sigma + t)) = x(\sigma + t) - h(\sigma + t)$. Suppressing t in the notation we find

$$x_\sigma = F(\sigma) + h_\sigma - R * (F(\sigma) + h_\sigma)$$

and hence

$$\begin{aligned} B * x_\sigma &= B * (F(\sigma) + h_\sigma) - B * R * (F(\sigma) + h_\sigma) \\ &= -R * (F(\sigma) + h_\sigma). \end{aligned}$$

A combination of the last two identities yields (4.1). So we conclude that (4.1) and (4.3) are "equivalent" in the sense that (4.2) expresses F in terms of x such that (4.1) implies (4.3), whereas (4.4) expresses x in terms of F such that (4.3) implies (4.1). This kind of equivalence between a convolution equation in $\mathbb{R}^n$ and an integral-operator equation in X will be used repeatedly.

Formal differentiation of (4.3) yields the inhomogeneous ordinary differential equation

$$\frac{dF}{ds}(s) = AF(s) + Bh(s).$$

So we have shown that there is a one-to-one correspondence between the mild solutions of this o.d.e. and solutions of (4.1).

Our next objective is to apply the result to a nonlinear problem.

THEOREM 4.2. *Let the solution x of*

$$x_\sigma = B * g(x_\sigma) + F(\sigma) \tag{4.5}$$

be defined for $t \in [\sigma, \omega)$. Define F: $[\sigma, \omega) \to X$ by

$$F(s) = \Pi(F(\sigma), s - \sigma). \tag{4.6}$$

Then F is continuous and

$$F(s) = T(s - \sigma)\, F(\sigma) + \int_\sigma^s T(s - \tau)\, Br(\alpha(F(\tau))\, d\tau, \tag{4.7}$$

where

$$r(x) := g(x) - x. \tag{4.8}$$

Conversely, let $F\colon [\sigma, \omega) \to X$ *be a continuous function which satisfies* (4.7). *Then* x *defined by*

$$x(s) = \alpha(F(s)) \tag{4.9}$$

satisfies (4.5).

Proof. The continuity of F defined by (4.6) is a consequence of Theorem 2.1. Define $y = g(x)$, then (4.5) can be written as

$$y_\sigma = B * y_\sigma + F(\sigma) + r(x_\sigma) \tag{4.10}$$

which is of the form (4.1) with $h = r(x)$. By (4.4), $x = \alpha(F)$ and hence we obtain (4.7) from (4.3). Conversely, by the discussion following Theorem 4.1, (4.7) implies $y_\sigma = B * y_\sigma + F(\sigma) + r(\alpha(F))_\sigma$, where $y(s) = \alpha(F(s)) + r(\alpha(F(s))) = g(\alpha(F(s)))$. Hence (4.9) leads to (4.5). ∎

COROLLARY 4.3. *If* $g(x) = x + o(|x|)$, $|x| \downarrow 0$, *then* $T(s)$ *is the Fréchet derivative of* $\Pi(\cdot, s)$ *in* $f = 0$.

Once again formal differentiation yields an o.d.e. in X:

$$\frac{dF}{ds} = AF + Br(\alpha(F)) = F' + Bg(\alpha(F)). \tag{4.11}$$

In the third part we will analyse the situation where B depends on some parameter μ. In particular it will be necessary to linearize with respect to both the "state" and the parameter, say at $\mu = 0$. In terms of the o.d.e. (4.11) the procedure is as follows. Let for each $\mu \in \mathbb{R}^m$, $B(\mu, \cdot)$ be an $n \times n$ matrix valued L_2-function with support contained in $[0, b]$, and let $g(\mu, \cdot)$ be a mapping of $\mathbb{R}^n$ into itself. Defining $r(\mu, x) = g(\mu, x) - x$ we regroup the terms in (4.11):

$$\begin{aligned}\frac{dF}{ds} &= F' + B(0, \cdot)\, \alpha(F) + B(0, \cdot)\, r(\mu, \alpha(F)) + (B(\mu, \cdot) - B(0, \cdot))\, g(\mu, \alpha(F)) \\ &= AF + B(0, \cdot)\, r(\mu, \alpha(F)) + (B(\mu, \cdot) - B(0, \cdot))\, g(\mu, \alpha(F)),\end{aligned}$$

where A denotes the generator corresponding to $\mu = 0$. This procedure suggests a version of the variation-of-constants formula which we now prove.

THEOREM 4.4. *Let* $T(s)$ *denote the semigroup associated with* $B(0, \cdot)$. *Let* $\sigma \in \mathbb{R}$ *and* $F(\sigma) \in X$ *be given. Let the solution* x *of*

$$x_\sigma = B(\mu, \cdot) * g(\mu, x_\sigma) + F(\sigma) \tag{4.12}$$

be defined for $t \in [\sigma, \omega)$. Define F: $[\sigma, \omega) \to X$ by

$$F(s) = \Pi(\mu, F(\sigma), s - \sigma) \tag{4.13}$$

then F is continuous and

$$F(s) = T(s-\sigma)\, F(\sigma) + \int_\sigma^s T(s-\tau)\{B(0, \cdot)\, r(\mu, \alpha(F(\tau)))$$
$$+ (B(\mu, \cdot) - B(0, \cdot))\, g(\mu, \alpha(F(\tau)))\}\, d\tau. \tag{4.14}$$

Conversely, let F: $[\sigma, \omega) \to X$ be a continuous function which satisfies (4.14), *then x defined by*

$$x(s) = \alpha(F(s)) \tag{4.15}$$

satisfies (4.12).

Proof. The general formula can be obtained from the special case $\sigma = 0$ via the semigroup property. So without loss of generality we put $\sigma = 0$. Moreover, we write $F(0) = f$.

Let R denote the resolvent associated with $B(0, \cdot)$ then, by definition (2.3) of Π and the representation (3.5) of $T(s)$,

$$\begin{aligned} F(s)(t) &= f(t+s) + (B_t(\mu, \cdot) * g(\mu, x))(s) \\ &= (T(s) f)(t) + (B_t(\mu, \cdot) * g(\mu, x))(s) \\ &\quad - (B_t(0, \cdot) - B_t(0, \cdot) * R) * f(s) \end{aligned}$$

(here a subscript denotes translation with respect to the second variable). In this formula we substitute $f = x - B(\mu, \cdot) * g(\mu, x)$ and we regroup the terms as follows:

$$\begin{aligned} F(s)(t) &= (T(s) f)(t) + ((B_t(\mu, \cdot) + (B_t(0, \cdot) - B_t(0, \cdot) * R) * B(\mu, \cdot)) \\ &\quad * g(\mu, x))(s) - ((B_t(0, \cdot) - B_t(0, \cdot) * R) * x)(s) \\ &= (T(s) f)(t) + \int_0^s (T(s-\tau)\, B(\mu, \cdot))(t)\, g(\mu, x(\tau))\, d\tau \\ &\quad - \int_0^s (T(s-\tau)\, B(0, \cdot)(t)\, x(\tau)\, d\tau. \end{aligned}$$

Using $x(s) = \alpha(F(s))$ (by (4.13)) and $g(\mu, x) = x + r(\mu, x)$ we obtain (4.14), with $\sigma = 0$ and $F(\sigma) = f$.

Next, let F be defined by (4.14), with $\sigma = 0$ and $F(\sigma) = f$, and x by (4.15). Applying α to (4.14) and recalling (3.11) and (3.12) we obtain

$$x = f - R * f - R * r(\mu, x) + (B(\mu, \cdot) - R * B(\mu, \cdot) + R) * g(\mu, x).$$

This identity is of the form $y = R * y$ with $y = f + B(\mu, \cdot) * g(\mu, x) - x$. A local contraction argument shows that necessarily $y = 0$ and consequently (4.12), with $\sigma = 0$ and $F(\sigma) = f$, is satisfied. ∎

In conclusion of this section we discuss the decomposition of the variation-of-constants formula. The projection operators described in Theorem 3.3 commute with the integral from σ to s in (4.3), (4.7) and (4.14) (indeed, this boils down to the interchanging of the order of two integrations) and with $T(\tau)$ for any τ. Thus, for instance, applying the projection P_- on X_- to (4.3) we obtain

$$P_- F(s) = T(s - \sigma)\, P_- F(\sigma) + \int_\sigma^s T(s - \tau)\, P_- Bh(\tau)\, d\tau.$$

Similar identities are valid for any of the projections and each version of the variation-of-constants formula.

PART II: THE SADDLE POINT PROPERTY

If A has no spectrum on the imaginary axis, the decomposition of X as described in Theorem 3.3 gives a very clear picture of the asymptotic behaviour, as $s \to \pm\infty$, of solutions of a linear equation. In this part we shall show that, locally near zero, this picture remains valid for a nonlinear equation with $g(x) = x + o(|x|)$, $|x| \downarrow 0$. An analysis of the case where A does have spectrum on the imaginary axis is postponed to the third part.

In the nonlinear situation invariant manifolds replace the invariant linear subspaces of Theorem 3.3. We shall construct these manifolds and determine their properties in a number of steps. Our argumentation is inspired by Hale [20, III.6; 21, 9.2].

5. The Unstable Manifold

Our first step is the construction of a pseudo-inverse related to the linear problem for bounded functions on $\mathbb{R}_-$. More precisely, for given $h \in BC(\mathbb{R}_-; \mathbb{R}^n)$ we want to describe those functions $F \in BC(\mathbb{R}_-; X)$ which satisfy

$$F(s) = T(s - \sigma)\, F(\sigma) + \int_\sigma^s T(s - \tau)\, Bh(\tau)\, d\tau, \qquad -\infty < \sigma \leqslant s \leqslant 0. \quad (5.1)$$

THEOREM 5.1. *The expression*

$$(\mathscr{K} h)(s) = \int_0^s T(s - \tau)\, P_+ Bh(\tau)\, d\tau + \int_{-\infty}^s T(s - \tau)\, P_- Bh(\tau)\, d\tau \quad (5.2)$$

defines a continuous linear operator $\mathscr{K}: BC(\mathbb{R}_-; \mathbb{R}^n) \to BC(\mathbb{R}_-; X)$. *Moreover,* $F = \mathscr{K}h$ *satisfies* (5.1) *and any* $F \in BC(\mathbb{R}_-; X)$ *which satisfies* (5.1) *is of the form* $F = T(\cdot)\,\phi + \mathscr{K}h$ *for some* $\phi \in X_+$.

