MATHEMATICAL CENTRE TRACTS
5

GENERALIZED MARKOVIAN DECISION PROCESSES
APPLICATIONS

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PREFACE

During the last ten years there has been a rapid growth of interest in the theory of Markovian Decision Processes. The interest in this subject was generated by R.A. HOWARD's book, Dynamic Programming and Markov Processes [5]. In this book the most simple Markov-programming models are treated. HOWARD's results have been generalized by W.S. JEWELL, G. DE LEVE and others [1,6,7].

The generalized techniques developed by DE LEVE in his thesis Generalized Markovian Decision Processes are also applicable to continuous time models and/or models with a non-denumerable state space. These techniques are not "ready-made" techniques and their final form depends heavily on the structure of the decision problem considered. The decision situations we will consider have an infinite planning horizon and the problem is to find a strategy which minimizes the expected average costs per unit of time.

The only purpose of this book is to demonstrate how the generalized techniques can be applied in practice. This is done by means of a number of practical applications. For these applications the final form of the solving techniques turns out to be rather simple.

An effort has been made to write the text in such a way that it can be read independently of the above mentioned thesis. To accomplish this, the first chapter contains a survey of the generalized Markov-programming techniques we will use. Proofs about the efficiency of these methods and conditions imposed on the models to set up the theory are omitted and can be found in [7]. Furthermore, it is demonstrated in chapter 1 that the iterationmethods of HOWARD and JEWELL follow from the generalized iterationmethod.

The solving techniques are of two types. An optimal strategy may be found by a direct approach (functional equations) or by an iterative approach.

Five applications are given. In chapters 2 and 3 problems with a discrete state space are considered. In chapter 2 a productionproblem has been solved in an iterative way. In chapter 3 we have determined in a direct way the expected average costs per unit of time of (S,s)- and (Q,s)-strategies for continuous time inventory models. Furthermore an iteration-procedure has been indicated, which exploits the simple properties of an (S,s)-strategy.

In chapters 4, 5 and 6 problems with a non-denumerable state space are considered. In chapter 4 the continuous version of the automobile replacement problem of HOWARD [4, pp. 54] has been treated. The problem has been solved iteratively. In chapter 5 an insurance problem of a motorist has been solved by a direct approach. In chapter 6 another production-problem has been solved in an iterative way.

The problems treated can be read independently of each other.

CONTENTS

1. Markovian decision processes.

1.1. The general case.

A physical system will be considered which is controlled by a decisionmaker. At each point of time the system is in some state. In the mathematical model a state of the system is represented by a point in a finite dimensional Cartesian space. The set of all possible states will be called the state space X.

In case the decisionmaker does not intervene it is supposed that for each initial state the evolution of the system (called the natural process) can be described by a homogeneous strong Markov-process. The natural process, however, is not the only source of changes in the state of the system. The decisionmaker influences the natural process by interventions. We restrict ourselves to models with a finite number of interventions in each finite period of time. An intervention results in a random transition in the state of the system. A transition is assumed to take no time. We suppose that each intervention is defined by the probability distribution of the state into which the system is transferred by the intervention. We differentiate between decisions called null-decisions and decisions which are interventions. A so-called null-decision is made at each point of time the decisionmaker does not intervene. A null-decision "transfers" the system with probability one into its present state. The introduction of the concept of null-decision enables us to state that at each point of time a decision is made. It is assumed that the behaviour of the system in the time interval between two successive interventions is described by a natural process. The initial state of that process will be the state into which the system was transferred by the intervention at the beginning of the interval concerned.

For each state x of the system we have a set D(x) of feasible decisions d. We consider only stationary strategies, i.e. strategies which base their decisions on the present state only and associate unambigiously with each state x a decision $d \in D(x)$. Let Z be the class of the stationary strategies. The decision assigned by a strategy $z \in Z$ to state x will be denoted by z(x).
The result of the natural process and a strategy $z \in Z$ is called a

decisionprocess. Under general conditions it can be shown that each decisionprocess is also a homogeneous strong Markov-process.

In our decisionproblems we suppose that we are dealing with costs only. The costfunctions have to satisfy some regularity conditions. The costs are not discounted. Our criterion for an optimal strategy will be the expected average costs per unit of time when the system is considered for an infinite period of time.

Let A_z be the set of all states at which strategy $z \in Z$ dictates an intervention. It is supposed that the intersection

$$A_0 = \bigcap_{z \in Z} A_z \tag{1.1}$$

is not empty. Furthermore it is assumed that in the natural process with probability one the set A_0 is reached from each initial state within a finite time. Observe that each strategy $z \in Z$ dictates an intervention at each state of A_0.
Choose the non-empty sets

$$A_{0,1} \subseteq A_0 \quad \text{and} \quad A_{0,2} \subseteq A_0 \tag{1.2}$$

such that in the natural process with probability one each of these sets is reached from each initial state within a finite time. For each $i = 1,2$ there corresponds to every state x and decision $d \in D(x)$ two random walks $\underline{w}_{0,i}$ and $\underline{w}_{d,i}$. The walk $\underline{w}_{0,1}$ (resp. $\underline{w}_{0,2}$) has x as initial state and during this walk the system is subjected to the natural process. The walk $\underline{w}_{0,1}$ (resp. $\underline{w}_{0,2}$) ends as soon as the system assumes a state of $A_{0,1}$ (resp. $A_{0,2}$). The walk $\underline{w}_{d,1}$ (resp. $\underline{w}_{d,2}$) has x as initial state too. But now in state x decision d is made, by which the system is transferred (instantaneously!) into a random state and from that state on the system is subjected to the natural process. The walk $\underline{w}_{d,1}$ (resp. $\underline{w}_{d,2}$) ends as soon as the system assumes a state of $A_{0,1}$ (resp. $A_{0,2}$). Let $k_0(x)$ and $k_1(x;d)$ be the expected costs incurred during $\underline{w}_{0,1}$ (resp. $\underline{w}_{d,1}$). The costs of the decision d are included in $k_1(x;d)$. Let $t_0(x)$ and $t_1(x;d)$ be the expected duration of $\underline{w}_{0,2}$ (resp. $\underline{w}_{d,2}$). Define the $(x;d)$-functions

$$(1.3)\qquad k(x;d) = k_1(x;d) - k_0(x) \quad \text{for} \quad d \in D(x),\ x \in X$$

and

$$(1.4)\qquad t(x;d) = t_1(x;d) - t_0(x) \quad \text{for} \quad d \in D(x),\ x \in X.$$

The walks $\underline{w}_{d,i}$ and $\underline{w}_{0,i}$ are identical if d is a null-decision. Hence

$$(1.5)\qquad k(x;d) = t(x;d) = 0 \quad \text{if d is a null-decision, } x \in X.$$

It follows from their definition that $k(x;d)$ and $t(x;d)$ do not depend on any particular strategy. Hence we need only once and for all to determine the functions $k(x;d)$ and $t(x;d)$.

Suppose now that a strategy $z \in Z$ is applied and let x be the initial state of the system. Let $\{\underline{I}_n = \underline{I}_n(z;x),\ n \geq 1\}$ be the sequence of <u>future</u> interventionstates. It can be proved under rather general conditions that this sequence constitutes a homogeneous Markov-process in A_z with discrete time parameter. Although the interventionstates of strategy z belong to A_z, we take X as state space of this Markov-process.*) Let $p^{(k)}(A;z;x)$ be the probability that $\underline{I}_k$ belongs to a Borelset A . Under the Doeblin condition there is a $q(A;z;x)$ for which

$$(1.6)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p^{(k)}(A;z;x) = q(A;z;x).$$

The function $q(A;z;x)$ defines for each $x \in X$ a stationary probability distribution on X, which satisfies

$$(1.7)\qquad q(A;z;x) = \int p^{(1)}(A;z;y)\ q(dy;z;x)$$

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*) It is assumed that for each $x \in X$ the Markov-process $\{\underline{I}_n\}$ satisfies the Doeblin condition. For the meaning of this condition and the concepts "ergodic sets, transient sets, cyclically moving subsets" the reader is referred to DOOB [2]. We, however, prefer the name "simple ergodic set" to the term "ergodic set". A decomposition of X in a finite number of simple ergodic sets and a transient set is not always unique. However it is supposed throughout that disjoint simple ergodic sets $E_1,\ldots,E_m$ and a transient set F have been chosen in some definite way.

Furthermore

$$q(A;z;x_1) = q(A;z;x_2) \tag{1.8}$$

if x_1 and x_2 belong to the same simple ergodic set.

Let $\underline{k}_T(z;x)$ be the costs incurred during the period $[0,T)$. Under weak conditions

$$\lim_{T\to\infty} \underline{k}_T(z;x)/T \tag{1.9}$$

exists with probability one for each initial state $x \in X$. This limit represents the average costs per unit of time and with probability one equals

$$\int_{A_z} k(I;z(I))\ q(dI;z;x) \Big/ \int_{A_z} t(I;z(I))\ q(dI;z;x) \tag{1.10}$$

if the initial state x belongs to a simple ergodic set. For each state x belonging to a simple ergodic set E_j we define the function

$$g(z;x) = \int_{A_z} k(I;z(I))\ q(dI;z;x) \Big/ \int_{A_z} t(I;z(I))\ q(dI;z;x). \tag{1.11}$$

For the states x of the transient set F we define

$$g(z;x) = \int_{A_z} g(z;y)\ q(dy;z;x). \tag{1.12}$$

Observe, by (1.10) and (1.11), that the average cost per unit of time assumes with probability one the value $g(z;x)$ if the initial state x belongs to a simple ergodic set. Furthermore by (1.8) the function $g(z;x)$ is constant on a simple ergodic set. If the initial state x belongs to the transient set F the average cost per unit of time is a random variable with expectation $g(z;x)$.

The criterionfunction $g(z;x)$ may be determined without using the stationary distribution $q(A;z;x)$. Consider the following functional equations in the functions $r(z;x)$ and $c(z;x)$

$$r(z;x) = Er(z;\underline{I}_1) \qquad \text{for } x \in X \tag{1.13}$$

and

(1.14) $c(z;x) = k(x;z(x)) - r(z;x)\ t(x;z(x)) + Ec(z;\underline{I}_1)$ for $x \in X$,

where $\underline{I}_1$ is the first future interventionstate when x is the initial state. E stands for the expectationsymbol, hence

$$(1.15) \qquad Er(z;\underline{I}_1) = \int_{A_z} r(z;I)\ p^{(1)}(dI;z;x).$$

Observe that by (1.5) equations (1.14) for $x \notin A_z$ reduce to

$$(1.16) \qquad c(z;x) = Ec(z;\underline{I}_1) \qquad \text{for } x \notin A_z.$$

It can be shown that the functional equations (1.13) and (1.14) have a solution and that for each solution $(r(z;x),c(z;x))$ we have

$$(1.17) \qquad r(z;x) = g(z;x) \qquad \text{for } x \in X.$$

Furthermore it can be proved that by choosing in each simple ergodic set E_j a state e_j and by adding to (1.13) and (1.14) the conditions

$$(1.18) \qquad c(z;e_j) = 0 \qquad \text{for } j = 1,\ldots,m$$

the resulting system of functional equations has a unique solution. The function $r(z;x)$ equals $g(z;x)$; hence $r(z;x)$ is the expected average costs per unit of time if x is the initial state.The function $c(z;x)$ may be given a physical interpretation too. If the decisionproces has no cyclically moving subsets it can be shown that for states x_1 and x_2 in the same simple ergodic set $c(z;x_1) - c(z;x_2)$ is equal to the decrease in total expected costs caused by starting in state x_1 rather than in state x_2.

By means of the functions $r(z;x)$ and $c(z;x)$ an iterationprocedure can be given, which yields a sequence of strategies $\{z^{(i)}, i \geqq 1\}$ for which, under certain conditions, the following interesting properties can be proved:

$$(1.19) \qquad r(z^{(i+1)};x) \leqq r(z^{(i)};x) \qquad \text{for } x \in X;\ i = 1,2,\ldots$$

and

(1.20) $$\lim_{i\to\infty} r(z^{(i)};x) = \min_{z\in Z} r(z;x) \qquad \text{for } x\in X.$$

We shall now give an explanation of the iterationprocedure. Suppose that in the initial state x the decisionmaker chooses a feasible decision d and that he applies strategy $z\in Z$ thereafter. For this mixed strategy d.z the expected average cost per unit of time is given by

(1.21) $$r(d.z;x) \overset{\text{def}}{=} Er(z;\underline{u}),$$

where $\underline{u}$ is the state into which the system is transferred by decision d in x. Obviously the decisionmaker chooses in state x a decision $d\in D(x)$ which minimizes r(d.z;x). We assume that such a decision exists. Let $D_z(x)$ be the set of r(d.z;x) minimizing decisions d; hence

(1.22) $$D_z(x) = \{d \mid d\in D(x),\ r(d.z;x) = \min_{d'\in D(x)} r(d'.z;x)\}$$

If $D_z(x)$ contains more than one decision, we minimize

(1.23) $$c(d.z;x) \overset{\text{def}}{=} k(x;d) - r(d.z;x)\ t(x;d) + Ec(z;\underline{u})$$

with respect to $d\in D_z(x)$. We assume that the minimum is attained. We associate now to state x a decision $d\in D_z(x)$ which minimizes c(d.z;x). We adopt the convention that we choose d = z(x) if z(x) belongs to $D_z(x)$ and minimizes c(d.z;x). If we have associated in this way with each state x a decision d, we have constructed a strategy $z_1\in Z$. The following important result can be proved,

(1.24) $$r(z_1;x) \leqq r(z;x) \qquad \text{for } x\in X.$$

From the definitions (1.21) and (1.23) and from (1.5) it follows that both for d = null-decision and d = z(x)

(1.25) $$r(d.z;x) = r(z;x) \text{ and } c(d.z;x) = c(z;x) \qquad \text{for } x\in X.$$

Our convention and relation (1.25) have as a consequence that each intervention of strategy z is an interventionstate of z_1 too.

Hence

(1.26) $$A_{z_1} \supseteq A_z .$$

It will be clear that we need a mechanism which may cancel an intervention*). We shall now give a cutting mechanism which reduces strategy z_1 to a strategy $z_2 \in Z$ with $A_{z_2} \subseteq A_{z_1}$. Let strategy $z \in Z$ and let strategy $z_1 \in Z$ be obtained as described above. The mixed strategy $(z_1)z$ dictates first an intervention in accordance with z_1 and then interventions in accordance with z. For abbreviation let $\hat{z} = (z_1)z$ and

(1.27) $$r(\hat{z};x) = r(z_1(x).z;x) = \min_{d \in D(x)} r(d.z;x) \qquad \text{for } x \in A_{z_1}$$

and

(1.28) $$c(\hat{z};x) = c(z_1(x).z;x) = \min_{d \in D_z(x)} c(d.z;x) \qquad \text{for } x \in A_{z_1} .$$

Let A be a closed set of states satisfying

(1.29) $$A_0 \subseteq A \subseteq A_{z_1} .$$

Let the mixed strategy $A.\hat{z}$ interdicts any intervention up to (but not including) the moment that the system assumes a state of A for the first time. From that time onwards the mixed strategy $\hat{z}$ is applied.
Define

(1.30) $$r(A.\hat{z};x) = Er(\hat{z};\underline{a}) \qquad \text{for } x \in X$$

and

(1.31) $$c(A.\hat{z};x) = Ec(\hat{z};\underline{a}) \qquad \text{for } x \in X,$$

where $\underline{a}$ is the first state in A taken on by the system if the mixed strategy $A.\hat{z}$ is applied and x is the initial state. Observe that the probability distribution of $\underline{a}$ depends only on the natural process.

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*) Such a mechanism is not needed if $A_z = A_0$ for each $z \in Z$.

Clearly

(1.32) $\quad r(A.\hat{z};x) = r(\hat{z};x)$ and $c(A.\hat{z};x) = c(\hat{z};x) \qquad$ for $x \in A$.

Furthermore it will be obvious that $r(A.\hat{z};x)$ can be interpreted as the expected average cost per unit of time if the mixed strategy $A.\hat{z}$ is applied and x is the initial state of the system. The class $\mathcal{X}(\hat{z})$ of closed sets A is defined as follows. Each set A from $\mathcal{X}(\hat{z})$ satisfies (1.29) and has the property that for each state $x \in A_{z_1}$ we have

either

(1.33) $$r(A.\hat{z};x) \leqq r(\hat{z};x)$$

or

(1.34) $$r(A.\hat{z};x) = r(\hat{z};x) \text{ and } c(A.\hat{z};x) \leqq c(\hat{z};x),$$

Using (1.32) it follows that

(1.35) $$A_{z_1} \in \mathcal{X}(\hat{z}).$$

It can be shown that the intersection of any finite number of sets from $(\hat{z})$ belongs also to this class. Define $A'_{\hat{z}}$ as the intersection of all sets of $(\hat{z})$, hence

(1.36) $$A'_{\hat{z}} = \bigcap_{A \in \mathcal{X}(\hat{z})} A.$$

If $A'_{\hat{z}} \in \mathcal{X}(\hat{z})$ it can be shown that strategy $z_2 \in Z$ defined by

(1.37) $$z_2(x) = \begin{cases} z_1(x) & \text{for } x \in A'_{\hat{z}} \\ \text{null-decision} & \text{otherwise} \end{cases}$$

satisfies

(1.38) $$r(z_2;x) \leqq r(z;x) \qquad \text{for } x \in X.$$

From (1.35), (1.36) and (1.37) it follows that

(1.39) $$A_{z_2} \subseteq A_{z_1}.$$

It can be shown that a strategy $z \in Z$ is optimal if it satisfies

(1.40) $$\min_{d \in D(x)} r(d.z;x) = r(z;x) \qquad \text{for } x \in X,$$

(1.41) $$\min_{d \in D_z(x)} c(d.z;x) = c(z;x) \qquad \text{for } x \in X,$$

(1.42) $$A'_z = A_z.$$

These formulas present us a direct approach for determining an optimal strategy. However in the most cases an optimal strategy will be determined by the following iterationprocedure:

Prepatory part

Determine the (x;d)-functions k(x;d) and t(x;d).

Iterative part

Let $z = z^{(n-1)}$ be the strategy obtained at the end of the $(n-1)^{th}$ step of the iterationprocedure (start in step 1 of the iterationprocedure with an arbitrary strategy of Z). The n^{th} step runs as follows:

1) Determine the unique solution of the functional equations (1.13), (1.14) and (1.18).

