Acyclic edge-colouring of planar graphs*

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Abstract

A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph G, denoted $\chi'_a(G)$, is the minimum k such that G admits an *acyclic edge-colouring* with k colours. We conjecture that if G is planar and $\Delta(G)$ is large enough then $\chi'_a(G) = \Delta(G)$. We settle this conjecture for planar graphs with girth at least 5. We also show that $\chi'_a(G) \leq \Delta(G) + 12$ for all planar G, which improves a previous result by Fiedorowicz et al. [12].

1 Introduction

A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph G, denoted $\chi'_a(G)$, is the minimum k such that G admits an *acyclic edge-colouring* with k colours. Fiamčik [9] and later Alon, Sudakov and Zaks [2] conjecture that $\Delta(G)+2$ colours are enough.

Conjecture 1 (Fiamčik [9]–Alon, Sudakov and Zaks [2]) *For every graph G, $\chi'_a(G) \leq \Delta(G)+2$.*

This conjecture would be tight as there are cases where more than $\Delta+1$ colours are needed. Consider for example a graph G on $2n$ vertices with at least $2n^2-2n+2$ edges. The union of two perfect matchings is a cycle factor and thus contains a cycle. Thus, in an acyclic edge-colouring, at most one colour class contains n edges. Hence there are at least $1+\left\lceil \frac{2n^2-3n+2}{n-1} \right\rceil = 2n+1$ colours. So $\chi'_a(G) \geq \Delta(G)+2$.

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Clearly, every graph with maximum degree at most 2 has acyclic chromatic index at most 3. If $\Delta(G) \leq 3$ then its line-graph $L(G)$ has maximum degree at most 4. Thus by Burnstein's results [7] $\chi_a(L(G)) \leq 5$ and so $\chi'_a(G) \leq 5$. So Conjecture 1 holds for $\Delta(G) \leq 3$. In 1980, Fiamčik [10] conjectured that K_4 is the only cubic graph requiring five colours in an acyclic edge-colouring (and actually gave an uncorrect proof of it). More generally, Alon, Sudakov and Zaks [2] conjectured that if G is a Δ-regular graph then $\chi'_a(G) = \Delta + 1$ unless $G = K_{2n}$.

However as noted by Fiamčik [11], these two conjectures are false as $\chi'_a(K_{3,3}) = 5$. In addition, Basavaraju, Chandran and Kummini [5] showed that all d-regular graphs with $2n$ vertices and $d > n$, require at least $d+2$ colours to be acyclically edge-coloured and for every odd n, $\chi'_a(K_{n,n}) = n+2$. They also showed that for every d,n such that $d \geq 5$, $n \geq 2d+3$ and dn even, there exist d-regular graphs which require at least $d+2$-colours to be acyclically edge-coloured.

Alon, Sudakov and Zaks [2] showed that Conjecture 1 is true for almost all regular graphs. This was later improved by Nešetřil and Wormald [19] who proved that the acyclic edge-chromatic number of a random Δ-regular graph is asymptotically almost surely equal to $\Delta+1$. Alon, McDiarmid and Reed [1] showed an upper bound of $64\Delta(G)$ for $\chi'_a(G)$ which was later improved to $16\Delta(G)$ by Molloy and Reed [16]. For graphs with large girth, better upper bounds are known. Muthu et al [17] showed that, if G has girth at least 9, then $\chi'_a(G) \leq 6\Delta(G)$, and, if it has girth at least 220, then $\chi'_a(G) \leq 4.52\Delta(G)$. Finally, Alon, Sudakov and Saks also showed that Conjecture 1 is true for graphs with girth at least $C\Delta\log(\Delta)$ for some fixed constant C.

Muthu et al [18] proved that $\chi'_a(G) \leq \Delta(G)+1$ for outerplanar graphs. Fiedorowicz et al. [12] proved that $\chi'_a(G) \leq 2\Delta(G)+29$ if G is planar and $\chi'_a(G) \leq \Delta(G)+6$ if G is planar and triangle-free. This bound has been improved for planar graphs with larger girth. Recall that the *girth* of a graph is the minimum length of a cycle it contains or $+\infty$ if it has no cycles. Hou et al. [14] showed that if G is a planar graph G then $\chi'_a(G) \leq \Delta(G)+2$ if G has girth at least 5, $\chi'_a(G) \leq \Delta(G)+1$ if G has girth at least 7 and $\chi'_a(G) \leq \Delta(G)$ if G has girth at least 16 and $\Delta(G) \geq 3$.

Sanders and Zhao [20] showed that planar graphs with maximum degree $\Delta \geq 7$ have chromatic index Δ. A conjecture of Vizing [21] asserts that planar graphs of maximum degree 6 are also 6-edge-colourable. This would be best possible as for any $\Delta \in \{2,3,4,5\}$, there are some planar graphs with maximum degree Δ with chromatic index $\Delta+1$ [21].

We propose a conjecture analogous to the above one of Vizing.

Conjecture 2 There exists Δ_0 such that every planar graph with maximum degree $\Delta \geq \Delta_0$ has an acyclic edge-colouring with Δ colours.

In this paper, we give some evidences to this conjecture. Firstly, in Section 2, we show that every planar graph G has an acyclic edge-colouring with $\Delta(G)+12$ colours thus improving the $2\Delta(G)+29$ bound of Fiedorowicz et al. [12]. In Section 3, we show that Conjecture 2 holds for planar graphs of girth at least 5 (with $\Delta_0 = 19$) thus improving the results of Hou et al. [14] and Borowiecki and Fiedorowicz [6]. More generally, we settle Conjecture 2 for graphs with maximum average degree less than $4-\varepsilon$ for any $\varepsilon > 0$. The *maximum average degree* of G is $Mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} \mid H \text{ is a subgraph of } G\}$. It is well known that a planar graph of girth g has maximum average degree less than $2+\frac{4}{g-2}$. Conjecture 2 holds for outerplanar graphs with $\Delta_0 = 5$ as shown by Hou et al. [15]. Note that $\sup\{Mad(G) \mid G \text{ is outerplanar}\} = 4$.

Our proofs are constructive and yield efficient polynomial time algorithms. We present the proofs in a non-algorithmic way. But it is easy to extract the underlying algorithms from them.

2 Planar graphs

In this section we will prove the following result.

Theorem 3 $\chi'_a(G) \leq \Delta(G) + 12$ *for all planar graphs G.*

The proof of Theorem 3 relies on the following theorem of van den Heuvel and McGuiness [13] which establishes a set of unavoidable configurations in planar graphs.

Lemma 4 (van den Heuvel and McGuiness [13]) *Let G be a planar graph with minimum degree at least two. Then there exists a vertex v in G with exactly $d(v) = k$ neighbours $v_1, v_2, \ldots, v_k$ with $d(v_1) \leq d(v_2) \leq \ldots \leq d(v_k)$ such that at least one of the following is true:*

(A1) $k = 2$,

(A2) $k = 3$ *and* $d(v_1) \leq 11$,

(A3) $k = 4$ *and* $d(v_1) \leq 7$, $d(v_2) \leq 11$,

(A4) $k = 5$ *and* $d(v_1) \leq 6$, $d(v_2) \leq 7$, $d(v_3) \leq 11$.

Sketch of the proof of Theorem 3: Let G be a minimum counter-example with respect to the number of vertices and edges for the statement in Theorem 3. Trivially G has minimum degree at least 2. Indeed, it has no vertex v of degree 0 because any acyclic edge-colouring of $G - v$ is an acyclic edge-colouring of G, and it has no vertex v with a unique neighbour u, since any acyclic edge-colouring of $G - v$ on at least $\Delta(G)$ colours may be extended to an acyclic edge-colouring of G by assigning to uv a colour not already assigned to an edge incident to u. From Lemma 4, we know that there exists a vertex v in G such that it belongs to one of the configurations $A1$–$A4$. If there is a configuration A_2, A_3 and A_4 in G, we show in Subsection 2.2 how to derive an acyclic edge-colouring with $\Delta(G) + 12$ colours of G from one of $G \setminus vv_1$. Hence, we assume that there is no such configurations. In such case, we select an appropriate edge uu' and show again how to derive an acyclic edge-colouring of G with $\Delta(G) + 12$ colours from one of $G \setminus uu'$. This gives a final contradiction. See Subsection 2.3.

In order to show how to extend an acyclic edge-colouring of $G \setminus e$ for some edge e into an acyclic edge-colouring of G,we first establish some preliminaries.