Recalling that in the nonlinear problem h is replaced by $r(\alpha(F))$ (see Theorem 4.2; $r(x) = g(x) - x = o(|x|)$, $|x| \downarrow 0$) we introduce a mapping

$$\begin{aligned} &\mathscr{F}: BC(\mathbb{R}_-; X) \times X_+ \to BC(\mathbb{R}_-; X),\\ &\mathscr{F}(F, \phi) = F - T(\cdot)\,\phi - \mathscr{K}(r(\alpha(F))). \end{aligned} \tag{5.3}$$

Clearly $\mathscr{F}(F, \phi) = 0$ if and only if

$$F(s) = T(s-\sigma)\,F(\sigma) + \int_\sigma^s T(s-\tau)\,Br(\alpha(F(\tau)))\,d\tau, \qquad -\infty < \sigma \leqslant s \leqslant 0, \tag{5.4}$$

and $P_+F(0) = \phi$. We now show that the $BC(\mathbb{R}_-; X)$ solutions can be (locally near zero) parametrized by $\phi \in X_+$.

THEOREM 5.2. $\mathscr{F} \in C^k$, $\mathscr{F}(0, 0) = 0$ *and* $\partial\mathscr{F}(0, 0)/\partial F = \mathrm{Id}$. *Consequently the implicit function theorem implies the existence and (local) uniqueness of a* C^k*-function* $F^*(\phi)$, *defined for* ϕ *sufficiently small, such that* $\mathscr{F}(F^*(\phi), \phi) = 0$.

Proof. The term $\mathscr{K}(r(\alpha(F)))$ is the composition of two continuous linear operators $\mathscr{K}$ and α and the substitution operator $r: BC(\mathbb{R}_-; \mathbb{R}^n) \to BC(\mathbb{R}_-; \mathbb{R}^n)$ defined by $r(h)(s) = r(h(s))$. Clearly r is C^k in this sense as well and its derivative in zero vanishes. Furthermore, $\mathscr{F}$ is linear and continuous in ϕ. ∎

The unstable manifold is the image of the mapping $\mathscr{U}: X_+ \to X$ defined by $\mathscr{U}(\phi) = F^*(\phi)(0)$. Since F^* is C^k, so is $\mathscr{U}$. Furthermore $P_+\mathscr{U}(\phi) = \phi$.

THEOREM 5.3. (i) $\mathrm{Im}(\mathscr{U})$ *is invariant in the sense that for* ϕ *sufficiently small*

$$F^*(\phi)(s) = \mathscr{U}(P_+F^*(\phi)(s)). \tag{5.5}$$

(ii) $\mathrm{Im}(\mathscr{U})$ *is tangent to* X_+ *at zero*: $d\mathscr{U}(0)\,\psi/d\phi = \psi$.

(iii) *There exists a positive constant* L *such that, for sufficiently small* $\delta > 0$, $\mathscr{U}$ *is a diffeomorphism of the ball of radius* δ *in* X_+ *onto the set* $\{f \in X \mid \|P_+f\| \leqslant \delta$ *and there exists a solution of* (5.4) *on* $\mathbb{R}_-$ *such that* $F(0) = f$ *and* $\|F(s)\| \leqslant L\delta,\ s \leqslant 0\}$.

We say that $F(0)$ admits a *backward continuation* if we can find $\sigma < 0$ and $F(\sigma) \in X$ such that (5.4) holds for $\sigma \leqslant s \leqslant 0$. We have found that a suitably bounded backward continuation on $\mathbb{R}_-$ exists if and only if $F(0) \in \operatorname{Im}(\mathscr{U})$ and that there is at most one such continuation. However, in general backward continuation is not unique (not even in the linear case) and there may be many backward continuations on finite intervals.

In the linear case bounded solutions in fact decay exponentially. This property is, as we now state, preserved under nonlinear perturbations.

THEOREM 5.4. *For each $\varepsilon > 0$ there exists $M = M(\varepsilon)$ such that for $\|\phi\| \leqslant \delta$ with δ sufficiently small*

$$\|F^*(\phi)(s)\| \leqslant M\,\|\phi\|\, e^{(\gamma_+ - \varepsilon - \nu(\delta))s}, \qquad s \leqslant 0.$$

Here $\nu(\delta) \downarrow 0$ as $\delta \downarrow 0$.

F^* is a C^k-function of ϕ but, as our next result shows, likewise $F^*(\phi)$ is a C^k-function of s. This is due to the fact that $T(s)$ is differentiable on the finite-dimensional space X_+.

THEOREM 5.5

$$F^*(\phi) \in C^k(\mathbb{R}_-; X).$$

Proof. Put $y(s) = P_+ F^*(\phi)(s)$, then

$$y(s) = T(s)\, y(0) + \int_0^s T(s-\tau)\, P_+ Br(\alpha(\mathscr{U}(y(\tau))))\, d\tau,$$

and consequently

$$y'(s) = Ay(s) + P_+ Br(\alpha(\mathscr{U}(y(s)))).$$

In terms of coordinates with respect to a basis for X_+, this is a nonlinear o.d.e. with C^k right-hand side and therefore the solution is C^{k+1}. Since $F^*(\phi)(s) = \mathscr{U}(y(s))$ and $\mathscr{U} \in C^k$, the result follows. ∎

Finally, we make ends meet and interpret our results in terms of solutions of the so-called limiting equation

$$x(t) = \int_0^b B(\tau)\, g(x(t-\tau))\, d\tau. \tag{5.6}$$

THEOREM 5.6. (i) *Let x be a solution of (5.6) on $(-\infty, 0]$ with $|x(t)| \leqslant \delta$ for some δ sufficiently small. Define $f \in X$ by $f(t) = \int_t^b B(\tau)\, g(x(t-\tau))\, d\tau$. Then $f \in \operatorname{Im}(\mathscr{U})$ and $x(s) = \alpha(F^*(P_+ f)(s))$, $s \leqslant 0$.*

(ii) *Conversely, let $f \in \operatorname{Im}(\mathscr{U})$ be given, then x defined by $x(s) = \alpha(F^*(P_+ f)(s))$ satisfies (5.6) on $(-\infty, 0]$.*

This theorem is nothing but a combination of Theorem 4.2 and Theorem 5.3 and it gives a parametrization of those solutions of (5.6) which are defined and suitably bounded on $\mathbb{R}_-$. The translation invariance of (5.6) is reflected in the invariance (5.5) of $\mathscr{U}$.

Theorems 5.4 and 5.5 imply that these "infinitely old" solutions of (5.6) decay exponentially and are C^k-smooth. It seems to be difficult to arrive at the smoothness result by other means (at least without additional assumptions on the kernel B).

6. The Stable Manifold

Starting from the continuous linear mapping

$$\mathscr{L}: BC(\mathbb{R}_+; \mathbb{R}^n) \to BC(\mathbb{R}_+; X),$$

$$(\mathscr{L}h)(s) = \int_0^s T(s-\tau)\, P_-\, Bh(\tau)\, d\tau + \int_\infty^s T(s-\tau)\, P_+\, Bh(\tau)\, d\tau,$$

one can give a characterization of the $BC(\mathbb{R}_+; X)$ solutions of (5.4) in a neighbourhood of zero. The analysis proceeds along the same lines and we confine ourselves to a formulation of the final result.

THEOREM 6.1. *There exists a C^k-mapping $\mathscr{S}$ of a neighbourhood of zero in X_- into X such that*

(i) *there exists a positive constant L such that for small $\delta > 0$, $\mathscr{S}$ is a diffeomorphism of the ball of radius δ in X_- onto the set*

$$\{f \in X \mid \|P_- f\| \leqslant \delta,\ \Pi(f, s) \text{ is defined for all } s \geqslant 0 \text{ and } \|\Pi(f, s)\| \leqslant L\delta\},$$

(ii) $\mathrm{Im}(\mathscr{S})$ *is tangent to X_- at zero*: $d\mathscr{S}(0)\, \psi/d\phi = \psi$,

(iii) $\mathrm{Im}(\mathscr{S})$ *is invariant under Π*:

$$\Pi(f, s) = \mathscr{S}(P_- \Pi(f, s))$$

for all $f \in \mathrm{Im}(\mathscr{S})$ with $\|P_- f\|$ sufficiently small,

(iv) $\|\Pi(f, s)\| \leqslant M \|f\| \exp((\gamma_- + \varepsilon + \nu(\delta))\, s), \quad s \geqslant 0, \quad f \in \mathrm{Im}(\mathscr{S})$, *provided $\|P_- f\| \leqslant \delta$. Here $\nu(\delta) \downarrow 0$ as $\delta \downarrow 0$,*

(v) *the solution of $x = B * g(x) + f$ is defined for all $t \geqslant 0$ and satisfies $|x(t)| \leqslant \delta$, for some δ sufficiently small, if and only if $f \in \mathrm{Im}(\mathscr{S})$.*

COROLLARY 6.2 (exponential asymptotic stability). *If the spectrum of A belongs to the left half plane, then any solution of $x = B * g(x) + f$ with $f \in X$ and $\|f\|$ sufficiently small converges exponentially to zero.*

We remark that the solution of $x = B * g(x) + f$ with $f \in X$ satisfies the limiting equation (5.6) for $t \geqslant b$. So also the stable manifold $\mathrm{Im}(\mathscr{S})$ gives a parametrization of solutions of (5.6), notably of those which are defined and suitably bounded on $\mathbb{R}_+$.

The subspaces X_+ and X_- admit, in general, a further decomposition in $T(s)$-invariant subspaces characterized by growth rates of $T(\cdot)\phi$ (see the remark following Theorem 3.3). Similarly the manifolds $\mathrm{Im}(\mathscr{U})$ and $\mathrm{Im}(\mathscr{S})$ contain submanifolds which are in one to one correspondence with solutions in some BC^η-space. Again the construction of such a submanifold can be based on operators like $\mathscr{K}$ and $\mathscr{L}$, defined on the appropriate spaces. We formulate one result which we need in the next section. We emphasize that this result holds equally well if some of the eigenvalues lie on the imaginary axis.