2a) Determine the functions r(d.z;x) and c(d.z;x) by using the definitions (1.21) and (1.23).

b) Determine for each state $x \in X$ the set $D_z(x)$ of r(d.z;x) minimizing decisions $d \in D(x)$.

c) Minimize for each $x \in X$ the d-funxtion c(d.z;x) with respect to $d \in D_z(x)$.

d) Associate with each state x a solution of c). If z(x) is a solution of c), this decision will be associated with state x (this convention is made for reasons of convergence).

As soon as operation d) has been performed a strategy $z_1 \in Z$ has been constructed.

3) Determine the set A'_z (c.f.(1.36)). The strategy $z^{(n)} \in Z$ is defined by

$$(1.43)\qquad z^{(n)}(x) = \begin{cases} z_1(x) & \text{for } x \in A_2' \\ \text{null-decision} & \text{otherwise.} \end{cases}$$

End of the n^{th} step.

If $z^{(n)} = z^{(n-1)}$ the iterationprocedure has converged to an optimal strategy and it stops; otherwise the $(n+1)^{th}$ step of the iterationprocedure starts with strategy $z^{(n)}$.

Notes

a) Computations may be reduced considerably when it is realized that the criterionfunction $r(z;x)$ is constant on a simple ergodic set.

b) The structure of the functionalequations (1.13) and (1.14) implies that the amount of computation needed to solve the functionalequations is primarily determined by the structure of A_z and the number of simple ergodic sets. By a proper choice of the states e_j for which we punt $c(z;e_j) = 0$ (c.f. (1.18)) the solving of the functional equations may be simplified considerably. In problems with a non-denumerable state space a successful use may be made of the states in A_z with a probability concentration.

c) The way in which the set A_2' can be determined depends heavily on the structure of the decisionproblem considered. In the boundary points of this set it will sometimes be indifferent whether to intervene or not. This property may be useful to construct an optimal strategy.

The wellknown iterationprocedure of HOWARD and JEWELL can be derived from our iterationprocedure, as will be shown in section 1.2.

1.2. The models of HOWARD and JEWELL as special cases of the general case.

The so-called Markov-renewal programming model introduced by JEWELL [6] is briefly reviewed (the model of JEWELL is a direct generalisation of the models of HOWARD [5]). There is a finite imbedded Markov-chain whose states are points where the decisionmaker has to choose interventions from finite sets. These states are called the distinguished states. At other points the process cannot be affected by the decisionmaker. Each distinguished state is occupied only for a moment during which an intervention is taken. The process returns then to limbo for a random time where it remains until the next distinguished state is reached. The intervention taken determines the distribution of transit time through limbo, the distribution of costs incurred during this transit and the one-step transition probabilities for the next distinguished state. Getting specific; let $\{1,...,n\}$ be the set of the distinguished states and let $D(i)$ be the finite set of feasible interventions for state i. Let p_{ij}^{d} be the probability that the next observed state is j given the initial state i and intervention $d \in D(i)$. Let $F_{ij}^{d}(t)$ be the conditional distribution function of the time the system remains in limbo given that the next transition is to state j. It is assumed that $F_{ij}^{d}(0) = 0$, for all i, j and d (this condition can be relaxed). The first moment of $F_{ij}^{d}(t)$ will be denoted by ν_{ij}^{d}. Let

(1.44) $$\nu_{i}^{d} = \sum_{j=1}^{n} \nu_{ij}^{d} < \infty \qquad \text{for } d \in D(i),\ i = 1,...,n.$$

If for all i and d

(1.45) $$\nu_{i}^{d} = 1$$

we have the familiar discrete time case, which is firstly treated by HOWARD [5]. If for an appropriate $\lambda_{i}^{d} > 0$

(1.46) $$F_{ij}^{d}(t) = 1 - e^{-\lambda_{i}^{d} t} \qquad \text{for all } i,j,d \text{ and } t \geqq 0$$

and

(1.47) $$p_{ii}^{d} = 0 \qquad \text{for all } i \text{ and } d,$$

we have a finite continous-time Markov decision process, which is also treated by HOWARD [5].

Returning to the general case it can be verified that JEWELL's model can be translated to our model. We shall omit details. By choosing

(1.48) $$A_{0,1} = A_{0,2} = A_0 = \{1,\dots,n\}$$

we shall need of the information concerned the costs and the transition times only the expected time to transition ν_i^k and the expected costs to transition q_i^d given state i and intervention d. It can easily be verified that from our iterationprocedure the following policy iteration scheme can be derived. We describe the n^{th} step.

a) Let $z = z^{(n-1)}$ be the strategy obtained at the end of the $(n-1)^{th}$ step. Determine first the unique simple ergodic sets, say $E_1,\dots,E_m$, of the Markov-chain $((p_{ij}^{z(i)}))$. Choose in each set E_i a state e_i. Next determine the unique solution of the following system of linear equations in $r_i(z)$ and $c_i(z)$:

(1.49) $$r_i(z) = \sum_{j=1}^{n} p_{ij}^{z(i)} \; r_j(z) \qquad \text{for } 1 \leqq i \leqq n$$

(1.50) $$c_i(z) = q_i^{z(i)} - r_i(z)\,\nu_i^{z(i)} + \sum_{j=1}^{n} p_{ij}^{z(i)} \; c_j(z) \quad \text{for } 1 \leqq i \leqq n$$

(1.51) $$c_{e_i}(z) = 0 \qquad \text{for } 1 \leqq i \leqq m.$$

b) Determine for each $i = 1,\dots,n$ the set $D_z(i)$ of decisions which minimize

$$\sum_{j=1}^{n} p_{ij}^{d} \; r_j(z)$$

with respect to $d \in D(i)$. Associate with state i a decision from $D_z(i)$ which minimizes

$$(1.53) \qquad q_i^d - \sum_{j=1}^{n} p_{ij}^d \, r_j(z) \, \nu_i^d + \sum_{j=1}^{n} p_{ij}^d \, c_j(z)$$

with respect to $d \in D_z(i)$. If z(i) is such a decision, choose then z(i). In this way we obtain a strategy $z^{(n)}$.

<u>End n^{th} step</u>

If strategy $z^{(n)} = z$ then the iterationprocedure has converged to an optimal strategy and it stops; otherwise the $(n+1)^{th}$ step of the iteration-procedure starts with strategy $z^{(n)}$. It can be shown that this iteration-procedure converges in a finite number of steps to an optimal strategy, a complete proof is given by DENARDO and FOX [1].

2 A discrete productionproblem.

2.1 Introduction.

In order to satisfy the demand for a single item a firm sets up non-overlapping productionseries occasionaly. A productionserie of d units takes $T_d \geqq 0$ units of time and costs $\phi(d)$, T_d is fixed. After completion there is a fixed idle time $\tau_d \geqq 0$ during which no production can be started. When the idle time is passed the starting point of a new production can be chosen freely. Awaiting the sale the finished units are stocked. At most M units can be carried in inventory. For each unit carried in inventory there are inventory costs $c_1 t$ which depend on the length of time t for which the unit remains in inventory. Customers arrive at the firm according to a Poisson process with rate λ. Independently of the arrival process each customer demands k units with probability p_k, $k \geqq 0$. The demands of the customers are mutually independent and have a finite and positive expectation. When the demand of a customer exceeds the on hand inventory it is assumed that the overshoot is satisfied by an emergency purchase; costs c_2 per unit.

Using the expected average costs criterion for an infinite planning horizon the production manager looks for a strategy that leads to an optimal production schedule.

The solution of this problem starts in section 2.2. In this section the probabilistic background for both this problem and the next problem is given. In section 2.3 the state space, the natural process and the feasible decisions are defined. The functions k(x;d) and t(x;d) are determined in section 2.4. Section 2.5 has been devoted to an iteration-procedure for an optimal strategy. An numerical example can be found in section 2.6. Finally some generalizations are suggested in section 2.7.

2.2 Preliminaries.

Suppose customers for a single product arrive at a firm according to a Poisson process $\{\underline{w}(t), t \geq 0\}$ with rate λ. Independently of the arrival process each customer demands k units with probability p_k, $k \geq 0$. It is assumed that the demands of the customers are mutually independent random variabeles with a finite and positive expectation.

Firstly we review three important properties of the Poisson process:

(a) the number of arrivals in any time interval $(t,t+h]$ has a Poisson distribution with mean λh. Hence,

$$P\{\underline{w}(t+h) - \underline{w}(t) = n\} = e^{-\lambda h} \frac{(\lambda h)^n}{n!} \quad \text{for } n = 0,1,\ldots \quad (2.1)$$

(b) the interval from 0 up to the first arrival and thereafter the intervals between two successive arrivals, are independently distributed and have an common exponential distribution with mean $1/\lambda$.

(c) given an arbitrary but fixed point of time, the waiting time to the first future arrival has also an exponential distribution with mean $1/\lambda$, irrespective of the "past".

Define

$$\underline{v}(0) = 0 \quad (2.2)$$

and for $t > 0$

$$\underline{v}(t) = \text{number of units demanded in the interval } (0,t]. \quad (2.3)$$

Our assumptions imply (1) for any $t,s \geq 0$ the random variables $\underline{v}(t+s)-\underline{v}(t)$ and $\underline{v}(s)$ are identically distributed (2) if $0 \leq t_1 < t_2 \ldots < t_n$ $(n \geq 3)$ the differences $\underline{v}(t_2) - \underline{v}(t_1), \ldots, \underline{v}(t_n) - \underline{v}(t_{n-1})$ are mutually independent. Let

$$a_n(t) = \{P\ \underline{v}(t) = n\} \quad \text{for } n = 0,1,\ldots, t \geq 0. \quad (2.4)$$

It is wellknown that the generating function of $\underline{v}(t)$ is given by [3]

(2.5) $$g_t(s) = \sum_{n=0}^{\infty} a_n(t)s^n = e^{-\lambda t(1 - \sum_{n=0}^{\infty} p_n s^n)} \qquad \text{for } |s| \leq 1,\ t \geq 0$$

and that

(2.6) $$E\underline{v}(t) = g_t'(1) = \lambda t \sum_{n=1}^{\infty} np_n \qquad \text{for } t \geq 0.$$

The power-series expansion of the righthand member of (2.5) may enable us to write down an analytical expression for the probabilities $a_n(t)$. These probabilities can also be calculated in another way. From (2.5) it follows that with regard to the determination of the probabilities $a_n(t)$ it can be equivalently stated that the customers arrive according to a Poisson process with rate $\lambda(1-p_0)$ and that the demand per customer has $\{p_i/(1-p_0),\ i \geq 1\}$ as probability distribution. Suppose for the moment that we are dealing with this demand process. Obviously we have that

(2.7) $$a_0(t) = P\{\text{no arrival in } (0,t]\} = e^{-\lambda(1-p_0)t}$$

and

(2.8) $$a_n(t) = \sum_{k=1}^{n} P\{k \text{ customers arrive in } (0,t] \text{ and the cumulative demand of these customers is } n\} =$$

$$= \sum_{k=1}^{n} c_n(k)\ e^{-\lambda(1-p_0)t}\ \frac{(\lambda(1-p_0))^k}{k!} \qquad \text{for } n \geq 1,$$

where

(2.9) $c_n(k)$ = probability that the cumulative demand of k customers is n.

Needless to say that $c_n(k) = 0$ for $n < k$. Observe that for k fixed $\{c_n(k),\ n \geq k\}$ is the probability distribution of the sum of k independent random variables, which have $\{p_i/(1-p_0),\ i \geq 1\}$ as common probability distribution. We have for $c_n(k)$ the recursionformula

$$(2.10) \qquad c_n(k) = \sum_{i=1}^{n-k+1} c_{n-i}(k-1)\, p_i/(1-p_0) \qquad \text{for } n \geqq k,\ k \geqq 1,$$

where $c_0(0) = 1$ and $c_n = (0) = 0$ for $n \neq 0$.

Returning to the original demand process, let $\underline{t}_k$ be the interval from 0 up to the epoch on which the k^{th} unit is demanded, mathematically

$$(2.11) \qquad \underline{t}_k = \inf \{t \mid \underline{v}(t) \geqq k\} \qquad \text{for } k \geqq 1,$$

and let

$$(2.12) \qquad \underline{v}_k = \text{number of units demanded in } (0, \underline{t}_k] \qquad \text{for } k \geqq 1.$$

For notational convenience we define

$$(2.13) \qquad \underline{t}_k = \underline{v}_k = 0 \qquad \text{for } k \leqq 0.$$

We shall now derive a recursion formula for $E\underline{t}_k$. The waiting time $\underline{u}$ from 0 up to the arrival of the first customer has an exponential distribution with mean $1/\lambda$. Under the condition that the first customer demands i units the random variable $\underline{t}_k$ has the same distribution as $\underline{u} + \underline{t}_{k-i}$. By applying the theorem of total expectation,

$$(2.14) \qquad Ex = \int E(\underline{x} \mid y = y)\, dP\{\underline{y} \leqq \underline{y}\} \ ,$$

it follows that

$$(2.15) \qquad E\underline{t}_k = \sum_{i=0}^{k-1} p_i\, E\underline{t}_{k-i} + 1/\lambda \qquad \text{for } k \geqq 1.$$

Analogeously it can be shown that

$$(2.16) \qquad E\underline{v}_k = \sum_{i=0}^{k-1} p_i\, E\underline{v}_{k-i} + \sum_{i=1}^{\infty} i p_i \qquad \text{for } k \geqq 1.$$

From (2.15) and (2.16) it follows that

$$(2.17) \qquad E\underline{v}_k = \lambda \sum_{n=1}^{\infty} n p_n\, E\underline{t}_k \qquad \text{for } k \geqq 1.$$

We see that the numbers $E\underline{t}_k$ can be computed successively. Needless to say that $E\underline{t}_k = k/\lambda$ and $E\underline{v}_k = k$ if $p_1 = 1$. Observe that (2.15) and (2.16) are discrete renewal equations [3].

The generating function T(s) of the numbers $E\underline{t}_k$ is given by

$$(2.18)\qquad T(s) = \sum_{k=1}^{\infty} E\underline{t}_k\, s^k = \frac{s}{\lambda(1-s)(1-\sum_{n=0}^{\infty} p_n\, s^n)} \qquad \text{for } |s| < 1.$$

In special cases an analytical expression for $E\underline{t}_k$ may be obtained from the power-series development of T(s) (e.g. if the demand of a customer has a geometric distribution).

Define

$$(2.19)\qquad p_k(1) = p_k \qquad \text{for } k = 0,1, \dots$$

and define for each fixed $n \geqq 2$

$$(2.20)\qquad p_k(n) = \sum_{i=0}^{k} p_{k-i}(n-1)p_i \qquad \text{for } k = 0,1, \dots .$$

We can interprete $p_k(n)$ as the probability that the cumulative demand of n customers equals k. Define the renewal function

$$(2.21)\qquad u_k = \sum_{n=1}^{\infty} p_k(n) \qquad \text{for } k = 0,1, \dots .$$

We can interprete $\sum_{j=0}^{k} u_j$ as the expected number of customers before the cumulative demand exceeds k. This interpretation may be justified as follows: Let $\underline{x}_n = 1$ if the cumulative demand of the first n customers is $\leqq k$, and let $\underline{x}_n = 0$ otherwise. Then

$$E(\sum_{n=1}^{\infty} \underline{x}_n) = \sum_{n=1}^{\infty} \sum_{j=0}^{k} p_j(n) = \sum_{j=0}^{k} u_j .$$

From (2.20) and (2.21) it follows that u_k satisfies the discrete renewal equation

$$(2.22)\qquad u_k = p_k + \sum_{i=0}^{k} u_{k-i}\, p_i \qquad \text{for } k = 0,1, \dots .$$

The generating function U(s) of the numbers u_k is given by

$$(2.23) \qquad U(s) = \sum_{k=0}^{\infty} u_k s^k = \frac{\sum_{n=0}^{\infty} p_n s^n}{1 - \sum_{n=0}^{\infty} p_n s^n} \qquad \text{for } |s| < 1.$$

We note that $E\underline{t}_k$ and $E\underline{v}_k$ can be expressed in the quantities u_k. From (2.15) and (2.22) it follows easily that

$$(2.24) \qquad E\underline{t}_k = \frac{1}{\lambda}\left(1 + \sum_{j=0}^{k-1} u_j\right) \qquad \text{for } k \geqq 1,$$

as will be intuitively clear from the physical interpretation of the u_k. Define for each fixed $k \geqq 1$ the probabilities (c.f. (2.12))

$$(2.25) \qquad f_n(k) = P\{\underline{v}_k = n\} \qquad \text{for } n = k, k+1, \ldots .$$

Using a standard probabilistic argument it follows that

$$(2.26) \qquad f_n(k) = P\{\text{the demand of the first customer is } n\} + \sum_{n=1}^{\infty} \sum_{j=0}^{k-1} P\{\text{the cumulative demand of the first n customers is j and the demand of the } (n+1)^{th} \text{ customer is } n-j\} =$$

$$= p_k + \sum_{n=1}^{\infty} \sum_{j=0}^{k-1} p_n(j)\, p_{n-j} =$$

$$= p_k + \sum_{j=0}^{k-1} u_j\, p_{n-j} \qquad \text{for } n \geqq k.$$

Finally we define the ι-function

$$(2.27) \qquad \iota(x) = \begin{cases} 1 & \text{for } x \geqq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

This definition implies that

$$(2\ 28) \qquad E\,\iota(t-\underline{t}_k) = P\{\underline{t}_k \leqq t\} = P\{\underline{v}(t) \geqq k\} = 1 - \sum_{j=0}^{k-1} a_j(t) \qquad \text{for } k \geqq 1,\ t \geqq 0.$$

Next we prove that

$$(2.29)\qquad E\{(\underline{t}_k - t)\ \iota(\underline{t}_k - t)\} = \sum_{j=0}^{k-1} a_j(t)\ E\underline{t}_{k-j} \qquad \text{for } k \geq 1,\ t \geq 0.$$

Given an arbitrary but fixed point of time the waiting time to the arrival of the first future customer has the same distribution as the interval between two successive arrivals of customers, irrespective of the "past". Hence under the condition that the total demand in $(0,t]$ is j units we have that the lefthand member of (2.29) equals $E\underline{t}_{k-j}$. By applying now the theorem of total expectation (2.29) follows.