2.1 Preliminaries

Partial edge-colouring: Let H be a subgraph of G. Then an edge-colouring c' of H is also a *partial edge-colouring* of G. Note that H can be G itself. Thus an edge-colouring c of G itself can be considered a partial edge-colouring. A partial edge-colouring c of G is said to be a *proper partial edge-colouring* if c is proper. A proper partial edge-colouring c is called *acyclic* if there are no bichromatic cycles in the graph. Note that with respect to a partial edge-colouring c, $c(e)$ may not be defined for an edge e. So, whenever we use $c(e)$, we are considering an edge e for which $c(e)$ is defined, though we may not always explicitly mention it.

Let c be a partial edge-colouring of G. We denote the set of colours in c by $C = \{1, 2, \ldots, k\}$. For any vertex $u \in V(G)$, we define $F_u(c) = \{c(uz) \mid z \in N_G(u)\}$, with $N_G(u)$ denotes the set of vertices adjacent tot u. For an edge $ab \in E$, we define $S_{ab}(c) = F_b(c) - \{c(ab)\}$. Note that $S_{ab}(c)$ need not be the same as $S_{ba}(c)$. We will abbreviate the notation to F_u and S_{ab} when the edge-colouring c is understood from the context.

The following definitions arise out of our attempt to understand what may prevent us from extending a partial edge-colouring of $G \setminus e$ to G.

Maximal bichromatic Path: An *(α,β)-maximal bichromatic path* with respect to a partial edge-colouring c of G is a maximal path consisting of edges that are coloured using the colours α and β alternatingly. An *(α,β,a,b)-maximal bichromatic path* is an (α,β)-maximal bichromatic path which starts at the vertex a with an edge coloured α and ends at b. We emphasize that the edge of the (α,β,a,b)-maximal bichromatic path incident on vertex a is coloured α and the edge incident on vertex b can be coloured either α or β. Thus the notations (α,β,a,b) and (α,β,b,a) have different meanings. Also note that any maximal bichromatic path will have at least two edges. The following fact is obvious from the definition of proper edge-colouring.

Fact 5 *Given a pair of colours α and β of a proper edge-colouring c of G, there is at most one maximal (α,β)-bichromatic path containing a particular vertex v, with respect to c.*

A colour $\alpha \neq c(e)$ is a *candidate* for an edge e in G with respect to a partial edge-colouring c of G if none of the adjacent edges of e is coloured α. A candidate colour α is *valid* for an edge e if assigning the colour α to e does not result in any bichromatic cycle in G.

Let $e = ab$ be an edge in G. Note that any colour $\beta \notin F_a \cup F_b$ is a candidate colour for the edge ab in G with respect to the partial edge-colouring c of G. A sufficient condition for a candidate colour being valid is captured in the lemma below.

Lemma 6 (Basavaraju and Chandran [4]) *A candidate colour for an edge $e = ab$ is valid if $(F_a(c) \cap F_b(c)) \setminus \{c(ab)\} = S_{ab}(c) \cap S_{ba}(c) = \emptyset$.*

Now even if $S_{ab}(c) \cap S_{ba}(c) \neq \emptyset$, a candidate colour β may be valid. But if β is not valid, then what may be the reason? It is clear that colour β is not *valid* if and only if there exists $\alpha \neq \beta$ such that a (α,β)-bichromatic cycle gets formed if we assign colour β to the edge e. In other words, if and only if, with respect to edge-colouring c of G there existed an (α,β,a,b)-maximal bichromatic path with α being the colour given to the first and last edge of this path. Such paths play an important role in our proofs. We call them *critical paths*. It is formally defined below.

Critical Path: Let $ab \in E$ and c be a partial edge-colouring of G. Then an (α,β,a,b)-maximal bichromatic path which starts out from the vertex a via an edge coloured α and ends at the vertex b via an edge coloured α is called an *(α,β,a,b)-critical path*. Note that any critical path will be of odd length. Moreover the smallest length possible is three.

Let $a \in N_{G \setminus vv_1}(x)$ and let $c(x,a) = \alpha$. Let $\beta \in S_{xa}$. colour β is said to be *actively present* in a set S_{xa}, if there exists a (α,β,xy) critical path.

A natural strategy to extend a acyclic partial edge-colouring c of G would be to try to assign one of the candidate colours to an uncoloured edge e. The condition that a candidate colour is not valid for the edge e is captured in the following fact.

Fact 7 *Let c be a partial edge-colouring of G. A candidate colour β is not valid for the edge $e = ab$ if and only if for some colour $\alpha \in S_{ab} \cap S_{ba}$, there is an (α,β,a,b)-critical path in G with respect to c.*

Colour exchange: Let c be a partial edge-colouring of G. Let $u,v,w \in V(G)$ and $uv,uw \in E(G)$. We define *colour exchange* with respect to the edge uv and uw, as the modification of the current partial edge-colouring c by exchanging the colours of the edges uv and uw to get a partial edge-colouring c', i.e., $c'(uv) = c(uw)$, $c'(uw) = c(uv)$ and $c'(e) = c(e)$ for all other edges e in G. The colour exchange with respect to the edges uv and uw is said to be *proper* (resp. *acyclic*) if the edge-colouring obtained after the exchange is proper (resp. acyclic). The following fact is obvious.

Fact 8 *Let c' be the partial edge-colouring obtained from an acyclic partial edge-colouring c by the colour exchange with respect to the edges uv and uw. Then c' is proper if and only if $c(uv) \notin S_{uw}$ and $c(uw) \notin S_{uv}$.*

The colour exchange is useful in breaking some critical paths as is clear from the following lemma.

Lemma 9 (Basavaraju and Chandran [4, 3]) *Let u, v, w, a and b be vertices of G such that uv, uw and ab are edges. Also let α and β be two colours such that $\{\alpha,\beta\} \cap \{c(uv), c(uw)\} \neq \emptyset$ and $\{v,w\} \cap \{a,b\} = \emptyset$. Suppose there exists a (α,β,a,b)-critical path that contains vertex u, with respect to an acyclic partial edge-colouring c of G. Let c' be the partial edge-colouring obtained from c by the colour exchange with respect to the edges uv and uw. If c' is proper, then there is no (α,β,a,b)-critical path in G with respect to c'.*

Multisets and Multiset Operations: Recall that a *multiset* is a generalized set where a member can appear multiple times. If an element x appears t times in the multiset S, then we say that the *multiplicity* of x in S is t. In notation $mult_S(x) = t$. The cardinality of a finite multiset S, denoted by $\| S \|$, is defined as $\| S \| = \sum_{x \in S} mult_S(x)$. Let S_1 and S_2 be two multisets. The reader may note that there are various possible ways to define union of S_1 and S_2. For the purpose of this paper we define one such union notion- which we call as the *join* of S_1 and S_2, denoted as $S_1 \uplus S_2$. The multiset $S_1 \uplus S_2$ have all the members of S_1 as well as S_2. For a member $x \in S_1 \uplus S_2$, $mult_{S_1 \uplus S_2}(x) = mult_{S_1}(x) + mult_{S_2}(x)$. Clearly $\| S_1 \uplus S_2 \| = \| S_1 \| + \| S_2 \|$.

2.2 There exists a Configuration $A2$, $A3$ or $A4$

We now can resume the proof of Theorem 3. Suppose by way of contradiction that there exists a Configuration A_2, A_3 or A_4 in G. Let v, v_1, v_2 and v_3 be the vertices as described in Lemma 4.

In all the propositions of this subsection, we start with an acyclic edge-colouring c' of $G \setminus vv_1$. So the abbreviations F_u and S_{ab} stand for $F_u(c')$ and $S_{ab}(c')$ respectively.

Proposition 10 *For any acyclic edge-colouring c' of $G \setminus vv_1$, $|F_v \cap F_{v_1}| \geq 2$.*

Proof. Suppose by way of contradiction that there is an acyclic edge-colouring c' of $G \setminus vv_1$ with a set C of $\Delta + 12$ colours such that $|F_v \cap F_{v_1}| \leq 1$.

Assume first that $|F_v \cap F_{v_1}| = 0$. The reader can verify from close examination of Configurations $A2$, $A3$ and $A4$ that $|F_v \cup F_{v_1}|$ will be maximum for Configuration $A2$ and therefore $|F_v \cup F_{v_1}| = |F_v| + |F_{v_1}| \leq 2 + 10 = 12$. Thus there are Δ candidate colours for the edge vv_1 and by Lemma 6 all the candidate colours are valid, a contradiction to the assumption that G is a counter-example.

