THE GENERALIZED ABEL TRANSFORM FOR SL(2,$\mathbb{C}$)

by

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ABSTRACT

We study the generalized Abel transform for SL(2,$\mathbb{C}$) in the case of equal left and right, fixed K-type. We rewrite this transform as an integral transform of classical type. Then it involves a double integration with kernel expressed in terms of a Chebyshev polynomial of the second kind. We obtain the inversion formula in a similar form and we completely characterize the image in the C^{∞}-case. As a corollary we prove the subquotient theorem for SL(2,$\mathbb{C}$) by a global approach.

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0. INTRODUCTION

In earlier papers [13], [14] the second author formulated a program for a global approach to the representation theory of noncompact semisimple Lie groups G and he carried it out for $SL(2,\mathbb{R})$. "Global" means that no use of the Lie algebra and universal enveloping algebra of G is made. Instead, the analysis is based on a more or less explicit knowledge of the canonical matrix elements of the principal series representations of G with respect to a K-basis, K being a maximal compact subgroup of G. Furthermore, in the case of $SL(2,\mathbb{R})$ it turned out that the subquotient theorem (i.e. the Naimark equivalence of K-finite irreducible representations of G to subquotients of principal series representations) can be proved by use of the generalized Abel transform.

It is the purpose of the present paper to give a global proof of the subquotient theorem for $G = SL(2,\mathbb{C})$ by use of the generalized Abel transform. Let $I^{\infty}_{c,\delta}(G)$ be the commutative topological convolution algebra of K-central C^{∞}-functions with compact support on G which behave as the irreducible representation δ of K under left or right action of K. Then the generalized Abel transform is an algebra isomorphism of $I^{\infty}_{c,\delta}(G)$ onto a convolution algebra of certain vector-valued C^{∞}-functions with compact support on $\mathbb{R}$. The subquotient theorem follows from a knowledge of all continuous characters on this image algebra. So we have to know this image algebra. This method was earlier used by NAIMARK [19] and the characterization of the image algebra follows from WANG's [26] Paley-Wiener theorem. However, we will give a probably new proof with side results of independent interest. Namely, we write the generalized Abel transform as an integral transform of "classical" type and we obtain the inversion formula in a similar form:

$$F(\tau,t) = (2\pi)^{-1} \int_0^{2\pi} \int_t^{\infty} f(\phi,w) \cdot \tag{0.1}$$

$$\cdot U_{2\ell}\left(\frac{\operatorname{ch} t}{\operatorname{ch} w} \cos\phi \cos\tau + \frac{\operatorname{sh} t}{\operatorname{sh} w} \sin\phi \sin\tau\right) 2\operatorname{sh}2w \, d\phi \, dw,$$

$$(0.2)\qquad f(\phi,w) = (4\pi \mathrm{sh}2w)^{-1} \int_0^{2\pi} \int_w^{\infty} \left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial t^2}\right) F(\tau,t) \cdot$$

$$\cdot\ U_{2\ell}\left(\frac{\mathrm{ch}\ t}{\mathrm{ch}\ w} \cos\phi\ \cos\tau + \frac{\mathrm{sh}\ t}{\mathrm{sh}\ w} \sin\phi\ \sin\tau\right) d\tau\ dt.$$

In this integral transform pair, ℓ is in $\{0,\frac{1}{2},1,\frac{3}{2},\ldots\}$, $U_{2\ell}$ is a Chebyshev polynomial of the second kind, and f and F are C^∞-functions with compact support, $f(\psi,x)$ and $F(\psi,x)$ both having the form

$$\sum_{m\in\{-\ell,-\ell+1,\ldots,\ell\}} c_m(x) e^{-2im\psi},$$

with certain additional conditions.

Let us summarize the contents of the paper. In section 1 we give basic results about the global approach to finding irreducible subquotients and proving Naimark relatedness. Section 2 contains general theorems for the generalized Abel transform on a semisimple Lie group, section 3 a discussion of earlier results for $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$. Section 4 gives preliminaries on $SL(2,\mathbb{C})$ and the representation theory of $SU(2)$ which will be needed. The main work of the paper is done in sections 5,6,7: the derivation of (0.1) and an integral representation for related spherical functions of type δ in §5, the derivation of the inversion formula (0.2) (also leading to a new proof of the Plancherel formula for $SL(2,\mathbb{C})$) in §6, a characterization of the image algebra in §7. Finally, the subquotient theorem is derived in §8 and we state without proof some further results in §9.

Notation. rep means representation,
irr. means irreducible.

1. IRREDUCIBLE SUBQUOTIENTS, NAIMARK RELATEDNESS AND THE ALGEBRAS $I_{c,\delta}(G)$

In this section we collect some results which are relevant for the global approach to the representation theory of a general locally compact group.

Let G be a locally compact group satisfying the second axiom of countability and let K be a compact subgroup of G. Let τ be a *K-unitary Hilbert rep* of G, i.e. a strongly continuous rep τ of G on a separable Hilbert space $H(\tau)$ such that $\tau|_K$ is a unitary rep of K on $H(\tau)$. Then

$$(1.1) \qquad \tau|_K = \bigoplus_{\delta\in\hat{K}} m_\delta\ \delta,$$

where the multiplicity m_δ equals $0,1,2,\ldots$ or ω (countably infinite). Let $H_\delta(\tau)$ be the closed subspace of $H(\tau)$ which is the representation space of $m_\delta\ \delta$ in (1.1) and let P_δ be the orthogonal projection of $H(\tau)$ onto $H_\delta(\tau)$. For γ,δ in $M(\tau)$ define the canonical matrix elements $\tau_{\gamma\delta}$ of τ:

$$(1.2) \qquad \tau_{\gamma\delta}(g) := P_\gamma \tau(g)\ |_{H_\delta(\tau)}, \qquad g \in G.$$

Then $\tau_{\gamma\delta}(g)$ is a linear operator of $H_\delta(\tau)$ to $H_\gamma(\tau)$. The operator $\tau(g)$ can be written as a (usually infinite) block matrix with blocks $\tau_{\gamma\delta}(g)$. Define the K-*content* $M(\tau)$ of τ by

$$(1.3) \qquad M(\tau) := \{\delta \in \hat{K} \mid m_\delta \neq 0\}.$$

The rep τ is called K-*finite* if $m_\delta < \infty$ for all δ in $\hat{K}$ and τ is called K-*multiplicity free* if $m_\delta = 0$ or 1 for all δ in $\hat{K}$.

DEFINITION 1.1. Let τ be a Hilbert rep of G, H_0 a closed subspace of $H(\tau)$ and P_0 the orthogonal projection of $H(\tau)$ onto H_0. Let $\tau_0(g) := P_0\tau(g)|_{H_0}$ $(g\in G)$. If $\tau_0(g_1 g_2) = \tau_0(g_1)\tau_0(g_2)$ $(g_1,g_2\in G)$ then τ_0 is called a *subquotient rep* of τ on H_0.

THEOREM 1.2 (cf.[13,§3.2]). *Let τ be a* K*-multiplicity free rep of* G. *For γ,δ in $M(\tau)$ write $\gamma \sim \delta$ iff $\tau_{\gamma\delta} \neq 0$ and $\tau_{\delta\gamma} \neq 0$. Then $\sim$ is an equivalence relation on $M(\tau)$ and τ_0 is an irr. subquotient rep of τ iff $\tau_0|_K = \bigoplus_{\gamma|\gamma\sim\delta} \gamma$ for some δ in $M(\tau)$.*

DEFINITION 1.3. Two Hilbert reps σ and τ of G are called *Naimark related* if there is a closed (possibly unbounded) injective linear operator A of $H(\sigma)$ to $H(\tau)$ with domain $D(A)$ dense in $H(\sigma)$ and range $R(A)$ dense in $H(\tau)$ such that $D(A)$ is σ-invariant and $A\sigma(g)v = \tau(g)Av$ $(v\in D(A), g\in G)$. Notation: $\sigma \simeq \tau$ or $\sigma \overset{A}{\simeq} \tau$.

Naimark relatedness is an equivalence relation (called *Naimark equivalence*) on the class of K-finite Hilbert reps of G (cf.[13, Theorem 4.4]).

LEMMA 1.4. *Let* σ *and* τ *be irr. Hilbert reps of* G. *If, for certain nonzero* v *in* $H(\sigma)$ *and* w *in* $H(\tau)$, $(\sigma(g)v,v) = (\tau(g)w,w)$ *for all* g *in* G, *then* $\sigma \simeq \tau$.

PROOF. Define $\tilde{\sigma}(g) := (\sigma(g^{-1}))^*(g\in G)$, and similarly $\tilde{\tau}(g)$. Then $\tilde{\sigma}$ and $\tilde{\tau}$ are also irr. Hilbert reps of G. Define a linear operator A with $\mathcal{D}(A) := \mathrm{Span}\{\sigma(g)v \mid g \in G\}$ and $\mathcal{R}(A) := \mathrm{Span}\{\tau(g)w \mid g \in G\}$ by

$$A\left(\sum_{j=1}^{n} \alpha_j\sigma(g_j)v\right) := \sum_{j=1}^{n} \alpha_j\tau(g_j)w$$

for arbitrary n in $\mathbb{N}$, $\alpha_1,\ldots,\alpha_n$ in $\mathbb{C}$ and $g_1,\ldots,g_n$ in G. By irreducibility of σ and τ, $\mathcal{D}(A)$ is dense in $H(\sigma)$ and $\mathcal{R}(A)$ is dense in $H(\tau)$. For the proof that A is one-valued and injective note that the following equalities are equivalent:

$$\sum_{j=1}^{n} \alpha_j\sigma(g_j)v = 0,$$

$$\left(\sum_{j=1}^{n} \alpha_j\sigma(g_j)v,\tilde{\sigma}(g)v\right) = 0 \quad \forall g \in G,$$

$$\left(\sum_{j=1}^{n} \alpha_j\sigma(g^{-1}g_j)v,v\right) = 0 \quad \forall g \in G,$$

$$\left(\sum_{j=1}^{n} \alpha_j\tau(g^{-1}g_j)w,w\right) = 0 \quad \forall g \in G,$$

$$\left(\sum_{j=1}^{n} \alpha_j\tau(g_j)w,\tilde{\tau}(g)w\right) = 0 \quad \forall g \in G,$$

$$\sum_{j=1}^{n} \alpha_j\tau(g_j)w = 0.$$

Clearly, $\mathcal{D}(A)$ is σ-invariant and $A\sigma(g) = \tau(g)A$ on $\mathcal{D}(G)$ for g in G. For the proof that the closure $\bar{A}$ of A is one-valued and injective let

$$\sum_{j=1}^{n(k)} \alpha_{j,k}\sigma(g_{j,k})v \to v_0 \text{ and}$$

$$\sum_{j=1}^{n(k)} \alpha_{j,k}\, \tau(g_{j,k})w \to w_0 \quad \text{as } k \to \infty.$$

Then $v_0 = 0$ iff $w_0 = 0$, by a similar argument as above. Finally apply [13, Lemma 4.3]. □

THEOREM 1.5. *Let σ and τ be irr. K-unitary reps and let some δ in $\hat{K}$ have multiplicity 1 in both σ and τ. Then σ and τ are Naimark related iff*

$$\text{(1.4)} \qquad \operatorname{tr} \sigma_{\delta\delta}(g) = \operatorname{tr} \tau_{\delta\delta}(g) \quad \textit{for all } g \textit{ in } G.$$

PROOF. If $\sigma \simeq \tau$ and $I_\delta\colon \mathcal{H}_\delta(\sigma) \to \mathcal{H}_\delta(\tau)$ is a K-intertwining isometry then

$$\tau_{\delta\delta}(g) = I_\delta \sigma_{\delta\delta}(g) I_\delta^{-1}$$

(cf.[13, Theorem 4.5]). This proves (1.4). Conversely assume (1.4) and choose orthonormal bases $e_1,\dots,e_{d_\delta}$ for $\mathcal{H}_\delta(\tau)$ and $f_1,\dots,f_{d_\delta}$ for $\mathcal{H}_\delta(\tau)$ such that

$$\delta_{ij}(k) := (\sigma(k)e_j,e_i) = (\tau(k)f_j,f_i), \quad k \in K.$$

In (1.4) replace g by $gk(g\in G, k\in K)$. Then we obtain:

$$\sum_{i,j=1}^{d_\delta} \delta_{ij}(k)(\sigma(g)e_i,e_j) = \sum_{i,j=1}^{d_\delta} \delta_{ij}(k)(\tau(g)f_i,f_j).$$