Finally we note that from (2.29) and the identity $E(\underline{t}_k - t) = E\{(\underline{t}_k - t)\ \iota(\underline{t}_k - t)\} + E\{(\underline{t}_k - t)\ \iota(t - \underline{t}_k)\}$ it follows that

$$(2.30)\qquad E\{(t - \underline{t}_k)\ \iota(t - \underline{t}_k)\} = \sum_{j=0}^{k-1} a_j(t)\ E\underline{t}_{k-j} - E\underline{t}_k + t \qquad \text{for } k \geq 1,\ t \geq 0.$$

2.3 The state space, the natural process and the feasible decisions.

At each point of time the following information will be of interest:

(a) the inventory

(b) whether a production is running or not; if a production is running the productionsize and the time that the production is already running; if there is not a production running the time elapsed since the last production has been completed and the size of that production.

We take as state space

$$(2.31)\qquad X = \{i \mid 0 \leq i \leq M\} \cup \{(i,d,t,0) \mid 0 \leq i < M,\ 1 \leq d \leq M-i,\ 0 \leq t < T_d\}$$

$$\cup\ \{(i,0,d,\tau) \mid 0 \leq i \leq M,\ 1 \leq d \leq M,\ 0 \leq t < \tau_d\}.$$

The state i corresponds to the situation that the inventory is i and that a production can be started if desired. The state (i,d,t,0) corresponds to the situation that the inventory is i and a production of d units is running since t units of time.

The state $(i,0,d,\tau)$ corresponds to the situation that the inventory is i, no production can be started, τ units of time are elapsed since the last production was completed and the size of that production is d.

Next the natural process is described. The natural process can start in each of the state space X. Hence also in a state $(i,d,t,0)$, i.e. a production is running. However in the natural process the decisionmaker does not intervene, thus in the natural process no production is started. If the natural process has i as initial state the system remains in state i until the first future customer arrives. By the demand $\underline{b}$ of that customer the system assumes state $\max(i-\underline{b},0)$. If the natural process starts in state $(i,0,d,\tau^*)$, $0 \leq \tau^* < \tau_d$ the system runs next through the states $(\max[i-\underline{v}(\tau-\tau^*);0],0,d,\tau)$, $\tau^* < \tau < \tau_d$ (recall that $\underline{v}(t)$ is the total demand in a time interval of the length t). On the moment that the idle time τ_d is passed the system assumes state $\max[i-\underline{v}(\tau_d-\tau^*);0]$. If the natural process starts in state $(i,d,t^*,0)$, $0 \leq t^* < T_d$ the system runs next through the states $(\max[i-\underline{v}(t-t^*);0],d,t,0)$, $t^* < t < T_d$. On the moment that the production is completed the system assumes state $(\max[i-\underline{v}(T_d-t^*);0],0,d,0)$.

Finally the decisionmechanism is described. The decisionmaker can only intervence in the states $0,1,\ldots,M-1$. In the other states only null-decisions can be made. It is no restriction to assume that in state 0 the decisionmaker has always to intervene. Put for abbreviation

(2.32) $$I = \{i \mid 0 \leq i \leq M\}.$$

A decision will be represented by the size of the production (d = 0 is the null-decision). We take the set of feasible decisions in state x equal to

(2.33) $$D(x) = \begin{cases} \{d \mid 1 \leq d \leq M\} & \text{for } x = 0 \\ \{d \mid 0 \leq d \leq M-i\} & \text{for } x = i,\ 1 \leq i < M \\ \{\text{null-decision}\} & \text{for } x = M \text{ or } x \notin I. \end{cases}$$

If in state i the intervention $d \geq 1$ is made the system is transferred instantaneously into state (i,d,0,0). By the null-decision made in state x the system is "transferred" into state x itself.

For each strategy z from the class Z of stationary strategies we have that the set A_z of interventionstates is finite, because $A_z \subset I$. Furthermore it will obvious that

(2.34) $$A_0 = \bigcap_{z \in Z} A_z = \{0\}.$$

2.4 Determination of the functions k(x;d) and t(x;d).

The set A_0 consists of the state 0 only, so we choose

(2.35) $$A_{0,1} = A_{0,2} = \{0\}.$$

Hence both the walks $\underline{w}_{0,1}$ and $\underline{w}_{0,2}$ and the walks $\underline{w}_{d,1}$ and $\underline{w}_{d,2}$ are identical. Put for abbreviation $\underline{w}_0 = \underline{w}_{0,1} = \underline{w}_{0,2}$ and $\underline{w}_d = \underline{w}_{d,1} = \underline{w}_{d,2}$. The function k(x;d) is the difference in expected costs incurred during $\underline{w}_d$ and $\underline{w}_0$ and the function t(x;d) represents the difference in expected duration of $\underline{w}_d$ and $\underline{w}_0$ (c.f. (1.3) and (1.4)). To determine these functions we define

(2.36) $q_{ij}(d)$ = probability that the inventory is j on the moment that the idle time associated with a production of d units started in state i has been passed

for $i = 0,\dots,M-1$; $d \in D(i)$ and $d \geq 1$.

For notational convenience we define for $i,j = 0,\dots,M$

(2.37) $$q_{ij}(0) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

Using the fact that the numbers of units demanded in disjoint periods of time are mutually independent it is easily seen that

$$(2.38) \quad q_{ij}(d) = \sum_{k=0}^{i-1} a_k(T_d)a_{i-k+d-j}(\tau_d) + \sum_{k=i}^{\infty} a_k(T_d)a_{d-j}(\tau_d)$$

for $j = 1,\ldots,i+d;\ d \geq 1,\ i = 0,\ldots,M-1$

and

$$(2.39) \quad q_{i0}(d) = \sum_{k=0}^{i-1} a_k(T_d) \sum_{h=i-k+d}^{\infty} a_h(\tau_d) + \sum_{k=i}^{\infty} a_k(T_d) \sum_{h=d}^{\infty} a_h(\tau_d)$$

for $d \geq 1,\ i = 0,\ldots,M-1,$

where

$$(2.40) \quad a_n(t) = 0 \qquad \text{for } n < 0,\ t \geq 0.$$

Consider the walk $\underline{w}_0$ having i as initial state. During the walk $\underline{w}_0$ the system is subjected to the natural process and the walk ends as soon as the system assumes state 0. It will be obvious that in the walk $\underline{w}_0$ only inventory costs and stockout costs are incurred. The expectation of these costs are given by (c.f. (2.11), (2.12) and (2.13)):

$$(2.41) \quad k_0(i) = c_1 \sum_{k=1}^{i} E\underline{t}_k + c_2\ E(\underline{v}_i - i) \qquad \text{for } 0 \leq i < M.$$

The expected duration of the walk $\underline{w}_0$ is equal to

$$(2.42) \quad t_0(i) = E\underline{t}_i \qquad \text{for } 0 \leq i < M.$$

For the walk $\underline{w}_d$ having i as initial state we have that the intervention $d \geq 1$ in state i transfers the system into state (i,d,0,0). The costs of the intervention d are $\phi(d)$. From state (i,d,0,0) on the behaviour of the system during the walk $\underline{w}_d$ is described by the natural process having (i,d,0,0) as initial state. The walk $\underline{w}_d$ ends as soon as the system assumes state 0. To determine the expected stockout costs incurred during the walk $\underline{w}_d$ we consider three phases of the walk, namely the production time T_d, the idle time τ_d and the remainder of the walk.

It is now easily seen that the expected stockout costs incurred during the walk $\underline{w}_d$ are given by (c.f. (2.3) (2.4) and (2.36)):

$$(2.43)\qquad c_2 \sum_{k=i}^{\infty} (k-i)a_k(T_d) + $$
$$+ c_2\{ \sum_{k=0}^{i-1} a_k(T_d) \sum_{j=i-k+d}^{\infty} (j-i+k-d)\ a_j(\tau_d) + $$
$$+ \sum_{k=i}^{\infty} a_k(T_d) \sum_{j=d}^{\infty} (j-d)\ a_j(\tau_d)\} + $$
$$+ c_2 \sum_{j=1}^{i+d} q_{ij}(d)\ E(\underline{v}_j-j).$$

It is easily verified that the expected inventory costs incurred during the walk $\underline{w}_d$ are given by

$$(2.44)\qquad c_1\{ \sum_{k=1}^{i} E\underline{t}_k + \sum_{k=0}^{i-1} a_k(T_d) \sum_{j=i-k+1}^{i-k+d} E\underline{t}_j + \sum_{k=i}^{\infty} a_k(T_d) \sum_{j=1}^{d} E\underline{t}_j\}.$$

Hence the expected costs incurred during the walk $\underline{w}_d$ are given by

$$(2.45)\qquad k_1(i;d) = \phi(d) + (2.43) + (2.44) \qquad \text{for } 1 \leq d \leq M-i,\ 0 \leq i < M.$$

It is easily seen that the expected duration of $\underline{w}_d$ is equal to

$$(2.46)\qquad t_1(i;d) = T_d + \tau_d + \sum_{j=1}^{i+d} q_{ij}(d)\ E\underline{t}_j \qquad \text{for } 1 \leq d \leq M-i,\ 0 \leq i < M.$$

Hence

$$(2.47)\qquad k(i;d) = \phi(d) + c_1\{ \sum_{k=0}^{i-1} a_k(T_d) \sum_{j=i-k+1}^{i-k+d} E\underline{t}_j + \sum_{k=i}^{\infty} a_k(T_d) \sum_{j=1}^{d} E\underline{t}_j\} + $$
$$+ c_2\{ \sum_{k=i}^{\infty} \left[k-i+ \sum_{j=d}^{\infty} (j-d)a_j(\tau_d)\right] a_k(T_d) + $$
$$+ \sum_{k=0}^{i-1} a_k(T_d) \sum_{j=i-k+d}^{\infty} (j-i+k-d)\ a_j(\tau_d) + $$
$$+ \sum_{j=1}^{i+d} q_{ij}(d)\ E(\underline{v}_j-j) - E(\underline{v}_i-i)\}$$
$$\text{for } 1 \leq d < M-i,\ 0 \leq i < M-1,$$

and

$$(2.48) \qquad t(i;d) = T_d + \tau_d + \sum_{j=1}^{i+d} q_{ij}(d)\ E\underline{t}_j - E\underline{t}_i$$

for $1 \leqq d \leqq M-i,\ 0 \leqq i < M-1$.

Furthermore we have (c.f. (1.5))

$$(2.49) \qquad k(x;d) = t(x;d) = 0 \qquad \text{if } d \text{ is a null-decision, } x \in X.$$

Finally we note that the infinite summations in (2.47) can be reduced to finite ones by using the identity (c.f. (2.6)).

$$(2.50) \qquad \sum_{k=i}^{\infty} (k-i)\ a_k(t) = \lambda t \sum_{n=1}^{\infty} np_n - i + \sum_{k=0}^{i-1} (i-k)\ a_k(t).$$

2.5 Determination of an optimal strategy by an iterationprocedure.

Define the one-step transition probabilities

(2.51) $p_{xy}(z)$ = probability that y is the first future interventionstate taken on by the system when the initial state is x and strategy $z \in Z$ is applied.

Needless to say that $p_{xy}(z) = 0$ for $y \notin A_z$. Recall that

$$(2.52) \qquad A_z \subset I = \{i \mid 0 \leqq i \leqq M\} \qquad \text{for each } z \in Z.$$

The functionalequations (1.13) and (1.14) become for our problem (c.f. (1.16))

$$(2.53) \qquad r(z;i) = \sum_{j \in A_z} p_{ij}(z)\ r(z;j) \qquad \text{for } i \in A_z,$$

$$(2.54) \qquad c(z;i) = k(i;z(i)) - r(z;i)\ t(i;z(i)) + \sum_{j \in A_z} p_{ij}(z)\ c(z;j) \qquad \text{for } i \notin A_z,$$

$$(2.55) \qquad r(z;x) = \sum_{j \in A_z} p_{xj}(z)\ r(z;j) \qquad \text{for } x \notin A_z,$$

and

$$(2.56)\quad c(z;x) = \sum_{j \in A_z} p_{xj}(z)\ c(z;j) \qquad \text{for } x \in A_z.$$

Next we specify the functions $r(d.z;x)$ and $c(d.z;x)$ (c.f. (1.21) and (1.23)). We have (c.f. (1.25))

$$(2.57)\quad r(d.z;x) = r(z;x)\ ;\ c(d.z;x) = c(z;x) \quad \text{if } d \text{ is a null-decision.}$$

Only in the states i, $0 \leq i < M$ can be intervened. When in state i the intervention $d \geq 1$ is made the system is transferred into state $(i,d,0,0)$. Hence for $1 \leq d < M-i$, $0 \leq i < M$ we have that

$$(2.58)\quad r(d.z;i) = r(z;(i,d,0,0))$$

and

$$(2.59)\quad c(d.z;i) = k(i;d) - r(d.z;i)\ t(i;d) + c(z;(i,d,0,0)).$$

From the definitions (2.36) and (2.51) it will be obvious that

$$(2.60)\quad p_{(i,d,0,0),j}(z) = q_{ij}(d) + \sum_{k \in A_z} q_{ik}(d)\ p_{kj}(z) \qquad \text{for } j \in A_z.$$

Hence (use (2.55))

$$(2.61)\quad r(d.z;i) = \sum_{j \in A_z} \{q_{ij}(d) + \sum_{k \notin A_z} q_{ik}(d)\ p_{kj}(z)\}\ r(z;j) =$$

$$= \sum_{j \in A_z} q_{ij}(d)\ r(z;j) + \sum_{k \notin A_z} q_{ik}(d) \sum_{j \in A_z} p_{kj}(z)\ r(z;j) =$$

$$= \sum_{j \in A_z} q_{ij}(d)\ r(z;j) + \sum_{k \notin A_z} q_{ik}(d)\ r(z;k) =$$

$$= \sum_{j=0}^{i+d} q_{ij}(d)\ r(z;j) \qquad \text{for } 1 \leq d \leq M-i,\ 0 \leq i < M.$$

In the same way it can be shown that for $1 \leq d \leq M-i$, $0 \leq i < M$

$$(2.62)\quad c(d.z;i) = k(i;d) - r(d.z;i) + \sum_{j=0}^{i+d} q_{ij}(d)\ c(z;j).$$

From (2.37), (2.49) and (2.57) it follows that formulas (2.61) and (2.62) are also true for $d = 0$.

Using the fact that in the states $x \notin I = \{i \mid 0 \leq i \leq M\}$ only null-decisions can be made (i.e. $A_z \subset I$ for each $z \in Z$) and using formulas (1.27) up to and including (1.37) and formulas (2.53) up to and including (2.62) it can be easily seen that we can restrict ourselves to the states of I for solving the problem. From now on only the states $0, \ldots, M$ are considered. In order to derive formulas for $p_{ij}(z)$, $r(A.\hat{z};i)$ and $c(A.\hat{z};i)$ we shall introduce probabilities which depend on the natural process only. Let S be a set of states such that

(2.63) $\quad S \subset I \quad \text{and} \quad 0 \in S.$

Define

(2.64) $\beta_{ij}(S)$ = probability that j is the first state in the set S taken on by the system when the system starts in state i and is subjected to the natural process.

Because state $0 \in S$ we have that

(2.65) $$\sum_{j \in S} \beta_{ij}(S) = 1.$$

The definition implies that $\beta_{ij}(S) = 0$ for $j > i$ or $j \notin S$. If $i \in S$ then $\beta_{ii}(S) = 1$. Observe that if $p_0 + p_1 = 1$ (each customer demands at most one unit) $\beta_{ij}(S) = 1$ for the largest integer $j \in S$ which is smaller than or equal to i. Observe furthermore that in case the set $S = \{i \mid 0 \leq i \leq s\}$ for some integer s the probability $\beta_{ij}(S)$ equals $f_{i-j}(i-s)$ for $i \notin S$ and $1 \leq j \leq s$ (c.f. (2.25)). However if the set S has not this simple form we have to determine $\beta_{ij}(S)$ in another way. The following recursionformula will be obvious

(2.66) $$\beta_{ij}(S) = p_{i-j} + \sum_{k=0}^{i-j-1} p_k \, \beta_{i-k,j}(S) \quad \text{for } i \notin S,\ j \in S,\ j \geq 1.$$

From (2.65) it follows that

(2.67) $$\beta_{i0}(S) = 1 - \sum_{j \geq 1} \beta_{ij}(S).$$

We are now in position to give formulas for $p_{ij}(z)$, $r(A.\hat{z};i)$ and $c(A.\hat{z};i)$. For the probabilities $p_{ij}(z)$, $j \in A_z$, we have

$$(2.68) \qquad p_{ij}(z) = \begin{cases} \beta_{ij}(A_z) & \text{for } i \notin A_z \\ q_{ij}(z(i)) + \sum_{k \notin A_z} q_{ik}(z(i))\, \beta_{kj}(A_z) & \text{for } i \in A_z . \end{cases}$$

For the functions $r(A.\hat{z};i)$ and $c(A.\hat{z};i)$ we have (recall that A has to satisfy $A_0 \subseteq A \subseteq A_{z_1}$, thus $0 \in A$)

$$(2.69) \qquad r(A.\hat{z};i) = \sum_j \beta_{ij}(A)\, r(\hat{z};j) \text{ and } c(A.\hat{z};i) = \sum_j \beta_{ij}(A)\, c(\hat{z};j)$$

$$\text{for } i \in A_{z_1} .$$

We note that $r(A.\hat{z};i) = r(\hat{z};i)$ and $c(A.\hat{z};i) = c(\hat{z};i)$ if $i \in A$.
We shall now construct the set $A'_{\hat{z}}$ (c.f. (1.36)). Firstly we observe that the set A_{z_1} contains only a finite number of states because $A_{z_1} \subset I$. Hence the class $\mathfrak{X}(\hat{z})$ consists of a finite number of sets. It can be shown that the intersection of any finite number of sets from $\mathfrak{X}(\hat{z})$ belongs also to this class. Hence

$$(2.70) \qquad A'_{\hat{z}} \in \mathfrak{X}(\hat{z}) .$$

In other words $A'_{\hat{z}}$ is the smallest subset of A_{z_1} which belongs to $\mathfrak{X}(\hat{z})$. This property and the fact that in the natural process from a state i only states $j < i$ are reached will be used to construct the set $A'_{\hat{z}}$. The construction will now be described. We begin with a set S which consists initially of the state 0 only. The set S may be enlarged by the following procedure. Starting in state 0 we go from state to state in A_{z_1} such that we have visited each state $j < i$ before we visit state i. If at the visit in state i it appears that we have (c.f. (1.33) and (1.34))

either

$$(2.71) \qquad r(S.\hat{z};i) > r(\hat{z};i)$$

or

(2.72) $$r(S.\hat{z};i) = r(\hat{z};i) \; ; \; c(S.\hat{z};i) > c(\hat{z};i)$$

then the set S is enlarged with state i, otherwise the set S remains unaltered (observe that if i is added to S that then $r(S.\hat{z};i) = r(\hat{z};i)$ and $c(S.\hat{z};i) = c(\hat{z};i)$). When we have visited each state of A_{z_1} the finally obtained set $S = S^*$ equals $A'_{\hat{z}}$, because the above construction implies that $S^* \in \mathcal{X}(\hat{z})$ and that by removing any state from the set S^* the remaining set does not belong to the class $\mathcal{X}(\hat{z})$. Hence S is the smallest subset of A_{z_1} which belongs to $\mathcal{X}(\hat{z})$, in other words $S^* = A'_{\hat{z}}$.