THEOREM 6.3. *Let η be any positive number such that A has no spectrum on the line* $\mathrm{Re}\,\lambda = \eta$. *Let Q denote the direct sum of the generalized eigenspaces corresponding to the eigenvalues of A with real part exceeding η. There exists a C^k-mapping $\mathscr{W}$ of a neighbourhood of zero in Q into X such that* $\mathrm{Im}(\mathscr{W})$ *coincides with the set of $f \in X$ which admit a backward continuation on $\mathbb{R}_-$ belonging to some neighbourhood of zero in* $BC^\eta(\mathbb{R}_-; X)$. *Moreover,* $\mathrm{Im}(\mathscr{W})$ *is invariant in the sense that for* $f \in \mathrm{Im}(\mathscr{W})$ *and $\|P_Q f\|$ sufficiently small*

$$F^*(f)(s) = \mathscr{W}(P_Q F^*(f)(s)).$$

Here $F^*(f)$ denotes the backward continuation through f and P_Q the canonical projection on Q. Finally, $\mathrm{Im}(\mathscr{W})$ *is tangent to Q at zero.*

7. Infinitely Old Positive Solutions

In some problems from population biology (cf. [2, 12]) one has a scalar equation ($n = 1$) with a non-negative kernel B such that

$$\int_0^b B(\tau)\, d\tau > 1.$$

In that case the characteristic equation

$$\int_0^b B(\tau)\, e^{-\lambda\tau}\, d\tau = 1$$

has a real, positive, simple root λ^* (sometimes called the Malthusian parameter) and all other roots satisfy $\mathrm{Im}\,\lambda \neq 0$ and $\mathrm{Re}\,\lambda < \lambda^*$.

The eigenfunction of A corresponding to λ^* is

$$\phi(t) = e^{\lambda^* t} \int_t^b B(\tau)\, e^{-\lambda^* \tau}\, d\tau. \tag{7.1}$$

The corresponding solution of the linear equation

$$x(t) = \int_0^b B(\tau)\, x(t-\tau)\, d\tau \tag{7.2}$$

is $x(t) = \alpha(T(t)\,\phi) = e^{\lambda^* t}$ and this is, modulo translation, the only positive solution of (7.2) defined and bounded for all time $t \leqslant 0$ (this follows from Theorem 3.3 and the fact that $\operatorname{Im} \lambda \neq 0$ for all $\lambda \in \sigma(A) \backslash \{\lambda^*\}$).

Choose $\eta \in (0, \lambda^*)$ such that no eigenvalue λ other than λ^* satisfies $\operatorname{Re} \lambda \geqslant \eta$. According to Theorem 6.3 we can associate with the nonlinear equation

$$x(t) = \int_0^b B(\tau)\, g(x(t-\tau))\, d\tau \tag{7.3}$$

(with $g(x) = x + o(|x|)$) a one-dimensional invariant manifold $\operatorname{Im}(\mathscr{W})$ describing the $BC^{\eta}(\mathbb{R}_-; X)$-solutions.

THEOREM 7.1. *$\operatorname{Im}(\mathscr{W})$ consists of two submanifolds and each of these is in one-to-one correspondence with the translates of a sign-definite $BC^{\eta}(\mathbb{R}_-; \mathbb{R})$ solution of* (7.3).

Proof. We concentrate on positive solutions.

Step 1. Since $\operatorname{Im}(\mathscr{W})$ is tangent to $\operatorname{span}\{\phi\}$, we have $\|\mathscr{W}(\varepsilon\phi) - \varepsilon\phi\| = o(|\varepsilon|)$, $\varepsilon \to 0$. Therefore

$$\alpha(\mathscr{W}(\varepsilon\phi)) = \varepsilon + o(|\varepsilon|) > 0$$

for small positive ε.

Step 2. Let, for some $\delta > 0$, $F^*(\mathscr{W}(\delta\phi))$ denote the backward continuation in $BC^{\eta}(\mathbb{R}_-; X)$ through $\mathscr{W}(\delta\phi)$. Let P denote the projection onto ϕ along $\mathscr{R}(A - \lambda^* I)$. Define $\zeta(s)$ by

$$P(F^*(\mathscr{W}(\delta\phi))(s)) = \zeta(s)\,\phi,$$

then $\zeta\colon \mathbb{R}_- \to \mathbb{R}$ is a continuous function and $\zeta(0) = \delta$. Furthermore, ζ can not vanish since then, by the invariance relation

$$F^*(\mathscr{W}(\delta\phi))(s) = \mathscr{W}(\zeta(s)\,\phi),$$

$F^*(\mathscr{W}(\delta\phi))$ would be identically zero. So $\zeta(s) > 0$ and $\zeta(s) \to 0$ as $s \to -\infty$. (In fact ζ is the solution of an ordinary differential equation $\zeta' = \lambda^*\zeta + o(\zeta)$,

$\zeta(0)=\delta$ and consequently ζ is monotone increasing in a neighbourhood of the origin).

Step 3. Combining steps 2 and 3 we find

$$\alpha(F^*(\mathscr{W}(\delta\phi))(s))=\alpha(\mathscr{W}(\zeta(s)\,\phi))=\zeta(s)+o(|\zeta(s)|)>0$$

for s sufficiently close to $-\infty$. So the backward continuation is positive for those values of s. Finally, the fact that $(0,\delta]\subset\mathscr{R}(\zeta)$ implies that $F^*(\mathscr{W}(\delta\phi))(s)$ fills "half" of the manifold. More precisely, we conclude that the backward continuation through $\mathscr{W}(\varepsilon\phi)$ with $0<\varepsilon\leqslant\delta$ is obtained by an appropriate translation of $F^*(\mathscr{W}(\delta\phi))$. Upon application of α we obtain the stated result. ∎

In the linear case we found that every positive solution, defined and bounded on $\mathbb{R}_-$, is a translate of $\exp(\lambda^*t)$. In the nonlinear case we now have a similar existence and uniqueness result, but the uniqueness is, untill now, limited to positive $BC^\eta(\mathbb{R}_-;\mathbb{R})$-solutions. However, it so happens that any positive solution "on" the unstable manifold is in this class. In [12] the necessary a priori exponential estimate was given, but as was pointed out to one of us by H. Thieme, the proof contains an error. In [15] Tauberian methods are used to prove the estimate for integral equations which are not necessarily of Volterra type. Here we shall give another proof, based on ideas of H. Thieme, but completely rewritten in the language we have developed in the preceding sections. We refer to a forthcoming paper of H. Thieme for similar results in the much more general case of Banach-space valued functions [49].

We start with some auxiliary results. We shall write $f\geqslant 0$ for some $f\in X$ if $f(t)\geqslant 0$ for all $t\in[0,b]$. Throughout we use that $B\geqslant 0$.

LEMMA 7.2. *The resolvent R is nonpositive and there exist $\xi\geqslant 0$ and $\delta>0$ such that*

$$-R(\tau)\geqslant\delta\qquad \textit{for}\quad \tau\in[\xi,\xi+2b].$$

"*Proof*": See the proof of Lemma 2.1 in [15]. The basic idea is that $-R=\sum_{k=0}^{\infty}B^{k*}$, where B^{k*} denotes the k-times iterated convolution of B with itself. For the present inequalities the sense of convergence of the infinite sum is irrelevant.

LEMMA 7.3. *If $f_1\geqslant f_2$, then $T(s)f_1\geqslant T(s)f_2$.*

Proof. This follows directly from the explicit representation (3.4), the nonnegativity of B and the nonpositivity of R. ∎

LEMMA 7.4. *If $f \geqslant 0$, then, with ξ and δ as in Lemma 7.2,*

$$T(s+\xi+2b)f \geqslant \delta \int_0^b f(\tau)\,d\tau\, e^{\lambda^* s}\phi.$$

Proof. By the representation (3.4) and Lemma 7.2 we have

$$\begin{aligned}(T(\xi+2b)f)(t) &\geqslant -B_t * R * f(\xi+2b)\\ &= \int_0^b \int_0^{\xi+2b-\tau} B(t+\xi+2b-\tau-\sigma)(-R(\sigma))\,d\sigma\, f(\tau)\,d\tau\\ &\geqslant \delta \int_0^b f(\tau)\,d\tau \int_t^b B(\tau)\,d\tau \geqslant \delta \int_0^b f(\tau)\,d\tau\,\phi(t).\end{aligned}$$

The result then follows from the semigroup property, Lemma 7.3 and the fact that $T(s)\,\phi = \exp(\lambda^* s)\,\phi$. ∎

THEOREM 7.5. *Assume B is bounded and let x be any positive solution of* (7.3) *defined on $\mathbb{R}_-$ and such that* $\lim_{t\to-\infty} x(t)=0$. *Then $x \in BC^\eta(\mathbb{R}_-;\mathbb{R})$.*

Proof. Choose $\varepsilon > 0$ such that the positive real root $\lambda^*(\varepsilon)$ of

$$(1-\varepsilon)\int_0^b B(\tau)\,e^{-\lambda\tau}\,d\tau = 1$$

exceeds η. Choose $\bar{y}$ such that $g(y) \geqslant (1-\varepsilon)\,y$ for $0 \leqslant y \leqslant \bar{y}$. We may assume that $0 \leqslant x(t) \leqslant \bar{y}$ for $-\infty < t \leqslant b$ (if not we first translate x). Define for $s \leqslant 0$

$$F(s)(t) = \int_t^b B(\tau)\, g(x(s+t-\tau))\,d\tau.$$

Let $T_\varepsilon(s)$ denote the semigroup associated with the kernel $(1-\varepsilon)\,B$ and define similarly ϕ_ε as in (7.1) and $\delta(\varepsilon)$, $\xi(\varepsilon)$ as in Lemma 7.2. From the variation-of-constants formula (4.7) induced by $g(x) = (1-\varepsilon)\,x + g(x) - (1-\varepsilon)\,x$ we conclude that

$$F(s) \geqslant T_\varepsilon(s-\sigma)\,F(\sigma)$$

and by Lemma 7.4 we subsequently obtain

$$F(s) \geqslant \delta(\varepsilon) \int_0^b F(\sigma)(\tau)\,d\tau\, e^{\lambda^*(\varepsilon)(s-\sigma-\xi(\varepsilon)-2b)}\phi_\varepsilon.$$

Application of α then yields (put $s=0$)

$$\int_0^b F(\sigma)(\tau)\,d\tau \leqslant \text{const} \cdot e^{\lambda^*(\varepsilon)\sigma}.$$

It remains to convert this estimate into an estimate for x. Let L be a Lipschitz constant for g on $(0, \bar{y}]$, and let $\tilde{R}$ denote the resolvent associated with the kernel LB. Then $x_s \leqslant LB * x_s + F(s)$ and therefore $x_s \leqslant F(s) - \tilde{R} * F(s)$. Taking $t = b$ we obtain

$$\begin{aligned} x(s+b) &\leqslant \int_0^b \tilde{R}(b-\tau)\, F(s)(\tau)\, d\tau \\ &\leqslant \text{const} \int_0^b F(s)(\tau)\, d\tau \\ &\leqslant \text{const} \cdot e^{\lambda^*(\varepsilon)s} \end{aligned}$$

(here we use that, since B is bounded, $\tilde{R}$ is bounded on $[0, b]$). ∎

One can sharpen Theorem 7.5 by stating conditions which guarantee that $\lim_{t \to -\infty} x(t) = 0$. Let $\tilde{y}$ be defined by

$$\tilde{y} = \inf \left\{ y > 0 \;\middle|\; g(y) < \left(\int_0^b B(\tau)\, d\tau \right)^{-1} y \right\}.$$

Then any solution of (7.3) which also satisfies $0 < x(t) < \tilde{y}$ for $-\infty < t \leqslant 0$ necessarily has $\lim_{t \to -\infty} x(t) = 0$ (see the proof of Lemma 4.2 of [15]).