Hence $(\sigma(g)e_1,e_1) = (\tau(g)f_1,f_1)(g\in G)$ and $\sigma \simeq \tau$ by Lemma 1.4. □

The spaces $C(G)$ and, if G is a Lie group, $C_c^\infty(G)$ are algebras under convolution and, provided with the usual inductive limit topology, they become topological algebras. Consider the following closed subalgebras $I_c^\infty(G)$, $I_{c,\delta}(G)(\delta\in\hat{K})$ of $C_c(G)$ and $I_c^\infty(G)$, $I_{c,\delta}(G)(\delta\in\hat{K})$ of $C_c^\infty(G)$:

$$\text{(1.5)} \qquad I_c^{(\infty)}(G) := \{f \in C_c^{(\infty)}(G) \mid f(kgk^{-1}) = f(g),\ g \in G,\ k \in K\},$$

$$\text{(1.6)} \qquad I_{c,\delta}^{(\infty)}(G) := \{f \in I_c^{(\infty)}(G) \mid d_\delta \int_K f(gk)\chi_\delta(k)\,dk = f(g),\ g \in G,\ k \in K\}.$$

The present definition of $I^{(\infty)}_{c,\delta}(G)$ corresponds to the definition of $I^{(\infty)}_{c,\check{\delta}}$ ($\check{\delta}$ being contragredient to δ) in [13, p.43]. Here we follow the definition in WARNER [27,§4.5.1].

PROPOSITION 1.6. (cf. WARNER [27, Theor. 6.1.1.2, Prop. 6.1.1.6]). *Let* $\delta \in \hat{\hat{K}}$ *and let the algebra* $I_{c,\delta}(G)$ *be commutative. Let* τ *be an irr.* K*-unitary rep of* G *in which* δ *occurs with finite nonzero multiplicity* m_δ. *Then* $m_\delta = 1$ *and the linear functional*

$$f \mapsto d_\delta^{-1} \int_G f(g) \operatorname{tr} \tau_{\delta\delta}(g)\,dg \tag{1.7}$$

is a nonzero continuous character on $I_{c,\delta}(G)$. *Furthermore, this character completely determines* $\operatorname{tr}\tau_{\delta\delta}(.)$ *as a function on* G.

PROOF. By [3, Lemma 5.1] $m_\delta = 1$ iff the rep 1 of

$$K^* := \{(k,k) \in G \times K \mid k \in K\} \tag{1.8}$$

has multiplicity 1 in the rep $\tau \otimes \check{\delta}$ of $G \times K$. By restriction to $G \times \{e\}$ the algebra $C_c(K^*\backslash G{\times}K/K^*)$ is mapped isomorphically onto $I_c(G)$. Under this mapping the algebra

$$A := \{f \in C_c(K^*\backslash G{\times}K/K^*) \mid d_\delta \int_K f(g,k)\chi_\delta(k^{-1})\,dk = f(g,e), g \in G, k \in K\}$$

corresponds to $I_{c,\delta}(G)$.
For f in $C_c(G{\times}K)$ define $f^\#$ by

$$f^\#(g,k) := d_\delta \int_K \int_K \int_K f(k_1 g k_2, k_1 k k_3 k_2)\ \chi_\delta(k_3^{-1})\,dk_1\ dk_2\ dk_3,\ g \in G,\ k \in K.$$

Then $f \mapsto f^\#$ is a projection of $C_c(G{\times}K)$ onto A. Let P_1 be the orthogonal projection of $\mathcal{H}(\tau\otimes\check{\delta})$ onto $\mathcal{H}_1(\tau\otimes\check{\delta})$. One easily verifies that, for v in $\mathcal{H}_1(\tau\otimes\check{\delta})$, $f \in C_c(G{\times}K)$:

$$(\tau\otimes\check{\delta})(f^{\#})\ v = P_1(\tau\otimes\check{\delta})(f)\ v.$$

By irreducibility of τ, $(\tau\otimes\check{\delta})(C_c(G\times K))\ v$ is a dense subspace of $H(\tau\otimes\check{\delta})$. Hence, since $H_1(\tau\otimes\check{\delta})$ is finite dimensional,

$$(\tau\otimes\check{\delta})(A)\ v = P_1(\tau\otimes\check{\delta})(C_c(G\times K))\ v = H_1(\tau\otimes\check{\delta}).$$

Thus $\tau\otimes\check{\delta}$ is an irr. rep of the commutative algebra A on $H_1(\tau\otimes\check{\delta})$, so $H_1(\tau\otimes\check{\delta})$ has dimension 1.

For the proof of the second statement note that, for f in $C_c(G\times K)$ and v in $H_1(\tau\otimes\check{\delta})$ with $\|v\| = 1$, we have

$$\int_G\int_K f(g,k)((\tau\otimes\check{\delta})(g,k)v,v)dg\ dk = d_\delta^{-1}\int_G f^{\#}(g,e)\ \mathrm{tr}\ \tau_{\delta\delta}(g)dg.$$

Finally the third statement follows from the observation that

$$\int_G f(g)\ \mathrm{tr}\ \tau_{\delta\delta}(g)dg = d_\delta\int_G\Big[\int_K\int_K f(k_1gk_2k_1^{-1})\chi_\delta(k_2)dk_1\,dk_2\Big]\mathrm{tr}\tau_{\delta\delta}(g)dg,$$
$$f\in C_c(G). \qquad \square$$

The function $\mathrm{tr}\ \tau_{\delta\delta}(.)$ is called a *spherical trace function of type* δ. The theory of these functions goes back to GODEMENT [5].

COROLLARY 1.7. *Let* $\delta\in\hat{K}$ *and let* $I_{c,\delta}(G)$ *be commutative. Let* σ *and* τ *be irr. K-unitary reps of G in which* δ *has finite nonzero multiplicity. Then* $\sigma\approx\tau$ *iff the corresponding characters on* $I_{c,\delta}(G)$ *(or* $I^\infty_{c,\delta}(G)$ *if G is a Lie group) defined by* (1.7) *coincide.*

PROOF. Use Theorem 1.5 and Prop. 1.6. $\square$

The pair (G,K) is called a *Gelfand pair* if the algebra $C_c(K\backslash G(K)$ is commutative.

COROLLARY 1.8. *If* $(G\times K,K^*)$ *is a Gelfand pair then each irr. K-finite Hilbert rep of G is K-multiplicity free.*

PROOF. Use Prop. 1.6 and the correspondence between $C_c(K^*\backslash G*K/K^*)$ and $I_c(G)$. □

2. THE GENERALIZED ABEL TRANSFORM

Let us restrict attention now to the case that G is a noncompact connected real semisimple Lie group with finite center and that K is a maximal compact subgroup of G. Choose subgroups A and N of G such that G = KAN is an Iwasawa decomposition of G and let g in G be accordingly factorized as

$$g = u(g)e^{H(g)}n(g), \tag{2.1}$$

where $H(g) \in \mathfrak{a}$, the Lie algebra of A. Let M be the centralizer of A in K. For ξ in $\hat{M}$ and λ in $\mathfrak{a}_c^*$ (the complex linear dual of $\mathfrak{a}$) we define the *principal series rep* $\pi_{\xi,\lambda}$ as the rep of G induced by the rep

$$man \mapsto e^{\lambda(\log a)}\ \xi(m),\ m \in M,\ a \in A,\ n \in N,$$

of the subgroup MAN of G. Let $\mathfrak{n}$ be the Lie algebra of N and let ρ in $\mathfrak{a}$ be defined by $\rho(H) := \frac{1}{2}\mathrm{tr}(\mathrm{ad}H|_{\mathfrak{n}})$, $H \in \mathfrak{a}$. In the *compact picture* the rep $\pi_{\xi,\lambda}$ is realized on the Hilbert space $L^2_\xi(K;\mathcal{H}(\xi))$ consisting of all $\mathcal{H}(\xi)$ -valued L^2 - functions f on K such that $f(km) = \xi(m^{-1})f(k)$, $k \in K$, $m \in M$. Then

$$(\pi_{\xi,\lambda}(g)f)(K) = e^{-(\rho+\lambda)H(g^{-1}k)}f(u(g^{-1}k)), \tag{2.2}$$

$$g \in G,\ k \in K,\ f \in L^2_\xi(K;\mathcal{H}(\xi)).$$

The rep $\pi_{\xi,\lambda}$ is a K-finite Hilbert rep of G.

We would like to attempt a global approach to HARISH-CHANDRA's [7, Theorem 4], [8, Theorem 4] *subquotient theorem:*

THEOREM 2.1. *Every* K*-finite irr. Hilbert rep of* G *is Naimark equivalent to some irr. subquotient rep of some principal series rep.*

Choose a Haar measure dn on N. For f in $I^\infty_{c,\delta}(G)$ ($\delta\in\hat{K}$) define

(2.3) $$F_f(k,a) := e^{\rho(\log a)} \int_N f(kan)dn, \; k \in K, \; a \in A,$$

(2.4) $$F_f^\delta(a) := e^{\rho(\log a)} \int_K \int_N f(kan)\, \check{\delta}(k^{-1})dk\, dn = \int_K F_f(k,a)\, \check{\delta}\, (k^{-1})dk, \; a \in A,$$

the latter function F_f^δ being $L(H(\check{\delta}))$-valued. For some choice of an orthonormal basis of $H(\check{\delta})$, F_f can be expressed in terms of F_f^δ by

(2.5) $$F_f(k,a) = d_\delta \sum_{i,j=1}^{d_\delta} \check{\delta}_{ij}\, (k)(F_f^\infty(a))_{ji}, f \in I_{c,\delta}^\infty(G).$$

The transform $f \mapsto F_f$ or $f \mapsto F_f^\delta$ is called the *generalized Abel transform* (cf. WARNER [27, §6.2.2]).

From now on assume that δ is M-multiplicity free. Then $\check{\delta}$ is also M-multiplicity free. Let $M_M(\check{\delta})$ denote the M-content of $\check{\delta}$. Note that

$$F_f^\delta(a) = F_f^\delta(mam^{-1}) = \check{\delta}(m)F_f^\delta(a)\check{\delta}(m^{-1}), \; a \in A, \; m \in M.$$

Hence,

$$(F_f^\delta(a))_{\xi,\eta} = \xi(m)(F_f^\delta(a))_{\xi,\eta}\, \eta(m^{-1}), \; \xi, \; \eta \in M_M(\check{\delta})$$

(where $(F_f^\delta(a))_{\xi,\eta}$ is the matrix block of $F_f^\delta(a)$ corresponding to (ξ,η)). Since $\check{\delta}$ is M-multiplicity free, this implies

$$(F_f^\delta(a))_{\xi,\eta} = \begin{cases} 0 \text{ if } \xi \neq \eta, \\ F_{f,\xi}^\delta(a).\mathrm{id} \text{ if } \xi = \eta, \end{cases}$$

for certain functions $F_{f,\xi}^\delta(\xi \in M_M(\check{\delta}))$ in $C_c^\infty(A)$. Combining this with (2.5) we get

(2.6) $$F_f(k,a) = d_\delta \sum_{\xi \in M_M(\check{\delta})} \mathrm{tr}\, \check{\delta}_{\xi\xi}(k)F_{f,\xi}^\delta(a), \; f \in I_{c,\delta}^\infty(G).$$

It can easily be proved that for f in $I_{c,\delta}^\infty(G)$ we have a similar formula: there exist functions f_ξ in $C_c^\infty(A)$ such that

$$(2.7)\qquad f(k_1ak_2) = d_\delta \sum_{\xi\in M_M(\check{\delta})} \operatorname{tr} \check{\delta}_{\xi\xi}(k_2k_1)f_\xi(a),\ k_1,\ k_2 \in K,\ a \in A.$$

Note that f and $F_f(f\in I^\infty_{c,\delta}(G))$ are completely determined by their restrictions to $M \times A$.