We are now in position to specify the on pages 9 and 10 described iteration procedure for this problem.

<u>Prepatory part</u>

Compute the values of the functions $k(i;d)$ and $t(i;d)$ and the probabilities $q_{ij}(d)$ by using (2.37), (2.38), (2.39), (2.47), (2.48) and (2.49).

<u>Iterative part</u>

Let $z = z^{(n-1)}$ be the strategy obtained at the end of the $(n-1)^{th}$ step. The n^{th} step runs as follows.

1) Determine first the unique simple ergodic sets, say $E_1, \ldots, E_m$, of the Markov-chain $((p_{ij}(z)))$, $i,j \in A_z$. Choose in each set E_i a state e_i. Next determine the unique solution of the following system of linear equations in $r(z;i)$ and $c(z;i)$, $i \in A_z$:

$$r(z;i) = \sum_{j \in A_z} p_{ij}(z)\, r(z;j)$$

(2.73) $$c(z;i) = k(i;z(i)) - r(z;i)\, t(i;z(i)) + \sum_{j \in A_z} p_{ij}(z)\, c(z;j)$$

$$c(z;e_i) = 0 \qquad \text{for } 1 \leq i \leq m.$$

If these lineair equations are solved, determine next

(2.74) $r(z;i) = \sum_{j \in A_z} p_{ij}(z)\, r(z;j)$; $c(z;i) = \sum_{j \in A_z} p_{ij}(z)\, c(z;j)$

for $i \notin A_z$.

2) Determine for each $i = 0,1,\ldots,M-1$ the set $D_z(i)$ of decisions which minimize

$$(2.75) \qquad r(d.z;i) = \sum_{j=0}^{i+d} q_{ij}(d)\, r(z;j)$$

with respect to $d \in D(i)$. Associate with state i a decision d from $D_z(i)$ which minimizes

$$(2.76) \qquad c(d.z;i) = k(i;d) - r(d.z;i)\, t(i;d) + \sum_{j=0}^{i+d} q_{ij}(d)\, c(z;j)$$

with respect to $d \in D_z(i)$. If $z(i)$ is such a decision, choose then $z(i)$. In this way we obtain strategy z_1.

3) Let the set S consists initially of the state 0 only. Starting in state 0 go from state to state in A_{z_1} such that each state $j < i$ has been visited before state i is visited. If at the visit in state i it appears that

either

$$(2.77) \qquad r(S.\hat{z};i) > r(\hat{z};i)$$

or

$$(2.78) \qquad r(S.\hat{z};i) = r(\hat{z};i) \quad \text{and} \quad c(S.\hat{z};i) > c(\hat{z};i)$$

then state i is added to the set S, otherwise the set S remains unaltered. When each state of A_{z_1} is visited the obtained set S equals $A'_{\hat{z}}$. Define the strategy $z^{(n)}$ as follows

$$z^{(n)}(i) = \begin{cases} z_1(i) & \text{if } i \in A'_{\hat{z}} \\ \text{null-decision} & \text{otherwise.} \end{cases}$$

<u>End n^{th} step</u>

Notes

a) We recall that $r(z;i) = r(z;j)$ if the states i and j belong to the same simple ergodic set. If the Markov-chain $((p_{ij}(z)))$, $i,j \in A_z$ has only one simple ergodic set we have that the criterionfunction $r(z;i)$ is constant, say $r(z)$, on the set of all states. We have in that case

(2.79) $\quad r(z;i) = r(d.z;i) = r(z) \qquad$ for $d \in D(i)$, $i = 0,\ldots,M$.

Furthermore the iterationprocedure is then simplified considerably, because in step 2) we have only to minimize $c(d.z;i)$ with respect to $d \in D(i)$ and in step 3) we have only to compare $c(S.\hat{z};i)$ and $c(\hat{z};i)$.

b) If

(2.80) $\quad \tau_d > 0 \qquad$ for $d = 1,\ldots,M$

each Markov-chain $((p_{ij}(z)))$, $i,j \in A_z$ has only one simple ergodic set, because state 0 can be reached from every other state.

5.6 Numerical example

Suppose the following numerical data are given

(2.81) $\quad \lambda = 1,\ p_1 = 1,\ c_1 = 1,\ c_2 = 10,\ M = 4,$
$\qquad \phi(d) = 4d,\ T_d = 1,\ \tau_d = \frac{1}{2}$ for $d = 1,\ldots,4$.

The values of the strategy-independent quantities $k(i;d)$, $t(i;d)$ and $q_{ij}(d)$ are given in tabel 2.1. Before starting the iterationprocedure we note that the probabilities $\beta_{ij}(S)$ are easy to determine. Since $p_1 = 1$ we have that $\beta_{ij}(S) = 1$ for the largest integer $j \in S$ which is $\leq i$. Since $\tau_d > 0$ for all d notes a) and b) apply. In step 2) of the iterative part we have only to minimize $c(d.z;i)$ with respect to $d \in D(i)$ and in step 3) we have only to compare the $c(\hat{z};i)$ (observe that $c(S.\hat{z};i) = c(\hat{z};j)$ for the largest integer $j \in S$ which is less than i).

i	d	$q_{i0}(d)$	$q_{i1}(d)$	$q_{i2}(d)$	$q_{i3}(d)$	$q_{i4}(d)$	k(i;d)	t(i;d)
0	1	.393	.607	0	0	0	16.065	2.107
	2	.090	.303	.607	0	0	21.163	3.016
	3	.014	.076	.303	.607	0	28.019	4.002
	4	.002	.012	.076	.303	.607	36.002	5.000
1	0	1	0	0	0	0	0	0
	1	.282	.495	.223	0	0	9.780	1.441
	2	.062	.220	.495	.223	0	15.525	2.379
	3	.010	.052	.220	.495	.223	22.795	3.369
2	0	0	0	1	0	0	0	0
	1	.142	.300	.335	.223	0	8.225	1.139
	2	.030	.112	.300	.335	.223	15.766	2.109
3	0	0	0	0	1	0	0	0
	1	.054	.137	.251	.335	.223	8.852	1.036

Tabel 2.1

The quantities k(i;d), t(i;d) and $q_{ij}(d)$.

We start the iterationprocedure with strategy

(2.82) $z = (z(0), z(1), z(2), z(3), z(4)) = (1,0,0,0,0).$

1^{st} step

a) The solution of the system of lineair equations

(2.83)
$$c(z;0) = 16.065 - 2.107\ r(z) + c(z;0)$$
$$c(z;0) = 0$$

is given by

(2.84) $r(z) = 7.60 \ ; \ c(z;0) = 0.$

The values of the other $c(z;i)$ are

(2.85) $\quad c(z;1) = c(z;2) = c(z;3) = c(z;4) = c(z;0) = 0.$

b) The values of the test quantity

(2.86) $$c(d.z;i) = k(i;d) - r(z)\ t(i;d) + \sum_{j=0}^{i+d} q_{ij}(d)\ c(z;j)$$

are given in tabel 2.2.

i	d	c(d.z;i)	
0	1	0	
	2	-1.76	
	3	-2.40	←
	4	-2.00	
1	0	0	
	1	-1.17	
	2	-2.56	
	3	-2.81	←
2	0	0	
	1	-.43	←
	2	-.27	
3	0	0	←
	1	.98	

Tabel 2.2

The test quantity c(d.z;i).

Hence the minimalization of $c(d.z;i)$ with respect to $d \in D(i)$ results in the strategy

(2.87) $$z_1 = (3,3,1,0,0)$$

In addition we find

(2.88) $\quad c(\hat{z};0) = -2.40\ ;\ c(\hat{z};1) = -2.81\ ;\ c(\hat{z};z) = -.43.$

c) Since $c(\hat{z};0) > c(\hat{z};1)$ state 1 belongs to $A'_{\hat{z}}$. Since $c(\hat{z};1) \leqq c(\hat{z};2)$ is, state 2 does not belong to the set $A'_{\hat{z}}$. Hence

(2.89) $$A'_{\hat{z}} = \{0,1\}$$

and

(2.90) $$z^{(1)} = (3,3,0,0,0).$$

<u>End 1st step</u>

<u>2nd step</u>

a) Let z be the strategy $z^{(1)} = (3,3,0,0,0)$. The solution of the system of lineair equations

(2.91)
$$\begin{aligned} c(z;0) &= 28.019 - 4.002\ r(z) + 0.014\ c(z;0) + 0.986\ c(z;1) \\ c(z;1) &= 22.795 - 3.369\ r(z) + 0.010\ c(z;0) + 0.990\ c(z;1) \\ c(z;1) &= 0 \end{aligned}$$

is given by

(2.93) $r(z) = 6.76$; $c(z;0) = .97$; $c(z;1) = 0.$

The values of the other $c(z;1)$ are

(2.92) $c(z;2) = c(z;3) = c(z;4) = c(z;1) = 0.$

b) The minimization of the test quantity $c(d.z;i)$ with respect to $d \in D(i)$ results in the strategy

(2.94) $$z_1 = (3,2,0,0,0).$$

In addition we find

(2.95) $c(\hat{z};0) = .97$; $c(\hat{z};1) = -.62.$

c) Since $c(\hat{z};0) > c(\hat{z};1)$ state 1 belongs to $A'_{\hat{z}}$. Hence

(2.96) $$A'_{\hat{z}} = \{0,1\}$$

and

(2.97) $$z^{(2)} = (3,2,0,0,0).$$

<u>End 2^{nd} step</u>

<u>3^{rd} step</u>

a) Let strategy z be equal to $z^{(2)}$. The solution of the system of lineaire equations

$$\begin{aligned} c(z;0) &= 28.019 - 4.002\ r(z) + 0.014\ c(z;0) + 0.986\ c(z;1) \\ c(z;1) &= 15.525 - 2.379\ r(z) + 0.062\ c(z;0) + 0.938\ c(z;1) \\ c(z;1) &= 0 \end{aligned} \qquad (2.98)$$

is given by

(2.99) $r(z) = 6.47 \quad c(z;0) = 2.16\ ;\ c(z;1) = 0.$

The other $c(z;i)$, $i = 2,3,4$ are equal to $c(z;1) = 0$.

b) The minimization of $c(d.z;i)$ with respect to $d \in D(i)$ results in the strategy

(2.100) $$z_1 = (3,2,0,0,0).$$

In addition we find that

(2.101) $c(\hat{z};0) = 2.16\ ;\ c(\hat{z};0) = 0.$

c) Since $c(\hat{z};0) > c(\hat{z};1)$ state 1 belongs to $A_{\hat{z}}'$. Hence

(2.102) $$A_{\hat{z}}' = \{0,1\}$$

and

(2.103) $$z^{(3)} = (3,2,0,0,0).$$

<u>End 3^{rd} step</u>

Since $z^{(3)} = z^{(2)}$ the iterationprocedure has converged to the optimal strategy

(2.104) $$z^{*} = (3,2,0,0,0).$$

5.7 Generalizations.

1) In the problem considered we have assumed that the production times and idle times are fixed. Assume now that a productionserie of d units takes a random time $\underline{T}_d$ and that the associated idle time is a random variable $\underline{\tau}_d$, which is independent of $\underline{T}_d$. It is supposed that the random variables $\underline{T}_d$ and $\underline{\tau}_d$ are independent of the demand process. It is easily seen that this new problem can be solved analogeously. Let $F_d(t)$ and $G_d(t)$ be the distributionfunctions of $\underline{T}_d$ and $\underline{\tau}_d$ respectively. In order to obtain the appropriate formulas for the new problem we have to replace both in the formulas for k(i;d) and t(i;d) and in the formulas for the probabilities $q_{ij}(d)$

(2.105) $\quad a_k(T_d) \quad$ by $\quad \displaystyle\int_0^\infty a_k(t)\, F_d(dt)$

(2.106) $\quad a_k(\tau_d) \quad$ by $\quad \displaystyle\int_0^\infty a_k(\tau)\, G_d(d\tau)$

(2.107) $\quad T_d$ and τ_d by $E\underline{T}_d$ and $E\underline{\tau}_d$ respectively.

2) The single-item productionproblem considered can be generalized to the following multi-item problem. Consider a firm which manufactures n different items. The firm sets up occasionaly non-overlapping productionseries. A productionserie may consist of more than one item. A feasible productionserie $(d) = (d_1,\ldots,d_n)$ (that means $d_i \geqq 0$ units of item i are produced) takes $\underline{T}_{(d)}$ units of time and costs $\phi((d))$. After completion there is a random idle time $\underline{\tau}_{(d)}$ during which no production can be started. When the idle time is passed the starting point of a new productionserie can be freely chosen. The times between demands for item i are generated by a Poisson process with rate λ_i, $i = 1,\ldots,n$. These n Poisson processes are assumed to be mutually independent. A random number of units are demanded each time a demand occurs. Excess demand is satisfied by an emergency purchase. For each item separately we have the same assumptions about the inventory and stockout costs as in the single-item problem considered.

This multi-item productionproblem can be solved in an analogous way as the single-item problem has been solved. However the state space becomes soon too large for numerical computations.

3. (S,s)-strategies for continuous time inventory models.

3.1 Introduction.

Consider an inventory model which can be described as follows. A Poisson proces with rate λ generates the times between demands and the number of units demanded per demand is a discrete random variable with probability distribution $\{p_i, i \geq 0\}$. The sizes of the demands are both mutually independent and independent of the Poisson process. Each time a demand occurs the decision is made whether or not to place an order. The procurement lead time is a constant $\tau > 0$. The total cost of placing an order for k units is $\phi(k)$. The costs of carrying a unit in inventory are directly proportional to the length of time for which the unit remains in inventory. The constant of proportionality is c_1. In section 3.2 it is assumed that each unit demanded which cannot be met from the on hand inventory is backordered. For each unit backordered there is a fixed cost c_2 plus a variable cost $c_3 t$ which depends on the length of time t for which the backorder exists. The subsequent delivery discipline is "first-come-first-served" and each unit backordered is delivered subsequently on the moment that there is on hand inventory available. The operating doctrine for the system is based on the economic inventory, which is defined as the inventory on hand plus on order minus backorders. The operating doctrine considered is an (S,s)-strategy ($S > s \geq 0$), i.e. if the inventory level falls to i, $i \leq s$ on some demand, a quantity S-i is ordered, otherwise no order is placed. An explicit expression for the expected average costs per unit of time is obtained in section 3.2.3. In section 3.2.4 the special case that units are demanded one at a time is considered. The (S,s)-strategy becomes then the familiar (Q,s)-strategy, i.e. if the economic inventory reaches the reorder level s a quantity $Q = S-s$ is ordered. Inequalities satisfied by the optimal Q^* and s^* are given.
In section 3.3 it is assumed that the units are demanded one at a time and that excess demand is lost. Furthermore it is stipulated that there is never more than a single outstanding order.

However the procurement lead time is taken random. The (Q,s)-strategy ($Q \geq s$) is considered, i.e. if the on hand inventory assumes the level s, a quantity $Q \geq s$ is ordered. An explicit expression for the average costs per unit of time is obtained in section 3.3. In addition inequalities satisfied by the optimal Q^* and s^* are given.

We note that the probabilistic results derived in section 2.2 of chapter 2 are also needed in this chapter.

3.2 The (S,s)-strategy for the backorder case.

3.2.1 Definition of the state space, the natural process and the feasible decisions.

Suppose S and s fixed, $S > s \geq 0$. We take as state space

$$X = \{i \mid i \text{ integer}, i \leq S\}. \tag{3.1}$$

The state i corresponds to the situation that the economic inventory is i. We shall see in sections 3.2.2 and 3.2.3 that by a proper choice of the set $A_{0,1}$ we can confine ourselves to this simple state space.
The natural process is also very simple. If the natural process has i as initial state the system remains in that state until the first future demand occurs. In case the size of that demand is b the system is transferred into state i-b.

Next the feasible decisions are defined. Each decision will be defined by the economic inventory just after the decision. We take the set of feasible decisions in state i equal to

$$D(i) = \begin{cases} \{S\} & \text{for } i \leq s \\ \{i\} & \text{for } s < i \leq S. \end{cases} \tag{3.2}$$

Hence in state $i \leq s$ the only feasible decision is to order a quantity S-i and in state i, $s < i \leq S$ only the null-decision can be made. If in state $i \leq s$ the intervention $d = S$ is made the system is transferred into state S. The null-decision $d = i$ made in state i "transfers" the system into state i itself. By definition (3.2) the class Z of stationary strategies consists of the strategy $z = (S,s)$ only. Obviously (c.f. (1.1))

(3.3) $$A_0 = \{i \mid i \leq s\}.$$

3.2.2 Determination of the functions k(i;d) and t(i;d).

We choose (c.f. (1.2))

(3.4) $$A_{0,1} = \{i \mid i \leq 0\}$$

and

(3.5) $$A_{0,2} = \{i \mid i \leq s\}.$$

The walk $\underline{w}_{0,1}$ (resp. $\underline{w}_{0,2}$) having i as initial state ends as soon as the economic inventory assumes a value ≤ 0 (resp. $\leq s$) and during this walk the system is subjected to the natural process. In the initial state i of the walk $\underline{w}_{d,1}$ (resp. $\underline{w}_{d,2}$) decision d is made by which the system is transferred into state d. From that state on the behaviour of the system during the walk $\underline{w}_{d,1}$ (resp. $\underline{w}_{d,2}$) is described by the natural process. The walk $\underline{w}_{d,1}$ (resp. $\underline{w}_{d,2}$) ends as soon as the economic inventory assumes a value ≤ 0 (resp. $\leq s$). The function k(i;d) is the difference in expected costs incurred during the walks $\underline{w}_{d,1}$ and $\underline{w}_{0,1}$. The function t(i;d) is the difference in expected duration of the walks $\underline{w}_{d,2}$ and $\underline{w}_{0,2}$. In the states i, $s < i \leq S$ only the null-decision can be made, hence for these states we have that $k(i;i) = t(i;i) = 0$.