Assume now that $|F_v \cap F_{v_1}| = 1$. It is easy to see that $|F_v \cup F_{v_1}| = |F_v| + |F_{v_1}| - |F_v \cap F_{v_1}| \leq 11$ and hence there are at least $\Delta + 1$ candidate colours for the edge vv_1. Let $F_v \cap F_{v_1} = \{\alpha\}$ and let $u \in N(v)$ be a vertex such that $c'(vu) = \alpha$. Now if none of the $\Delta + 1$ candidate colours is valid for the edge vv_1, then by Fact 7, for each $\gamma \in C \setminus (F_v \cup F_{v_1})$, there exists an (α,γ,v,v_1)-critical path. Since $c'(vu) = \alpha$, we have all the critical paths passing through the vertex u and hence $S_{vu} \subseteq C \setminus (F_v \cup F_{v_1})$. This implies that $|S_{vu}| \geq |C \setminus (F_v \cup F_{v_1})| \geq (\Delta + 12) - 11 = \Delta + 1$, a contradiction since $|S_{vu}| \leq \Delta - 1$. Thus we have a valid colour for the edge vv_1, a contradiction to the assumption that G is a counter-example. □

Let S_v be the multiset defined by $S_v = S_{vv_2} \uplus S_{vv_3} \uplus \ldots \uplus S_{vv_k}$.

Proposition 11 *For any acyclic edge-colouring c' of $G \setminus vv_1$, $|F_v \cap F_{v_1}| \neq 2$.*

Proof. Suppose not. Let $F_v \cap F_{v_1} = \{\alpha_1, \alpha_2\}$ and let $v', v'' \in N_{G \setminus vv_1}(v)$ and $u', u'' \in N_{G \setminus vv_1}(v_1)$ be such that $c'(vv') = c'(v_1u') = \alpha_1$ and $c'(vv'') = c'(v_1u'') = \alpha_2$. It is easy to see that $|F_v \cup F_{v_1}| \leq 10$. Thus there are at least $\Delta + 2$ candidate colours for the edge vv_1. If any of the candidate colours is valid for the edge vv_1, we are done. Thus none of the candidate colours is valid for the edge vv_1. This implies that there exists a $(\alpha_1, \theta, v, v_1)$- or $(\alpha_2, \theta, v, v_1)$-critical path for each candidate colour θ.

Claim 11.1 *The multiset S_v contains at least $|F_{v_1}| - 1$ colours from F_{v_1}.*

Proof. Suppose not. Then there are at least two colours in F_{v_1} which are not in S_v. Let ν and μ be any two such colours. Now assign colours ν and μ to the edges vv' and vv'' respectively to get an edge-colouring c''. Now since $\nu, \mu \notin S_v$, we have $\nu \notin S_{vv'}$ and $\mu \notin S_{vv''}$. Moreover $\mu, \nu \notin F_v(c') \setminus \{\alpha_1, \alpha_2\}$. Thus the edge-colouring c'' is proper. Now we claim that the edge-colouring c'' is acyclic also. Suppose not. Then there has to be a bichromatic cycle containing at least one of the colours ν and μ. Clearly this cannot be a (ν, μ)-bichromatic cycle since $\mu \notin S_{vv'}$. Therefore it has to be a (ν, λ)- or (μ, λ)-bichromatic cycle where $\lambda \in F_v(c'') \setminus \{\nu, \mu\}$. Let u be a vertex such that $c''(vu) = \lambda$. This means that there was already a (λ, ν, v, v')- or (λ, μ, v, v'')-critical path with respect to the edge-colouring c'. This implies that $\nu \in S_{vu}$ or $\mu \in S_{vu}$, implying that $\nu \in S_v$ or $\mu \in S_v$, a contradiction. Thus the edge-colouring c'' is acyclic. Let $u_1, u_2 \in N_{G \setminus vv_1}(v_1)$ be such that $c''(v_1u_1) = \nu$ and $c''(v_1u_2) = \mu$.

Note that $|F_v \cup F_{v_1}| \leq 10$ (The maximum value of $|F_v \cup F_{v_1}|$ is attained when the graph has Configuration $A2$). Therefore there are at least $\Delta + 2$ candidate colours for the edge vv_1. If any of the candidate colours are valid for the edge vv_1, then we are done as this is a contradiction to the assumption that G is a counter-example. Thus none of the candidate colours is valid for the edge vv_1 and therefore there exist either a (ν, θ, v, v_1)-critical or a (μ, θ, v, v_1)-critical path for each candidate colour θ. Let C_ν and C_μ respectively be the set of candidate colours which are forming critical paths with colours ν and μ. Then clearly $C_\nu \subseteq S_{v_1u_1}$ and $C_\mu \subseteq S_{v_1u_2}$ since $c''(v_1u_1) = \nu$ and $c''(v_1u_2) = \mu$. Now we exchange the colours of the edges vv' and vv'' to get a modified edge-colouring c. Note that c is proper since $\mu \notin S_{vv'}$ and $\nu \notin S_{vv''}$. By Lemma 9, all (ν, β, v, v_1)-critical paths where $\beta \in C_\nu$ and all (μ, γ, v, v_1)-critical paths where $\gamma \in C_\mu$ are broken. Now if none of the colours in C_ν are valid for edge vv_1, then it means that for each $\beta \in C_\nu$, there exists a (μ, β, v, v_1)-critical path with respect to the edge-colouring c, implying that $C_\nu \subseteq S_{v_1u_2}$. Since the recolouring involved no candidate colours, we still have $C_\mu \subseteq S_{v_1u_2}$. Thus we have $(C_\nu \cup C_\mu) \subseteq S_{v_1u_2}$. But $|C_\nu \cup C_\mu| \geq \Delta + 2$ which implies that $|S_{v_1u_2}| \geq \Delta + 2$, a contradiction since $|S_{v_1u_2}| \leq \Delta - 1$. □

Claim 11.2 *There exists at least two colours β_1 and β_2 in $C \setminus F_{v_1}$ with multiplicity at most one in S_v.*

Proof. In view of Claim 11.1 we have $\sum_{x \in C \setminus F_v} mult_{S_v}(x) = \| S_v \| - (|F_v| - 1)$. Thus if $\| S_v \| - (|F_{v_1}| - 1) \leq 2|(C \setminus F_{v_1})| - 3$, then there exist at least two colours β_1 and β_2 in $C \setminus F_{v_1}$ with multiplicity at most one in S_v. Thus it is enough to prove $\| S_v \| \leq 2|C| - |F_{v_1}| - 4 \leq 2\Delta + 24 - |F_{v_1}| - 4 = 2\Delta + 20 - |F_{v_1}|$. Now we can easily verify that $\| S_v \| + |F_{v_1}| \leq 2\Delta + 20$ for Configurations $A2$, $A3$ and $A4$ as follows:

- For $A2$, $\| S_v \| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + |F_{v_1}| = (\Delta - 1) + (\Delta - 1) \ + 10 = 2\Delta + 8$.
- For $A3$, $\| S_v \| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) + |F_{v_1}| = 10 + (\Delta - 1) + (\Delta - 1) \ + 6 = 2\Delta + 14$.
- For $A4$, $\| S_v \| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) + (d(v_5) - 1) + |F_{v_1}| = 6 + 10 + (\Delta - 1) + (\Delta - 1) \ + \ 5 = 2\Delta + 19$.

□

The colours β_1 and β_2 of Claim 11.2 are crucial to the proof. Now we make another claim regarding β_1 and β_2:

Claim 11.3 β_1 *and* $\beta_2 \in F_v$.

Proof. Without loss of generality, let $\beta_1 \notin F_v$. Then recalling that $\beta_1 \notin F_{v_1}$, β_1 is a candidate for the edge vv_1. If it is not valid, then there exists either an $(\alpha_1, \beta_1, vv_1)$- or $(\alpha_2, \beta_1, v, v_1)$-critical path with respect to c'. Since the multiplicity of β_1 in S_v is at most one, we have the colour β_1 in exactly one of $S_{vv'}$ or $S_{vv''}$. Without loss of generality let $\beta_1 \in S_{vv''}$. Hence there exists an $(\alpha_2, \beta_1, v, v_1)$-critical path with respect to c'.

Now recolour the edge vv' with colour β_1 to get an edge-colouring c. Then c is proper since $\beta_1 \notin F_v$ and $\beta_1 \notin S_{vv'}$. We shall prove that is is acyclic. Suppose, by way of contradiction, that there is a bichromatic cycle with respect to c. Then it has to be a (β_1, γ)-bichromatic cycle for some $\gamma \in F_v(c) \setminus c(vv')$. Let $a \in N_{G\setminus vv_1}(v)$ be such that $c(va) = \gamma$. Then the (β_1, γ)-bichromatic cycle should contain the edge va and therefore $\gamma \in S_{va}(c)$. But we know that v'' is the only vertex in $N_{G\setminus vv_1}(v)$ such that $\beta_1 \in S_{vv''}$. Therefore $a = v''$. This implies that $\gamma = \alpha_2$ and there existed an $(\alpha_2, \beta_1, v, v')$-critical path with respect to the edge-colouring c'. This is a contradiction to Fact 5 since there already existed an $(\alpha_2, \beta_1, v, v_1)$-critical path with respect to the edge-colouring c'.