Choose a Haar measure da on A and normalize the Haar measure dg on G such that

$$(2.8)\qquad \int_G f(g)dg = \int_{K\times A\times N} f(kan)e^{2\rho(\log a)}dk\, da\, dn,\ f \in C_c(G).$$

THEOREM 2.2. *Let* $\delta \in \hat{K}$ *such that* δ *is* M*-multiplicity free. Then the transform*

$$f \mapsto \{F^\delta_{f,\xi}\}_{\xi\in M_M(\check{\delta})} : I^\infty_{c,\delta} \to \mathbb{C}^\infty_c(A;\mathbb{C}^{|M_M(\delta)|})$$

has the following properties:

(i) *it is continuous;*

(ii) *it is injective if* G *is a linear Lie group;*

(iii) *it is an algebra homomorphism, i.e.*

$$(2.9)\qquad F^\delta_{f_1*f_2,\xi}(a) = \int_A F^\delta_{f_1,\xi}(a_1)F^\delta_{f_2,\xi}(a_1^{-1}a)da_1,\ \xi \in M_M(\check{\delta}),$$
$$f_1,\ f_2 \in I^\infty_{c,\delta}(G);$$

(iv) *for each* f *in* $I^\infty_{c,\delta}(G)$, ξ *in* $M_M(\delta)$ *and* λ *in* $\mathfrak{a}^*_c$ *we have*

$$(2.10)\qquad d_\delta^{-1}\int_G f(g)\operatorname{tr}\pi_{\xi,\lambda;\delta,\delta}(g)dg = \int_A F^\delta_{f,\check{\xi}}(a)e^{\lambda(\log a)}da;$$

(v) $I^\infty_{c,\delta}(G)$ *is a commutative algebra.*

PROOF. The proof of (i) is straightgorward. See [13, Theorem 5.17]for the proof of (ii) and WARNER [27, §6.2.2] for the proof of (iii). Combination of (ii) and (iii) proves (v). Let us prove (iv). From WALLACH [25, Lemma 8.3.11] we have

$$\pi_{\xi,\lambda}(g) = (\pi_{\xi,-\bar{\lambda}}(g^{-1}))^*,$$

hence

$$(*) \qquad \operatorname{tr} \pi_{\xi,\lambda;\delta,\delta}(g) = \overline{\operatorname{tr} \pi_{\xi,-\bar{\lambda};\delta,\delta}(g^{-1})}.$$

Choose an orthonormal basis $e_1,\dots,e_{d_\delta}$ of $H(\check{\delta})$ such that $e_1,\dots,e_{d_\xi}$ is an orthonormal basis of $H_{\check{\xi}}(\check{\delta})$. Realize $\pi_{\xi,\lambda}$ on the Hilbert space $H(\pi_{\xi,\lambda}) = L^2(K;H(\xi))$. Then the vector-valued functions $f_i (i=1,\dots,d_\delta)$ defined by

$$f_i(k) := \left(\frac{d_\delta}{d_\xi}\right)^{\frac{1}{2}} (\check{\delta}_{i1}(k),\dots,\check{\delta}_{id_\xi}(k)),\ k \in K,$$

form an orthonormal basis for $H_\delta(\pi_{\xi,\lambda})$. It follows from (2.2) that

$$\sum_{i=1}^{d_\delta} (\pi_{\xi,\lambda}(g)f_i,f_i) = \frac{d_\delta}{d_\xi} \sum_{i=1}^{d_\delta} \sum_{j=1}^{d_\xi} \int_K e^{-(\rho+\lambda)H(g^{-1}k)}\, \check{\delta}_{ij}(u(g^{-1}k))\, \overline{\check{\delta}_{ij}(k)}\,dk.$$

Hence

$$(2.11) \qquad \operatorname{tr} \pi_{\xi,\lambda;\delta,\delta}(g) = d_\delta d_\xi^{-1} \int_K e^{-(\rho+\lambda)H(g^{-1}k)} \operatorname{tr} \delta_{\xi\xi}((u(g^{-1}k))^{-1}k)\,dk.$$

Combination of (*) and (2.11) yields:

$$d_\delta^{-1} \int_G f(g) \operatorname{tr} \pi_{\xi,\lambda;\delta,\delta}(g)\,dg = d_\xi^{-1} \int_G \int_K f(g) e^{(\lambda-\rho)H(gk)} \operatorname{tr} \delta_{\xi\xi}(k^{-1}u(gk))\,dg\,dk.$$

Next make the transformation of variables $g \mapsto gk^{-1}$ and substitute (2.8) into the right hand side:

$$\text{LHS of (2.10)} = d_\xi^{-1} \int_K \int_A \int_N f(kan) \operatorname{tr} \delta_{\xi\xi}(k) e^{(\lambda+\rho)\log a} dk\,da\,dn.$$

In view of (2.4) this yields (2.10). □

COROLLARY 2.3. *If* $(K\times M,M^*)$ *is a Gelfand pair then* $(G\times K,K^*)$ *is a Gelfand pair,* $I^{\infty}_{c,\delta}(G)$ *is a commutative algebra and* δ *is* M*-multiplicity free for each* δ *in* $\hat{K}$*, and each irr.* K*-finite Hilbert rep of* G *is* K*-multiplicity free.*

PROOF. Use Cor. 1.8 and Theorem 2.2 (v). □

In the following examples $(K\times M,M^*)$ and (hence) $(G\times K,K^*)$ are Gelfand pairs:

G	K	M
$SL(2,\mathbb{R})$	$SO(2)$	$O(1)$
$SL(2,\mathbb{C})$	$SU(2)$	$U(1)$
$SO(n,1)$	$SO(n)$	$SO(n-1)$
$SU^{\circ}(n,1)$	$U(n)$	$U(n-1)$

Note that the cases $G = SL(2,\mathbb{R}), SO_{\circ}(2,1), SU(1,1)$ are locally isomorphic and also $G = SL(2,\mathbb{C}), SO_{\circ}(3,1)$. Since, under the assumption that $(G\times K,K^*)$ is a Gelfand pair, each finite-dimensional irr. rep of G is K-multiplicity free, it follows from KRÄMER [17] that almost all cases with $(K\times M,M^*)$ being a simple linear Lie group occur is the above table.

We can now formulate the program for a global approach to Harish-Chandra's subquotient theorem:

(a) Let $\delta \in \hat{K}$ such that δ is M-multiplicity free. Let A_δ be the image of $I^{\infty}_{c,\delta}(G)$ under $f \mapsto \{F^{\delta}_{f,\xi}\}$ and provide A_δ with the topology which makes this transform an homeomorphism. Determine A_δ completely, also topologically, and prove that each continuous character on A_δ has the form

$$F \mapsto \int_A F_{\xi}^{\vee}(a)e^{\lambda(\log a)}da$$

for some ξ in $M_M(\delta)$, λ in $\mathfrak{a}^*_c$. Then in view of (2.10) and Cor. 1.7, we conclude that each irr. K-unitary rep τ in which δ has finite nonzero multiplicity is Naimark related to an irr. subquotient rep (namely the one containing δ) of some $\pi_{\xi,\lambda}$.

(b) In particular, study $f|_{MA} \mapsto F_f|_{M\times A}$ $(f \in I^\infty_{c,\delta}(G))$ as a "classical" integral transform (i.e., as an integral transform given in analytic form without group variables) and determine its inversion formula.

Clearly, δ is M-multiplicity free if $\delta = 1$ (the spherical case). Then $I^\infty_{c,\delta}(G) = C^\infty_c(K\backslash G/K)$. Its image under the generalized Abel transform is known by GANGOLLI's [3] Paley-Wiener theorem: the space of all Weyl group invariant C^∞-functions on A with compact support. However, part (b) of the above program in the spherical case has been done only in the rank one case. Then $f \mapsto F_f$ can be written as a Weyl type fractional integral transform or a composition of two such transforms (cf. KOORNWINDER [11]).

The above programs has been completed for all δ in the case $G = SL(2,\mathbb{R})$ (cf. KOORNWINDER [13], [14], TAKAHASHI [20]). Then $A = \left\{a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\right\}$,

$$K = \left\{u_\theta := \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\right\}.$$

$\hat{K}$ consists of the reps $\delta_n (n \in \mathbb{Z})$, where $\delta_n(u_\theta) := e^{in\theta}$. In considering the transform $f \mapsto F_f$ for f in $I^\infty_{c,\delta_n}(G)$, we can restrict f and F_f to A and we obtain

$$(2.12) \qquad F_f(a_t) = \int_t^\infty f(a_w) T_{|n|}\left(\frac{\operatorname{ch} t}{\operatorname{ch} w}\right)\left(\operatorname{ch}^2 w - \operatorname{ch}^2 t\right)^{-\frac{1}{2}} \operatorname{sh} 2w \, dw,$$

where $T_{|n|}$ is the *Chebyshev polynomial* of degree $|n|$:

$$(2.13) \qquad T_n(\cos\theta) := \cos n\,\theta.$$

The inversion formula to (2.12) is

$$(2.14) \qquad f(a_w) = -\pi^{-1} \int_w^\infty F_f'(a_t) T_{|n|}\left(\frac{\operatorname{ch} t}{\operatorname{ch} w}\right)\left(\operatorname{ch}^2 t - \operatorname{ch}^2 w\right)^{-\frac{1}{2}} dw.$$

The correspondences $f \leftrightarrow f|_A \leftrightarrow F_f$ identify the spaces $I^\infty_{c,\delta_n}(G)$, $\mathcal{D}_{\text{even}}(\mathbb{R})$ and $\mathcal{D}_{\text{even}}(\mathbb{R})$, respectively, with each other. Note that in the spherical case (n=0) the pair (2.12), (2.14) becomes the classical Abel transform together with its inversion formula (cf. GODEMENT [6]). There are three different proofs that (2.14) is the inversion formula to (2.12).

(i) by Mellin transform techniques (cf. MATSUSHITA [18]);

(ii) by specializing the inversion formula for the Euclidean Radon transform on $\mathbb{R}^2$ to functions which behave according to rep δ_n of SO(2)(cf. DEANS [2]);

(iii) by using generalized fractional integrals (cf. KOORNWINDER [13, §5.9]).

3. SL(2,$\mathbb{C}$), DISCUSSION OF EARLIER RESULTS

Let us now try to deal with the generalized Abel transform for SL(2,$\mathbb{C}$) in the same spirit as for SL(2,$\mathbb{R}$) above. A global approach to the representation theory of SL(2,$\mathbb{C}$) can already be found in NAIMARK [19]. He determined irreducibility properties of principal series reps by using Theorem 1.2 (cf. [19, Ch. 3, §9, no. 15]) and he used the generalized Abel transform for proving the subquotient theorem 2.1 (ibidem, no. 16). (In fact, he considered the generalized Abel transform not on the algebras $I^{\infty}_{c,\delta}(G)$ but on certain algebras denoted by X^k_j which are isomorphic to them (ibidem, no. 6).) However, there are certain unsatisfactory points in his approach: (i) the formula for the generalized Abel transform is not very explicit (cf. ibidem, no. 10, formula (1)), with integration variables defined in an implicit way; (ii) the inversion formula (ibidem, no. 10, formula (9)) is derived by using the Plancherel formula; (iii) the image under the generalized Abel transform is not completely characterized (ibidem, no. 10, IV) but a subalgebra of the image is obtained which is big enough to prove that the characters on it have the desired form. See BRUMMELHUIS [1] for a more detailed discussion of Naimark's approach.

KOSTERS [16] studied irreducibility, Naimark equivalence and unitarizability for subquotients of the principal series of SL(2,$\mathbb{C}$) by using the global methods developed in [13], but he did not give a global approach to the subquotient theorem 2.1.