For each backorder it is known in advance the length of time the backorder remains on the books. We agree that the backorder costs associated with a unit backordered are incurred on the moment the backorder arises. Consider now the walks $\underline{w}_{0,1}$ and $\underline{w}_{S,1}$ having $i \leq s$ as initial state. Some reflection shows that the fact the leadtime is constant, the choice of $A_{0,1}$ and the assumptions made about the inventory costs and the backorder costs have as a consequence that in the two walks both the expected inventory costs associated with the i units from the initial economic inventory i and the expected backorder costs associated with these units are the same.

Because state $i \leq 0$ belongs to $A_{0,1}$ no costs are incurred during the walk $\underline{w}_{0,1}$ having $i \leq 0$ as initial state. The walk $\underline{w}_{0,1}$ having $i > 0$ as initial state ends in state $i - \underline{v}_i$ of $A_{0,1}$ (c.f. (2.12)). The expectation of the backorder costs incurred in the walk $\underline{w}_{0,1}$ on the moment the system enters $A_{0,1}$ is thus equal to

$$(3.6) \qquad (c_2+c_3\tau)\ E(\underline{v}_i-i) \qquad \text{for } i > 0.$$

On the moment the walk $\underline{w}_{S,1}$ starts there are $-\min(i,0)$ backorders on the books. These backorders will be filled by $-\min(i,0)$ units from the order of $S-i$ units placed at the start of the walk $\underline{w}_{S,1}$. Consider now the other $S-i+\min(i,0) = S-\max(i,0)$ units from that order. For a unit belonging to this lot of units there are inventory (resp. backorder) costs incurred during the walk $\underline{w}_{S,1}$ if and only if that unit is needed after (resp. before) the delivery of the last order. The expectation of the inventory (resp. backorder) costs associated with the $S-\max(i,0)$ units considered is equal to (c.f. (2.11) and (2.27))

$$(3.7) \qquad \sum_{k=\max(i,0)+1}^{S} E\{(\underline{t}_k-\tau)\,\iota(\underline{t}_k-\tau)\}$$

resp.

$$(3.8) \qquad \sum_{k=\max(i,0)+1}^{S} E\{(c_2 + c_3(\tau-\underline{t}_k))\,\iota(\tau-\underline{t}_k)\}.$$

Observe that (3.7) is equal to the difference in expected inventory costs incurred during the walks $\underline{w}_{S,1}$ and $\underline{w}_{0,1}$. The walk $\underline{w}_{S,1}$ ends in state $S - \underline{v}_S$ of $A_{0,1}$. The expected backorder costs incurred on the moment the walk $\underline{w}_{S,1}$ ends are thus equal to

$$(3.9) \qquad (c_2+c_3\tau)\ E(\underline{v}_S-S).$$

We can now state that the difference in expected backorder costs incurred during $\underline{w}_{S,1}$ and $\underline{w}_{0,1}$ having i as initial state are given by

$$(3.10) \qquad \begin{cases} (3.8) + (3.9) - (3.6) & \text{for } i > 0 \\ (3.8) + (3.9) & \text{for } i \leq 0 \end{cases}$$

The costs associated with the intervention S at the start of the walk $\underline{w}_{S,1}$ having i as initial state are equal to

$$\phi(S-i) \qquad \text{for } i \leqq s. \tag{3.11}$$

Summarizing the foregoing we see that

$$k(i;S) = (3.11) + (3.7) + (3.10) \qquad \text{for } i \leqq s. \tag{3.12}$$

Using (2.17), (2.28), (2.29) and (2.38) we find that

$$\begin{aligned} k(i;S) = \phi(S-i) + \sum_{k=i+1}^{S} \{ (c_1+c_3) \sum_{j=0}^{k-1} a_j(\tau)\, E\underline{t}_{k-j} + \\ + c_3(\tau - E\underline{t}_k) + c_2(1 - \sum_{j=0}^{k-1} a_j(\tau))\} + \\ + (c_2+c_3\tau)(\lambda \sum_{n=1}^{\infty} np_n (E\underline{t}_S - E\underline{t}_i) + S - i) \qquad \text{for } 0 \leqq i \leqq s \end{aligned} \tag{3.13}$$

and

$$k(i;S) = \phi(S-i) - \phi(S) + k(0,S) \qquad \text{for } i < 0. \tag{3.14}$$

Consider next the walks $\underline{w}_{0,2}$ and $\underline{w}_{S,2}$ having $i \leqq s$ as initial state. The duration of the walk $\underline{w}_{0,2}$ is zero because state $i \in A_{0,2}$. The expected duration of the walk $\underline{w}_{S,2}$ is equal to $E\underline{t}_{S-s}$, hence

$$t(i;S) = E\underline{t}_{S-s} \qquad \text{for } i \leqq s. \tag{3.15}$$

3.2.3. The average costs per unit of time for the (S,s)-strategy.

The Markov-chain $\{\underline{I}_n, n \geqq 1\}$ of future interventionstates has only one simple ergodic set, because state S can be reached from every other state Hence the criterionfunction r(z;i) is constant, say r(s,S), on the set of all states. In other words the average costs per unit of time are with probability one equal to r(s,S) irrespective of the initial state of the system. The unknown r(s,S) satisfies the following system of linear equations in r(s,S) and c(z;i) (c.f. (1.13), (1.14) and (1.16)):

$$c(z;i) = k(i;S) - r(s,S)t(i;S) + Ec(z;\underline{I}_1) \qquad \text{for } i \leqq s \tag{3.16}$$

(3.17) $$c(z;i) = Ec(z;\underline{I}_1) \qquad \text{for } s < i \leq S,$$

where $\underline{I}_1$ is the first future interventionstate assumed by the system when the initial state is i and strategy $z = (S,s)$ is applied. Denote by $\underline{I}^*$ the first future interventionstate given initial state S. The interventionstates are the states i, $i \leq s$. In each interventionstate the same intervention $d = S$ is taken, by which the system is transferred into state S. Hence the probability distribution of the first future interventionstate given the initial state $i \leq s$ is equal to the probability distribution of $\underline{I}^*$. This observation and the relations (3.16) and (3.17) imply that

(3.18) $$c(z;S) = Ec(z;\underline{I}^*) = $$
$$= E\{k(\underline{I}^*;S) - r(s,S)t(\underline{I}^*;S)\} + Ec(z;\underline{I}^*).$$

Hence

(3.19) $$r(s,S) = \frac{Ek(\underline{I}^*;S)}{Et(\underline{I}^*;S)} .$$

It only remains to determine the probability distribution of $\underline{I}^*$. This probability distribution can be found in section 2.2 of chapter 2. From definition (2.25) it follows that

(3.20) $$P\{\underline{I}^* = j\} = f_{S-j}(S-s) \qquad \text{for } j \leq s.$$

Hence (c.f. (3.15))

(3.21) $$r(s,S) = \frac{\sum_{j\leq s} f_{S-j}(S-s)k(j;S)}{E\underline{t}_{S-s}} .$$

<u>Notes</u>

a) If we choose the set $A_{0,2}$ equal to $\{i \mid i \leq 0\}$ we obtain for $Et(\underline{I}^*;S)$ the expression $\sum_{j\leq s} f_{S-j}(S-s)(E\underline{t}_S - E\underline{t}_j)$. From (3.19) and (3.21) it follows that

(3.22) $$E\underline{t}_{S-s} = \sum_{j\leq s} f_{S-j}(S-s)(E\underline{t}_S - E\underline{t}_j),$$

as can also be verified by using the theorem of total expectation.
b) If for some m we have $p_i = 0$ for $i > m$ and thus $f_{S-j}(S-s) = 0$ for $j \leq S-m$ the summation in (3.21) is a finite one. Furthermore the summation can be reduced to a finite one, when the ordering costs satisfy

$$\text{(3.23)} \qquad \phi(k) = ck + K\delta(k) \qquad \text{for } k \geq 0,$$

where $\delta(k) = 1$ for $k > 0$ and $\delta(0) = 0$. Because excess demand is backordered we may assume that $c = 0$. The expected average costs per unit of time are then only reduced with an amount $c\lambda \sum_{n=1}^{\infty} np_n$ (= c times the expected demand per unit of time) and this term does not depend on the strategy considered. If $\phi(k) = K\delta(k)$ it follows from (3.13) and (3.14) that $k(i;S) = k(j;S)$ for $i,j \leq 0$. In that case the summation in (3.21) can be reduced to a finite one.
c) An iterationprocedure can be given which exploits the simple properties of an (S,s)-strategy and converges in most cases to such a strategy. We shall indicate briefly in which way an iterationprocedure can be drawn up. Suppose a lower bound $L \geq 0$ for the optimal s and an upper bound U for the optimal S are given, i.e. $s \geq L$ and $S \leq U$. The state space is defined as follows. It consists of the points i, i integer and $i \leq U$ and the points (i,S), i and S integers, $i \leq S$ and $S = L+1,\dots,U$. Both the state i and the state (i,S) correspond to the situation that the economic stock is i. Next the feasible decisions are defined. Both in state i and state (i,S) the null-decision (do not order) transfers the system into the present state itself. The intervention $d > i$ in state i (order d-i units) transfers the system into state (d,d); for state i the intervention d has to satisfy $L+1 \leq d \leq U$. The intervention $d = S$ in state (i,S), $i < S$ (order S-i units) transfers the system into state (S,S). The intervention $d = S$ is the only feasible intervention in state (i,S). The decisionmaker has always to intervene in the states $i \leq L$ and the states (i,S), $i \leq L$. The sets $A_{0,1}$ and $A_{0,2}$ are chosen equal to the set which consists of the states $i \leq 0$ and the states (i,S), $i \leq 0$, $S = L+1,\dots,U$. For each strategy $z \in Z$ we have that the states (i,S), $i \leq S$ and S fixed, constitute a simple ergodic set. By adding to (1.13) and (1.14) the conditions $c(z;(S,S)) = 0$ for $S = L+1,\dots,U$

(c.f. (1.18)) the resulting system of lineair equations becomes very simple. The unknowns $r(z;x)$ and $c(z;x)$ can be computed without solving any equation, they can be computed one by one. Since we have punt $c(z;(S,S)) = 0$ the expressions for the functions $r(d.z;x)$ and $c(d.z;x)$ become also very simple. The set $A'_{\hat{z}}$ can be determined in the same way it is done in chapter 2 (since for each S the set $\{(i,S) \mid i \leq S\}$ constitutes a simple ergodic set and it can be shown that $r(\hat{z};i) \leq r(\hat{z};j)$ for $i < j$, $i,j \in A_{z_1}$ we need for the determination of $A'_{\hat{z}}$ only the function $c(\hat{z};x)$, $x \in A_{z_1}$).

3.2.4. The (Q,s)-strategy for the backorder case.

Assume that

$$p_1 = 1, \tag{3.24}$$

i.e. the units are demanded one at a time. The (S,s)-strategy is now called an (Q,s)-strategy, i.e. when the economic inventory becomes s (there is no overshoot of the reorder point s!) an order of the size $Q = S-s$ is placed. It is easily shown that an (Q,s)-strategy is optimal. We now have

$$a_k(\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!} \qquad \text{for } k = 0,1,\ldots \tag{3.25}$$

$$E\underline{t}_k = \frac{k}{\lambda} \qquad \text{for } k = 0,1,\ldots \tag{3.26}$$

and

$$f_{S-i}(S-s) = \begin{cases} 1 & \text{for } i = s \\ 0 & \text{for } i < s. \end{cases} \tag{3.27}$$

From (3.13) and (3.21) it is now easily derived that the expected average costs per unit of time for the (Q,s)-strategy are equal to

$$r(s,Q) = \frac{k(s;s+Q)}{t(s;s+Q)} = \tag{3.28}$$

$$= \frac{\lambda}{Q}\,\phi(Q) + \frac{(c_1+c_3)}{Q}\sum_{k=s+1}^{s+Q}\sum_{j=0}^{k}(k-j)a_j(\tau) +$$

$$+ c_3(\lambda\tau - s - \frac{Q}{2} - \frac{1}{2}) + c_2 \frac{\lambda}{Q} \sum_{k=s+1}^{s+Q} P(k,\lambda\tau) ,$$

where

$$(3.29) \qquad P(k,\lambda\tau) = \sum_{j=k}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^j}{j!} \qquad \text{for } k \geqq 0.$$

The optimal Q* and s* satisfy the inequalities

$$(3.30) \qquad r(s^*,Q^* \underline{+} 1) \geqq r(s^*,Q^*) \quad \text{and} \quad r(s^* \underline{+} 1,Q^*) \geqq r(s^*,Q^*).$$

We note that these inequalities can also be deduced from the relation $r(s,Q) = k(s;s+Q)/t(s;s+Q)$and the steps 2) and 3) from the on pages 9 and 10 described iterationprocedure.
Using wellknown properties of the Poisson probabilities we find after some calculations that the optimal Q* and s* satisfy

$$(3.31) \qquad \nu_1(s^*,Q^*+1) \leqq \frac{c_1}{2} \leqq \nu_1(s^*,Q^*)$$

and

$$(3.32) \qquad \nu_2(s^*+1,Q^*) \leqq c_1 \leqq \nu_2(s^*,Q^*),$$

where

$$(3.33) \quad \nu_1(s,Q) = \frac{1}{Q(Q-1)} \Big[\lambda Q\phi(Q-1) - \lambda(Q-1)\phi(Q) +$$
$$+ \{\lambda c_2 s - \frac{(c_1+c_3)}{2} (s(s+1) - Q(Q-1))\} P(s+Q,\lambda\tau) +$$
$$+ \{(c_1+c_3)\lambda\tau s - \lambda^2\tau c_2\}\{P(s+Q-1,\lambda\tau) - P(s,\lambda\tau)\} +$$
$$- \frac{(\lambda\tau)^2}{2} (c_1+c_3) \{P(s+Q-2,\lambda\tau) - P(s-1,\lambda\tau)\} +$$
$$+ \{\frac{(c_1+c_3)}{2} s(s+1) - \lambda c_2 s\} P(s+1,\lambda\tau)\Big]$$

and

$$(3.34)\quad \nu_2(s,Q) = \frac{1}{Q}\Big[\lambda\tau(c_1+c_3)\ \{P(s-1,\lambda\tau) - P(s+Q-1,\lambda\tau)\} +$$

$$+ \{(c_1+c_3)(s+Q) - \lambda c_2\}\ P(s+Q,\lambda\tau) +$$

$$- \{(c_1+c_3)s - \lambda c_2\}\ P(s,\lambda\tau)\Big] .$$

A numerical procedure for finding Q^* and s^* is as follows. Take an initial estimate for Q^*, say Q_1 (if $\phi(k) = ck + K\delta(k)$, take $Q_1 = \sqrt{2K\lambda/c_1}$). Then use Q_1 in (3.32) to compute the integer s_1 which satisfies (3.32). The s_1 so obtained is used in (3.31) to compute Q_2. The Q_2 is used to compute s_2, etc. Continue until there is no change in Q and s. It is theoretical possible that we find a relative minimum which is not the absolute minimum. We note that a similiar numerical procedure can be deduced from the on pages 9 and 10 described iterationmethod.

The formula (3.28) can be found in a different but equivalent form in HADLEY and WHITIN [4, page 187]. In [4] the expected average costs per unit of time are determined by computing the steady state probabilities for the economic inventory and the on hand inventory. Using wellknown properties of the Poisson probabilities formula (3.28) can be rewrited to

$$(3.35)\quad r(s,Q) = \frac{\lambda}{Q}\ \phi(Q) + c_1\left(\frac{Q+1}{2} + s - \lambda\tau\right) + c_2\ \frac{\lambda}{Q}\ (\alpha(s) - \alpha(s+Q)) +$$

$$+ \frac{(c_1+c_3)}{Q}\ (\beta(s) - \beta(s+Q)) ,$$

where

$$(3.36)\qquad \alpha(k) = \sum_{j=k+1}^{\infty} P(k,\lambda\tau) = \lambda\tau P(k,\lambda\tau) - kP(k+1,\lambda\tau)$$

and

$$(3.37)\qquad \beta(k) = \sum_{u=k+1}^{\infty} (u-k-1)P(k,\lambda\tau) = \frac{(\lambda\tau)^2}{2}\ P(k-1,\lambda\tau) +$$

$$- \lambda\tau kP(k,\lambda\tau) + \frac{k(k+1)}{2}\ P(k+1,\lambda\tau) .$$

It is noted in [4, page 188] that in problems of practical interest it is usually a very good approximation to neglect the terms $\alpha(s+Q)$ and $\beta(s+Q)$ in (3.35). These terms are important only if there is a significant probability that the lead time demand exceeds s+Q. When $\alpha(s+Q)$ and $\beta(s+Q)$ are neglected a considerable simplification of $r(s,Q)$ occurs and simple inequalities satisfied by Q^* and s^* can be given [4, page 189 and 190]. When $\phi(k) = ck + K\delta(k)$ we have approximately for the optimal Q^*,

$$Q^* = \sqrt{\frac{2\lambda}{c_1}\{K + c_2\alpha(s^*) + \frac{c_1+c_3}{\lambda}\beta(s^*)\}} \ . \tag{3.38}$$

3.3 The (Q,s)-strategy (Q $\geq$ s) for the lost sales case.

Consider again a situation in which a Poisson process generates the times between demands, units are demanded one at a time and the mean rate of demand is λ units per unit of time. Demands occurring when the system is out of stock are lost. The cost of each lost sale is a constant c_2. The costs of carrying a unit in inventory are directly proportional to the length of time for which the unit remains in inventory. The constant of proportionality is c_1. At each point of time an order can be placed. However it is stipulated that there is never more than a single order outstanding. The total cost of placing an order for k units is $\phi(k)$. The procurement lead time is a random variable $\underline{\tau}$ with finite expectation. Let $F(\tau)$ be the distribution function of the lead time. The operating doctrine for the system is a (Q,s)-strategy ($Q \geq s$), i.e. if the on hand inventory reaches the reorder level s, a quantity $Q \geq s$ is ordered.