Thus the edge-colouring c is acyclic and $|F_v(c) \cap F_{v_1}(c)| = 1$, a contradiction to Proposition 10. □

Note that $\{\beta_1, \beta_2\} \cap \{\alpha_1, \alpha_2\} = \emptyset$ since $\beta_1, \beta_2 \notin F_{v_1}$. In view of Claim 11.3, we have $\{\alpha_1, \alpha_2, \beta_1, \beta_2\} \subseteq F_v$ and thus $|F_v| \geq 4$, which implies that $d(v) \geq 5$. Thus the vertex v belongs to Configuration $A4$. Therefore $d(v) = 5$ and $F_v = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. There are at least $\Delta + 12 - (5 + 4 - 2) = \Delta + 5$ candidate colours for the edge vv_1. Also recall that $d(v_2) \leq 7$, $c'(vv') = c'(v_1u') = \alpha_1$ and $c'(vv'') = c'(v_1u'') = \alpha_2$.

Claim 11.4 $v_2 \notin \{v', v''\}$.

Proof. Suppose not. Then, without loss of generality, $v_2 = v'$ and $c'(vv_2) = \alpha_1$. Now if none of the $\Delta + 5$ candidate colours is valid for the edge vv_1, then they all are in critical paths that contain either the edge vv' or the edge vv''. Now $|S_{vv'}| + |S_{vv''}| \leq 6 + \Delta - 1 = \Delta + 5$. Since each of the $\Delta + 5$ candidate colours has to be present in either in $S_{vv'}$ or $S_{vv''}$, we infer that $S_{vv''} \cup S_{vv'}$ is exactly the set of candidate colours, i.e., $|S_{vv'}| + |S_{vv''}| = \Delta + 5$. This requires that $|S_{vv'}| = 6$, $|S_{vv''}| = \Delta - 1$ and $S_{vv''} \cap S_{vv'} = \emptyset$. Since for each $\gamma \in S_{vv''}$, we have $(\alpha_2, \gamma, v, v_1)$-critical path containing u'', we can infer that $S_{vv''} \subseteq S_{v_1u''}$ (Recall that $c'(v_1u'') = \alpha_2$). But since $|S_{v_1u''}| \leq \Delta - 1$, we have $S_{vv''} = S_{v_1u''}$. Thus $S_{v_1u''} \cap S_{vv'} = S_{vv''} \cap S_{vv'} = \emptyset$.

Now we exchange the colours of the edges vv' and vv'' to get an edge-colouring c. Hence $c(vv') = \alpha_2$ and $c(vv'') = \alpha_1$. The edge-colouring c is proper since $\alpha_2 \notin S_{vv'}$ and $\alpha_1 \notin S_{vv''}$ (Recall that $S_{vv'}$ and $S_{vv''}$ contain only candidate colours). We shall prove that c is also acyclic: A bichromatic cycle with respect to c has to be an (α_1, η)- or (α_2, η)-bichromatic cycle for some $\eta \in F_v$. Clearly it cannot be an (α_1, α_2)-bichromatic cycle since $\alpha_1 \notin S_{vv'}(c)$ and therefore $\eta \in \{\beta_1, \beta_2\}$ (Recall that $F_v = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$). This implies that either β_1 or β_2 belongs to $S_{vv'} \cup S_{vv''}$. But we know that $S_{vv'} \cup S_{vv''}$ is exactly the set of candidate colours for the edge vv_1, a contradiction since $\beta_1, \beta_2 \in F_v$ cannot be candidate colours for the edge vv_1.

Therefore the edge-colouring c is acyclic. By Lemma 9, all the existing critical paths are broken. Now consider a colour $\gamma \in S_{vv'}$. If it is still not valid then there has to be a $(\alpha_2, \gamma, v, v_1)$-critical path since $c(vv') = \alpha_2$ and $\gamma \notin S_{vv''}(c)$. This implies that $\gamma \in S_{v_1u''}(c)$, a contradiction since $S_{v_1u''}(c) \cap$

$S_{vv'}(c) = \emptyset$. Thus we have a valid colour for the edge vv_1, a contradiction to the assumption that G is a counter-example. $\square$

From Claim 11.4, we infer that $c'(vv_2) \notin F_v \cap F_{v_1}$ since $F_v \cap F_{v_1} = \{c'(vv'), c(vv'')\} = \{\alpha_1, \alpha_2\}$. Therefore we have $c(vv_2) \in \{\beta_1, \beta_2\}$ since $F_v = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. Without loss of generality let $c(vv_2) = \beta_1$. We know that the colour β_2 can be in at most one of $S_{vv'}$ and $S_{vv''}$ by Claim 11.2. Now let v' be such that $\beta_2 \notin S_{vv'}$. Note that $C \setminus (S_{vv'} \cup F_v \cup F_{v_1}) \neq \emptyset$ since $|S_{vv'} \cup F_v \cup F_{v_1}| \leq \Delta - 1 + 4 + 5 - 2 = \Delta + 6$. Assign a colour $\theta \in C \setminus (S_{vv'} \cup F_v \cup F_{v_1})$ to the edge vv' to get an edge-colouring c''. Now $|F_v(c'') \cap F_{v_1}(c'')| = 1$. Thus in view of Proposition 10, the edge-coloring c'' is not acyclic. Hence there is a bichromatic cycle with respect to c''. This bichromatic cycle should involve one of the colours α_2, β_1, β_2 along with θ. Since the bichromatic cycle contains a colour from $S_{vv'}$ and $\beta_2 \notin S_{vv'}$, it cannot be a (θ, β_2)-bichromatic cycle. Now with respect to the edge-colouring c', colour θ was not valid for the edge vv_1 implying that there existed a $(\alpha_1, \theta, v, v_1)$- or $(\alpha_2, \theta, v, v_1)$-critical path. But $(\alpha_1, \theta, v, v_1)$-critical path was not possible since $\theta \notin S_{vv'}$ by the choice of θ. Thus there existed an $(\alpha_2, \theta, v, v_1)$-critical path with respect to c'. Thus by Fact 5, there cannot be an $(\alpha_2, \theta, v, v')$-critical path with respect to c' and hence there cannot be an (α_2, θ)-bichromatic cycle in c'' formed due to the recolouring. Thus if there is a bichromatic cycle formed, then it has to be a (β_1, θ)-bichromatic cycle, which implies that $\beta_1 \in S_{vv'}$.

Now taking into account the fact that β_1 is in $S_{vv'}$ as well as F_v, we get $|S_{vv'} \cup F_v \cup F_{v_1}| \leq \Delta - 1 + 4 + 5 - 2 - 1 = \Delta + 5$ and therefore $|S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2}| \leq \Delta + 5 + 6 = \Delta + 11$. Thus $C \setminus (S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2}) \neq \emptyset$. Now recolour the edge vv' using a colour $\gamma \in C \setminus (S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2})$ to get an edge-colouring c. Clearly this edge-colouring is proper. It is also acyclic since if a bichromatic cycle gets formed it has to be a (β_1, γ) bichromatic cycle (Note that the (α_2, γ) and (β_2, γ) bichromatic cycles are argued out as before). But $\gamma \notin S_{vv_2}$, a contradiction. Thus the edge-colouring c is acyclic.

But $|F_v(c) \cap F_{v_1}(c)| = 1$, a contradiction to Proposition 10. This completes the proof of Proposition 11. $\square$

Proposition 12 *For any acyclic edge-colouring c' of $G \setminus vv_1$, $|F_v \cap F_{v_1}| \neq 3$.*

Proof. Suppose not. Let c' be an acyclic edge-colouring of $G \setminus vv_1$ such that $|F_v \cap F_{v_1}| = 3$. Then $|F_v| \geq 3$ and therefore $d(v) \geq 4$. Thus v belongs to either configuration $A3$ or $A4$. Let S'_v be the multiset defined by $S'_v = S_v \setminus (F_{v_1} \cup F_v$. Let $v', v'', v''' \in N_{G \setminus vv_1}(v)$ be such that $\{c(vv'), c(vv''), c(vv''')\} = F_v \cap F_{v_1}$. Also let $c(vv') = \alpha_1$, $c(vv'') = \alpha_2$ and $c(vv''') = \alpha_3$.

Claim 12.1 $\parallel S'_v \parallel \leq 2\Delta + 11$.