In the population models the only relevant solutions are the non-negative ones. The results of this section show that there is, modulo time-translation, one and only one biologically meaningful way to leave a neighbourhood of an unstable trivial equilibrium (here "trivial" means that there is no population at all). In [12] it was shown that in a certain well-defined sense (involving translation) solutions of $x = B * g(x) + \varepsilon f$ with $f \geqslant 0$ converge to the positive solution of (7.3) as $\varepsilon \downarrow 0$. This approximation can be used to derive that the positive solution is monotone increasing, at least on an interval of the form $(-\infty, t_0]$. (See [12].) Whether or not it remains increasing (and positive) as $t \to +\infty$ depends on global properties of g.

PART III: A CENTER MANIFOLD AND THE HOPF BIFURCATION THEOREM

If A has some eigenvalues on the imaginary axis, the stability character of the zero solution is sensitive to perturbations in both the linear and the nonlinear part of the equation. We shall introduce a parameter μ into the equation in order to incorporate perturbations of the linear part.

We shall construct a nonlinear analogue of X_0, called a center manifold. The construction requires a modification of the nonlinearity outside some neighbourhood of zero and, as a consequence, some matters like uniqueness, differentiability and relationship with the original equation are much more complicated than in the case of the stable or unstable manifold (see [44] for a detailed analysis of these questions).

Nevertheless a center manifold and the o.d.e. associated with it are, as is well known, convenient tools in the analysis of the Hopf bifurcation problem (cf. [4; 5; 27; 35; 39, pp. 195–206; 40]).

Although most of our results are well known in the context of other dynamical systems, they seem to be new for Volterra integral equations.

8. Definition and Analysis of a Pseudo-inverse

Let Ω be a compact subset of $\mathbb{R}^m$ with zero in its interior. For each $\mu \in \Omega$, let $B(\mu, \cdot)$ be an $n \times n$-matrix valued L_2-function with support contained in $[0, b]$. Let $T(s)$ and A denote the linear semigroup and generator associated with $B(0, \cdot)$. Assume that some of the eigenvalues of A lie on the imaginary axis. Define X, P and γ with lower indices $+$, $-$ or 0 as in Theorem 3.3. Choose some number η in the interval $(0, \min\{\gamma_+, -\gamma_-\})$.

Motivated by formula (4.14) we introduce the inhomogeneous linear equation

$$F(s) = T(s-\sigma)\,F(\sigma) + \int_\sigma^s T(s-\tau)\,B(0,\cdot)\,h_1(\tau)\,d\tau + \int_\sigma^s T(s-\tau)(B(\mu,\cdot) - B(0,\cdot))\,h_2(\tau)\,d\tau, \qquad s \geqslant \sigma, \tag{8.1}$$

and a mapping

$$\begin{aligned}\mathscr{K}(\mu, h_1, h_2)(s) = &\int_{-\infty}^s T(s-\tau)\,P_-B(0,\cdot)\,h_1(\tau)\,d\tau\\ &+ \int_0^s T(s-\tau)\,P_0B(0,\cdot)\,h_1(\tau)\,d\tau\\ &+ \int_\infty^s T(s-\tau)\,P_+B(0,\cdot)\,h_1(\tau)\,d\tau\\ &+ \int_{-\infty}^s T(s-\tau)\,P_-(B(\mu,\cdot) - B(0,\cdot))\,h_2(\tau)\,d\tau\\ &+ \int_0^s T(s-\tau)\,P_0(B(\mu,\cdot) - B(0,\cdot))\,h_2(\tau)\,d\tau\\ &+ \int_\infty^s T(s-\tau)\,P_+(B(\mu,\cdot) - B(0,\cdot))\,h_2(\tau)\,d\tau. \end{aligned} \tag{8.2}$$

THEOREM 8.1. *For fixed* μ, $\mathscr{K}(\mu, \cdot, \cdot)$ *is a continuous linear mapping of* $BC^\eta(\mathbb{R}; \mathbb{R}^n) \times BC^\eta(\mathbb{R}; \mathbb{R}^n)$ *into* $BC^\eta(\mathbb{R}; X)$. *Moreover,* $F = \mathscr{K}(\mu, h_1, h_2)$ *satisfies* (8.1) *and any* $F \in BC^\eta(\mathbb{R}; X)$ *which satisfies* (8.1) *is of the form* $F = T(\cdot)\,\phi + \mathscr{K}(\mu, h_1, h_2)$ *for some* $\phi \in X_0$.

The proof is based on Theorem 3.3 and Lemma 3.5.

In subsequent applications we shall need continuity and even differentiability with respect to μ. Of course one can make the dependence on μ as smooth as one wants by making suitable assumptions concerning the dependence of B on μ. However, we want to include in our analysis the case where some parameter is a "delay" (for example, take $n = m = 1$ and $B(\mu, \cdot)$ is the characteristic function of $[0, \mu]$; also see Hale [22] and Hale and Oliveira [23]). For that reason we want to have weak assumptions concerning the μ-dependence of the kernel.

For each $\mu \in \Omega$, $B(\mu, \cdot)$ defines a mapping of $C([0, b]; \mathbb{R}^n)$ into $\mathbb{R}^n$ as follows:

$$\psi \longmapsto \int_0^b B(\mu, \tau)\, \psi(\tau)\, d\tau.$$

We assume that $\mu \longmapsto B(\mu, \cdot)$ is differentiable in this sense or, more precisely, we make the

Assumption.

$$\mu \longmapsto \int_0^\sigma B(\mu, \tau)\, d\tau, \qquad \sigma \in [0, b],$$

is a C^k-smooth ($k \geqslant 1$) mapping of Ω into $NBV([0, b]; \mathbb{R}^{n \times n})$.

The projections P_+ and P_0 are, with respect to some basis for X_+ and X_0, given by integral operators with a continuous (even analytical) kernel (see Section 10 for an example and [13] for the general case). Hence our assumption implies that $\mu \longmapsto P_+ B(\mu, \cdot)$ and $\mu \longmapsto P_0 B(\mu, \cdot)$ are C^k-smooth. We conclude that all difficulties are concentrated in X_--components.

Our assumption is not strong enough to achieve that $\mathscr{K}$ is differentiable with respect to μ. However, it so happens that the mapping

$$\alpha\mathscr{K}\colon \Omega \times BC^\eta(\mathbb{R}; \mathbb{R}^n) \times BC^\eta(\mathbb{R}; \mathbb{R}^n) \to BC^\eta(\mathbb{R}; \mathbb{R}^n)$$

is at the same time more important than $\mathscr{K}$ and more smooth than $\mathscr{K}$. We can prove

THEOREM 8.2. $\alpha\mathscr{K}$ *is* C^k.

Proof. We know already that $\alpha\mathscr{K}$ is linear and continuous in h_1 and h_2 and that $P_+ B(\mu, \cdot)$ and $P_0 B(\mu, \cdot)$ are differentiable. It remains to prove that the mapping of $\Omega \times BC^\eta(\mathbb{R}; \mathbb{R}^n)$ into $BC^\eta(\mathbb{R}; \mathbb{R}^n)$ defined by

$$(\mu, h) \mapsto \int_0^\infty \alpha(T(\tau) P_- B(\mu, \cdot)) h(s-\tau) d\tau$$

is differentiable.

Because of Lemma 3.5(ii) (take $t=0$) the integral consists of two terms. A proof of the differentiability of the first term $\int_0^\infty \alpha(P_- U(\tau) B(\mu, \cdot)) \cdot h(s-\tau) d\tau$ can be patterned after the corresponding proof for the second term

$$\int_0^\infty \int_0^\tau R^-(\tau-\sigma) B(\mu, \sigma) h(s-\tau) d\sigma d\tau$$

$$= \int_0^b \int_0^\infty R^-(\tau) B(\mu, \sigma) h(s-\tau-\sigma) d\tau d\sigma.$$

So we only give the latter. Moreover, we confine ourselves to the case $k=1$.

Let $\zeta(\mu, \cdot) \in NBV([0, b]; \mathbb{R}^{n \times n})$ be the derivative of $\mu \mapsto \int_0^\sigma B(\mu, \tau) d\tau$. Then for each $\varepsilon > 0$ we can find $\bar{\delta} = \bar{\delta}(\varepsilon)$ such that

$$e^{-\eta|s|} \left| \int_0^b \int_0^\infty d\tau R^-(\tau) d_\sigma \left[\frac{1}{\delta} \int_0^\sigma (B(\mu+\delta, \rho) - B(\mu, \rho)) d\rho \right.\right.$$

$$\left.\left. - \zeta(\mu, \sigma) \right] h(s-\tau-\sigma) \right|$$

$$\leqslant \operatorname*{Var}_{[0,b]} \left(\frac{1}{\delta} \int_0^\sigma (B(\mu+\delta, \rho) - B(\mu, \rho)) d\rho - \zeta(\mu, \sigma) \right)$$

$$\times \sup_{0 \leqslant \sigma \leqslant b} \int_0^\infty |R^-(\tau)| |h(s-\tau-\sigma| d\tau \cdot e^{-\eta|s|}$$

$$\leqslant \varepsilon \int_0^\infty |R^-(\tau)| e^{\eta\tau} d\tau \cdot e^{\eta b} \cdot |||h|||^\eta$$

provided $\delta \leqslant \bar{\delta}$. ∎

9. Construction of a Center Manifold

Let $g(\cdot, \cdot)$ be a C^k-mapping ($k \geqslant 1$) of $\Omega \times \mathbb{R}^n$ into $\mathbb{R}^n$ such that $g(\mu, x) = x + r(\mu, x)$ with $r(\mu, 0) = 0$ and $\partial r(\mu, 0)/\partial x = 0$. Recalling formula (4.14) we are tempted to introduce, for instance,

$$(\mu, F) \to \mathscr{K}(\mu, r(\mu, \alpha(F)), g(\mu, \alpha(F))).$$

However, in general growth restrictions on F are *not* reproduced by $r(\mu, \alpha(F))$ and $g(\mu, \alpha(F))$ and consequently this prescription does not define a mapping between the right spaces.