The expected average costs per unit of time for the (Q,s)-strategy ($Q \geq s$) will now be derived by using Markov-programming. Suppose Q and s fixed, $Q \geq s$.

We take as state space

$$X = \{i \mid 0 \leq i \leq s+Q\} \cup \{(i,t) \mid 0 \leq i \leq s,\ t \geq 0\} \ . \tag{3.39}$$

The state i corresponds to the situation that the on hand inventory is i and no order is outstanding. The state (i,t) corresponds to the situation that the on hand inventory is i and since t units of time an order of Q units is outstanding. Next the natural process is described. The natural process has i as initial state the system remains in state i until the first future demand occurs. By that demand the system is transferred into state max(i-1,0). If the natural process has (i,t_0) as initial state then the system runs next through the states $(\max[i-\underline{v}(t-t_0);0],t)$, $t > t_0$ until the moment the outstanding order arrives. On that moment the system assumes state $\underline{i}^* + Q$, where $\underline{i}^*$ is the on hand inventory just before the arrival of the order ($\underline{v}(t)$ is the cumulative demand in an interval of the length t).

Each intervention will be represented by the size of the order placed. In the states i, $0 \leqq i \leqq s$ we take the intervention d = Q as the only feasible decision and for the other states the null-decision is the only feasible decision. Therefore the class Z of stationary strategies consists of the (Q,s)-strategy only. Obviously we have that (c.f. (1.1))

(3.40) $$A_0 = \{i \mid 0 \leqq i \leqq s\} .$$

For the determination of the k- and t-functions we choose

(3.41) $$A_{0,1} = \{0\}$$

and

(3.42) $$A_{0,2} = \{i \mid 0 \leqq i \leqq s\} .$$

Define

(3.43) $$a_k = \int_0^\infty e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} F(d\tau) \qquad \text{for } k = 0,1,\ldots,$$

i.e. a_k is the probability that the lead time demand is k. It is easily verified that (c.f. (1.3), (1.4) and (2.11))

(3.44) $$k(s;Q) = k_1(s;Q) - k_0(s) =$$

$$= \phi(Q) + c_1 \sum_{k=1}^{s} E\underline{t}_k + c_1 \sum_{k=0}^{s-1} a_k \sum_{j=s-k+1}^{s-k+Q} E\underline{t}_j +$$

$$+ c_1 \sum_{k=s}^{\infty} a_k \sum_{j=1}^{Q} E\underline{t}_j + c_2 \sum_{k=s}^{\infty} (k-s)a_k - c_1 \sum_{k=1}^{s} E\underline{t}_k$$

and

(3.45) $$t(s;Q) = t_1(s;Q) - t_0(s) =$$

$$= E\underline{\tau} + \sum_{k=0}^{s-1} a_k \, E\underline{t}_{Q-k} + \sum_{k=s}^{\infty} a_k \, E\underline{t}_{Q-s} - 0 \; .$$

Using the fact that $E\underline{t}_k = k/\lambda$ and $\sum_1^{\infty} ka_k = \lambda E\underline{\tau}$ we can simplify the formulas (3.44) and (3.45),

(3.46) $$k(s;Q) = \phi(Q) + \frac{c_1}{\lambda} \left(\frac{1}{2} Q(Q+1) + s - \lambda E\underline{\tau}\right) + \\ + \left(\frac{c_1 Q}{\lambda} + c_2\right) \sum_{k=s}^{\infty} (k-s) \; a_k$$

and

(3.47) $$t(s;Q) = \frac{Q}{\lambda} + \frac{1}{\lambda} \sum_{k=s}^{\infty} (k-s) \; a_k \; .$$

The Markov-chain $\{\underline{I}_n, \; n \geqq 1\}$ of future interventionstates associated with strategy $z = (Q,s)$ has only one simple ergodic set, because state s can be reached from every other state. We have even that state s is the only interventionstate assumed in the decision process considered, because $Q \geqq s$ and the units are demanded one at a time. Hence the criterionfunction $r(z;x)$ is constant, say $r(s,Q)$, on the set of all states. If the initial state is s, the first future interventionstate is s again, thus (c.f. (1.14))

(3.48) $$c(z;s) = k(s;Q) - r(s,Q)t(s;Q) + c(z;s).$$

Hence

(3.49) $$r(s,Q) = \frac{\phi(Q) + \frac{c_1}{\lambda} \left(\frac{1}{2} Q(Q+1)+s-\lambda E\underline{\tau}\right) + \left(\frac{c_1 Q}{\lambda} + c_2\right) \sum_{k=s}^{\infty} (k-s) \; a_k}{\frac{Q}{\lambda} + \frac{1}{\lambda} \sum_{k=s}^{\infty} (k-s) \; a_k} \; .$$

Define a cycle as the period of time between two successive orderings. t can be easily verified that $t(s;Q)$ is the expected length of a cycle and that k(s;Q) is the expectation of the costs incurred during a cycle. Hence we have the wellknown formula (c.f. (3.49))

$$(3.50)\qquad r(s,Q) = \frac{\text{expected costs incurred during a cycle}}{\text{expected length of a cycle}} .$$

With the aid of relation (3.50) the expected average costs per unit of time are determined in [4, page 200].

The optimal Q* and s* have to satisfy the inequalities $r(s^*\underline{+}1,Q^*) \geqq r(s^*,Q^*)$ and $r(s^*,Q^*\underline{+}1) \geqq r(s^*,Q^*)$. From these inequalities it follows after some calculations (provided that $r(s^*,Q^*) < c_1Q^* + \lambda c_2$)

$$(3.51)\qquad \sum_{k=s^*+1}^{\infty} a_k \leqq \frac{c_1Q^*}{c_1Q^* + \lambda c_2 - r(s^*,Q^*)} \leqq \sum_{k=s^*}^{\infty} a_k$$

and

$$(3.52)\qquad \phi(Q^*) - \phi(Q^*-1) + \frac{c_1}{\lambda} Q^* \leqq \frac{1}{\lambda} r(s^*,Q^*) - \frac{c_1}{\lambda} \sum_{j=0}^{s^*} (s^*-j)\, a_j \leqq$$
$$\leqq \phi(Q^*+1) - \phi(Q^*) + \frac{c_1}{\lambda} (Q^*+1).$$

From (3.51) and (3.52) the optimal Q* and s* can be iteratively computed. Finally we note that from (2.30) it easily follows that the expected length of time out of stock per cycle is

$$(3.53)\qquad \sum_{j=0}^{s-1} \frac{(s-j)}{\lambda} a_j - \frac{s}{\lambda} + E\underline{\tau} = \frac{1}{\lambda} \sum_{k=s}^{\infty} (k-s)\, a_k.$$

If the expected length of time out of stock per cycle is a very small fraction of the total length of the cycle, the probability distribution of the lead time demand can be approximated by the distributionfunction H(x) with density h(x) and if $\phi(k) = ck + K\delta(k)$, the optimal Q* and s* satisfy approximately [4, page 169 and 200]

$$(3.54)\qquad Q^* = \sqrt{\frac{2\lambda}{c_1} (K + c_2\eta(s^*))}$$

$$(3.55)\qquad H(s^*) = \frac{c_1Q^*}{\lambda c_2 + c_1Q^*} ,$$

where

$$\eta(x) = \int_x^{\infty} th(t)dt - xH(x). \tag{3.56}$$

The equations (3.54) and (3.55) can be iteratively solved.

4. The automobile replacement problem.

4.1 Introduction.

This problem was used as an example by Howard in [5]. Here the solution by our method will be discussed. The problem is to find the optimal replacement strategy for a particular type of car, which is replaced by a specimen of the same type. The lifetimes of the new cars are mutually independent random variables with common distribution function F(y). The following costfunctions of the age x are given:

a) the price : p(x),

b) the trade-in value : q(x),

c) the operating costs per time unit : e(x).

In section 4.2 the strategy-independent notions like the state space, the natural process, the set of feasible decisions and the set A_0 are discussed. In the same section expressions for the functions k(x;d) and t(x;d) are derived. In sections 4.3 the iterative approach to the optimal strategy will be discussed. Finally in section 4.4 a numerical example will be given.

4.2 The strategy-independent notions.

The state of the system will be the age x of the car. Let L denote the age of the car at which the trade-in value equals its scrapvalue. Then for the state into which the system is transferred immediately after a breakdown also state L can be taken. The state space X will be defined as follows:

(4.1) $$X = \{ x \mid 0 \leq x \leq L \}$$

During the natural process with initial state $x_0 \in X$ either the car breaks down somewhere in the interval $x_0 \leq x \leq L$ at which a transition to state L takes place or all states in the interval $x_0 \leq x \leq L$ are successively taken on. As soon as state L is taken on, the natural process remains in that state forever.

Decisions in this problem lead to deterministic transformations. Hence a decision may be denoted by the state into which the system is transformed by that decision. Further the decision to replace a car by a car of the same age is considered to be infeasible. The null-decision in state x is denoted by d = x. The set of feasible decisions in state x is denoted by D(x) and is given by

$$D(x) = \{ d \mid 0 \leq d < L \}. \tag{4.2}$$

From the definitions of the problem it follows that each strategy will dictate an intervention in state L. Consequently the set A_0 is given by:

$$A_0 = \{ L \}. \tag{4.3}$$

In this problem we take:

$$A_{0,1} = A_{0,2} = A_0 \tag{4.4}$$

and consequently we have:

$$\underline{w}_{0,1} = \underline{w}_{0,2} \tag{4.5}$$

$$\underline{w}_{d,1} = \underline{w}_{d,2}. \tag{4.6}$$

To abbreviate the notation the walks corresponding to (4.5) and (4.6) are denoted respectively by $\underline{w}_0$ and $\underline{w}_d$. The walk $\underline{w}_0$ with initial state x is subjected to the natural process from state x on and ends in state L. During the walk $\underline{w}_d$ with initial state x the system is transformed to state d by the decision $d \in D(x)$. After this transformation the walk is subjected to the natural process from state d on and ends in state L.

Let G(y;x) denote the conditional probability that a car of age x has a breakdown before age y is reached. G(y;x) is obtained from the distribution function of the lifetime of a new car in the following way :

$$(4.7) \qquad G(y;x) = \begin{cases} \dfrac{F(y) - F(x)}{1 - F(x)} & y \geq x, \\ 0 & y < x. \end{cases}$$

For the expected duration $t_0(x)$ of the walk $\underline{w}_0$ with initial state x, it follows:

$$(4.8) \qquad t_0(x) = \int_x^L (y-x)\, G(dy;x) + (L-x)(1-G(L;x)) \qquad \text{for } 0 \leq x < L,$$

$$(4.9) \qquad t_0(L) = 0.$$

The expected costs $k_0(x)$ incurred during the $\underline{w}_0$ - walk consist of the expected operating costs until state L is taken on and are given by:

$$(4.10) \qquad k_0(x) = \int_x^L G(dy;x) \int_x^y e(u)du + (1-G(L;x)) \int_x^L e(u)du$$

$$\text{for } 0 \leq x \leq L,$$

$$(4.11) \qquad k_0(L) = 0.$$

Because the replacement does not take time we have for $t_1(x;d)$, the expected duration of the walk $\underline{w}_d$ with initial state x and $d \neq x$:

$$(4.12) \qquad t_1(x;d) = t_0(d) \qquad \text{for } 0 \leq x \leq L \text{ and } 0 \leq d < L.$$

The expected costs $k_1(x;d)$ of the $\underline{w}_d$ - walk with initial state x consist of the costs of replacement of a car of age x by a specimen of age d and the expected operating costs from state d on until the walk ends in state L. Hence we have for $d \neq x$:

$$(4.13) \qquad k_1(x;d) = p(d) - q(x) + k_0(d) \qquad \text{for } 0 \leq x \leq L. \text{ and } 0 \leq d < L$$

For the functions $k(x;d)$ and $t(x;d)$ we obtain according to (1.3) and (1.4) respectively:

$$k(x;d) = k_0(d) + p(d) - q(x) - k_0(x) \quad (4.14)$$

$$t(x;d) = t_0(d) - t_0(x) \quad (4.15)$$

for $0 \leq x \leq L$, $0 \leq d < L$ and $d \neq x$. For $d = x$ (null-decision) both functions are identical to zero.

4.3 The iterative approach to the optimal strategy.

The possibility of strategies which subdivide the state space into two or more disjoint simple ergodic sets is verified primarily. In spite of the initial state x and the applied strategy z, state L will be assumed with probability one in a finite time. Hence for each strategy $z \in Z$ the associated Markov-process in A_z has only one simple ergodic set. So we may write for every strategy z and feasible decision d:

$$r(d.z;x) = r(z;x) = r(z). \quad (4.16)$$

The three steps of the iterative approach to the optimal strategy are given in chapter 1. The first step is to solve functional equations (1.14) and (1.18) for $c(z;x)$. Thereafter the strategy is improved by means of step 2 and 3. The specific form the functional equations (1.14) and (1.18) take on in a particular problem depends strongly on the structure of the set of intervention states A_z for a given strategy z. The structure of A_z is especially important in problems with a non-denumerable state space. In these problems the functional equation (1.14) does not in general take the form of a set of linear equations. To obtain more information about the structure of the strategies that emerge from the strategy-improvement steps 2 and 3, these steps are discussed primarily. Thereafter the functional equations (1.14) and (1.18) are specified for this problem.

Suppose that the function $c(z;x)$ is known for a given strategy z and that this strategy is going to be improved. By substitution of (4.14) and

(4.15) into (1.23), it follows:

$$c(d.z;x) = p(d) + k_0(d) - q(x) - k_0(x) - r(z)\{t_0(d) - t_0(x)\} + c(z;d) \quad \text{for } 0 \leq d < L,\ d \neq x. \tag{4.17}$$

For d = x (null-decision in x) we have:

$$c(d.z;x) = c(z;x). \tag{4.18}$$

Observe that the right member of (4.17) is a separable function of d and x. Denote the part which only depends on d by h(d).
Let d_1 be one of the values of d for which h(d) assumes an absolute minimum on the interval $0 \leq d \leq L$. Usually the absolute minimum will be unique, otherwise each of the corresponding d may be chosen.

By comparing the numerical values of $c(d_1.z;x)$ obtained from (4.17) for $d = d_1$ and c(z;x) for each state x it is decided whether state x is an intervention state of strategy z_1 or not. To be more specific we have:

$$x \in A_{z_1} \qquad \text{if } c(d_1.z;x) \leq c(z;x) \tag{4.19}$$

and

$$x \notin A_{z_1} \qquad \text{if } c(d_1.z;x) > c(z;x). \tag{4.20}$$

Hence strategy z_1 subdivides the state space into a finite number of intervals of intervention states alternated by intervals of non-intervention states. In all intervention states the same decision d_1 is dictated.

Next the cutting mechanism (step 3) is applied on the strategy z_1 obtained in step 2. Let A be a closed set of states satisfying:

$$A_0 \subseteq A \subseteq A_z. \tag{4.21}$$

Consider the natural proces with initial state x. Denote the state $y \in A$ with the smallest number $y > x$ by a. The first state $\underline{a}$ taken on in the set A by the natural process with initial state $x \in \bar{A} \cap A_{z_1}$ will be either state a or state L. Hence we have:

(4.22) $\quad c(A.\hat{z};x) = (1-G(a;x))\ c\ (\hat{z};a) + G(a;x)\ c\ (\hat{z};L) \qquad \text{for } x \in \bar{A} \cap A_{z_1}$

and in agreement with (1.32) we have :

(4.23) $\quad c(A.\hat{z};x) = c(\hat{z};x) \qquad \text{for } x \in A.$

It will be shown now how the intersection $A'_{\hat{z}}$ of the class of all sets A obeying the inequality

(4.24) $\quad c(A.\hat{z};x) \leq c(\hat{z};x) \qquad \text{for } x \in A_{z_1}$

is obtained, (cf(1.34)).

By subtraction of $c(\hat{z};L)$ from both sides of (4.22), substitution of (4.7) and denoting $c(A.\hat{z};x) - c(\hat{z};L)$ and $c(\hat{z};a) - c(\hat{z};L)$ respectively by $\bar{c}(A.\hat{z};x)$ and $\bar{c}(\hat{z}.a)$ we obtain:

(4.25) $\quad \bar{c}(A.\hat{z};x) = \dfrac{1-F(a)}{1-F(x)}\ \bar{c}(\hat{z};a) \qquad \text{for } x \in \bar{A} \cap A_{z_1}.$

By subtraction of $c(\hat{z};L)$ from both sides of (4.24), substitution of (4.25) and multiplying the result with $(1-F(x)) > 0$ the requirement (4.24) for $x \in \bar{A} \cap A_{z_1}$ becomes:

(4.26) $\quad (1-F(a))\ \bar{c}(\hat{z};a) \leq (1-F(x))\ \bar{c}(\hat{z};x).$

Denote the important function $(1-F(x))\ \bar{c}(\hat{z};x)$ by $f(x;\hat{z})$. Then (4.23) and (4.26) are equivalent to:

(4.27) $\quad f(a;\hat{z}) \leq f(x;\hat{z}) \qquad \text{for } x \in A_{z_1}.$

Clearly the behavior of the function $f(x;\hat{z})$ on the set $x \in A_{z_1}$ determines whether an arbitrary state $u \in A_{z_1}$ is also contained in $A'_{\hat{z}}$. A given state $u \in A_{z_1}$ does not belong to the set $A'_{\hat{z}}$ if there exists a state $a \in A_{z_1}$ with $a > u$ satisfying (4.27) for $x = u$. If a state a with this property cannot be found then $u \in A'_{\hat{z}}$.

After application of the cutting mechanism the basic structure of the strategy z_2 (c.f.(1.37)) is still the same as the strategy z_1. There will in the general case be a finite number of intervals in which the same decision is dictated, alternated by intervals in which null decisions are dictated.