Proof. When $d(v) = 4$, it is clear that $\parallel S'_v \parallel \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) \leq 10 + \Delta - 1 + \Delta - 1 = 2\Delta + 8$. On the other hand when $d(v) = 5$, try to recolour one of the edges vv', vv'', vv''' using a colour in $C \setminus (F_v \cup F_{v_1})$. There are $\Delta + 6$ colours in $C \setminus (F_v \cup F_{v_1})$. If any of these colours is valid for one of vv', vv'' or vv''', then recolouring this edge with this colour, we obtain an acyclic edge-colouring c'' satisfying $|F_v(c'') \cap F_{v_1}(c'')| = 2$. This contradicts Proposition 11. Hence there has to be a bichromatic cycle formed during each recolouring. Since such a bichromatic cycle has to be a (γ_1, γ_2)-bichromatic cycle where γ_1 is the colour used in the recolouring and $\gamma_2 \in F_v \setminus \{\gamma_1\}$, we infer that $S_{vv'}$, $S_{vv''}$ and $S_{vv'''}$ contain at least one colour from F_v. Thus we have $\parallel S'_v \parallel \leq \parallel S_v \parallel -3 \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) + (d(v_5) - 1) - 3 \leq 6 + 10 + \Delta - 1 + \Delta - 1 - 3 = 2\Delta + 11$. $\square$

Claim 12.2 *There exists at least one colour $\beta \in C \setminus (F_v \cup F_{v_1})$ with multiplicity at most one in S'_v.*

Proof. Since v belongs to either configuration $A3$ or configuration $A4$, we have $|F_v \cup F_{v_1}| \leq 9-3=6$. Thus $|C \setminus (F_v \cup F_{v_1})| \leq \Delta+6$. By *Claim* 12.1 we have $\| S'_v \| \leq 2\Delta+11$ and from this it is easy to see that there exists at least one colour $\beta \in C \setminus (F_v \cup F_{v_1})$ with multiplicity at most one in S'_v. □

Note that $\beta \in C \setminus (F_v \cup F_{v_1})$, where β is the colour from Claim 12.2 is a candidate colour for the edge vv_1. If it is not valid then there has to be a (θ, β, v, v_1)-critical path, where $\theta \in \{\alpha_1, \alpha_2, \alpha_3\}$. By Claim 12.2, β can be present in at most one of $S_{vv'}$, $S_{vv''}$ and $S_{vv'''}$. Without loss of generality let $\beta \in S_{vv''}$. Thus there exists an $(\alpha_2, \beta, v, v_1)$-critical path with respect to the edge-colouring c'. Recolour the edge vv' using the colour β to get an edge-colouring c. Clearly c is proper since $\beta \notin S_{vv'}$ and $\beta \notin F_v$. Let us show that it is also acyclic. A bichromatic cycle (with respect to c) has to contain the colour β as well as a colour $\gamma \in F_v(c) \setminus \{\beta\}$. If $\gamma = c(vw)$, then $\beta \in S_{vw}$, for the (β, γ)-bichromatic cycle to get formed. But v'' is the only vertex in $N_{G \setminus vv_1}(v)$ such that $\beta \in S_{vv''}$. Thus $w = v''$, $\gamma = \alpha_2$ and the cycle is an (α_2, β)-bichromatic cycle. This means that there existed an (α_2, β, v, v')-critical path with respect to the edge-colouring c', a contradiction to Fact 5 since there already existed an $(\alpha_2, \beta, v, v_1)$-critical path with respect to the edge-colouring c'. Thus the edge-colouring c is acyclic.

But $|F_v(c) \cap F_{v_1}(c)| = 2$, a contradiction to Proposition 11. This completes the proof of Proposition 12. □

Proposition 13 *For any acyclic edge-colouring c' of $G \setminus vv_1$, $|F_v \cap F_{v_1}| \neq 4$.*

Proof. Suppose not. Let c' be an acyclic edge-colouring of $G \setminus vv_1$ such that $|F_v \cap F_{v_1}| = 4$. Then $|F_v| \geq 4$ and since $d(v) \leq 5$, we have $d(v) = 5$. Hence v belongs to Configuration $A4$. Let S'_v be the multiset defined by $S'_v = S_v \setminus (F_{v_1} \cup F_v)$. Also let $c(vv_2) = \alpha_1$, $c(vv_3) = \alpha_2$, $c(vv_4) = \alpha_3$ and $c(vv_5) = \alpha_4$.

Now try to recolour an edge incident on v with a candidate colour from $C \setminus (F_v \cup F_{v_1})$. If the obtained edge-colouring c'' is acyclic then $|F_v(c'') \cap F_{v_1}(c'')| = 3$, a contradiction to Proposition 12. Hence there has to be a bichromatic cycle created due to recolouring with one of the colours from F_v. This implies that $F_v \cap S'_v \neq \emptyset$. Thus we have $\| S'_v \| \leq \| S_v \| -1 \leq (d(v_2)-1)+(d(v_3)-1)+(d(v_4)-1)+(d(v_5)-1) \leq 6+10+\Delta-1+\Delta-1-1 = 2\Delta+13$. Now since there are $|C \setminus (F_v \cup F_{v_1})| \geq \Delta+12-(4+5-4) = \Delta+7$ candidate colours and $\| S'_v \| \leq 2\Delta+13$, it is easy to see that there exists at least one candidate colour β with multiplicity at most one in S'_v.

Note that $\beta \in C \setminus (F_v \cup F_{v_1})$ is a candidate colour for the edge vv_1. If it is not valid then there has to be a (θ, β, v, v_1)-critical path, where $\theta \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. We know that β can be present in at most one of S_{vv_2}, S_{vv_3}, S_{vv_4} and S_{vv_5}. Without loss of generality let $\beta \in S_{vv_3}$. Thus there exists an $(\alpha_2, \beta, v, v_1)$-critical path with respect to the edge-colouring c'. Recolour the edge vv_2 using the colour β to get an edge-colouring c. Clearly c is proper since $\beta \notin S_{vv_2}$ and $\beta \notin F_v$. Let us now show that it is acyclic. A bichromatic cycle with respect to c has to contain the colour β as well as a colour $\gamma \in F_v(c) \setminus \{\beta\}$. If $\gamma = c(vw)$, then $\beta \in S_{vw}$, for the (β, γ) bichromatic cycle to get formed. But v_3 is the only vertex in $N_{G \setminus vv_1}(v)$ such that $\beta \in S_{vv_3}$. Thus $w = v_3$, $\gamma = \alpha_2$ and it has to be a (β, α_2) bichromatic cycle. This means that there existed an $(\alpha_2, \beta, v, v_2)$-critical path with respect to the edge-colouring c', a contradiction to Fact 5 since there already existed an $(\alpha_2, \beta, v, v_1)$-critical path with respect to the edge-colouring c'. Thus the edge-colouring c is acyclic.

But $|F_v(c) \cap F_{v_1}(c)| = 3$, a contradiction to Proposition 12. □

By Lemma 4, $d_{G \setminus vv_1}(v) \leq 4$. Thus $|F_v \cap F_{v_1}| \leq |F_v| \leq 4$. Then Propositions 10, 11, 12 and 13 gives a contradiction to the assumption that G contains a Configuration $A2$, $A3$ or $A4$.

2.3 There is no Configuration $A2$, $A3$ or $A4$

In the previous subsection, we showed that G contains no Configuration $A2$, $A3$ or $A4$. Then by Lemma 4, there is a Configuration $A1$, that is a vertex v such that $d(v) = 2$. Now delete all the degree 2 vertices from G to get a graph H. Now since the graph H is also planar, there exists a vertex v' in H such that v' belongs to one of the configurations $A1$, $A2$, $A3$ or $A4$, say A'. The vertex v' was not already in Configuration A' in G. This means that the degree of at least one of the vertices of the configuration A' i.e., $\{v'\} \cup N_H(v')$, got decreased by the removal of 2-degree vertices. Let $P = \{x \in \{v'\} \cup N_H(v') : d_H(x) < d_G(x)\}$. Let u be the minimum degree vertex in P in the graph H. Now it is easy to see that $d_H(u) \leq 11$ since v' did not belong to A' in G.

Let $N'(u) = \{x | x \in N_G(u) \text{ and } d_G(u) = 2\}$. Let $N''(u) = N_G(u) - N'(u)$. It is obvious that $N''(u) = N_H(u)$.

Since $u \in P$ and $d_H(u) \leq 11$, we have $|N'(u)| \geq 1$ and $N''(u) \leq 11$. In G let $u' \in N'(u)$ be a two degree neighbour of u such that $N(u') = \{u, u''\}$. Now by minimality of G, the graph $G \setminus uu'$ admits an acyclic edge-colouring c' using a set C of $\Delta + 12$ colours. Let $F'_u = \{c'(ux) | x \in N'(u)\}$ and $F''_u = \{c'(ux) | x \in N''(u)\}$. Now if $c(u'u'') \notin F_u$ we are done since $|F_u \cup F_{u'}| \leq \Delta$ and thus there are at least 12 candidate colours which are also valid by Lemma 6.