In order to remedy this shortcoming we modify $r(\mu, \cdot)$ and $g(\mu, \cdot)$ outside some neighbourhood of zero in $\mathbb{R}^n$. Of course we then need to interpret the final results with the requisite care.

Let $\xi: \mathbb{R}_+ \to \mathbb{R}$ be a C^∞-function such that

(i) $\xi(y) = 1$ for $0 \leqslant y \leqslant 1$,

(ii) $0 \leqslant \xi(y) \leqslant 1$ for $1 \leqslant y \leqslant 2$,

(iii) $\xi(y) = 0$ for $y \geqslant 2$.

Define, for some positive parameter δ which will be chosen to satisfy suitable bounds later,

$$\begin{aligned} \tilde{r}(\mu, x) &= r(\mu, x)\, \xi\left(\frac{|x|}{\delta}\right), \\ \tilde{g}(\mu, x) &= g(\mu, x)\, \xi\left(\frac{|x|}{\delta}\right). \end{aligned} \tag{9.1}$$

We are interested in "small" solutions of

$$x(t) = \int_0^b B(\mu, \tau)\, g(\mu, x(t-\tau))\, d\tau, \qquad -\infty < t < +\infty, \tag{9.2}$$

and therefore it is reasonable to first study solutions of the modified equation

$$\begin{aligned} x(t) = &\int_0^b (B(0, \tau)\, x(t-\tau))\, d\tau \\ &+ \int_0^b (B(\mu, \tau) - B(0, \tau))\, \tilde{g}(\mu, x(t-\tau))\, d\tau \\ &+ \int_0^b B(0, \tau)\, \tilde{r}(\mu, x(t-\tau))\, d\tau. \end{aligned} \tag{9.3}$$

According to Theorem 4.4 we can equally well study

$$\begin{aligned} F(s) = T(s-\sigma)\, F(\sigma) + \int_\sigma^s T(s-\tau)\{&B(0, \cdot)\, \tilde{r}(\mu, \alpha(F(\tau))) \\ &+ (B(\mu, \cdot) - B(0, \cdot))\, \tilde{g}(\mu, \alpha(F(\tau)))\}\, d\tau, \qquad \sigma \leqslant s. \end{aligned} \tag{9.4}$$

Subsequently Theorem 8.1 shows that this problem is equivalent to

$$F = T(\cdot)\, \phi + \mathscr{K}(\mu, \tilde{r}(\mu, \alpha(F)), \tilde{g}(\mu, \alpha(F))). \tag{9.5}$$

However, as stated before, in this form we are not assured of differentiable dependence on μ.

Suppose F satisfies (9.5), then $x = \alpha(F)$ satisfies

$$x = \alpha(T(\cdot)\,\phi) + \alpha \mathscr{K}(\mu, \tilde{r}(\mu, x), \tilde{g}(\mu, x)). \tag{9.6}$$

Conversely, suppose x satisfies (9.6); then F defined by $F = T(\cdot)\,\phi + \mathscr{K}(\mu, \tilde{r}(\mu, x), \tilde{g}(\mu, x))$ satisfies (9.5). Therefore we concentrate on (9.6) which may be viewed as a rewriting of (9.3) with emphasis on $BC^{\eta}(\mathbb{R}; \mathbb{R}^n)$-solutions.

Define

$$\mathscr{G}: BC^{\eta}(\mathbb{R}; \mathbb{R}^n) \times \Omega \times X_0 \to BC^{\eta}(\mathbb{R}; \mathbb{R}^n)$$

by

$$\mathscr{G}(x, \mu, \phi) = x - \alpha(T(\cdot)\,\phi) - \alpha \mathscr{K}(\mu, \tilde{r}(\mu, x), \tilde{g}(\mu, x)).$$

Since the substitution operators are not C^k-smooth in BC^{η} spaces, it is impossible to apply the implicit function theorem. However, for small μ and δ and for all $\phi \in X_0$, $x - \mathscr{G}(x, \mu, \phi)$ defines a contraction and consequently there exists a unique fixed point $x^* = x^*(\mu, \phi)$, smooth as a function of μ. The differentiability with respect to ϕ, which we need, can be demonstrated as follows. For $0 < \rho < \eta$ and μ, δ small enough, x^* is an element of BC^{η_1}, $\eta_1 = k^{-1}(\eta - \rho)$, as well. The substitution operators $\tilde{r}$ and $\tilde{g}$ are C^k smooth from BC^{η_1} into BC^{η} (Lemma 4.1 in [45]). Using this fact one can conclude that the fixed point of the linear variational equation is in fact the derivative of x^* with respect to ϕ viewed as a mapping of X_0 into BC^{η}. By induction (see Theorem 2.1 in [45] for the details) we arrive at the following:

THEOREM 9.1. *There exists a unique C^k-function $x^* = x^*(\mu, \phi)$ defined on a neighbourhood of* $(0, 0)$ *in* $\Omega \times X_0$ *such that* $\mathscr{G}(x^*(\mu, \phi), \mu, \phi) = 0$.

Remark 1. We emphasize that x^* gives a parametrization by μ and ϕ of "small" $BC^{\eta}(\mathbb{R}; \mathbb{R}^n)$ solutions of (9.3). These are also solutions of (9.2) if and only if in addition $|x^*(\mu, \phi)(s)| \leqslant \delta$, $-\infty < s < \infty$. Note that $x^*(\mu, 0) = 0$. We define

$$F^*(\mu, \phi) = T(\cdot)\,\phi + \mathscr{K}(\mu, \tilde{r}(\mu, x^*(\mu, \phi)), \tilde{g}(\mu, x^*(\mu, \phi))) \tag{9.7}$$

and, finally, the center manifold as the image of

$$\mathscr{C}: \Omega \times X_0 \to X,$$

where

$$\mathscr{C}(\mu, \phi) = F^*(\mu, \phi)(0). \tag{9.8}$$

Note that F^* and $\mathscr{C}$ are C^k with respect to ϕ but not necessarily with respect to μ. On the other hand $\alpha\mathscr{C}(\mu, \phi) = \alpha(F^*(\mu, \phi)(0)) = x^*(\mu, \phi)(0)$ is C^k with respect to both μ and ϕ.

The tangency relation $\partial\mathscr{C}(0, 0)\, \psi/\partial\phi = \psi$ and the invariance relation

$$F^*(\mu, \phi)(s) = \mathscr{C}(\mu, P_0 F^*(\mu, \phi)(s)), \tag{9.9}$$

are easy consequences of the definition. Note that now

$$F^*(\mu, \phi)(s) = \tilde{\Pi}(\mu, F^*(\mu, \phi)(\sigma), s - \sigma), \qquad s \geqslant \sigma,$$

where $\tilde{\Pi}$ is the semiflow associated with the modified equation (9.3). So "invariant" refers to $\tilde{\Pi}$.

In order to arrive at an ordinary differential equation we put

$$y(s) = P_0 F^*(\mu, \phi)(s). \tag{9.10}$$

Then

$$\begin{aligned} y(s) = T(s)\, y(0) + \int_0^s T(s-\tau)\, P_0\{&B(0, \cdot)\, \tilde{r}(\mu, \alpha\mathscr{C}(\mu, y(\tau))) \\ &+ (B(\mu, \cdot) - B(0, \cdot))\, \tilde{g}(\mu, \alpha\mathscr{C}(\mu, y(\tau)))\}\, d\tau \end{aligned}$$

and consequently

$$\begin{aligned} y'(s) = Ay(s) &+ P_0 B(0, \cdot)\, \tilde{r}(\mu, \alpha\mathscr{C}(\mu, y(s))) \\ &+ P_0(B(\mu, \cdot) - B(0, \cdot))\, \tilde{g}(\mu, \alpha\mathscr{C}(\mu, y(s))). \end{aligned} \tag{9.11}$$

In terms of coordinates with respect to a basis for X_0, this is a nonlinear o.d.e. with a right-hand side which is C^k in y (and also in μ). Hence y is a C^{k+1}-function of s and therefore $x^*(\mu, \phi)(s) = \alpha\mathscr{C}(\mu, y(s))$ is at least C^k-smooth as a function of s.

In certain situations the smoothness with respect to s adds to the smoothness with respect to μ. By this we mean the following. If $k = 1$ in the foregoing, then $\alpha\mathscr{C} \in C^1$ and

$$\begin{aligned} \frac{\partial \alpha\mathscr{C}}{\partial\mu}(\mu, \phi) &= \frac{\partial x^*}{\partial\mu}(\mu, \phi)(0) \\ &= -\left(\left(\frac{\partial\mathscr{G}}{\partial x}(x^*(\mu, \phi), \mu, \phi) \right)^{-1} \frac{\partial\mathscr{G}}{\partial\mu}(x^*(\mu, \phi), \mu, \phi) \right)(0). \end{aligned}$$

It is possible that the right-hand side is differentiable with respect to μ although $\mathscr{G}$ itself is *not* C^2 with respect to μ. For instance, this happens if $g(\cdot, \cdot)$ is C^2 and if the mapping

$$\mu \mapsto \int_0^b B(\mu, \tau)\, \psi(\tau)\, d\tau$$

is two times differentiable uniformly for ψ in any bounded subset of $C^1([0, b]; \mathbb{R}^n)$. Here "uniformly" refers to the convergence in the definition of differentiability and, in addition, it intends to express that the derivative depends continuously on the C^1-norm of ψ. (Again we have in mind examples like $n = m = 1$ and $B(\mu, \cdot)$ is the characteristic function of $[0, \mu]$.) Since $g(\cdot, \cdot) \in C^2$ also implies that $\alpha\mathscr{C}$ is C^2 with respect to ϕ we conclude, using (9.11) and the formula for $\partial\alpha\mathscr{C}/\partial\mu$ above, that under these assumptions $\alpha\mathscr{C}(\cdot, \cdot)$ is C^2. These arguments are inspired by similar ones in Hale [22].

In conclusion of this section we shall be concerned with the attractivity of the center manifold in the case that $X_+ = \{0\}$. This property is very helpful if one wants to show that for fixed $\mu = 0$ the stability of the zero solution is completely determined by the stability of the origin with respect to the o.d.e. (9.11) (cf. Hale [25], Hausrath [28] and the references given there) and, likewise, if one wants to show that the stability of a bifurcating periodic solution is determined by the stability of the corresponding solution of (9.11) (see Section 11 and [39, pp. 195–206; 40]).