Finally, a helpful reference to us was WANG [26], who derived a Paley-Wiener theorem characterizing the image of the algebras $I^{\infty}_{c,\delta}(SL(2,\mathbb{C}))$ under the group Fourier transform. Indeed, his result is equivalent to characterizing the image under the generalized Abel transform.

4. SL(2,$\mathbb{C}$), PRELIMINARIES

Let us fix an Iwasawa decomposition G = KAN for G = SL(2,$\mathbb{C}$) with

$$(4.1)\qquad K = SU(2) = \left\{k_{\alpha,\beta} := \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \mid \alpha,\beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\right\},$$

$$(4.2)\qquad A = \left\{a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R}\right\},$$

$$(4.3)\qquad N = \left\{n_{x+iy} := \begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix} \mid x,y \in \mathbb{R}\right\}.$$

Then

$$(4.4)\qquad M = \left\{m_\phi := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \mid 0 \le \phi < 2\pi\right\}.$$

We will also use special elements of K given by

$$(4.5)\qquad u_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

$\hat{K}$ consists of the reps T^ℓ ($\ell=0,\frac12,1,\ldots$) of dimension $2\ell+1$. A model for the representation space of T^ℓ is given by the space H_ℓ of homogeneous polynomials of degree 2ℓ in two complex variables with orthonormal basis consisting of the functions ψ^ℓ_n ($n=-\ell,-\ell+1,\ldots,\ell$):

$$(4.6)\qquad \psi^\ell_n(x,y) := \binom{2\ell}{\ell-n}^{\frac12} x^{\ell-n}\, y^{\ell+n}.$$

Then

$$(4.7)\qquad (T^\ell(k_{\alpha,\beta})f)(x,y) := f(\alpha x-\bar\beta y, \beta x+\bar\alpha y)$$

defines an irr. unitary rep of K on H_ℓ. Note that the orthonormal basis is an M-basis:

$$(4.8)\qquad T^\ell(m_\phi)\,\psi^\ell_n = e^{-2in\phi}\,\psi^\ell_n.$$

Let

(4.9) $$t^{\ell}_{mn}(k) := (T^{\ell}(k)\psi^{\ell}_{n}, \psi^{\ell}_{m})$$

denote the matrix elements of $T^{\ell}(k)$ with respect to this basis. From (4.6), (4.7) and (4.9) one obtains a generating function for these matrix elements:

(4.10) $$\binom{2\ell}{\ell-n}^{\frac{1}{2}} (\alpha x-\bar{\beta}y)^{\ell-n}(\beta x+\bar{\alpha}y)^{\ell+n} = \sum_{m=-\ell}^{\ell} t^{\ell}_{mn}\left(k_{\alpha,\beta}\right)\binom{2\ell}{\ell-m}^{\frac{1}{2}} x^{\ell-m}y^{\ell+m}.$$

From this one can obtain an explicit expression for $t^{\ell}_{mn}(k)$ in terms of Jacobi polynomials A more detailed account of the representation theory of SU(2) is, for instance, given in VILENKIN [24, Ch. III].

We will need two special functions associated with the reps T_{ℓ}. First, for the character χ_{ℓ} of T_{ℓ} we have

(4.11) $$\chi_{\ell}\left(k_{\alpha,\beta}\right) = U_{2\ell}(\mathrm{Re}\ \alpha),$$

where

(4.12) $$U_{n}(\cos\phi) := \frac{\sin(n+1)\phi}{\sin\phi}$$

is the *Chebyshev polynomial of the second kind*. Next, for the diagonal matrix element $t^{\ell}_{jj}(k)$ we have:

(4.13) $$t^{\ell}_{jj}\left(k_{\alpha,\beta}\right) = R_{\ell-j,\ell+j}(\bar{\alpha}),$$

where

(4.14) $$R_{m,n}(r\,e^{i\phi}) := P^{(0,|m-n|)}_{m\wedge n}(2r^{2}-1)r^{|m-n|}e^{i(m-n)\phi}$$

and $P^{(\alpha,\beta)}_{n}(x)$ denotes a Jacobi polynomial. $R_{m,n}(x+iy)$ is an orthogonal polynomial in the two variables x,y, a so-called *disk polynomial*. It can be completely characterized by the three conditions

$$(4.15)\qquad \begin{cases} R_{m,n}(x+iy) = c.(x+iy)^m(x-iy)^n + \text{polynomial of degree less than } m+n; \\ \displaystyle\iint_{x^2+y^2<1} R_{m,n}(x+iy)x^p y^q\,dx\,dy = 0 \text{ if } p+q < m+n; \\ R_{m,n}(1) = 1. \end{cases}$$

See KOORNWINDER [12]. It follows from (4.11) and (4.13) that

$$(4.16)\qquad U_{2\ell}(\mathrm{Re}\,\alpha) = \sum_{j=-\ell}^{\ell} R_{\ell-j,\ell+j}(\bar{\alpha}).$$

$\hat{M}$ consists of the reps ξ_j $(j\in\frac{1}{2}\mathbb{Z})$ defined by

$$(4.17)\qquad \xi_j(m_\phi) := e^{-2ij\phi}.$$

Let $\pi_{j,\lambda}$ $(j\in\frac{1}{2}\mathbb{Z},\lambda\in\mathbb{C})$ denote the principal series rep of G induced by the rep

$$m_\phi a_t n_{x+iy} \mapsto e^{-2ij\phi}e^{2\lambda t}$$

of MAN. Since T_ℓ has M-content $\{\xi_j \mid j=-\ell,-\ell+1,\ldots,\ell\}$, we obtain by Frobenius reciprocity that $\pi_{j,\lambda}$ has K-content $\{T_\ell \mid \ell=|j|,|j|+1,\ldots\}$.

For ρ we obtain:

$$(4.18)\qquad \rho(\log a_t) = 2t.$$

5. THE GENERALIZED ABEL TRANSFORM FOR SL(2,$\mathbb{C}$)

Write $I^\infty_{c,\ell}(G)$ instead of $I^\infty_{c,T_\ell}(G)$, $d_\ell := 2\ell+1$. Note that $\check{T}_\ell = T_\ell$. Normalize the Haar measure on N by $dn_{x+iy} := (2\pi)^{-1}dx\,dy$. Specialization of (2.3) to $G = SL(2,\mathbb{C})$ yields

$$(5.1)\qquad F_f(k,a_t) = (2\pi)^{-1}e^{2t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ka_t n_{x+iy})\,dx\,dy,\quad f\in I^\infty_{c,\ell}(G).$$

Formulas (2.7) and (2.6) can be written as

$$(5.2)\qquad f(k_1 a_w k_2) = d_\ell \sum_{m=-\ell}^{\ell} t^\ell_{mm}(k_2 k_1) f_m(w),$$

$$(5.3)\qquad F_f(k,a_t) = d_\ell \sum_{j=-\ell}^{\ell} t^\ell_{jj}(k) F^\ell_{f,j}(t),$$

for f in $I^\infty_{c,\ell}(G)$. From now on we fix ℓ and Σ_m will mean a sum with m running through the set $\{-\ell,-\ell+1,\dots,\ell\}$. Let us use the notation

$$f(\phi,w) := f(m_\phi\, a_w),$$

$$F(\tau,t) := F_f(m_\tau,a_t),$$

$$F_j(t) := F^\ell_{f,j}(t).$$

Then

$$(5.4)\qquad f(\phi,w) = d_\ell \sum_m e^{-2im\phi} f_m(w),$$

$$(5.5)\qquad F(\tau,t) = d_\ell \sum_j e^{-2ij\tau} F_j(t).$$

Note that, for $f \in I^\infty_{c,\ell}(G)$, the f_m's and F_j's are in $C^\infty_c(\mathbb{R})$. The function f satisfies an obvious symmetry because of the Weyl group action:

$$(5.6)\qquad f(\phi,w) = f(-\phi,-w).$$

Indeed, if $f \in I^\infty_c(G)$ then

$$f(m_\phi a_w) = f(u_{\pi/2} m_\phi a_w u_{-\pi/2}) = f(m_{-\phi} a_{-w}).$$

Next we want to rewrite (5.1) as a "classical" integral transform. An intermediate stage (essentially the same as in NAIMARK [19, Ch. 3, §9, no. 10]) is as follows:

<u>LEMMA 5.1.</u> *If* $f \in I^\infty_{c,\ell}(G)$ *then*

$$(5.7)\qquad F_j(t) = e^{2t} \int_0^\infty \sum_m f_m(w)\, t^\ell_{jm}(u_{\theta_1})\, t^\ell_{mj}(u_{-\theta_2})\, z\,dz,$$

where, for given z *and* t, w, θ_1 *and* θ_2 *are such that*

(5.8) $$a_t n_z = u_{\theta_1} a_w u_{-\theta_2}.$$

<u>PROOF.</u>

$$\begin{aligned}
F_j(t) &= \int_K F_f(k,a_t)\, t^{\ell}_{jj}(k^{-1})\,dk \\
&= (2\pi)^{-1} e^{2t} \int_K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(ka_t n_{x+iy})\, t^{\ell}_{jj}(k^{-1})\,dk\,dx\,dy \\
&= (2\pi)^{-1} e^{2t} \int_K \int_0^{\infty} \int_0^{2\pi} f(m_{-\psi/2} k m_{\psi/2} a_t n_z)\, t^{\ell}_{jj}(k^{-1})\,dk\,zdz\,d\psi \\
&= e^{2t} \int_K \int_0^{\infty} f(ka_t n_z)\, t^{\ell}_{jj}(k^{-1})\,dk\,zdz \\
&= e^{2t} \int_K \int_0^{\infty} f(u_{-\theta_2} k u_{\theta_1} a_w)\, t^{\ell}_{jj}(k^{-1})\,dk\,zdz \\
&= d_{\ell} e^{2t} \int_K \int_0^{\infty} \sum_m f_m(w)\, t^{\ell}_{mm}(u_{-\theta_2} k u_{\theta_1})\, t^{\ell}_{jj}(k^{-1})\,dk\,zdz \\
&= e^{2t} \int_0^{\infty} \sum_m f_m(w)\, t^{\ell}_{jm}(u_{\theta_1})\, t^{\ell}_{mj}(u_{-\theta_2})\,zdz. \qquad \square
\end{aligned}$$

A straightforward calculation shows that (5.8) is equivalent to

(5.9) $$\begin{cases} \cos(\theta_1-\theta_2)\ \mathrm{ch}\ w = \mathrm{ch}\ t, \\ \sin(\theta_1-\theta_2)\ \mathrm{ch}\ w = -\tfrac{1}{2} z\, e^t, \\ \cos(\theta_1+\theta_2)\ \mathrm{sh}\ w = \mathrm{sh}\ t, \\ \sin(\theta_1+\theta_2)\ \mathrm{sh}\ w = \tfrac{1}{2} z\, e^t, \end{cases}$$

and that it implies

$$(5.10)\qquad \mathrm{ch}2w = \mathrm{ch}2t + \tfrac{1}{2}e^{2t}z^2.$$

The final version of our Abel transform is given in the following theorem:

THEOREM 5.2. *If* $f \in I^{\infty}_{c,\ell}(G)$ *then*

$$(5.11)\qquad F(\tau,t) = (2\pi)^{-1}\int_0^{2\pi}\int_t^{\infty} f(\phi,w).$$
$$U_{2\ell}\left(\frac{\mathrm{ch}\, t}{\mathrm{ch}\, w}\cos\phi\,\cos\tau + \frac{\mathrm{sh}\, t}{\mathrm{sh}\, w}\sin\phi\,\sin\tau\right) 2\mathrm{sh}2w\, d\phi\, dw$$

and

$$(5.12)\qquad F(\tau,t) = F(-\tau,-t).$$

PROOF. By (5.7), (5.4) and (4.8) we obtain:

$$F_j(t) = \frac{e^{2t}}{2\pi d_\ell}\int_0^{2\pi}\int_0^{\infty} f(\phi,w)\sum_m t^{\ell}_{jm}(u_{\theta_1})t^{\ell}_{mm}(m_{-\phi})t^{\ell}_{mj}(u_{-\theta_2})d\phi\, zdz$$
$$= \frac{e^{2t}}{2\pi d_\ell}\int_0^{2\pi}\int_0^{\infty} f(\phi,w)t^{\ell}_{jj}(u_{\theta_1}m_{-\phi}u_{-\theta_2})d\phi\, zdz.$$