We know that $|F''_v| \leq 11$. If $c'(u'u'') \in F'_v$, then let $c = c'$. Else if $c'(u'u'') \in F''_v$, then recolour edge $u'u''$ using a colour from $C \setminus (S_{u'u''} \cup F''_v)$ to get an edge-colouring c (Note that $|C \setminus (S_{u'u''} \cup F''_v)| \geq \Delta + 12 - (\Delta - 1 + 11) = 2$ and since u' has degree one in $G - \{uu'\}$, c is acyclic). Now if $c(u'u'') \notin F_u$ the proof is already discussed. Thus $c(u'u'') \in F'_u$.

Let us now consider the edge-colouring c. Let $a \in N'(u)$ be such that $c(ua) = c(u'u'') = \alpha$. Now if none of the candidate colours in $C \setminus (F_u \cup F_{u'})$ are valid for the edge uu', then by Fact 7, for each $\gamma \in C \setminus (F_u \cup F_{u'})$, there exists an (α, γ, u, u')-critical path. Since $c'(ua) = \alpha$, we have all the critical paths passing through the vertex a and hence $S_{ua} \subseteq C \setminus (F_u \cup F_{u'})$. This implies that $|S_{ua}| \geq |C \setminus (F_u \cup F_{u'})| \geq \Delta + 12 - (1 + \Delta - 1 - 1) = 13$, a contradiction since $|S_{ua}| = 1$. Thus we have a valid colour for the edge uu', a contradiction to the assumption that G is a counter-example.

This final contradiction completes the proof of Theorem 3.

3 Planar graphs of girth at least 5

The aim of this section is to prove Conjecture 2 for planar graphs of girth at least 5. Actually, we prove the conjecture for a more general class of graphs: the graphs of maximum average degree at most $10/3$. The *average degree* of a graph G is $Ad(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} d(v) = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G is $Mad(G) = \max\{Ad(H) \mid H \text{ is a subgraph of } G\}$. It is well known that the girth and the maximum average degree of a planar graph are related to each other:

Proposition 14 *Let G be a planar graph of girth g.*

$$Mad(G) < 2 + \frac{4}{g-2}.$$

Theorem 15 *Let $\Delta \geq 19$ and G be a graph with maximum degree at most Δ and maximum average degree less than $\frac{10}{3}$. Then $\chi'_a(G) \leq \Delta$.*

Theorem 15 and Proposition 14 immediately yield the following.

Corollary 16 *Let $\Delta \geq 19$ and G be a planar graph with maximum degree at most Δ and girth at least 5. Then $\chi'_a(G) \leq \Delta$.*

More generally than Theorem 15, we show the following.

Theorem 17 *For any $\varepsilon > 0$, there exists an integer Δ_ε such that every graph G with maximum degree at most Δ with $\Delta \geq \Delta_\varepsilon$ and maximum average degree less than $4-\varepsilon$ is acyclically Δ-edge-colourable.*

In order to prove Theorems 15 and 17, we first establish some properties of Δ-*minimal graphs* which are graphs with maximum degree at most Δ, not acyclically Δ-edge-colourable but such that every proper subgraph is. Then, by the Discharging Method, we deduce that such a graph has maximum average degree at least $4-\varepsilon$ (resp. $10/3$) if Δ is at least Δ_ε (resp. 19). We will first prove, in Subsection 3.2, Theorem 17 for its discharging procedure is simpler because we only establish the existence of Δ_ε and make no attempt to minimize it. We then show Theorem 15 in Subsection 3.3.

A vertex of degree i is called an *i-vertex* and an *i-neighbour* of a vertex v is a neighbour of v having degree i.

3.1 Properties of Δ-minimal graphs

Proposition 18 *A Δ-minimal graph G is 2-connected. In particular, $\delta(G) \geq 2$.*

Proof. If G is not connected, it is the disjoint union of G_1 and G_2. Both G_1 and G_2 admits an acyclic Δ-edge-colouring by minimality of G. The union of these two edge-colourings is an acyclic Δ-edge-colouring of G.

Suppose now that G has a cutvertex v. Let C_i, for $1 \leq i \leq p$ be the components of $G-v$ and G_i the graph induced by $C_i \cup \{v\}$. By minimality of G, all the G_i admit an acyclic Δ-edge-colouring. Moreover, free to permute the colours we may assume that two edges incident to v get different colours. Hence the union of these edge-colourings is an acyclic Δ-edge-colouring of G because any cycle of G is entirely contained in one of the G_i. □

Proposition 19 *Let G be a Δ-minimal graph. For every vertex $v \in V(G)$, $\sum_{u\in N(v)} d(u) \geq \Delta+1$.*

Proof. Suppose by way of contradiction that there is a vertex v such that $\sum_{u\in N(v)} d(u) \leq \Delta$. Let w be a neighbour of v. By minimality of G, $G \setminus vw$ admits an acyclic edge-colouring with Δ colours. Now colour vw with a colour distinct from the ones of the edges incident to a neighbour of v. This is possible as there are at most $\Delta-1$ such edges distinct from vw. Doing so we clearly obtain a proper edge-colouring. Let us now show that there is no bicoloured cycle. A cycle that does not contain vw has edges of at least three colours as the edge-colouring of G was acyclic and a cycle containing vw must contain an edge vu and an edge tu with $u \in N(v) \setminus \{w\}$. By construction, the colours of tu, uv and vw are distinct. □

A *thread* is a path of length two whose internal vertex has degree 2.

Proposition 20 *Let $k \geq 2$ be an integer and G a Δ-minimal graph. In G, a Δ-vertex is the end of at most k threads whose other endvertex has degree at most k.*

To prove this proposition we need the following lemma.

Lemma 21 *Let $H = ((A,B),E)$ be a bipartite graph with $|A| = |B| = q$ such that for any vertex $a \in A$ $d(a) = 1$ and let $K_{A,B}$ be the complete bipartite graph with bipartition (A,B). If at least 3 vertices of B of degree at least one in H then there exists a perfect matching M of $K_{A,B}$ such that the bipartite graph $((A,B), E\cup M)$ has girth at least 6.*

Proof. Let m be the number of vertices of B of degree at least one. Let $b_1,\dots,b_q$ be the vertices of B with $d(b_i) \geq 1$ if $i \leq m$ and $d(b_i) = 0$ otherwise. And let $a_1,\dots,a_q$ be the vertices of A with $a_ib_i \in E$ for all $1 \leq i \leq m$. If $m \geq 3$, let $M = \{a_ib_{i+1} \mid 1 \leq i < m\} \cup \{a_mb_1\} \cup \{a_ib_i \mid m < i \leq q\}$. Then the unique cycle in $((A,B),E \cup M)$ is $C = (a_1,b_2,a_2,b_3,\dots,a_{m-1},b_m,a_1)$. It has length $2m \geq 6$. □

Proof of Proposition 20. Suppose for a contradiction that there is a Δ-vertex u with $q = k+1$ threads uv_iw_i, $1 \leq i \leq q$, such $d(w_i) \leq k$. Note that $q \geq 3$.

Set $A = \{v_1,\dots,v_q\}$. By Proposition 18, $w_i \notin A$ for all $1 \leq i \leq q$. By minimality of G, $G-A$ admits an acyclic Δ-edge-colouring.

Let us first extend it to the v_iw_i as follows. Let F be the set of colours assigned to the edges incident to u and to no vertex of A and for $1 \leq i \leq q$ let F_i be the set of colours assigned to the edges incident to w_i (and distinct from v_iw_i). Then $|F| = \Delta - q$ and $|F_i| \leq k-1$. For all $1 \leq i \leq q$, let S_i be the set of colours not in $F \cup F_i$. Since $|F| + |F_i| = \Delta - q + k - 1 = \Delta - 2$ then $|S_i| \geq 2$.

Assume first that $|\bigcup_{i=1}^{q} S_i| \geq 3$, then one can assign to each v_iw_i a colour in S_i in such a way that at least 3 colours appear on such edges and that different colours appear on v_iw_i and v_jw_j if$w_i = w_j$. We will now colour the edges uv_i for $1 \leq i \leq q$. Therefore let $H_1 = ((A,B),E_1)$ be the bipartite graph with B the set of q colours $\{b_1,\dots,b_q\}$ not in F and in which v_i is adjacent to b_j if $c(v_iw_i) = b_j$. As long as some v_i has degree 0 then add an edge between a_i and an isolated b_j to obtain a bipartite graph $H_2 = ((A,B),E_2)$. Because at least three colours appear on the v_iw_i, the graph H_2 fulfils the hypothesis of Lemma 21. So there exists a perfect matching M of $K_{A,B}$ such that $((A,B),E_2 \cup M)$ has girth at least 6. For $1 \leq i \leq q$, assign to each uv_i the colour to which v_i is linked in M.