THEOREM 9.2. *Let B be bounded and suppose that $X_+ = \{0\}$. For each $\varepsilon > 0$ there exists $\bar{\mu} = \bar{\mu}(\varepsilon)$ and $\bar{\delta} = \bar{\delta}(\varepsilon)$ such that, provided $|\mu| \leqslant \bar{\mu}$ and $\delta \leqslant \bar{\delta}$, any two solutions F_1 and F_2 of (9.4) on $[0, s]$ which satisfy $P_0 F_1(s) = P_0 F_2(s)$ necessarily also satisfy*

$$\| P_- F_1(s) - P_- F_2(s) \| \leqslant K e^{(\gamma_- + \varepsilon)s} \| P_- F_1(0) - P_- F_2(0) \|$$

for some constant $K = K(\varepsilon)$ independent of s.

The extra assumption that B is bounded is not absolutely necessary. However, it makes it possible to estimate $T(s)(B(\mu, \cdot) - B(0, \cdot))$, etc., in the supremum-norm and, as a consequence, the proof of Lemma 2.3 in [1] applies without modification.

LEMMA 9.3. *Let B be bounded. Then there exists $\bar{\mu} > 0$ such that $\|(\mathscr{C} - P_0\mathscr{C})(\mu, \phi)\|$ is bounded uniformly in ϕ and $|\mu| \leqslant \bar{\mu}$.*

Proof.

$$\begin{aligned}(\mathscr{C} - P_0\mathscr{C})(\mu, \phi) = \int_0^\infty & T(\tau)\{P_- B(0, \cdot)\, \tilde{r}(\mu, \alpha(F^*(\mu, \phi)))(-\tau) \\ & + P_-(B(\mu, \cdot) - B(0, \cdot))\, \tilde{g}(\mu, \alpha(F^*(\mu, \phi)))(-\tau)\}\, d\tau \\ & + \int_{-\infty}^0 T(\tau)\{P_+ B(0, \cdot)\, \tilde{r}(\mu, \alpha(F^*(\mu, \phi)))(-\tau) \\ & + P_+(B(\mu, \cdot) - B(0, \cdot))\, \tilde{g}(\mu, \alpha(F^*(\mu, \phi)))(-\tau)\}\, d\tau.\end{aligned}$$

We know that $\tilde{g}(\mu, \cdot)$ and $\tilde{r}(\mu, \cdot)$ are bounded by constants which depend continuously on δ (the modification parameter) and μ. Because B is bounded we may estimate $T(s) P_{\pm}(B(\mu, \cdot) - B(0, \cdot))$ in the supremum norm. ∎

COROLLARY 9.4 (attractivity of the center manifold). *Let B be bounded and suppose $X_+ = \{0\}$. There exist positive constants κ and $\bar{\mu}$ such that $\|\mathscr{C}(P_0 F(s)) - F(s)\| \leqslant K e^{-\kappa s}$, $s \in \mathbb{R}_+$ and $|\mu| \leqslant \bar{\mu}$, where F is a solution of* (9.4) *and K is a constant which depends only on $P_- F(0)$.*

Proof. First we observe that by the global Lipschitz character solutions of (9.4) can always be continued to $+\infty$. F and $F^*(\mu, P_0 F(s))(\cdot - s)$ are solutions of (9.4) and the conditions of Theorem 9.2 are satisfied. Hence

$$\begin{aligned}\|\mathscr{C}(\mu, P_0 F(s)) - F(s)\| &= \|F^*(\mu, P_0 F(s))(0) - F(s)\| \\ &= \|P_-(F^*(\mu, P_0 F(s))(0) - F(s))\| \\ &\leqslant K e^{(\gamma_- + \varepsilon)s} \|P_- F^*(\mu, P_0 F(s))(-s) - P_- F(0)\|,\end{aligned}$$

and for ε sufficiently small $\gamma_- + \varepsilon < 0$. Finally we use the preceeding lemma. ∎

Remark. If X_+ has positive dimension, one still has the same results for solutions which remain for all positive time in a certain neighbourhood of the origin. This is sometimes called "local" or "conditional" attractivity.

COROLLARY 9.5. *Let B be bounded and suppose $X_+ = \{0\}$. For μ and δ sufficiently small any solution of* (9.4) *defined on $\mathbb{R}$ lies on the center manifold, or, in other words, satisfies*

$$F(s) = \mathscr{C}(\mu, P_0 F(s)).$$

10. SIMPLE EIGENVALUES ON THE IMAGINARY AXIS

In this section we make some preparations for the study of the Hopf bifurcation problem. We omit the (elementary) proofs. In [13] some of the underlying ideas are presented systematically. First of all we derive a criterion for simple eigenvalues in terms of properties of the characteristic function

$$\Delta(\mu, \lambda) = I - \int_0^b B(\mu, \tau)\, e^{-\lambda\tau}\, d\tau. \qquad (10.1)$$

LEMMA 10.1. *The following two assertions are equivalent*:

(i) *$i\omega$ is a simple eigenvalue of A*,

(ii) *there exist a column n-vector* $p(0) \neq 0$ *and a row n-vector* $q(0) \neq 0$ *such that*

(α) $\Delta(0, i\omega)\, v = 0$ *implies* $v = cp(0)$ *for some* $c \in \mathbb{C}$,

(β) $w\Delta(0, i\omega) = 0$ *implies* $w = cq(0)$ *for some* $c \in \mathbb{C}$,

(γ) $q(0)\, \partial\Delta(0, i\omega)\, p(0)/\partial\lambda = 1$.

Throughout the rest of this section we assume that $i\omega$ is a simple eigenvalue and that $p(0)$ and $q(0)$ are as in the Lemma 10.1.

LEMMA 10.2

$$X = \mathcal{N}(A - i\omega I) \oplus \mathcal{R}(A - i\omega I)$$

and the associated projection onto $\mathcal{N}(A - i\omega I)$ *is given by*

$$Pf = p\langle q, f\rangle, \tag{10.2}$$

where

$$q(t) = e^{i\omega t} q(0), \tag{10.3}$$

$$p(t) = e^{i\omega t} \left(I - \int_0^t e^{-i\omega\tau} B(0, \tau)\, d\tau \right) p(0), \tag{10.4}$$

$$\langle q, f\rangle := \int_0^b q(-\tau) f(\tau)\, d\tau. \tag{10.5}$$

If $i\omega$ is a simple eigenvalue, so is $-i\omega$ (recall that $B(\cdot, \cdot)$ is real-valued) and the corresponding projection is given by $\bar{p}\langle \bar{q}, f\rangle$. The definitions

$$\Phi = (p, \bar{p}), \qquad \Psi = \begin{pmatrix} q \\ \bar{q} \end{pmatrix} \tag{10.6}$$

allow us to represent the canonical projection onto $\mathcal{N}(A - i\omega I) \oplus \mathcal{N}(A + i\omega I)$ by

$$Pf = \Phi\langle \Psi, f\rangle \tag{10.7}$$

We observe that

$$PB = \Phi\Psi(0). \tag{10.8}$$

Furthermore we note the relation

$$T(s)\, \Phi = \Phi \begin{pmatrix} e^{i\omega s} & 0 \\ 0 & e^{-i\omega s} \end{pmatrix}. \tag{10.9}$$

We end this section with a formula for the first variation of the simple eigenvalue with respect to μ.

LEMMA 10.3. *The equation*

$$\Delta(\mu, \lambda)(p(0) + \zeta) = 0, \qquad \lambda \in \mathbb{C}, \zeta \in \mathscr{R}(\Delta(0, i\omega)),$$

has a unique solution $\lambda = \lambda(\mu)$, $\zeta = \zeta(\mu)$ such that $\lambda(0) = i\omega$ and $\zeta(0) = 0$. These are C^k-functions and

$$\lambda'(0) = -q(0) \frac{\partial \Delta}{\partial \mu}(0, i\omega)\, p(0).$$

11. HOPF BIFURCATION

The most important conclusion of the analysis in Section 9 is that all small solutions of (9.2) are, for small μ, described by the o.d.e. (9.11). Consequently results from the theory of o.d.e.'s yield results for the Volterra integral equation (9.2). In particular the vast collection of results about Hopf bifurcation (see [4–7, 31, 35, 42, 43], several contributions in [39] and the references given there) carries over.

In Sections 8 and 9 we have formulated sufficient conditions on B and g such that the right-hand side of the o.d.e. (9.11) is a C^k-function of μ and ϕ (in particular we recall the remarks some lines below formula (9.11)). In this section we assume throughout that $k \geqslant 2$ and sometimes that $k \geqslant 3$. Moreover, we limit ourselves to the case $m = 1$, that is, μ is a scalar parameter. The only work that remains to be done is to relate other assumptions and quantities connected with the o.d.e. to assumptions and quantities expressed in terms of B and g. This is a matter of calculus and in this section we perform the calculations for the simplest case. We shall use the notation and the definitions of Section 10.

THEOREM 11.1. *Assume that*

(i) $k \geqslant 2$,

(ii) $\pm i\omega$ *are simple eigenvalues of A and $\pm i\omega l \notin \sigma(A)$ for $l = 0$, and $l = 2, 3, 4, \ldots$,*

(iii) $\operatorname{Re}(q(0)\, \partial\Delta(0, i\omega)\, p(0))/\partial\mu \neq 0$.

Then there exist C^1-functions $\mu^*(\varepsilon)$, $\phi^*(\varepsilon)$ and $\rho^*(\varepsilon)$ (with values in, respectively, $\mathbb{R}$, X_0 and $\mathbb{R}$ and defined for ε sufficiently small) such that $\mu^*(0) = 0$

and $p^*(0) = 2\pi\omega^{-1}$ *and such that* $x^*(\mu^*(\varepsilon), \phi^*(\varepsilon)) = \alpha\mathscr{C}(\mu^*(\varepsilon), P_0\Pi(\mu^*(\varepsilon), \phi^*(\varepsilon), \cdot))$ *is a* $p^*(\varepsilon)$*-periodic solution of the equation*

$$x(t) = \int_0^b B(\mu, \tau)\, g(\mu, x(t-\tau))\, d\tau$$

with $\mu = \mu^*(\varepsilon)$. *Moreover, if* x *is any small periodic solution of this equation with* μ *close to* 0 *and period close to* $2\pi\omega^{-1}$, *then necessarily* $\mu = \mu^*(\varepsilon)$, *the period is* $p^*(\varepsilon)$ *and modulo translation* $x = x^*(\mu^*(\varepsilon), \phi^*(\varepsilon))$.