Now, by (5.9), we have

$$u_{\theta_1}m_{-\phi}u_{-\theta_2} = k_{\alpha,\beta}$$

with

$$\alpha = \cos(\theta_1-\theta_2)\cos\phi - i\cos(\theta_1+\theta_2)\sin\phi =$$
$$= \frac{\mathrm{ch}\, t}{\mathrm{ch}\, w}\cos\phi - i\,\frac{\mathrm{sh}\, t}{\mathrm{sh}\, w}\sin\phi.$$

Hence, by using (5.5), (4.11) and (5.10) it follows that

$$F(\tau,t) = e^{2t}(2\pi)^{-1}\int_0^{2\pi}\int_0^{\infty} f(\phi,w).$$

$$\cdot U_{2\ell}\left(\frac{\mathrm{ch}\ t}{\mathrm{ch}\ w}\cos\phi\ \cos\tau + \frac{\mathrm{sh}\ t}{\mathrm{sh}\ w}\sin\phi\ \sin\tau\right) d\phi\ zdz =$$

$$= (2\pi)^{-1}\int_0^{2\pi}\int_{|t|}^{\infty} f(\phi,w).$$

$$\cdot\ U_{2\ell}\left(\frac{\mathrm{ch}\ t}{\mathrm{ch}\ w}\cos\phi\ \cos\tau + \frac{\mathrm{sh}\ t}{\mathrm{sh}\ w}\sin\phi\ \sin\tau\right) 2\mathrm{sh}2w\ d\phi\ dw.$$

This shows (5.12) and also (5.11) for $t \geq 0$. Finally, (5.11) holds for $t < 0$ because the right hand side of (5.11) with $\int_t^\infty$ replaced by $\int_{-t}^{t}$ $(t>0)$ equals 0 (use (5.6)). □

Because of (4.11) and (4.13) we have the following two variants of (5.11).

$$(5.13)\qquad F_j(\tau,t) = \frac{1}{2\pi d_\ell}\int_0^{2\pi}\int_t^{\infty} f(\phi,w).$$

$$\cdot R_{\ell-j,\ell+j}\left(\frac{\mathrm{ch}\ t}{\mathrm{ch}\ w}\cos\phi + i\frac{\mathrm{sh}\ t}{\mathrm{sh}\ w}\sin\phi\right) 2\mathrm{sh}2w\ d\phi\ dw,$$

$$(5.14)\qquad F(\tau,t) = d_\ell \int_t^\infty \sum_m f_m(w).$$

$$\cdot R_{\ell+m,\ell-m}\left(\frac{\mathrm{ch}\ t}{\mathrm{ch}\ w}\cos\tau + i\frac{\mathrm{sh}\ t}{\mathrm{sh}\ w}\sin\tau\right) 2\mathrm{sh}2w\ dw.$$

For a function F of the form (5.5) $(F_j \in C_c^\infty(\mathbb{R}))$ define

$$(5.15)\qquad \hat{F}(2j,2\lambda) := \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} F(\tau,t)e^{2ij\tau}e^{-2\lambda t}d\tau\ dt,$$

$$j \in \{-\ell,-\ell+1,\ldots,\ell\},\ \lambda \in \mathbb{C}.$$

Then, by Fourier inversion:

$$(5.16)\qquad F(\tau,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_j \hat{F}(2j,2i\lambda)e^{-2ij\tau}e^{2i\lambda t}d\lambda.$$

Normalize the Haar measure on A by $da_t := dt$ and the Haar measure dg on G by (2.8). It follows by specialization of (2.10) that

$$d_\ell^{-1}\int_G f(g)\mathrm{tr}\ \pi_{j,\lambda;\ell,\ell}(g)dg = \int_{-\infty}^{\infty} F_{-j}(t)e^{2\lambda t}dt,\ f \in I_{c,\ell}^\infty(G),$$

where we wrote $\pi_{j,\lambda;\ell,\ell}$ instead of $\pi_{j,\lambda;T_\ell,T_\ell}$. Hence, by combination with (5.5), (5.15) and (5.12):

$$\int_G f(g)\,\mathrm{tr}\,\pi_{j,\lambda;\ell,\ell}(g)dg = \hat{F}(2j,2\lambda). \tag{5.17}$$

Formulas (5.17) and (5.11) together will yield an integral representation for $\pi_{j,\lambda;\ell,\ell}|_{M\times A}$. We need a few preparations.

First observe from (HELGASON [9, Prop. X. 1.17] that, for $G = SL(2,\mathbb{C})$, the left hand side of (2.8) equals

$$c\int_0^\infty \int_K \int_K f(k_1a_wk_2)\,\mathrm{sh}^2 2w\; dw\; dk_1\; dk_2$$

for some positive constant c. It follows easily that:

<u>LEMMA 5.3.</u> *Let* $f_1 \in C_c(G)$, $f_2 \in C(G)$ *and let both functions have the form* (5.2). *Then*

$$\int_G f_1(g)f_2(g)dg = \int_G f_1(g)f_2(g^{-1})dg = \tag{5.18}$$
$$= \frac{c}{2\pi d_\ell} \int_0^{2\pi}\int_0^\infty f_1(m_\phi a_w)f_2(m_\phi a_w)\,\mathrm{sh}^2 2w\; d\phi\; dw.$$

By a closer look at the Cartan decomposition $G = KAK$ (cf. HELGASON [10, Ch. IX, §1]) we obtain:

<u>LEMMA 5.4.</u> *Let* $f_{-\ell}, f_{-\ell+1}, \ldots, f_\ell$ *be* C^∞*-functions on* $\mathbb{R}$ *with compact support included in* $(0,\infty)$. *Then* (5.2) *with* $k_1, k_2 \in K$, $w \geq 0$ *unambiguously defines a function* f *in* $I^\infty_{c,\ell}(G)$.

Apply (5.18) to (5.17) and use (5.12) and (5.11):

$$\frac{c}{2\pi d_\ell}\int_0^{2\pi}\int_0^\infty f(\phi,w)\,\mathrm{tr}\;\pi_{j,\lambda;\ell,\ell}(a_wm_\phi)\,\mathrm{sh}^2 2w\; d\phi\; dw =$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty F(\tau,t)(e^{2ij\tau}e^{-2\lambda t}+2^{-2ij\tau}e^{2\lambda t})d\tau\; dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} f(\phi,w)\left[\frac{1}{2\pi}\int_0^{2\pi}\int_0^{w} (e^{2ij\tau}e^{-2\lambda t}+e^{-2ij\tau}e^{2\lambda t}).\right.$$

$$\left. .U_{2\ell}\left(\frac{\text{ch } t}{\text{ch } w}\cos\phi\cdot\cos\tau + \frac{\text{sh } t}{\text{sh } w}\sin\phi\,\sin\tau\right)d\tau\, dt\right]2\text{sh } 2w\, d\phi\, dw.$$

The first and the last member of the above equalities are equal to each other for all f in $I^{\infty}_{c,\ell}(G)$. Hence, in view of Lemma 5.4, the expression in square brackets in the last member will be equal to

$$\frac{c\ \text{sh}2w}{2d_\ell}\ \text{tr}\ \pi_{j,\lambda;\ell,\ell}(a_w m_\phi).$$

Divide both sides by sh2w and put $w = \phi = 0$. We get $c = 2$. Thus we have derived:

<u>LEMMA 5.5.</u> *If* $f \in C_c(G)$ *then*

$$(5.19)\qquad \int_G f(g)dg := \frac{1}{2\pi}\int_K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ka_t n_{x+iy})e^{2t}dk\ dt\ dx\ dy =$$

$$= 2\int_0^{\infty}\int_K\int_K f(k_1 a_2 k_2)\ \text{sh}^2 2w\ dw\ dk_1\ dk_2.$$

<u>THEOREM 5.6.</u> *(integral representation).*

$$(5.20)\qquad d_\ell^{-1}\text{tr}\ \pi_{j,\lambda;\ell,\ell}(m_\phi a_w) = \frac{1}{2\pi\text{sh}2w}\int_0^{2\pi}\int_0^{w} (e^{2ij\tau}e^{-2\lambda t}+e^{-2ij\tau}e^{2\lambda t}).$$

$$.U_{2\ell}\left(\frac{\text{ch } t}{\text{ch } w}\cos\phi\ \cos\tau + \frac{\text{sh } t}{\text{sh } w}\sin\phi\ \sin\tau\right)d\tau\ dt.$$

6. THE INVERSION FORMULA

In order to invert the transformation $f \to F$ given by (5.11) we will first express f(e) (f $I^{\infty}_{c,\ell}(G)$) in term . Let

$$(6.1)\qquad a_m(\lambda) := \int_0^1 P^{(0,2|m|)}_{\ell-|m|}(2r^2-1)r^{2|m|+\lambda+1}dr,$$

$$(6.2)\qquad b_m(\lambda,\mu) := \int_0^{\frac{1}{2}\pi} e^{2im\psi}(\cos\psi)^{\lambda}(\sin\psi)^{\mu}d\psi,$$

where $P_n^{(\alpha,\beta)}$ denotes a Jacobi polynomial.

LEMMA 6.1. *If* $f \in I_{c,\ell}^{\infty}(G)$, $\mathrm{Re}\lambda$, $\mathrm{Re}\mu > -1$ *then*

$$(6.3)\qquad \int_0^\infty\int_0^{\frac{1}{2}\pi} F(\tau,t)(\mathrm{ch}\ t\ \cos\tau)^{\lambda}(\mathrm{sh}\ t\ \sin\tau)^{\mu}(\mathrm{ch}^2 t-\cos^2\tau)\,dt\,d\tau =$$

$$= 4d_\ell \sum_m a_m(\lambda+\mu)b_m(\lambda,\mu)\int_0^\infty f_m(w)(\mathrm{ch}\ w)^{\lambda+2}(\mathrm{sh}\ w)^{\mu+2}dw.$$

PROOF. It follows from (5.14) that the left hand side of (6.3) equals

$$4d_\ell \int_0^\infty \sum_m c_m(\lambda,\mu) f_m(w)(\mathrm{ch}\ w)^{\lambda+2}(\mathrm{sh}\ w)^{\mu+2}dw,$$

where

$$c_m(\lambda,\mu) := \int_0^w\int_0^{\frac{1}{2}\pi} R_{\ell+m,\ell-m}\left(\frac{\mathrm{ch}\ t}{\mathrm{ch}\ w}\cos\tau + i\frac{\mathrm{sh}\ t}{\mathrm{sh}\ w}\sin\tau\right).$$

$$\cdot\left(\frac{\mathrm{ch}\ t\ \cos\tau}{\mathrm{ch}\ w}\right)^{\lambda}\left(\frac{\mathrm{sh}\ t\ \sin\tau}{\mathrm{sh}\,w}\right)^{\mu}\frac{(\mathrm{ch}^2 t-\cos^2 t)}{\mathrm{ch}\ w\ \mathrm{sh}\ w}\,dt\,d\tau =$$

$$= \iint_{\substack{x^2+y^2<1\\ x,y>0}} R_{\ell+m,\ell-m}(x+iy)x^{\lambda}y^{\mu}dx\,dy = a_m(\lambda+\mu)b_m(\lambda,\mu)$$

by (4.14). □

There is some similarity of formula (6.3) with the formula in MATSUSHITA [18, p. 115] which is obtained by taking Mellin transforms at both sides of (2.12). Matsushita could invert his formula and thus, by taking inverse Mellin transforms, obtain (2.14). We did not succeed in inverting (6.3). However, we can prove:

PROPOSITION 6.2. *If* $f \in I_{c,\ell}^{\infty}(G)$ *then*

$$(6.4)\qquad \left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(0,0) = -4f(0,0).$$

PROOF. By integration by parts and application of (6.3) we obtain:

$$\int_0^\infty\int_0^{\frac12\pi}\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(\tau,t)(\mathrm{ch}\ t\ \cos\tau)^\lambda(\mathrm{sh}\ t\ \sin\tau)^\mu dt\ d\tau =$$

$$= 4d_\ell\int_0^\infty\int_0^{\frac12\pi} F(\tau,t)[\lambda(\lambda-1)(\mathrm{ch}\ t\ \cos\tau)^{\lambda-2}(\mathrm{sh}\ t\ \sin\tau)^\mu +$$

$$+ \mu(\mu-1)(\mathrm{ch}\ t\ \cos\tau)^\lambda(\mathrm{sh}\ t\ \sin\tau)^{\mu-2}](\mathrm{ch}^2 t-\cos^2\tau)dt\ d\tau =$$

$$= \sum_m a_m(\lambda+\mu-2)[\mu(\mu-1)b_m(\lambda,\mu-2)\int_0^\infty f_m(w)(\mathrm{ch}\ w)^{\lambda+2}(\mathrm{sh}\ w)^\mu dw +$$

$$+ \lambda(\lambda-1)b_m(\lambda-2,\mu)\int_0^\infty f_m(w)(\mathrm{ch}\ w)^\lambda(\mathrm{sh}\ w)^{\mu+2}dw] =$$

$$= 4d_\ell\sum_m a_m(\lambda+\mu-2)[(((\lambda-1)\ -4m^2)b_m(\lambda-2,\mu)\ +$$

$$-2im(\lambda-2)b_m(\lambda-3,\mu+1))\int_0^\infty f_m(w)(\mathrm{ch}\ w)^{\lambda+2}(\mathrm{sh}\ w)^\mu dw +$$

$$+ \lambda(\lambda-1)b_m(\lambda-2,\mu)\int_0^\infty f_m(w)(\mathrm{ch}\ w)^\lambda(\mathrm{sh}\ w)^{\mu+2}\ dw].$$

By analytic continuation in λ,μ, the first member of these equalities equals the last member for Re $\lambda > 2$, Re $\mu > -1$. Now let f have support inside $[0,2\pi]\times[-M,M]$, then the same holds for $\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F$. Divide the first and last member by

$$\int_0^M\int_0^{\frac12\pi}(\mathrm{sh}\ t\ \sin\tau)^\mu dt\ d\tau$$

and let $\mu \downarrow -1$. Then we obtain :

$$\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(0,0) = -4d_\ell\sum_m a_m(\lambda-3)(-\lambda+1-4m^2)f_m(0). \tag{6.5}$$

Let $\lambda\to\infty$ in this identity. Then

$$\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(0,0) = -4d_\ell\sum_m f_m(0) = -4f(0,0). \qquad \square$$

REMARK 6.3. It is evident from (6.5) that a system of functions $\{f_m\}$ determined by f in $I^\infty_{c,\ell}(G)$ must satisfy certain additional conditions at 0. In Prop. 9.1 we will make a more precise statement about this.

<u>PROPOSITION 6.4.</u> *If* $f_1, f_2 \in I^{\infty}_{c,\ell}(G)$ *then*

$$(6.6)\qquad \int_0^{2\pi}\int_0^{\infty} f_1(\phi,w)f_2(\phi,w)\,\mathrm{sh}^2 2w\,d\phi\,dw = -\frac{1}{4}\int_0^{2\pi}\int_0^{\infty} F_1(\tau,t)\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F_2(\tau,t)\,d\tau\,dt.$$

<u>PROOF.</u> First observe that (5.18) (with c=2) yields:

$$\frac{1}{\pi d_\ell}\int_0^{2\pi}\int_0^{\infty} f_1(\phi,w)f_2(\phi,w)\,\mathrm{sh}^2 2w\,d\phi\,dw = (f_1 * f_2)(e).$$

Next, by (2.9) and (5.5):

$$F^{\ell}_{f_1*f_2,j}(t) = \int_{-\infty}^{\infty} F^{\ell}_{f_1,j}(t_1)F^{\ell}_{f_2,j}(t-t_1)\,dt_1,$$

$$F_{f_1*f_2}(m_\tau a_t) = \frac{1}{2\pi d_\ell}\int_0^{2\pi}\int_{-\infty}^{\infty} F_{f_1}(m_{\tau_1}a_{t_1})F_{f_2}(m_{\tau-\tau_1}a_{t-t_1})\,d\tau_1\,dt_1.$$

Thus, by (6.4) and (5.12):

$$(f_1*f_2)(e) = -\frac{1}{4}\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F_{f_1*f_2}(m_\tau a_t)\,|_{\tau=t=0} =$$

$$= -\frac{1}{8\pi d_\ell}\int_0^{2\pi}\int_{-\infty}^{\infty} F_1(\tau_1,t_1)\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial t_1^2}\right)F_2(\tau_1,t_1)\,d\tau_1\,dt_1.$$

A second application of (5.12) yields (6.6). □

<u>THEOREM 6.5</u> (inversion formula).
If $f \in I^{\infty}_{c,\ell}(G)$ *then*

$$(6.7)\qquad f(\phi,w) = \frac{-1}{2\mathrm{sh}2w}\,\frac{1}{2\pi}\int_0^{2\pi}\int_0^{w}\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(\tau,t)\,\cdot U_{2\ell}\left(\frac{\mathrm{ch}\,t}{\mathrm{ch}\,w}\cos\phi\,\cos\tau+\frac{\mathrm{sh}\,t}{\mathrm{sh}\,w}\sin\phi\,\sin\tau\right)d\tau\,dt.$$

<u>PROOF.</u> Substitute (5.11) for $f = f_1$ into the right hand side of (6.6) and interchange the order of integration:

$$\int_0^{2\pi}\int_0^{\infty} f_1(\phi,w)f_2(\phi,w)\mathrm{sh}^2 2w\, d\phi\, dw =$$
$$= \int_0^{2\pi}\int_0^{\infty} f_1(\phi,w)\left[\frac{-1}{2\mathrm{sh}2w}\,\frac{1}{2\pi}\int_0^{2\pi}\int_0^{w}\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F_2(\tau,t).\right.$$
$$\left.\cdot U_{2\ell}\left(\frac{\mathrm{ch}\, t}{\mathrm{ch}\, w}\cos\phi\,\cos\tau + \frac{\mathrm{sh}\, t}{\mathrm{sh}\, w}\sin\phi\,\sin\tau\right)d\tau\, dt\right]\mathrm{sh}^2 2w\, d\phi\, dw.$$

For fixed f_2 in $I^{\infty}_{c,\ell}(G)$ this formula holds for all f_2 in $I^{\infty}_{c,\ell}(G)$. By Lemma 5.4 we conclude that (6.7) is valid. □

REMARK 6.6. The proof of the inversion formula (6.7) uses the group theoretic property that the generalized Abel transform is an homomorphism with respect to convolution. We did not succeed in finding a direct analytic proof for the inversion formula.

REMARK 6.7. Prop. 6.4 implies that $f \mapsto F_f$ is injective on $I^{\infty}_{c,\ell}(G)$, which we already knew from Theorem 2.2 (ii).

COROLLARY 6.8. *If* $f \in I^{\infty}_{c,\ell}(G)$ *then*

$$f(e) = \frac{1}{\pi}\sum_j \int_0^{\infty}\left[\int_G f(g)\,\mathrm{tr}\,\pi_{j,i\lambda;\ell,\ell}(g^{-1})dg\right](\lambda^2+j^2)d\lambda. \tag{6.8}$$

PROOF. It follows from (5.16) that

$$\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(0,0) = -\,2\pi^{-1}\int_{-\infty}^{\infty}\sum_j \hat{F}(2j,2i\lambda)(\lambda^2+j^2)d\lambda.$$

Now substitute (6.4) and (5.17) into this formula and use (5.12). □

The above corollary immediately implies the Plancherel formula for SL(2,$\mathbb{C}$). This formula was first obtained by GELFAND & NAIMARK [4], see also the very readable proof (for $SO_0(3,1)$) in TAKAHASHI [21].

7. CHARACTERIZATION OF THE IMAGE OF THE GENERALIZED ABEL TRANSFORM

Let A_ℓ denote the image of $I^{\infty}_{c,\ell}(G)$ under the transform $f \mapsto F$. We already know that all F in A_ℓ have the form (5.5) with F_j in $C^{\infty}_c(\mathbb{R})$ and that

F satisfies the symmetry (5.12). Now we will derive an additional condition satisfied by each F in A_ℓ.

PROPOSITION 7.1. *If* $F \in A_\ell$ *then*

$$\hat{F}(2p,2q) = \hat{F}(2q,2p), \quad p,\ q \in \{-\ell,-\ell+1,\ldots,\ell\}. \tag{7.1}$$

PROOF. We will prove that

$$\mathrm{tr}\ \pi_{p,q;\ell,\ell}(m_\phi a_w) = \mathrm{tr}\ \pi_{q,p;\ell,\ell}(m_\phi a_w), \quad p,\ q \in \{-\ell,-\ell+1,\ldots,\ell\}. \tag{7.2}$$

By (5.17) and Lemmas 5.3, 5.4 this is equivalent to the proposition. It follows from (5.20) and (4.16) that

$$d_\ell^{-1}(\mathrm{tr}\pi_{p,q;\ell,\ell}(m_\phi a_w) - \mathrm{tr}\pi_{q,p;\ell,\ell}(m_\phi a_w)) =$$
$$= \sum_m e^{-2im\phi}\pi^{-1}\left[\int_0^{2\pi}\int_0^w R_{\ell+m,\ell-m}\left(\frac{\mathrm{ch}\ t}{\mathrm{ch}\ w}\cos\tau + i\frac{\mathrm{sh}\ t}{\mathrm{sh}\ w}\sin\tau\right).\right.$$
$$\left.\cdot\left(e^{2ip\tau-2qt} + e^{-2ip\tau+2qt} - e^{2iq\tau-2pt} - e^{-2iq\tau+2pt}\right)\frac{d\tau\ dt}{2\mathrm{sh}2w}\right].$$

The part in square brackets, with new integration variables

$$x = \frac{\mathrm{ch}\ t}{\mathrm{ch}\ w}\cos\tau,\quad y = \frac{\mathrm{sh}\ t}{\mathrm{sh}\ w}\sin\tau,$$

equals

$$\iint_{x^2+y^2<1} R_{\ell+m,\ell-m}(x+iy)f_{p,q}(\tau,t)\,dxdy,$$

where

$$f_{p,q}(\tau,t) := \frac{\mathrm{ch}(2ip\tau-2qt)-\mathrm{ch}(2iq\tau-2pt)}{\mathrm{ch}2t-\mathrm{ch}2i\tau}.$$

Now, by using the recurrences

$$\mathrm{ch}(t+i\tau)f_{p,q}(\tau,t) = \tfrac{1}{2}(f_{p+\frac{1}{2},q-\frac{1}{2}}(\tau,t) + f_{p-\frac{1}{2},q+\frac{1}{2}}(\tau,t))$$

$$\mathrm{ch}(t-i\tau)f_{p,q}(\tau,t) = \tfrac{1}{2}(f_{p+\frac{1}{2},q+\frac{1}{2}}(\tau,t) + f_{p-\frac{1}{2},q-\frac{1}{2}}(\tau,t),$$

together with

$$\mathrm{ch}(t+i\tau) = x\ \mathrm{ch}\ w + iy\ \mathrm{sh}\ w,$$

$$\mathrm{ch}(t-i\tau) = x\ \mathrm{ch}\ w - iy\ \mathrm{sh}\ w,$$

we conclude that $f_{p,q}$ is a polynomial in x,y of degree $2|p|\vee|q|-1$. Now use the orthogonality property of the disk polynomials $R_{\ell+m,\ell-m}(x+iy)$. □

DEFINITION 7.2. Let $\mathcal{B}_\ell$ be the space of all functions F on $[0,2\pi]\times\mathbb{R}$ of the form

$$F(\tau,t) = \sum_j e^{-2ij\tau}F_j(t),$$

with $F_j \in C_c^\infty(\mathbb{R})$ $(j\in\{-\ell,-\ell+1,\ldots,\ell\})$, such that

(i) $F(\tau,t) = F(-\tau,-t)$,

(ii) $\hat{F}(2p,2q) = \hat{F}(2q,2p)$ $(p,q\in\{-\ell,-\ell+1,\ldots,\ell\})$.