Consider a strategy z with this structure. Now the functional equations (1.14) and (1.18) will be specified. Denote the decision dictated by z in each state $x \in A_z$ by d_z and let a_z denote the state $x \in A_z$ with the smallest number $x > d_z$. The interval $d_z \leq x \leq a_z$ and state L constitute the unique simple ergodic set of states for strategy z. From the states in this simple ergodic set a_z and L are the only intervention states.
For state L we define:

$$c(z;L) = 0. \tag{4.28}$$

For the intervention states a_z and L we have respectively:

$$c(z;a_z) = k(a_z;d_z) - r(z)\ t(a_z;d_z) + c(z;d_z) \tag{4.29}$$

and

$$c(z;L) = k(L;d_z) - r(z)\ t(L;d_z) + c(z;d_z). \tag{4.30}$$

For non-intervention states in the interval $d_z \leq x \leq a_z$ it follows (c.f.(1.16)):

$$c(z;x) = (1-G(a_z;x))\ c(z;a_z) + G(a_z;x)\ c(z;L). \tag{4.31}$$

From (4.31) we obtain for $x = d_z$:

$$c(z;d_z) = (1-G(a_z;d_z))\ c(z;a_z) + G(a_z;d_z)\ c(z;L). \tag{4.32}$$

From the four linear equations (4.28), (4.29), (4.30) and (4.32) the unknowns $c(z;d_z)$, $c(z;a_z)$, $c(z;L)$ and $r(z)$ can be solved. For the other ergodic states $c(z;x)$ is obtained from (4.31).

For transient states x belonging to A_z it follows:

(4.33) $$c(z;x) = k(x;d_z) - r(z)\ t(x;d_z) + c(z;d_z).$$

The values of $c(z;x)$ for transient states x not in A_z are obtained from relation (4.31) in which a_z has to be replaced by the intervention state $y \in A_z$ with the smallest number $y > x$.

The computations needed during one step of the iteration cycle are summarized as follows:

1. Solve $r(z)$ and $c(z;x)$ for $x \in X$ and the current strategy z by the relations (4.28) up to (4.33).
2. Determine the decision $d = d_1$ for which the function $h(d)$ being the part of $c(d \cdot z;x)$, which only depends on d, is minimized. The computation involved in this step consists of the search for the absolute minimum of $h(d)$.
3. Determine the set A_{z_1} by means of (4.19) and (4.20). The zeros of the function $c(d_1 \cdot z;x) - c(z;x)$ have to be obtained for this purpose.
4. Determine the set $A'_{\hat{z}}$. Let the set $A'_{\hat{z}}$ be given by the intervals $a_j \le x \le b_j$, $j = 1, 2, \ldots, n$. Then a_j and b_j are obtained in the following way. Let the locations of the local minima of $f(x;\hat{z})$ on $x \in A_{z_1}$ be given by x_i, $i = 1, \ldots, m$ with $x_1 < x_2 < \ldots < x_m$ and let y_i be equal to $f(x_i;\hat{z})$, $i = 1, \ldots, m$. Then $a_j = x_{i_j}$ where $i = i_j$ is the largest index i such that $y_{i_j} = \min_{i_{j-1} < i \le m} y_i$ and b_{j-1} is the smallest x such that $f(x;\hat{z}) = y_{ij}$ for $a_{j-1} \le x \le a_j$, $x \in A_{z_1}$. If $i_j = m$ then $j = n$ and $b_n = L$ and we are finished.

4.4 Numerical example

For numerical illustration Howards data [5] were used. In his example the age of a car was discretised in quarters of a year. The maximum age considered was L = 10 years. Only decisions could be taken in the states $x = i\Delta$, $i = 0, \ldots, 40$ with $\Delta = 0.25$ year.

The data are presented in table 4.1. At each age $x = i\Delta$, $i = 0, \ldots$ $\ldots, 40$ the price $p(i\Delta)$ and the trade-in value $q(i\Delta)$ are given. The expected operating costs in the interval $i\Delta \leq x < (i+1)\Delta$ are given by $e(i\Delta)$ and the probability that a breakdown does not occur during this interval is given by $u(i\Delta)$ in table 4.1.

i	p(iΔ)	q(iΔ)	e(iΔ)	u(iΔ)	i	p(iΔ)	q(iΔ)	e(iΔ)	u(iΔ)
0	2000	1600	50	1.000	21	345	240	115	0.925
1	1840	1460	53	0.999	22	330	225	118	0.919
2	1680	1340	56	0.998	23	315	210	121	0.910
3	1560	1230	59	0.997	24	300	200	125	0.900
4	1300	1050	62	0.996	25	290	190	129	0.890
5	1220	980	65	0.994	26	280	180	133	0.880
6	1150	910	68	0.991	27	265	170	137	0.865
7	1080	840	71	0.988	28	250	160	141	0.850
8	900	710	75	0.985	29	240	150	145	0.820
9	840	650	78	0.983	30	230	145	150	0.790
10	780	600	81	0.980	31	220	140	155	0.760
11	730	550	84	0.975	32	210	135	160	0.730
12	600	480	87	0.970	33	200	130	167	0.660
13	560	430	90	0.965	34	190	120	175	0.590
14	520	390	93	0.960	35	180	115	182	0.510
15	480	360	96	0.955	36	170	110	190	0.430
16	440	330	100	0.950	37	160	105	205	0.300
17	420	310	103	0.945	38	150	95	220	0.200
18	400	290	106	0.940	39	140	87	235	0.100
19	380	270	109	0.935	40	130	80	250	0
20	360	255	113	0.930					

Table 4.1 Automobile replacement data.

The price $p(i\Delta)$ and the trade-in value $q(i\Delta)$ in table 4.1 are assumed to be the values of the functions $p(x)$ and $q(x)$ in the states $x = i\Delta$, $i = 0, \ldots, 40$. From the $u(i\Delta)$ in table 4.1 the distribution function of the age $F(x)$ can be obtained by means of the recurrence relations:

$$F(0) = 0$$

(4.35)

$$F(i\Delta+\Delta) = 1 - (1 - F(i\Delta))u(i\Delta), \quad i = 0, \ldots, 39.$$

From the $e(i\Delta)$ the values of the function $k_0(x)$ in the states $x = i\Delta$, $i = 0, \ldots, 40$ follow directly from the recurrence relations:

$$k_0(40\Delta) = 0$$

(4.36)

$$k_0(i\Delta) = e(i\Delta) + u(i\Delta)k_0(i\Delta+\Delta), \quad i = 39, \ldots, 0,$$

and the values of the function $t_0(x)$ in these states from:

$$t_0(40\Delta) = 0$$

(4.37)

$$t_0(i\Delta) = \Delta + u(i\Delta)t_0(i\Delta+\Delta), \quad i = 39, \ldots, 0.$$

The function values in intermediate states were obtained by a third degree interpolation polynomial.

In table 4.2 the results of the iteration are presented. The initial strategy $z^{(0)}$ dictates decision $d_1 = 0$ only in the state $x = L$. The strategy $z^{(4)}$ dictates to replace a car of age 6.444 years by a car of 3 years old. The average costs per year are somewhat smaller than Howards results. The interval with null decisions $6.754 < x < 6.792$ has no theoretical significance, but is merely due to the numerical inaccuracy in the values of $c(d_1 \cdot z;x)$ and $c(z;x)$ caused by interpolation. The results were obtained by a computerprogram especially written for this problem.

z	$r(z)$	d_1	A_z
$z^{(0)}$	691.19	0	10
$z_1^{(0)}$		2.0	$[4.237;10]$
$z^{(1)}$	623.83	3.0	$[8.007;10]$
$z_1^{(1)}$		3.0	$[0;0.483]$ $[5.177;10]$
$z^{(2)}$	604.29	3.0	$[6.792;10]$
$z_1^{(2)}$		3.0	$[0;0.564]$ $[6.179;6.754]$ $[6.792;10]$
$z^{(3)}$	603.76	3.0	$[0;0.536]$ $[6.444;6.754]$ $[6.792;10]$
$z_1^{(3)}$		3.0	$[0;0.802]$ $[6.444;6.754]$ $[6.792;10]$
$z^{(4)}$	603.76	3.0	$[0;0.551]$ $[6.444;6.754]$ $[6.792;10]$

Table 4.2 The numerical results of the iteration.

5. The motorist problem.

5.1. Problemformulation.

A motorist has decided to effect an accident insurance under the following conditions. The insurance runs for one year. The premium for the first year amounts E_0. If no damages have been claimed during i successive years i = 1, 2 or 3 the premium is reduced to E_i. After four years of damagefree driving no further premium reduction is granted, so the premium remains E_3. The premium is due on the first day of the premium year.*) The own risk amounts a_0.

The number of accidents is assumed to be Poisson-distributed with a mean of λ per year. It is assumed that the damages caused by the accidents are mutually independent random variables, which have a common distribution function F(s) with finite mean and variance. Furthermore the damages are assumed to be independent of the Poisson-process, which generates the accidents.

The problem of the motorist will be to decide whether to claim a damage or not. The solution of the problem will be a strategy that specifies his decisions in every possible situation. This strategy will be optimal if it minimizes the expected average costs per year in the long run.

In view of the premium reduction, it will be unprofitable to claim damages which are not much larger than a_0. Once a damage is claimed, it will be profitable to claim all damages that exceed a_0 during the remaining part of the year. Hence his decisions will also depend on the time of the year and the premium paid at the beginning of that year. So we distinguish between four types of year, for each premium one.

Our task will be to determine for each premium year a function s(t) with the following property: If at time t an accident occurs with damage s and no damages have been claimed since the last payment of premium, then s

*) It is no restriction to assume this is Januari 1.

should be claimed if $s > s(t)$. The strategy is completely fixed by this function. The optimal strategy will be the function $s(t)$ that minimizes the (expected) average costs per year in the long run.

The solution of this problem by our method will start with the application of the strategy-independent notions in section 5.2. In this section the state space, the natural process, the feasible decisions, the set A_0 and the functions $k(x;d)$ and $t(x;d)$ will be determined. In section 5.3 the functional equations (1.13) and (1.14) are specified to the situation met in this problem, after which the optimal strategy is determined using the direct approach given by (1.40), (1.41) and (1.42). Finally in section 5.4 some numerical results will be given.

5.2. The strategy-independent notions.

In order to define the state space in this problem the relevant information at each point of time is considered. The following information will be of interest:

(1) whether an eventual damage is covered or not;
(2) whether an accident happens or not;
(3) the amount of the last paid premium E_i, $i = 0, 1, 2, 3$;
(4) the date and time of the day considered;
(5) the extend of the damage;
(6) whether a damage has been claimed since the last payment of premium or not.

In figure 5.1. the state space is presented.

At the t-axis we distinguish:

a) Four points: E_i, $i = 0, 1, 2, 3$. In these states the corresponding premium has to be paid; damages are no longer covered by insurance.
b) Four intervals of one year *) : $1i \leq t < 1i + 1$, $i = 1, 2, 3, 4$. The t-component of the state runs through $1i \leq t < 1i + 1$ if and only if

*) $1i = 11, 12, 13, 14$ if $i = 1, 2, 3$ and 4 respectively.

the last premium paid was E_{i-1}, one or more damages have been claimed that year and coming damages are still covered by insurance.

c) Four intervals of one year: $2i \leq t < 2i + 1$, $i = 1, 2, 3, 4$. The t-component of the state runs through $2i \leq t < 2i + 1$ if and only if the last premium paid was E_{i-1}, no damages have been claimed up to t since the last payment of premium and coming damages are still covered by insurance.

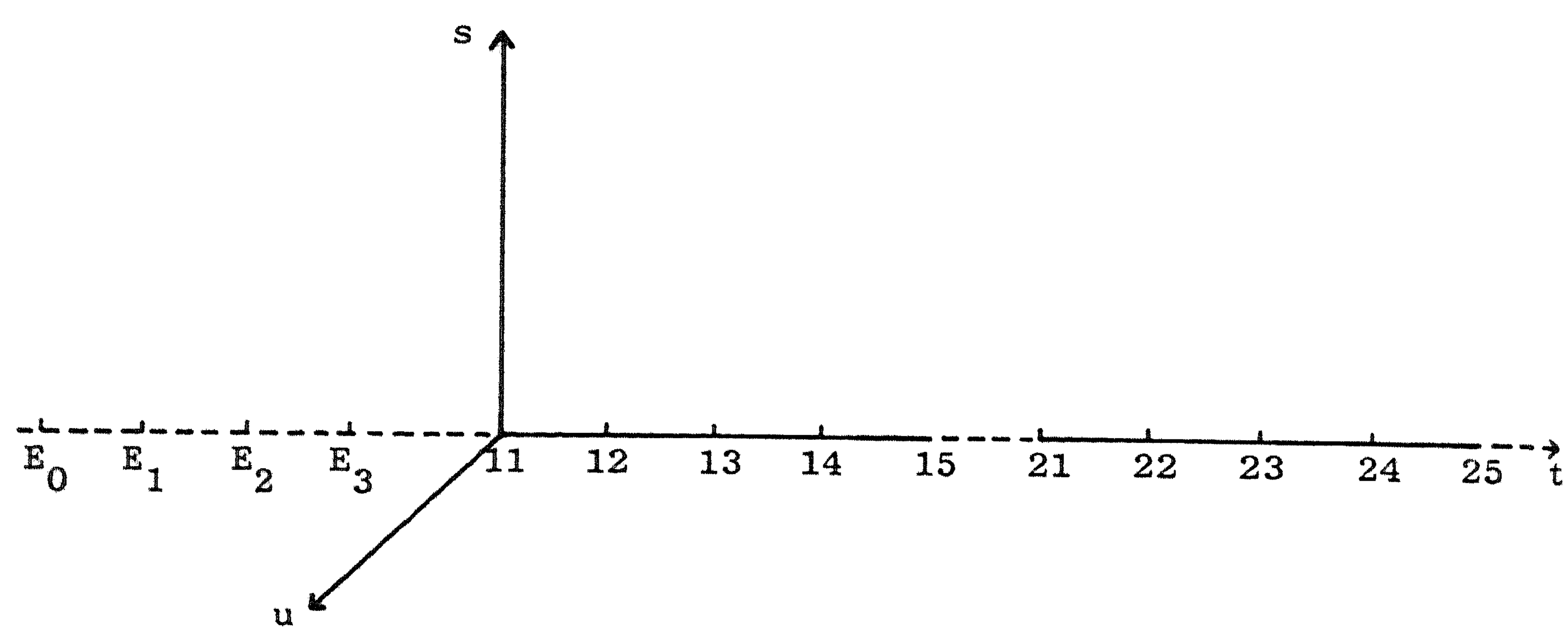

Figure 5.1 The state space

The s-variable is zero unless at least one damage has been claimed that year and moreover the coming damages are still covered by insurance. In that case the s-component denotes the extend of the last claim.

The u-variable is zero unless at least one damage has been claimed that year and coming damages are still covered by insurance. In that case the u-component denotes the time elapsed since the first claim that year.

Note that the s-component can only be different from zero if $1i \leq t < 1i + 1$, $i = 1, 2, 3, 4$. Consequently the state space consists of:

a) 4 points E_i $i = 0, 1, 2, 3$;

b) a 3-dimensional subspace (t,s,u) with $11 \leq t < 15$;

c) a 1-dimensional interval $21 \leq t < 25$.

Next the natural proces is described. This process can start in each state of the state space. In accordance with the premium paid the system runs through one of the time intervals $2i \leq t < 2i + 1$ $i = 1, 2, 3, 4$, if no damage has been claimed that year. If no accident happens during the rest of the year the system is transferred to E_i. Since in the natural process no premiums are paid the system will stay there forever. However, if at time t' during the year an accident occurs the system is transferred to (t'-10,s',0) where s' denotes the damage. Since during the natural process irrespective of their extends all damages are claimed the system will stay in the 3-dimensional part of the state space for the remaining part of the year. Then the u-component is increasing with time. The s-component only changes if a second, third, etc. accident happens. At the end of the year the system is transferred to E_0 where it stays forever.

The two feasible decisions in the states E_i $i = 0, 1, 2, 3$ are the null-decision and the decision involving the payment of the premium E_i. The respective transformations are $E_i \to E_i$ and $E_i \to (2i+1,0,0)$. In states (t,s,0) an accident has just occured and the decisionmaker can suppress the claim if he wants. In that case the respective transfermation is $(t,s,0) \to (t+10)$. Note that a claim corresponds with a null-decision. This is in accordance with the fact that in the natural process all damages are claimed. In the states (t,s,u) with $u > 0$ only null-decisions are feasible. If an accident occurs in a state with $u > 0$ the decision not to claim is of course a bad decision and is considered to be infeasible for that reason. Also in the states t with $21 \leq t < 25$ only null-decisions are feasible. In figure 5.2 states have been marked with two feasible decisions.

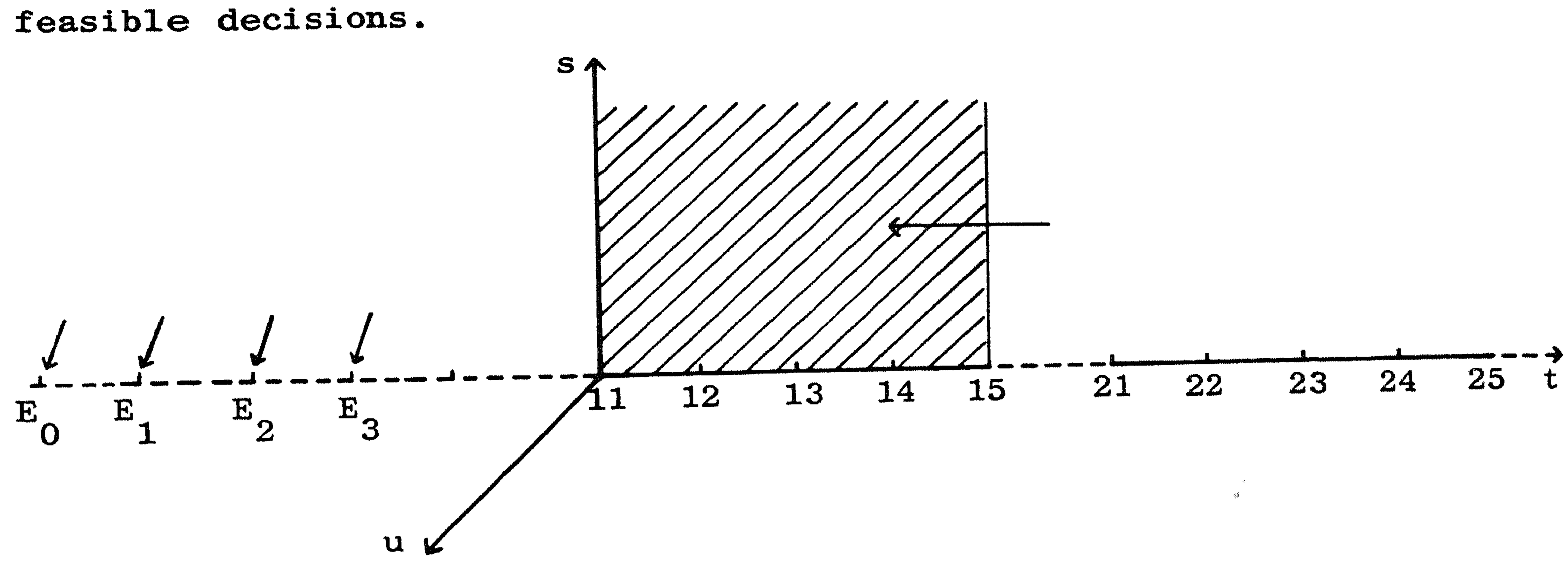

Figure 5.2 States with more than one feasible decision

From now on only strategies are considered which dictate payment of premium in the states E_i, i = 0, 1, 2, 3. By its definition (1.1) the set A_0 consists of the states in which each strategy dictates an intervention. In this problem the states E_i, i = 0, 1, 2, 3 and the states (t,s,u) with $11 \leq t < 15$, $s \leq a_0$ and u = 0 constitute the set A_0 because each strategy dictates the payment of premium and suppression of the claim if the damage does not exceed the own risk. So we have:

(5.1) $$A_0 = \{E_i, i = 0,1,2,3\} \cup \{11 \leq t < 15, s \leq a_0, u = 0\}.$$

The non-empty subsets $A_{0,1}$ and $A_{0,2}$ of the set A_0 are chosen in such a way that the most simple expressions for the functions k(x;d) and t(x;d) are obtained. Here we choose:

(5.2) $$A_{0,1} = A_{0,2} = \{E_i,\ i = 0,1,2,3\}$$

and consequently for the associated stochastic walks it follows:

(5.3) $$\underline{w}_{0,1} = \underline{w}_{0,2}$$

and

(5.4) $$\underline{w}_{d,1} = \underline{w}_{d,2}.$$

To abbreviate the notation we write $\underline{w}_0$ and $\underline{w}_d$ respectively.