Proof. According to the well-known Hopf bifurcation theorem for o.d.e.'s (see, for instance, Crandall and Rabinowitz [7]) we only have to check that (iii) implies that the relevant eigenvalue of the matrix $M(\mu)$, describing the linear part of the o.d.e. (9.11), crosses the imaginary axis with positive speed. With respect to a suitable basis for X_0 we find, using $\partial(\alpha\mathscr{C})(0, 0)\, \psi/\partial\phi = \alpha(\psi)$,

$$\begin{aligned} M_{11}(\mu) &= i\omega + \langle q, B(\mu, \cdot) - B(0, \cdot)\rangle\, p(0) + o(\mu) \\ &= i\omega + q(0)(\Delta(0, i\omega) - \Delta(\mu, i\omega))\, p(0) + o(\mu). \end{aligned}$$

As in Lemma 10.3 it follows that $M(\mu)$ has an eigenvalue $\lambda(\mu)$ such that $\lambda(0) = i\omega$ and

$$\lambda'(0) = -q(0)\frac{\partial\Delta}{\partial\mu}(0, i\omega)\, p(0).$$

(Of course this eigenvalue of $M(\mu)$ is identical to $\lambda(\mu)$ found in Lemma 10.3, but a proof of this fact seems to require many more arguments.) So indeed (iii) is the appropriate transversality condition. ∎

As in the case of o.d.e.'s we have the symmetry relations

$$\mu^*(-\varepsilon) = \mu^*(\varepsilon), \qquad p^*(-\varepsilon) = p^*(\varepsilon), \qquad \phi^*(-\varepsilon) = P_0\Pi(\mu^*(\varepsilon), \phi^*(\varepsilon), \tfrac{1}{2}p^*(\varepsilon))$$

(see [7]).

Our next goal is a formula for the third term in the Taylor expansion of $\mu^*(\varepsilon)$ (note that the second term necessarily vanishes). It is well known that the sign of this term determines whether or not the critical Floquet multiplier of the bifurcating periodic solution of the o.d.e. exceeds 1. Hence, under some further assumptions, the stability or instability of the bifurcating periodic solution can be concluded from an evaluation of the formula which we shall give. As a by-product we obtain a formula for the third term in the Taylor expansion of $p^*(\varepsilon)$.

It is known that the direction of bifurcation can be deduced from the stability of the origin of the o.d.e. with $\mu = 0$ (see Chafee [4, 5]). The stability analysis goes back to the work of Liapunov [33] (see also [34] and in the case of retarded equations Hale [25] and Hausrath [28] and the references given there).

Let $z(s)$ denote the coefficient of p in $y(s)$. Then from (9.11) with $\mu = 0$ we obtain

$$\frac{dz}{ds} = i\omega z + q(0)\,\tilde{r}(0, \alpha\mathscr{C}(0, pz + \bar{p}\bar{z} + v)). \tag{11.1}$$

Here v denotes the remaining component in X_0, if any. Let g_{ij} denote the coefficient of $z^i \bar{z}^j / i!\, j!$ in the Taylor expansion of the right-hand side at $z = 0$, $v = 0$. Following Hassard and Wan [27] we introduce the quantity

$$c_1 = \frac{i}{2\omega}\left(g_{20}\, g_{11} - 2g_{11}\,\overline{g_{11}} - \frac{1}{3}\, g_{02}\,\overline{g_{02}}\right) + \frac{1}{2}\, g_{21}. \tag{11.2}$$

THEOREM 11.2. *Under the assumptions of Theorem* 11.1, *but now with* $k \geqslant 3$, *we have*

$$\mu^*(\varepsilon) = -\frac{\operatorname{Re} c_1}{\operatorname{Re} \lambda'(0)}\, \varepsilon^2 + o(\varepsilon^2),$$

$$\rho^*(\varepsilon) = \frac{2\pi}{\omega}\left(1 - \left(\operatorname{Im} c_1 - \frac{\operatorname{Re} c_1}{\operatorname{Re} \lambda'(0)} \operatorname{Im} \lambda'(0)\right) \frac{\varepsilon^2}{\omega} + o(\varepsilon^2)\right),$$

$$\beta^*(\varepsilon) = e^{2 \operatorname{Re} c_1 \varepsilon^2} + o(\varepsilon^2),$$

where $\lambda'(0) = -q(0)\, \partial\Delta(0, i\omega)\, p(0)/\partial\mu$ *and where* $\beta^*(\varepsilon)$ *is a characteristic multiplier of the bifurcating periodic solution of* (9.12).

Proof. If $\pm i\omega$ are the only eigenvalues on the imaginary axis, the result follows from Hassard and Wan [27, formulas (9), (10), (13)]. If there are other eigenvalues on the imaginary axis which, however, satisfy the nonresonance condition (ii) of Theorem 11.1, one can follow the approach of Crandall and Rabinowitz [7]. This leads at first to formulas which look quite different. However, some perseverance, some knowledge of the theory of o.d.e.'s and a lot of paper is all that is needed to rewrite these in the required form. This is the way we proceeded. Alternatively, one can also transform the equation into normal form like in Schmidt [42, Sect. 4] and apply the inverse transformation to his formula for the characteristic multipliers. See also Chow and Mallet-Paret [6] and Stech [43]. ∎

COROLLARY 11.3. *Let the assumptions of Theorem* 11.2 *be satisfied and assume in addition that*

(i) $\pm i\omega$ *are the only eigenvalues on the imaginary axis,*

(ii) $\operatorname{Re} \lambda'(0) > 0$ *and* $\operatorname{Re} c_1 \neq 0$.

Then the bifurcating periodic solution is asymptotically orbitally stable with asymptotic phase as a solution of the o.d.e. (9.12) *if and only if* $\mu^*(\varepsilon) > 0$ *for small* $\varepsilon \neq 0$.

Proof. The o.d.e. is two-dimensional and β and 1 are the only characteristic multipliers corresponding to the linear variational equation associated with the periodic solution. The result follows for instance from Hale [20, Chap. VI]. ∎

By Corollary 9.4 we know that the center manifold is exponentially attractive if there are no eigenvalues of A in the right half plane (and if B is bounded). In this situation the stability of the periodic solution in a full neighbourhood of the origin is in fact determined by the stability within the center manifold. A very elegant and simple proof is given by Hassard, Kazarinoff and Wan [39, pp. 195–206]. Although the assumptions that they make are not satisfied here, the proof of this part of their work can be carried over literally. See also Negrini and Tesei [40].

Corollary 11.3 and the remarks above show that the Principle of Exchange of Stability holds. (See [7, 41].)

Finally, we shall express c_1 in terms of $p(0)$, $q(0)$, Fourier transforms of $B(0, \cdot)$ and derivatives of $r(0, \cdot)$.

LEMMA 11.4. *Assume* $k \geqslant 3$, *then*

$$g_{20} = q(0) \frac{\partial^2 r}{\partial x^2}(0, 0)\, p(0)^2,$$

$$g_{02} = q(0) \frac{\partial^2 r}{\partial x^2}(0, 0)\, \overline{p(0)}^{\,2},$$

$$g_{11} = q(0) \frac{\partial^2 r}{\partial x^2}(0, 0)(p(0), \overline{p(0)}),$$

$$\begin{aligned} g_{21} = {} & q(0) \frac{\partial^3 r}{\partial x^3}(0, 0)(p(0)^2, \overline{p(0)}) \\ & + 2q(0) \frac{\partial^2 r}{\partial x^2}(0, 0) \Big(p(0), \Delta(0, 0)^{-1}(I - \Delta(0, 0) \\ & - \hat{\Phi}(0)\, \Psi(0)) \frac{\partial^2 r}{\partial x^2}(0, 0)(p(0), \overline{p(0)}) \Big) \\ & + q(0) \frac{\partial^2 r}{\partial x^2}(0, 0) \Big(\overline{p(0)}, \Delta(0, 2i\omega)^{-1}(I - \Delta(0, 2i\omega) \\ & - \hat{\Phi}(2i\omega)\, \Psi(0)) \frac{\partial^2 r}{\partial x^2}(0, 0)\, p(0)^2 \Big). \end{aligned}$$

Proof.

Step 1. Using the definition of g_{ij} in terms of the Taylor expansion of the right-hand side of (11.1), the fact that $\partial r(0,0)/\partial x = 0$ and $\partial \alpha\mathscr{C}(0,0)\,\psi/\partial\phi = \alpha(\psi)$, we find from a straightforward calculation the first three identities as well as

$$\begin{aligned} g_{21} &= q(0)\frac{\partial^3 r}{\partial x^3}(0,0)(p(0)^2, \overline{p(0)}) \\ &\quad + 2q(0)\frac{\partial^2 r}{\partial x^2}(0,0)\left(p(0), \frac{\partial^2 \alpha\mathscr{C}}{\partial\phi^2}(0,0)(p(0), \overline{p(0)})\right) \\ &\quad + q(0)\frac{\partial^2 r}{\partial x^2}\left(\overline{p(0)}, \frac{\partial^2 \alpha\mathscr{C}}{\partial\phi^2}(0,0)\,p(0)^2\right). \end{aligned}$$

Step 2. So we have to determine the second term in the expansion of $\alpha\mathscr{C}(0,\phi)$ with respect to ϕ. From $\alpha\mathscr{C}(0,\phi) = x^*(0,\phi)(0)$ and the equation satisfied by x^* we deduce that

$$\begin{aligned} &\frac{\partial^2 \alpha\mathscr{C}}{\partial\phi^2}(0,0)(\psi_1,\psi_2) \\ &\quad = \alpha\left(\int_0^{\infty} T(\tau)\,P_-B(0,\cdot)\frac{\partial^2 r}{\partial x^2}(0,0)(\alpha(T(-\tau)\,\psi_1), \alpha(T(-\tau)\,\psi_2))\,d\tau \right. \\ &\qquad \left. + \int_0^{-\infty} T(\tau)\,P_+B(0,\cdot)\frac{\partial^2 r}{\partial x^2}(0,0)(\alpha(T(-\tau)\,\psi_1), \alpha(T(-\tau)\,\psi_2))\,d\tau\right). \end{aligned}$$

Step 3. Next we recall some identities from the linear theory. The formula $\int_0^\infty e^{-\lambda\tau}T(\tau)f\,d\tau = (\lambda I - A)^{-1}f$ holds in general for $\operatorname{Re}\lambda$ sufficiently large, but also in the particular case that $f \in X_-$ and $\operatorname{Re}\lambda \geqslant 0$. Similarly we have for $\operatorname{Re}\lambda \leqslant 0$ and $f \in X_+$ the identity $\int_0^{-\infty} e^{-\lambda\tau}T(\tau)f\,d\tau = (\lambda I - A)^{-1}f$ (use a basis for X_+ and evaluate the integral). Furthermore, we mention that $\alpha((\lambda I - A)^{-1}f) = \Delta(0,\lambda)^{-1}\hat{f}(\lambda)$ (see [13, formula (5.5)] or use the identity above, formula (3.5) and the resolvent equation (3.3)).