Clearly, $\mathcal{A}_\ell \subset \mathcal{B}_\ell$ (cf. (5.5), (5.12), (7.1)). If $F\in\mathcal{B}_\ell$ then define the function E_F on $[0,2\pi]\times\mathbb{R}$ by

(7.3) $$E_F(\phi,w) := \text{RHS of (6.7)}.$$

Thus, if $f\in I^\infty_{c,\ell}(G)$ and $F := F_f$ then $E_F = f|_{M\times A}$. We will show that the mapping $F\mapsto E_F$ is a bijection of $\mathcal{B}_\ell$ onto $I^\infty_{c,\ell}(G)$ (restricted to M×A). Thus it will turn out that $\mathcal{A}_\ell = \mathcal{B}_\ell$.

PROPOSITION 7.3. *Let* $F\in\mathcal{B}_\ell$, $f := E_F$. *Then*

(7.4) $$f(\phi,w) = \frac{1}{2\mathrm{sh}2w}\frac{1}{2\pi}\int_0^{2\pi}\int_w^\infty\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(\tau,t).$$

$$\cdot U_{2\ell}\left(\frac{\text{ch } t}{\text{ch } w}\cos\phi\ \cos\tau + \frac{\text{sh } t}{\text{sh } w}\sin\phi\ \sin\tau\right)d\tau\ dt,$$

and f *is the restriction to* $M \times A$ *of a function* f *on* G *belonging to* $I^{\infty}_{c,\ell}(G)$ *and given by*

$$(7.5) \qquad f(g) = \frac{1}{d_\ell \pi}\sum_j \int_0^\infty \hat{F}(2j,2i\lambda)\,\mathrm{tr}\ \pi_{j,i\lambda;\ell,\ell}(g)(\lambda^2+j^2)d\lambda.$$

PROOF. (7.4) follows from (7.3) because of condition (i) of Definition 7.2 and because

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(\tau,t)U_{2\ell}\left(\frac{\text{ch } t}{\text{ch } w}\cos\phi\ \cos\tau + \frac{\text{sh } t}{\text{sh } w}\sin\phi\ \sin\tau\right)d\tau\ dt = 0.$$

To prove this identity, observe that

$$\frac{\text{ch } t}{\text{ch } w}\cos\phi\ \cos\tau + \frac{\text{sh } t}{\text{sh } w}\sin\phi\ \sin\tau$$

is invariant under $(t,\tau) \to (i\tau,-it)$ and that $U_{2\ell}(-x) = (-1)^{2\ell}U_{2\ell}(x)$, so $U_{2\ell}(\ldots)$ is multiplied by $(-1)^{2\ell}$ under $\tau \to \tau + \pi$. Hence $U_{2\ell}(\ldots)$ is a finite linear combination of terms $e^{2ip\tau}e^{2qt} + e^{2iq\tau}e^{2pt}$ $(p,q \in \{-\ell,-\ell+1,\ldots\ell\})$ with coefficients depending on ϕ,w. Now if we write

$$\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial t^2}\right)F(\tau,t) = \sum_m H_m(t)e^{-2im\tau},$$

then condition (ii) of Def. 7.2 implies:

$$\int_{\mathbb{R}} H_p(t)e^{2qt}dt = -\int_{\mathbb{R}} H_q(t)e^{2pt}dt,$$

and our claim is clear.

Formula (7.5) is proved for $g \in MA$ by substituting (5.16) into (7.3) and next combining this with (5.20). Now observe that $\lambda \to \hat{F}(2j,2i\lambda)$ is rapidly decreasing $(j\in\{-\ell,-\ell+1,\ldots,\ell\})$, since $F \in C^\infty_c([0,2\pi]\times\mathbb{R})$, and that, by (2.11), all partial derivatives w.r.t. g of the function $(\lambda,g) \mapsto \mathrm{tr}\ \pi_{j,i\lambda;\ell,\ell}(g)$ exist and are of polynomial growth in λ. Hence (7.5) defines a function f in $C^\infty(G)$. Next, f is invariant under K-conjugation and behaves like the rep T_ℓ of K because of similar properties of the function $g \mapsto \mathrm{tr}\ \pi_{j,i\lambda;\ell,\ell}(g)$. Thus f will have the form (5.2). Finally, since $f_{|MA}$ has compact support because of (7.4), we conclude that f has compact support on G. Hence $f \in I^{\infty}_{c,\ell}(G)$. □

Next we want to prove that $F \to E_F$ is injective on $\mathcal{B}_\ell$. For this we need:

THEOREM 7.4.

(a) *If* $|\lambda| - |j| \notin \mathbb{N}$ *then* $\pi_{j,\lambda}$ *is irr. If* $|\lambda| - |j| \in \mathbb{N}$ *then* $\pi_{j,\lambda}$ *has two irr. subquotient reps* $\sigma_{j,\lambda}$ *and* $\tau_{j,\lambda}$, *with*

$$M(\sigma_{j,\lambda}) = \{T_{|j|}, T_{|j|+1}, \ldots T_{|\lambda|-1}\},$$

$$M(\tau_{j,\lambda}) = \{T_{|\lambda|}, T_{|\lambda|+1}, \ldots\}.$$

(b) *There exist precisely the following nontrivial Naimark equivalences between the above irr. reps:*

$$\pi_{j,\lambda} \simeq \pi_{-j,-\lambda} (|\lambda|-|j| \notin \mathbb{N}),$$

$$\sigma_{j,\lambda} \simeq \sigma_{-j,-\lambda}\ \tau_{j,\lambda} \simeq \tau_{-j,-\lambda} (|\lambda|-|j| \in \mathbb{N}),$$

$$\pi_{j,\lambda} \simeq \tau_{\lambda,j}, \pi_{j,\lambda} \simeq \tau_{-\lambda,-j} (|j|-|\lambda| \in \mathbb{N}).$$

PROOF. The matrix element $\pi_{j,\lambda;\ell_1,\ell_2;\ell_2,\ell_2}(a_t)$ $(\ell_1 \geq \ell_2)$ can be easily evaluated in terms of ${}_2F_1$ hypergeometric function (cf. KOSTERS [16, (3.11)]). This shows that, if $\pi_{j,\lambda;\ell_1,\ell_2} = 0$ then $|\lambda| - |j| \in \mathbb{N}$ and $\ell_1 \geq |\lambda|$, $\ell_2 < |\lambda|$ or $\ell_1 < |\lambda|$, $\ell_2 \geq |\lambda|$. Thus, by Theorem 1.2, $\pi_{j,\lambda}$ is irr. if $|\lambda| - |j| \notin \mathbb{N}$ and $\pi_{j,\lambda}$ has at most the irr. subquotient reps $\sigma_{j,\lambda}$ and $\tau_{j,\lambda}$ if $|\lambda| - |j| \in \mathbb{N}$. However, by (7.2) and Theorem 1.5 it follows that $\tau_{j,\lambda}(|\lambda|-|j| \in \mathbb{N})$ is indeed an irr. rep being Naimark equivalent to the irr. rep $\pi_{\lambda,j}$. This settles (a). The other equivalences in (b) follow from the evident identity

$$\mathrm{tr}\ \pi_{j,\lambda;\ell,\ell}(m_\phi a_w) = \mathrm{tr}\ \pi_{-j,-\lambda;\ell,\ell}(m_\phi a_w),$$

together with Theorem 1.5. Finally, to prove that this exhausts the possible Naimark equivalences, observe:

$$\pi_{j_1,\lambda_1;\ell,\ell;\ell,\ell} = \pi_{j_2,\lambda_2;\ell,\ell;\ell,\ell} \Rightarrow \lambda_1 j_1 = \lambda_2 j_2$$

(cf. KOSTERS [16, p. 16]). □

The above proof simplifies the proof by KOSTERS [16] because of our new proof of (7.2). The above proof that $\pi_{j,\lambda}$ is irr. if $|\lambda| - |j| \notin \mathbb{N}$ also occurs in NAIMARK [19, Ch. 3, §9, no. 15].

PROPOSITION 7.5. *The mapping* $F \mapsto E_F\colon \mathcal{B}_\ell \to I^\infty_{c,\ell}(G)$ *is injective.*

PROOF. Suppose $F_1 \in \mathcal{B}_\ell$, $f_1 := E_{F_1} = 0$, $f_2 \in I^\infty_{c,\ell}(G)$, $F_2 := F_{f_2}$ then

$$\int_0^{2\pi}\int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial t^2}\right) F_1(\tau,t)F_2(\tau,t)\,d\tau\,dt = 0.$$

To see this, substitute formula (5.11) with $f = f_2$ into the left hand side for F_2 and interchange the order of integration. Substitution of (5.16) into the above formula and application of (5.12) yields:

$$\sum_j \int_0^\infty \hat{F}_1(2j,2i\lambda)\hat{F}_2(2j,2i\lambda)(\lambda^2+j^2)\,d\lambda = 0.$$

Let $\hat{A}_\ell$ denote the set of functions $\hat{F}$ on $\{-2\ell,-2\ell+2,\ldots,2\ell\} \times i[0,\infty)$ for which $F \in A_\ell$. One easily verifies that $\hat{A}_\ell$ is an algebra under pointwise multiplication, consisting of continuous functions $\hat{F}$ which satisfy $\hat{F}(2j,2i\lambda) \to 0$ as $\lambda \to \infty$, and that $\hat{A}_\ell$ is closed under complex conjugation. Next, $\hat{A}_\ell$ separates points. Indeed, if $\hat{F}(2j_1,2i\lambda_1) = \hat{F}(2j_2,2i\lambda_2)$ for all $\hat{F} \in \hat{A}_\ell$ then

$$\operatorname{tr} \pi_{j_1,i\lambda_1;\ell,\ell}(g) = \operatorname{tr} \pi_{j_2,i\lambda_2;\ell,\ell}(g)$$

because of (5.17) and Lemmas 5.3, 5.4, so $(j_1,i\lambda_1) = (j_2,i\lambda_2)$ by Theorems 7.4 (b) and 1.2. Finally, another application of (5.17) and Lemmas 5.3, 5.4 shows that no $(2j,2i\lambda)$ is annihilated by all $\hat{F} \in \hat{A}_\ell$. Thus, by the version of the Stone-Weierstrass theorem for a locally compact Hausdorff space (cf. SIMMONS [23, §38, Theorem B]), each continuous function on

$\{-2\ell,\dots,2\ell\} \times i[0,\infty)$ vanishing at ∞ can be uniformly approximated by functions in $\hat{A}_\ell$. In particular, the (rapidly decreasing) function $\hat{\hat{F}}_1$ can be uniformly approximated by functions $\hat{F}_2 \in \hat{A}_\ell$, so

$$\sum_j \int_0^\infty |\hat{F}_1(2j,2i\lambda)|^2(\lambda^2+j^2)d\lambda = 0.$$

This shows that $F_1 = 0$. □

Define $\tilde{B}_\ell$ to be the algebra of all functions in $\mathcal{D}(\mathbb{R};\mathbb{C}^{2\ell+1})$ such that