Let us now prove that this edge-colouring of G is acyclic. It is obvious that it is proper since v_i is not linked to $c(v_iw_i)$ in M. Let us now prove that it is acyclic. Let C be a cycle of G. If it contains no vertex of A, then it contains edges of three different colours because the edge-colouring of $G-A$ is acyclic. Suppose now that C contains a unique vertex of A, say v_i. Then C contains w_iv_i, v_iu and ut with t a neighbour of u not in A. Then $c(ut) \in F$, so by construction, $c(w_iv_i) \neq c(ut)$. Hence the colours of w_iv_i, v_iu and ut are distinct. Suppose finally that C contains two vertices of A, say v_i and v_j. Then C contains w_iv_i, v_iu, w_jv_j and v_ju. Since $((A,B),E_2 \cup M)$ has girth at least 6, either $c(v_iu) \neq c(w_jv_j)$ or $c(v_ju) \neq c(w_iv_i)$. In both cases, C has edges of three different colours.

Asumme now that $|\bigcup_{i=1}^{q} S_i| < 3$. Then all the S_i are equal and of cardinality 2, say $S_i = \{a,b\}$ for all $1 \leq i \leq q$. Hence all the F_i are the same of cardinality $k-1$ and disjoint from F. Observe that this can happen only if all the w_i are distinct. Let us denote by $f_1,\dots,,f_{k-1}$ the elements of the F_i. Let us set $c(v_iw_i) = a$ for $1 \leq i \leq k$, $c(v_qw_q) = b$, $c(uv_i) = f_i$ for $1 \leq i \leq k-1$, $c(uv_k) = b$ and $c(uv_{k+1}) = a$. It is easy to check that the obtained edge-colouring is an acyclic edge-colouring of G.

□

Proposition 22 *Let k and l be two positive integers and G a Δ-minimal graph. In G, a $(\Delta - l)$-vertex is the end of at most $k-1-l$ threads whose other endvertex has degree at most k.*

To prove this proposition we need the following lemma.

Lemma 23 *Let $H = ((A,B),E)$ be a bipartite graph with $q = |A| < |B|$ such that for any vertex $a \in A$ $d(a) = 1$ and $K_{A,B}$ be the complete bipartite graph with bipartition (A,B).*

Then there exists a matching M of $K_{A,B}$ saturating A such that the bipartite graph $((A,B),E \cup M)$ has no cycle.

Proof. Let $q' = |B|$. Let $b_1, \dots, b_{q'}$ be the vertices of B with $d(b_i) \geq 1$ if $i \leq m$ and $d(b_i) = 0$ otherwise. And let $a_1, \dots, a_q$ be the vertices of A with $a_i b_i \in E$ for all $1 \leq i \leq m$. Let $M = \{a_i b_{i+1} \mid 1 \leq i \leq q\}$. This is well-defined since $q' > q$. Then $((A,B), E \cup M)$ has no cycle. □

Proof of Proposition 22. . Suppose for a contradiction that there is a $(\Delta - l)$-vertex u with $q = k - l$ threads uv_iw_i, $1 \leq i \leq q$, such $d(w_i) \leq k$.

Set $A = \{v_1, \dots, v_q\}$. By minimality of G, $G - A$ admits an acyclic Δ-edge-colouring. Let us first extend it to the v_iw_i as follows. Let F be the set of colours assigned to the edges incident to u and to no vertex of A and for $1 \leq i \leq q$ let F_i be the set of colours assigned to the edges incident to w_i (and distinct from v_iw_i). Then $|F| = \Delta - l - q$ and $|F_i| \leq k - 1$.

For all $1 \leq i \leq q$ colour v_iw_i with a colour not in $F \cup F_i$ and distinct from the colours. This is possible since $|F| + |F_i| = \Delta - l - q + k - 1 = \Delta - 1$.

We will now colour the edges uv_i for $1 \leq i \leq q$. Therefore let $H_1 = ((A,B), E_1)$ be the bipartite graph with B the set of $q + j$ colours $\{b_1, \dots, b_{q+j}\}$ not in F and in which v_i is adjacent to b_j if $c(v_iw_i) = b_j$. As long as some v_i has degree 0 then add an edge between a_i and an isolated b_j to obtain a bipartite graph $H_2 = ((A,B), E_2)$. Then H_2 fulfils the hypothesis of Lemma 23 so there exists a perfect matching M of $K_{A,B}$ such that $((A,B), E_2 \cup M)$ has no cycle. For $1 \leq i \leq q$, assign to each uv_i the colour to which v_i is linked in M.

In the same way as in the proof of Proposition 20, one shows that the obtained edge-colouring is acyclic. □

3.2 Proof of Theorem 17

Lemma 24 *Let $\varepsilon > 0$. There exists Δ_ε such that if $\Delta \geq \Delta_\varepsilon$ then any Δ-minimal graph has average degree at least $4 - \varepsilon$.*

Proof. The result for $\varepsilon = \frac{1}{2}$ implies the result for larger values of ε. Hence we assume that $\varepsilon \leq \frac{1}{2}$. Let us assign an initial charge of $d(v)$ to each vertex $v \in V(G)$ Set $d_\varepsilon = \lceil \frac{8}{\varepsilon} - 2 \rceil$.

We perform the following discharging rules.

R1: for $4 \leq d < d_\varepsilon$, every d-vertex sends $a(d) = 1 - \frac{4-\varepsilon}{d}$ to each neighbour.

R2: for $d_\varepsilon \leq d \leq \Delta + 1 - d_\varepsilon$ then every d-vertex sends $1 - \frac{\varepsilon}{2}$ to each neighbour.

R3: for $\Delta + 2 - d_\varepsilon \leq d \leq \Delta$ then every d-vertex sends

- $1 - \varepsilon$ to each 3-neighbour;
- $2 - \varepsilon$ to each 2-neighbour whose second neighbour has degree 2 or 3;
- $b(d) = 2 - \varepsilon - a(d)$ to each 2-neighbour whose second neighbour has degree d with $4 \leq d < d_\varepsilon$;
- $1 - \frac{\varepsilon}{2}$ to each 2-neighbour whose second neighbour has degree $d \geq d_\varepsilon$.

Let us now check that every vertex v has final charge $f(v)$ at least $4 - \varepsilon$.

If v is a 2-vertex then let u and w be its two neighbours with $d(u) \leq d(w)$. If $d(u) \leq 3$ then $d(w) \geq \Delta - 2$ by Proposition 19. Hence v receives $2 - \varepsilon$ from w by R3, so $f(v) \geq 2 + 2 - \varepsilon = 4 - \varepsilon$. If $4 \leq d(u) < d_\varepsilon$ then $d(w) > \Delta + 1 - d_\varepsilon$ by Proposition 19. Hence v receives $a(d)$ from u by R2 and

$b(d)$ from w by R3. So $f(v) = 4-\varepsilon$. If $d(u) \geq 10$ then v receives $1-\frac{\varepsilon}{2}$ from u and $1-\frac{\varepsilon}{2}$ from w by R3. So $f(v) = 4-\varepsilon$.

Suppose that v is a 3-vertex. Then by Proposition 19 it has at least two (≥ 8)-neighbours. Hence it receives at least $2 \times 1/2$ by R1, R2 or R3 because $\varepsilon \leq \frac{1}{2}$. So $f(v) \geq 4$.

Suppose $4 \leq d(v) < d_\varepsilon$. Then v sends $d(v)$ times $1-\frac{4-\varepsilon}{d(v)}$ so $f(v) \geq 4-\varepsilon$.

Suppose $d_\varepsilon \leq d(v) \leq \Delta+1-d_\varepsilon$. Then v sends at most $d(v)$ times $1-\frac{\varepsilon}{2}$ so $f(v) \geq d(v) \times \frac{\varepsilon}{2} \geq 4-\varepsilon$.