Step 4. Combining steps 2 and 3 and using $\alpha(T(-\tau)\,p) = e^{-i\omega\tau}p(0)$ and (10.8) we find

$$\begin{aligned} \frac{\partial^2 \alpha\mathscr{C}}{\partial\phi^2}(0,0)\,p^2 &= \alpha\left((2i\omega I - A)^{-1}(P_- + P_+)\,B(0,\cdot)\frac{\partial^2 r}{\partial x^2}(0,0)\,p(0)^2\right) \\ &= \Delta(0,2i\omega)^{-1}(I - \Delta(0,2i\omega) \\ &\quad - \hat{\Phi}(2i\omega)\,\Psi(0))\frac{\partial^2 r}{\partial x^2}(0,0)\,p(0)^2 \end{aligned}$$

and, using in addition that $\alpha(T(-\tau)\bar{p}) = e^{i\omega\tau}\overline{p(0)}$,

$$\frac{\partial^2 \alpha\mathscr{C}}{\partial\phi^2}(0,0)(p,\bar{p}) = \Delta(0,0)^{-1}(I - \Delta(0,0) - \hat{\Phi}(0)\,\Psi(0))\frac{\partial^2 r}{\partial x^2}(0,0)(p(0),\bar{p}(0)).$$

Substitution of these identities into the expression for g_{21} obtained in step 1 yields the desired result. ∎

In principle the formula for c_1 is obtained by plugging in the results of Lemma 11.4 into the definition (11.2). It turns out that some simplification is possible.

THEOREM 11.5

$$\begin{aligned} c_1 = {} & \frac{1}{2} q(0)\frac{\partial^3 r}{\partial x^3}(0,0)\left(p(0)^2, \overline{p(0)}\right) \\ & + q(0)\frac{\partial^2 r}{\partial x^2}(0,0)\left(p(0), (\Delta(0,0)^{-1} - I)\frac{\partial^2 r}{\partial x^2}(0,0)(p(0), \overline{p(0)})\right) \\ & + \frac{1}{2} q(0)\frac{\partial^2 r}{\partial x^2}(0,0)\left(\overline{p(0)}, (\Delta(0,2i\omega)^{-1} - I)\frac{\partial^2 r}{\partial x^2}(0,0)\,p(0)^2\right). \end{aligned}$$

Proof. The identity

$$\begin{aligned} & \frac{i}{2\omega}\left(g_{20}g_{11} - 2g_{11}\overline{g_{11}} - \frac{1}{3}g_{02}\overline{g_{02}}\right) \\ & \quad = q(0)\frac{\partial^2 r}{\partial x^2}(0,0)\left(p(0), \Delta(0,0)^{-1}\,\hat{\Phi}(0)\,\Psi(0)\frac{\partial^2 r}{\partial x^2}(0,0)(p(0), \overline{p(0)})\right) \\ & \quad\quad + \frac{1}{2} q(0)\frac{\partial^2 r}{\partial x^2}(0,0)\left(\overline{p(0)}, \Delta(0,2i\omega)^{-1}\,\hat{\Phi}(2i\omega)\,\Psi(0)\frac{\partial^2 r}{\partial x^2}(0,0)\,p(0)^2\right) \end{aligned}$$

is a straightforward consequence of the definition (10.6), the formulae

$$\hat{p}(\lambda) = (\lambda - i\omega)^{-1}\Delta(0,\lambda)\,p(0), \qquad \hat{\bar{p}}(\lambda) = (\lambda + i\omega)^{-1}\Delta(0,\lambda)\,\bar{p}(0)$$

and Lemma 11.4. ∎

Remark. The expressions given in Lemma 11.4 are, mutatis mutandis, the same as those for mappings derived in [46, p. 168]. In the case of evolution equations in Hilbert space an expression corresponding to c_1 is given in [47, formula (3.9)]. For retarded functional differential equations see [45; 48, formula (2.10)].

12. An Interesting Equation

Let

$$B_1(\tau) = \begin{cases} 1, & 0 \leqslant \tau \leqslant 1 \\ 0, & \tau > 1, \end{cases}$$

and let $B_2(\tau)$ be a non-negative L_2-function with support contained in $[0, 1]$ and such that $\int_0^1 B_2(\tau)\, d\tau = 1$. The equation

$$y(t) = \gamma \left(1 - \int_0^1 B_1(\tau)\, y(t-\tau)\, d\tau\right) \int_0^1 B_2(\tau)\, y(t-\tau)\, d\tau \qquad (12.1)$$

arises from a model for the spread, in a closed population, of an infectious disease which confers only temporary immunity. (See [17–19].) Here $B_2(\tau)$ describes the infectivity of an individual which was infected τ units of time ago and immunity is lost exactly one unit of time after infection. The constant γ is proportional to the total population size.

The equation admits the constant solutions

$$\bar{y}_1 = 0, \qquad \bar{y}_2 = 1 - \gamma^{-1}.$$

For $0 < \gamma < 1$, $\bar{y}_1$ is stable and $\bar{y}_2$ is unstable. As γ passes through one, (i.e., as the population size reaches a critical value) the endemic state $\bar{y}_2$ becomes positive and at the same time it takes over the stability of the state $\bar{y}_1$ in which the disease is absent from the population. This is the well-known threshold phenomenon.

In [17] the question was posed whether or not the endemic state would retain its stability as γ is further increased. The following answer was found.

THEOREM 12.1. *As γ increases from one to infinity, exactly as many pairs of conjugated roots of the characteristic equation*

$$\hat{B}_2(\lambda) + (1-\gamma)\, \hat{B}_1(\lambda) = 1$$

pass the imaginary axis as there are $n \in \mathbb{N}$ for which

$$b_n = \int_0^1 B_2(\tau) \sin(2\pi n \tau)\, d\tau > 0.$$

They cross from left to right with a positive velocity, the upper one in the interval $((2n-1)\pi, 2n\pi)$. Moreover, they are simple.

In [18] Gripenberg gives a bifurcation theorem which, after some minor modifications, converts the information of Theorem 12.1 into existence results for periodic solutions. Similar results are given by Cushing [9–11]. Gripenberg also gives a formula for the direction of bifurcation and he shows how this direction is related to the modulus of a quantity which may be interpreted as a Floquet multiplier.

We shall now demonstrate how the problem (12.1) fits into the framework of this paper. We are interested in the behaviour near the constant solution $\bar{y}_2$, so we introduce $z = y - \bar{y}_2$ and we rewrite (12.1) as

$$z(t) = (1-\gamma)\int_0^1 B_1(\tau)\, z(t-\tau)\, d\tau + \int_0^1 B_2(\tau)\, z(t-\tau)\, d\tau$$
$$-\gamma \int_0^1 B_1(\tau)\, z(t-\tau)\, d\tau \int_0^1 B_2(\tau)\, z(t-\tau)\, d\tau. \tag{12.2}$$

Next we introduce

$$\begin{cases} x_1(t) = (1-\gamma)\displaystyle\int_0^1 B_1(\tau)\, z(t-\tau)\, d\tau, \\ x_2(t) = \displaystyle\int_0^1 B_2(\tau)\, z(t-\tau)\, d\tau \end{cases} \tag{12.3}$$

and we find, using

$$z(t) = x_1(t) + x_2(t) - \gamma(1-\gamma)^{-1}\, x_1(t)\, x_2(t), \tag{12.4}$$

that (12.2) is equivalent to the two-system

$$x(t) = \int_0^1 B(\gamma, \tau)\, g(\gamma, x(t-\tau))\, d\tau \tag{12.5}$$

with

$$B(\gamma, \tau) = \begin{pmatrix} (1-\gamma) B_1 & (1-\gamma) B_1 \\ B_2 & B_2 \end{pmatrix} \tag{12.6}$$

and

$$g(\gamma, x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2}\gamma(1-\gamma)^{-1} \begin{pmatrix} x_1 x_2 \\ x_1 x_2 \end{pmatrix}. \tag{12.7}$$

(Note that the representation by a system is not unique but that this does not influence the chain of reasoning.) A simple calculation shows that the characteristic equation associated with the constant solution $x = 0$ is precisely the one in Theorem 12.1.

If $b_n > 0$ for some $n \in \mathbb{N}$, we are in a position to apply Theorem 11.1. Unfortunately, it is not known whether pairs can pass the imaginary axis simultaneously and "in resonance" (i.e., such that condition (ii) of Theorem 11.1 is not satisfied). However, if some pairs pass simultaneously, the largest one satisfies the nonresonance condition (note that at most finitely many roots can pass simultaneously) and Theorem 11.1 yields the existence and uniqueness of a bifurcating periodic solution (with initial period in the interval $(n^{-1}, (n-\frac{1}{2})^{-1})$). We conclude that at least one periodic solution bifurcates if at least one $b_n > 0$ and that countably many periodic solutions bifurcate if countably many $b_n > 0$ (note that all $b_n > 0$, if, for instance, B_2 is decreasing).

Theorem 11.2 applies as well and $\operatorname{Re}\lambda'(0)$ is always positive by Theorem 12.1. So it is useful to evaluate the expression for c_1 given in Theorem 11.5. Some straightforward calculations lead to

$$\begin{aligned} c_1 = \gamma^2 \Bigg\{ & \frac{1}{1-(1-\gamma)\,\hat{B}_1(2i\omega) - \hat{B}_2(2i\omega)} (\hat{B}_1(i\omega)\,\hat{B}_1(2i\omega)\,|\hat{B}_2(i\omega)|^2 \\ & + \hat{B}_2(i\omega)\,\hat{B}_2(2i\omega)\,|\hat{B}_1(i\omega)|^2) \\ & + \frac{2}{(1-\gamma)^2} (|\hat{B}_2(i\omega)|^2 - \operatorname{Re}\hat{B}_2(i\omega)) \left(\hat{B}_2(i\omega) + \frac{\hat{B}_1(i\omega)}{\hat{B}_1(0)}\right) \Bigg\} \\ & \times \frac{1}{-(1-\gamma)\,\hat{B}_1'(i\omega) - \hat{B}_2'(i\omega)}. \end{aligned} \tag{12.8}$$

Using the identification $b_0 = \gamma - 1$, $A = B_1$, $a = B_2$ we conclude that (12.8) and formula (2.18) in Gripenberg [18] are related by $c_1 = i\gamma^2 c(\omega)$. As a consequence the expressions for the second Taylor coefficient of the characteristic exponents differ by a positive factor (due to the different parametrizations; besides we think that a factor 2 dropped off in the formula for $\ddot{\gamma}(0)$ below (4.17) in [18]).

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