(i) $F_j(-t) = F_{-j}(t)$, $j = -\ell,-\ell+1,\dots,\ell$,

(ii) $\int_{-\infty}^{\infty} F_j(t)\, e^{-2kt}dt = \int_{-\infty}^{\infty} F_k(t)e^{-2jt}dt$, $j,k = -\ell,-\ell+1,\dots,\ell$,

with componentwise convolution as multiplication, and the topology inherited from $\mathcal{D}(\mathbb{R};\mathbb{C}^{2\ell+1})$. Note: $\tilde{B}_\ell$ and $\mathcal{B}_\ell$ are isomorphic as vector spaces under the mapping $F \mapsto \{F_j\}$ defined by (5.5). From (5.1) and (5.3) we have:

$$F^\ell_{f,j}(t) := (2\pi)^{-1}e^{2t} \int_K\int_{\mathbb{R}}\int_{\mathbb{R}} f(ka_tn_{x+iy})t^\ell_{jj}(k^{-1})dk\,dx\,dy. \tag{7.6}$$

Now we arrive at the main theorem:

<u>THEOREM 7.6.</u> *The mapping* $f \mapsto \{F^\ell_{f,j}\}\colon I^\infty_{c,\ell}(G) \to \tilde{B}_\ell$ *is an isomorphism of topological algebras.*

<u>PROOF.</u> The mapping is an injective continuous algebra homomorphism by Theorem 2.2 and it is surjective by Proposition 7.3 and 7.5. The only thing left to prove is that the inverse $E\colon \mathcal{B}_\ell \mapsto I^\infty_{c,\ell}(G)$ is continuous. This is done as follows. Denote by $\|g\|$ $(g\in SL(2,\mathbb{C}))$ the operator norm of g acting on the Hilbert space $\mathbb{C}^2$. The elements of $K(=SU(2))$ have norm 1, so, if $g = k_1a_tk_2$ then $\|g\| = \|a_t\| = \max\{e^t,e^{-t}\}$. If $R > 0$ then let $I^\infty_{c,\ell;R}(G)$ denote the space of functions in $I^\infty_{c,\ell}(G)$ which vanish on all g in G with $\log\|g\| > R$; similary, let $\mathcal{B}_{\ell;R}$ consist of all functions in $\mathcal{B}_\ell$ with support included in $[0,2\pi) \times [-R,R]$. Let $f \in I^\infty_{c,\ell}(G)$, $R > 0$. In view of (5.2), (5.11) and (7.4) we have:

$$f \in I^\infty_{c,\ell;R}(G) \iff \mathrm{Supp} f_{|M\times A} \subset [0,2\pi) \times [-R,R] \iff F_f \in \mathcal{B}_{\ell;R}.$$

so $f \mapsto F_f \colon I^{\infty}_{c,\ell;R}(G) \to \mathcal{B}_{\ell;R}$ is a continuous linear bijection with inverse $E|_{\mathcal{B}_{\ell;R}}$. Because of the open mapping theorem for Fréchet spaces (cf. RUDIN [22, Cor. 2.12(b)]), $E\colon \mathcal{B}_{\ell;R}(G) \to I^{\infty}_{c,\ell;R}(G)$ is also continuous. Since $I^{\infty}_{c,\ell}(G) = \bigcup_{R>0} I^{\infty}_{c,\ell;R}(G)$, endowed with inductive limit topology, and similar for $\mathcal{B}_{\ell}$, we conclude that $E\colon I^{\infty}_{c,\ell}(G) \to \mathcal{B}_{\ell}$ is continuous. □

8. THE SUBQUOTIENT THEOREM FOR SL(2,ℂ).

In order to derive the subqoutient theorem from Theorem 7.6 we have to find all continuous characters on $\tilde{\mathcal{B}}_{\ell}$. As a preparation we need two lemmas. Identify $\{F_j\}$ in $\tilde{\mathcal{B}}_{\ell}$ with F in $\mathcal{B}_{\ell}$ by means of (5.5). Thus (5.15) can be rewritten as

$$\hat{F}(2j,2\lambda) = d_{\ell} \int_{-\infty}^{\infty} F_j(t) e^{-2\lambda t} dt \tag{8.1}$$

and $F \mapsto d_{\ell}^{-1}\, \hat{F}(2j,2\lambda)$ defines a continuous character on $\tilde{\mathcal{B}}_{\ell}$ for each j j in $\{-\ell,-\ell+1,\ldots,\ell\}$, λ in $\mathbb{C}$.

LEMMA 8.1. *Let*

$$\tilde{\mathcal{B}}_{\ell,0} := \{F \in \tilde{\mathcal{B}}_{\ell} \mid \hat{F}(2p,2q) = 0 \quad \textit{for} \quad p,q = -\ell,-\ell+1,\ldots,\ell\}.$$

Then $\tilde{\mathcal{B}}_{\ell,0}$ *is a closed subalgebra of* $\tilde{\mathcal{B}}_{\ell}$ *and every nonzero continuous character on* $\tilde{\mathcal{B}}_{\ell,0}$ *has the form* $F \mapsto d_{\ell}^{-1}\, \hat{F}(2j,2\lambda)$ *for some j in* $\{-\ell,\ldots,\ell\}$, λ *in* $\mathbb{C}$.

PROOF. See NAIMARK [19, Ch. 3, §9, no. 13]. The lemma is easily reduced to the two problems of finding all nonzero continuous characters on the convolution algebras $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}_{\mathrm{even}}(\mathbb{R})$, respectively. In both cases these are of the form $f \mapsto \int_{\mathbb{R}} f(t) e^{-\lambda t} dt$ for some λ in $\mathbb{C}$ (see also [13, Prop. 5.6]). □

The next lemma is also given by NAIMARK, see [19, Ch. 3, §9, no. 14], but there is an error in his proof, so we will give here the full correct proof.

LEMMA 8.2. *Let* A *be an algebra,* $f, g, f_1,\ldots,f_n$ *multiplicative linear functionals on* A, $A_0 := \{x \in A \mid f_1(x) = \ldots = f_n(x) = 0\}$, $f|_{A_0} = g|_{A_0}$. *Then*

$f = g$ *or* $f = f_k$ *for some* k *in* $\{1,\dots,n\}$ *or* $f = 0$.

PROOF. Without loss of generality we may assume that the f_i's are linearly independent. By a simple inductive argument we can find x_k in A $(1 \le k \le n)$ such that $f_j(x_k) = \delta_{jk} (1 \le j,k \le n)$. If $x \in A$ then $x - \Sigma_{j=1}^n f_j(x)x_j \in A_0$, so

$$f(x) - \sum_{j=1}^{n} f_j(x)f(x_j) = g(x) - \sum_{j=1}^{n} f_j(x)g(x_j).$$

On putting $\alpha_j := f(x_j)-g(x_j)$ we see that

$$(*) \qquad f(x) = g(x) + \sum_{j=1}^{n} \alpha_j f_j(x).$$

Substitution of $x = yx_k$ with y in A_0 yields,

$$f(y)\,\alpha_k = 0 \quad (y \in A_0,\ 1 \le k \le n).$$

Now there are two possibilities:

(i) $\alpha_k = 0$ for all k. Then $f = g$ by $(*)$.

(ii) $\alpha_k \neq 0$ for some k. Then $f|_{A_0} = 0$ and

$(**)$ $f(x) = \Sigma_j f_j(x)f(x_j)$.

If $f \neq 0$ then $f(x_\ell) \neq 0$ for some ℓ and $(**)$ with $x = x_\ell x_i$ will yield $f(x_\ell)f(x_i) = \delta_{\ell i} f(x_\ell)$. So $f(x_i) = \delta_{i\ell}$ and $f = f_\ell$. □

PROPOSITION 8.3. *Each nonzero continuous character on* $\widetilde{B}_\ell$ *has the form* $F \mapsto d_\ell^{-1}\, \hat{F}(2j,2\lambda)$ *for some* j *in* $\{-\ell,-\ell+1,\dots,\ell\}$ *and some* λ *in* $\mathbb{C}$.

PROOF. Let χ be a nonzero continuous character on $\widetilde{B}_\ell$. Then $\chi_0 := \chi|_{\widetilde{B}_{\ell,0}}$ is a continuous character on $\widetilde{B}_{\ell,0}$. It follows from Lemma 8.1 that $\chi_0 = 0$ or $\chi_0(F) = d_\ell^{-1}\, \hat{F}(2j,2\lambda)$ $(F \in \widetilde{B}_{\ell,0})$ for some j in $\{-\ell,-\ell+1,\dots,\ell\}$ λ in $\mathbb{C}$. Now apply Lemma 8.2 with $f = \chi_0$ and $g\colon F \mapsto d_\ell^{-1}\, \hat{F}(2j,2\lambda)$, Then

$$\chi(F) = d_\ell^{-1}\, \hat{F}(2j,2\lambda) (F \in \widetilde{B}_\ell) \text{ or } \chi(F) = d_\ell^{-1}\, \hat{F}(2p,2q)$$

$(F \in \widetilde{B}_\ell)$ for some p,q in $\{-\ell,-\ell+1,\dots,\ell\}$. □

THEOREM 8.4 (subquotient theorem for $SL(2,\mathbb{C})$).
Let $G = SL(2,\mathbb{C})$, $K = SU(2)$. *Then every* K*-finite irr. Hilbert rep of* G *is Naimark equivalent with an irr. subquotient rep of a principal series rep.*

PROOF. Use Theorems 1.5, 2.2 (iv), 7.6 and Propositions 1.6 and 8.3. □

9. FURTHER REMARKS

Obviously, Theorem 7.6 together with (5.17) and the Paley-Wiener theorem for the classical Fourier transform yields a characterization of the image of $I^{\infty}_{c,\ell}(G)$ under the group Fourier transform. This provides a new proof of the Paley-Wiener theorem in WANG [26, Prop. 4.5].

Next we state without proof a characterization of functions f in $I^{\infty}_{c,\ell}(G)$ in terms of functions f_m.

PROPOSITION 9.1. *Formula* (5.2) *together with*

$$\phi_n(w)(\mathrm{sh}\ w)^{\ell+n}(\mathrm{ch}\ w)^{\ell-n} = \sum_m \frac{t^{\ell}_{mn}(u_{\pi/4})f_m(w)}{((\ell-m)!(\ell+m)!)^{\frac{1}{2}}} \tag{9.1}$$

defines a one - to - one correspondence $f \leftrightarrow (\phi_{-\ell},\phi_{-\ell+1},\dots,\phi_{\ell})$ *between* $I^{\infty}_{c,\ell}(G)$ *and* $(C^{\infty}_{c,\mathrm{even}}(\mathbb{R}))^{2\ell+1}$.

Finally, we would like to remark that similar results as in this paper for $SL(2,\mathbb{C})$ can be derived for the corresponding Cartan motion group $SU(2) \circ \mathbb{R}^3$. Let G be a noncompact connected real semisimple Lie group with finite center, K a maximal compact subgroups, $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebras, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. Consider the semidirect product $K \circ \mathfrak{p}$, where K acts on $\mathfrak{p}$ by Ad. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, $\mathfrak{a}^{\perp}$ the orthoplement of $\mathfrak{a}$ in $\mathfrak{p}$ w.r.t. the Killing form B on $\mathfrak{g}$. For δ in $\hat{K}$ let $I^{\infty}_{c,\delta}(G)$ be defined as in (2.3), (2.4). Then the analogue for $K \circ \mathfrak{p}$ of the generalized Abel transform (2.3) becomes

$$F_f(k,H) := \int_{\mathfrak{a}^{\perp}} f(k,H+Y)dY,\ k \in K,\ H \in \mathfrak{a},\ f \in I^{\infty}_{c,\delta}(G). \tag{9.2}$$

Now it can be proved that, if $G = SL(2,\mathbb{C})$, $f \in I^{\infty}_{c,\ell}(K\circ\mathfrak{p})$ and $\mathfrak{a} = \mathbb{R}\,H_0$ then

$$F_f(m_\tau, tH_0) = \text{const.} \int_0^{2\pi}\int_t^{\infty} f(m_\phi, wH_0) . \tag{9.3}$$

$$.U_{2\ell}\left(\cos\phi\ \cos\tau + \sin\phi\ \sin\tau\ \frac{t}{w}\right) d\phi\ w dw.$$

Compare this formula with (5.11).

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