Suppose now that $d(v) \geq \Delta+2-d_\varepsilon$. Then by Propositions 20 and 22, the most v can send is when it has three 2-neighbours with second neighbour of degree at most 3, one 2-neighbour with second neighbour of degree d for all $4 \leq d \leq d_\varepsilon - 1$ and $\Delta - d_\varepsilon + 1$ 2-neighbours with second neighbour of degree at least d_ε. Hence

$$\begin{aligned} f(v) &\geq \Delta+2-d_\varepsilon-3(2-\varepsilon)-\sum_{d=4}^{d_\varepsilon-1} b(d)-(\Delta-d_\varepsilon+1)(1-\frac{\varepsilon}{2}) \\ &\geq \Delta\frac{\varepsilon}{2}-S_\varepsilon \end{aligned}$$

with $S_\varepsilon = d_\varepsilon - 2 + 3(2-\varepsilon) + \sum_{d=4}^{d_\varepsilon-1} b(d) - (1-\frac{\varepsilon}{2})(d_\varepsilon - 1)$. Setting $\Delta_\varepsilon = \left\lceil \frac{2}{\varepsilon}(S_\varepsilon + 4 - \varepsilon) \right\rceil$, if $\Delta \geq \Delta_\varepsilon$, $f(v) \geq 4-\varepsilon$. □

Proof of Theorem 17. If Theorem 17 were false, then a minimum counterexample G would be a Δ-minimum graph. So by Lemma 24, its average degree would be at least $4-\varepsilon$, a contradiction. □

3.3 Proof of Theorem 15

Lemma 24 for $\varepsilon = 2/3$ yields that for $\Delta \geq \Delta_{2/3}$, a Δ-minimal graph G satisfies $Mad(G) \geq Ad(G) \geq 10/3$. The value of $\Delta_{2/3}$ given by the proof of Lemma 24 is 49. We now show that it could be decreased to 19.

Lemma 25 *Let $\Delta \geq 19$ and G be a Δ-minimal graph. Then $Mad(G) \geq Ad(G) \geq 10/3$.*

Proof. Let us assign an initial charge of $d(v)$ to each vertex $v \in V(G)$ and perform the following discharging rules.

R1: every 4-vertex sends 4/9 to each of its (≤ 3)-neighbours;

R2: every 5-vertex sends 7/12 to each 2-neighbour and $1/3$ to each 3-neighbour;

R3: for $6 \leq d \leq 9$, every d-vertex sends $1-10/3d$ to each neighbour.

R4: for $10 \leq d \leq \Delta-9$ then every d-vertex sends $2/3$ to each neighbour.

R5: for $\Delta-8 \leq d \leq \Delta$ then every d-vertex sends

- 2/3 to each d-neighbour with $3 \leq d \leq 5$;
- 4/3 to each 2-neighbour whose second neighbour has degree 2 or 3;
- 8/9 to each 2-neighbour whose second neighbour has degree 4;

- 9/12 to each 2-neighbour whose second neighbour has degree 5;
- $1/3+10/3d$ to each 2-neighbour whose second neighbour has degree d with $6 \le d \le 9$;
- 2/3 to each 2-neighbour whose second neighbour has degree $d \ge 10$.

Let us now check that every vertex v has final charge $f(v)$ at least $\frac{10}{3}$.

If v is a 2-vertex then let u and w be its two neighbours with $d(u) \le d(w)$. If $d(u) \le 3$ then $d(w) \ge \Delta-2$ by Proposition 19. Hence v receives 4/3 from w by R5, so $f(v) \ge 2+4/3 = 10/3$. If $d(u)=4$ then $d(w) \ge \Delta-3$ by Proposition 19. Hence v receives 4/9 from u by R1 and 8/9 from w by R5. So $f(v)=10/3$. If $d(u)=5$ then $d(w) \ge \Delta-4$ by Proposition 19. Hence v receives 7/12 from u by R2 and 9/12 from w by R5. So $f(v)=10/3$. If $6 \le d(u) \le 9$ then $d(w) \ge \Delta-8$ by Proposition 19. Hence v receives $1-10/3d$ from u by R3 and $1/3+10/3d$ from w by R5. So $f(v)=10/3$. If $d(u) \ge 10$ then v receives $2/3$ from u by R4 and $2/3$ from w by R5. So $f(v)=10/3$.

Suppose that v is a 3-vertex. Then, since $\Delta \ge 10$, by Proposition 19 it has either a (≥ 5)-neighbour or two 4-neighbours. Hence it receives either at least 1/3 by R2, R3, R4 or R5, or $2 \times 4/9 \ge 1/3$ by R1. In both cases, $f(v) \ge 3+1/3 = 10/3$.

Suppose that v is a 4-vertex. Then, since $\Delta \ge 18$, by Proposition 19, it has either three (≤ 3)-neighbours and one (≥ 10)-neighbour or at most two (≤ 3)-neighbours. In the first case, it sends 4/9 to each of its 3-neighbours and receives 2/3 form its (≥ 10)-neighbour. So $f(v) \ge 4-3\times\frac{4}{9}+\frac{2}{3} = 10/3$. In the second case, it sends 4/9 to at most 2 neighbours. So $f(v) \ge 4-2\times\frac{4}{9} > 10/3$.

Suppose that v is a 5-vertex.
Assume first that v has at most three (≤ 3)-neighbours. If it has at least one (3)-neighbour it sends at most $3/2$ so $f(v) \ge 5-3/2 > 10/3$. If not it has three 2-neighbours. Let u_1 and u_2 be the two (≥ 4)-neighbours of v. By Proposition 19, $d(u_1)+d(u_2) \ge 11$ since $\Delta \ge 16$. Hence one of these two vertices is a (≥ 6)-vertex and it sends at least 4/9 to u. Hence $f(v) \ge 5+4/9-7/4 > 10/3$.
Assume now that v has at least four (≤ 3)-neighbours. Let i be the number of 2-neighbours of v. Then by Proposition 19, v has exactly $4-i$ 3-neighbours and its fifth neighbour has degree at least $6+i$ since $\Delta \ge 17$. Hence $f(v) \ge 5-i.\frac{7}{12}-(4-i)\frac{1}{3}+1-\frac{10}{3(6+i)} > 10/3$.

Suppose $6 \le d(v) \le 9$. Then v sends $d(v)$ times $1-10/3d(v)$ so $f(v) \ge d(v)-d(v)(1-10/3d) = 10/3$.

Suppose $10 \le d(v) \le \Delta-10$. Then v sends at most $d(v)$ times $2/3$ so $f(v) \ge d(v)(1-2/3) \ge 10/3$.

Suppose that $d(v)=\Delta-l$ for $1 \le l \le 7$. By Proposition 22, v is incident to at most $\Delta-l-1$ threads so its has at least one (≥ 3)-neighbour to which it sends at most 2/3. Moreover the most it can send is when it has exactly one 2-neighbour with second neighbour of degree d for each $l+2 \le d \le 9$ and $\Delta-9$ 2-neighbours with second neighbour of degree at least 10. Hence its final charge is

$$\begin{aligned} f(v) &\ge \Delta-l-\left((\Delta-8)\frac{2}{3}+\sum_{d=l+2}^{9} s(d)\right) \\ &\ge \frac{1}{3}\Delta+\frac{16}{3}-\left(l+\sum_{d=l+2}^{9} s(d)\right) \end{aligned}$$

with $s(3) = 4/3$, $s(4) = 8/9$, $s(5) = 9/12$ and $s(d) = 1/3 + 10/3d$ for $6 \leq d \leq 9$. Since $s(3) > 1$ and $s(d) < 1$ when $d \geq 4$, then $l + \sum_{d=l+2}^{9} s(d)$ is minimum when $l = 2$. Hence

$$\begin{aligned} f(v) &\geq \frac{1}{3}\Delta + \frac{16}{3} - \left(2 + \sum_{d=4}^{9} s(d)\right) \\ &\geq \frac{1}{3}\Delta + \frac{61}{36} - \frac{10}{3}\sum_{d=6}^{9}\frac{1}{d} \\ &\geq \frac{1}{3}\Delta + \frac{61}{36} - \frac{10}{3} \times \frac{275}{504} \geq \frac{10}{3} \end{aligned}$$

because $\Delta \geq 11$.

Suppose $d(v) = \Delta$. By Proposition 20, the most it can send is when it has three 2-neighbours with second neighbour of degree at most 3, exactly one 2-neighbour with second neighbour of degree d for $4 \leq d \leq 9$ and $\Delta - 9$ 2-neighbours with second neighbour of degree at least 10. In this case it sends

$$\begin{aligned} 3 \times \frac{4}{3} + \frac{8}{9} + \frac{9}{12} + \sum_{d=6}^{9}\left(\frac{1}{3} + \frac{10}{3d}\right) + (\Delta - 9)\frac{2}{3} &= \frac{2}{3}\Delta + \frac{35}{36} + \frac{10}{3}\sum_{d=6}^{9}\frac{1}{d} \\ &= \frac{2}{3}\Delta + \frac{35}{36} + \frac{10}{3} \times \frac{275}{504} \\ &\leq \Delta - \frac{10}{3} \end{aligned}$$

because $\Delta \geq 19$. Hence $f(v) \geq \frac{10}{3}$.

Now $Ad(G) = \frac{1}{|V|}\sum_{v \in V(G)} d(v) = \frac{1}{|V|}\sum_{v \in V(G)} f(v) \geq \frac{10}{3}$. □

Proof of Theorem 15. If Theorem 15 would be false, a minimum counterexample G would be a Δ-minimum graph. So by Lemma 25, its average degree is at least $10/3$, a contradiction. □

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