Consider the $\underline{w}_0$-walk having $(1i+\tau, s, 0)$ as initial state. During the walk $\underline{w}_0$ the system is subjected to the natural process. In the natural process each damage is claimed. The damage s at time τ is thus claimed and the costs $\min(s, a_0)$ are incurred. For each damage which occurs in the natural process we have the expected costs

(5.5) $$k(a_0) = \int_0^{a_0} s\ F(ds) + a_0 \int_{a_0}^{\infty} F(ds)$$

The expected number of accidents in a fraction $1 - \tau$ of a year is equal to $\lambda(1-\tau)$. Hence the expected costs incurred during the walk $\underline{w}_0$ are given by:

$$(5.6) \qquad k_0(1i+\tau,s,0) = \lambda(1-\tau)\ k(a_0) + \min(s,a_0).$$

The expected duration of the walk $\underline{w}_0$ is obviously:

$$(5.7) \qquad t_0(1i+\tau,s,0) = 1 - \tau.$$

Since decisions lead to deterministic transitions in this problem they will be denoted by the resulting states.

During the $\underline{w}_d$-walk starting in $x = (1i+\tau,s,0)$ the claim is suppressed and the system is transformed to state $d = 2i + \tau$. After this transformation the system is subjected to the natural process up to the end of the year. At that moment either state E_0 is taken on if a second accident occurred or state E_i if no seoond accident occurred. The expected duration $t_1(x;d)$ of the $\underline{w}_d$-walk for $x = (1i+\tau,s,0)$ and $d = 2i + \tau$ is given by:

$$(5.8) \qquad t_1(1i+\tau,s,0;2i+\tau) = 1 - \tau,$$

and the expected costs by:

$$(5.9) \qquad k_1(1i+\tau,s,0;2i+\tau) = \lambda(1-\tau)\ k(a_0) + s.$$

By (5.6) ... (5.9) and referring to (1.3) and (1.4) the following relations are obtained for the functions $k(x;d)$ and $t(x;d)$ with $x = (1i+\tau,s,0)$ and $d = 2i + \tau$:

$$(5.10) \qquad k(1i+\tau,s,0;2i+\tau) = \max(s-a_0,0),$$

$$(5.11) \qquad t(1i+\tau,s,0;2i+\tau) = 0.$$

Finally the k- and t-functions for the states E_{i-1}, i = 1, 2, 3, 4 are determined. The $\underline{w}_0$-walk having E_{i-1} as initial state ends immediately in that state because $E_{i-1} \in A_{0,1} = A_{0,2}$. In the natural process no premiums are paid. Hence $k_0(E_{i-1}) = t_0(E_{i-1}) = 0$. Next we consider the $\underline{w}_d$-walk having E_{i-1} as initial state. The payment of premium in state E_{i-1} transforms the system into state 2i. The $\underline{w}_d$-walk is from state 2i on subjected to the natural process. At the end of the year the walk ends either in state E_0 or state E_i. The expected duration of the $\underline{w}_d$-walk is thus one year. The expected costs of the $\underline{w}_d$-walk consists of the premium E_{i-1} and the expected costs incurred during the year in the natural process. So we obtain

(5.12) $$t(E_{i-1},2i) = 1,$$

(5.13) $$k(E_{i-1},2i) = E_{i-1} + \lambda k(a_0).$$

5.3 Determination of the optimal strategy.

It is easily verified that for all strategies $z \in Z$ the Markov-process in A_z has only one simple ergodic set. Consequently for every strategy z and feasible decision d, we have;

(5.14) $$r(d.z;x) = r(z;x) = r(z) \qquad \text{for all } x.$$

Hence we need only to consider the functional equations concerning the function c(z;x). In order to obtain a unique solution, we put:

(5.15) $$c(z;E_0) = 0.$$

If an arbitrary strategy z dictates to claim in state x = (t,s,0) with $11 \leq t < 15$ and $s > a_0$ then the next intervention state is always E_0. So we have (c.f. (1.16)):

(5.16) $$c(z;t,s,0) = c(z;E_0) = 0.$$

If it is decided to suppress the claim in state $x = (t,s,0)$ then $d = t + 10$ and if future decisions are taken in accordance with strategy z the function $c(d.z;x)$ is given by (c.f. (1.23)):

$$c((t+10).z;t,s,0) = s - a_0 + c(z;t+10). \tag{5.17}$$

From now on only the optimal strategy z* is considered. Let the boundary of A_{z*} be given by the function $s = s(t)$. For z* we have (c.f. (1.41)):

$$c(z^*;x) = \min_{d \in D(x)} c(d.z^*;x). \tag{5.18}$$

For $x = (t,s,0)$ with $11 \leq t < 15$ and $a_0 < s \leq s(t)$ it will be profitable not to claim, so $d = z^*(x) = t + 10$. From (5.15) and (5.18) it follows:

$$c(z^*;t,s,0) = c((t+10).z^*;t,s,0) \leq c(z^*;E_0) = 0. \tag{5.19}$$

According to (5.17) $c((t+10).z^*;t,s,0)$ is a linear increasing function of s. Hence it will be indifferent on the boundary s(t) of A_{z*} to claim or not to claim. For $s = s(t)$ we have consequently:

$$c((t+10).z^*;t,s(t),o) = c(z^*;t,s(t),0) = 0 \tag{5.20}$$

and by (5.17) and (5.20)

$$c(z^*;t,s(t),0) = s(t) - a_0 + c(z^*;t+10) = 0. \tag{5.21}$$

From (5.21) it follows:

$$c(z^*;t+10) = a_0 - s(t) \tag{5.22}$$

and from (5.17) and (5.22) we obtain:

$$c(z^*;t,s,0) = s - s(t) \qquad \text{for } a_0 < s \leq s(t). \tag{5.23}$$

For $s \leq a_0$ by (5.10) and (5.11) we have (c.f. (1.14)):

$$c(z^*;t,s,0) = c(z^*;t+10) = a_0 - s(t). \tag{5.24}$$

Furthermore holds for $c(z^*;E_i)$, i = 1, 2, 3:

$$c(z^*;E_i) = \lim_{t\uparrow 2i+1} c(z^*;t), \tag{5.25}$$

or by (5.22) and (5.25):

$$c(z^*;E_i) = a_0 - \lim_{t\uparrow 1i+1} s(t). \tag{5.26}$$

Summarizing our results:

$$c(z^*;x) =$$

$$\begin{cases} 0 & \text{for } x \in E_0 \cup \{11 \leq t < 15, s > s(t), u = 0\} \\ & \cup \{11 \leq t < 15, s \geq 0, u > 0\}, \\ a_0 - \lim\limits_{t\uparrow 1i+1} s(t) & \text{for } x \in \bigcup\limits_{i=1}^{3} E_i, \\ a_0 - s(t) & \text{for } x \in \{11 \leq t < 15, s \leq a_0, u = 0\}, \\ s - s(t) & \text{for } x \in \{11 \leq t < 15, a_0 < s < s(t), u = 0\}, \\ a_0 - s(t-10) & \text{for } x \in \{21 \leq t < 25\}. \end{cases} \tag{5.27}$$

From functional equation (1.14) it follows for $x = E_{i-1}$, i = 1, 2, 3, 4:

$$c(z^*;E_{i-1}) = k(E_{i-1};2i) - r(z^*)\ t(E_{i-1};2i) + c(z^*;2i). \tag{5.28}$$

By substitution of (5.12) and (5.13) in (5.28) it follows:

$$(5.29) \qquad c(z^*;E_{i-1}) - c(z^*;2i) = E_{i-1} + \lambda k(a_0) - r(z^*).$$

From (5.27) and (5.29) we obtain:

$$(5.30) \qquad s(1i) = \begin{cases} E_0 + \lambda k(a_0) - r(z^*) + a_0 & \text{for } i = 1, \\ E_{i-1} + \lambda k(a_0) - r(z^*) + \lim\limits_{t \uparrow 1i} s(t) & \text{for } i = 2, 3, 4. \end{cases}$$

Furthermore we have the relation:

$$(5.31) \qquad \lim_{t \uparrow 14} s(t) = \lim_{t \uparrow 15} s(t).$$

For $x = (t,s,0)$ with $s > a_0$ and $t = 1i + \tau$ it follows (c.f. (1.14)):

$$(5.32) \qquad c(z^*;t,s,0) =$$

$$k(t,s,0;t+10) - r(z^*)\, t(t,s,0;t+10) +$$

$$+ \int_{1i+1-t}^{\infty} c(z^*;E_i)\, \lambda e^{-\lambda\tau_1}\, d\tau_1 +$$

$$+ \int_0^{1i+1-t} \lambda e^{-\lambda\tau_1}\, d\tau_1 \int_0^{s(t+\tau_1)} c(z^*;t+\tau_1,y,0)\, F(dy) +$$

$$+ \int_0^{1i+1-t} \lambda e^{-\lambda\tau_1}\, d\tau_1 \int_{s(t+\tau_1)}^{\infty} c(z^*;E_0)\, F(dy).$$

Substitution of (5.10), (5.11) and (5.27) in (5.32) leads to:

$$(5.33) \qquad c(z^*;t,s,0) =$$

$$s - a_0 + e^{-\lambda(1i+\tau-t)} (a_0 - \lim_{t \uparrow 1i+1} s(t)) +$$

$$+ \int_{0}^{li+1-t} e^{-\lambda\tau_1} d\tau_1 \int_{a_0}^{s(t+\tau_1)} (y-s(t+\tau_1))\, dF(y) \; +$$

$$+ \int_{0}^{li+1-t} \lambda e^{-\Lambda\rightarrow_1} d\tau_1 \int_{0}^{a_0} (a_0-s(t+\tau_1))\, dF(y).$$

After substitution of s = s(t) and (5.29) the differentiation of (5.33) with respect to t leads to:

$$\text{(5.34)} \qquad \frac{ds(t)}{dt} = \lambda \int_{a_0}^{\infty} (y-a_0)\, dF(y) - \lambda \int_{s(t)}^{\infty} (y-s(t))\, dF(y).$$

By partial integration this functional equation can be written in the more simple form:

$$\text{(5.35)} \qquad \frac{ds(t)}{dt} = \lambda \int_{a_0}^{s(t)} (1-F(y))\, dy.$$

Except a translation along the t-axis the boundary s(t) is determined by (5.35). In other words the boundary of A_{z*} for i = 1, 2, 3, 4 are in the t-direction translated parts of one curve satisfying (5.35). The location of each part on this curve has to be determined from the relations (5.30) and (5.31).

Suppose that r(z*) is known, then s(11) is solved from (5.30). From the curve s = s(t) we find $\lim_{t\uparrow 12} s(t)$. From (5.30) we obtain s(12). Similarly we compute $\lim_{t\uparrow 13} s(t)$, s(13), $\lim_{t\uparrow 14} s(t)$, s(14) and $\lim_{t\uparrow 15} s(t)$. If r(z*) is not known its value is determined by relation (5.31).

It should be noted that the functional equation (5.35) has an analytical solution in the case the damage per accident is exponentially distributed. We have then for $F(s) = 1 - e^{-\mu s}$:

$$(5.36) \qquad \frac{ds(t)}{dt} = \frac{\lambda}{\mu} e^{-\mu a_0} (1-e^{-\mu(s(t)-a_0)}).$$

The solution of (5.36) is given by:

$$(5.37) \qquad s(t) = a_0 + \frac{1}{\mu} \ln \{1+e^{\lambda(t+c_i)e^{-\mu a_0}}\},$$

where the c_i $i = 1, 2, 3, 4$ are integration constants each corresponding to the time intervals $1i \leq t < 1i + 1$, $i = 1, 2, 3, 4$. If the distribution of the damage is not exponential we have to solve (5.36) numerically in most cases.

5.4 Some numerical results.

The following numerical data are used:

$$E_0 = 1.6$$
$$E_1 = 1.4$$
$$E_2 = 1.2$$
$$E_3 = 1.1$$
$$a_0 = 0.4.$$

For these data and $\lambda = 2$ accidents per year five distributions with the same expectation were investigated. The type of distribution, its expectation and coefficient of variation are given in the following table:

Number of curve	Type of distribution	Expectation	Coefficient of variation
1	exponential	1	1
2	gamma	1	1/3
3	log normal	1	1
4	log normal	1	1/3
5	log normal	1	3

The densityfunctions are sketched in figure 5.3

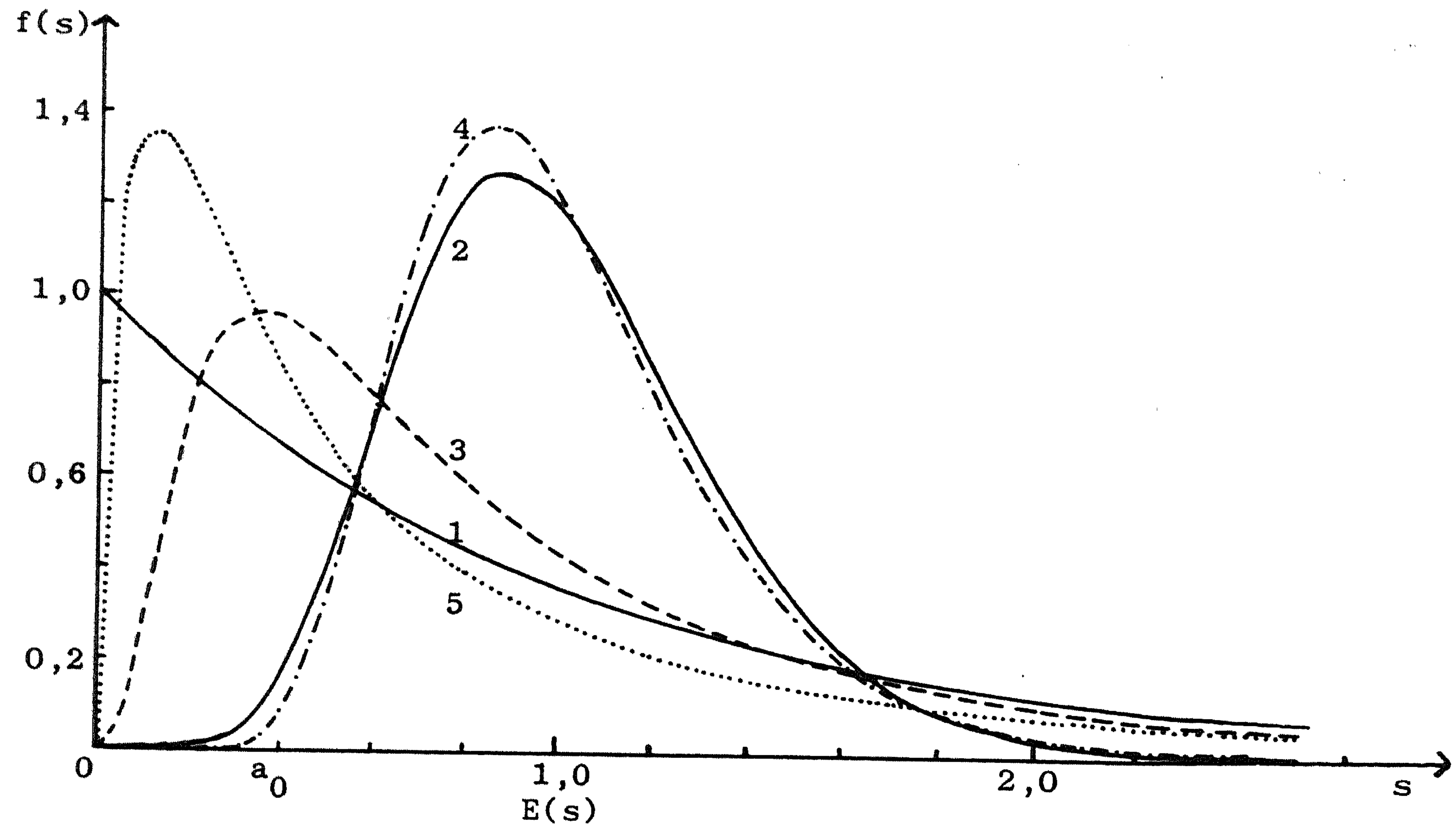

Figure 5.3. The five used damage distributions

The corresponding optimal strategies are presented in figure 5.4. From these results it can be deduced that for distributions with the same mean and variance the optimal strategy are nearly the same. Further, if the variance increases the boundary of A_{z*} moves upwards. The results were obtained by a computer program especially written for this problem.

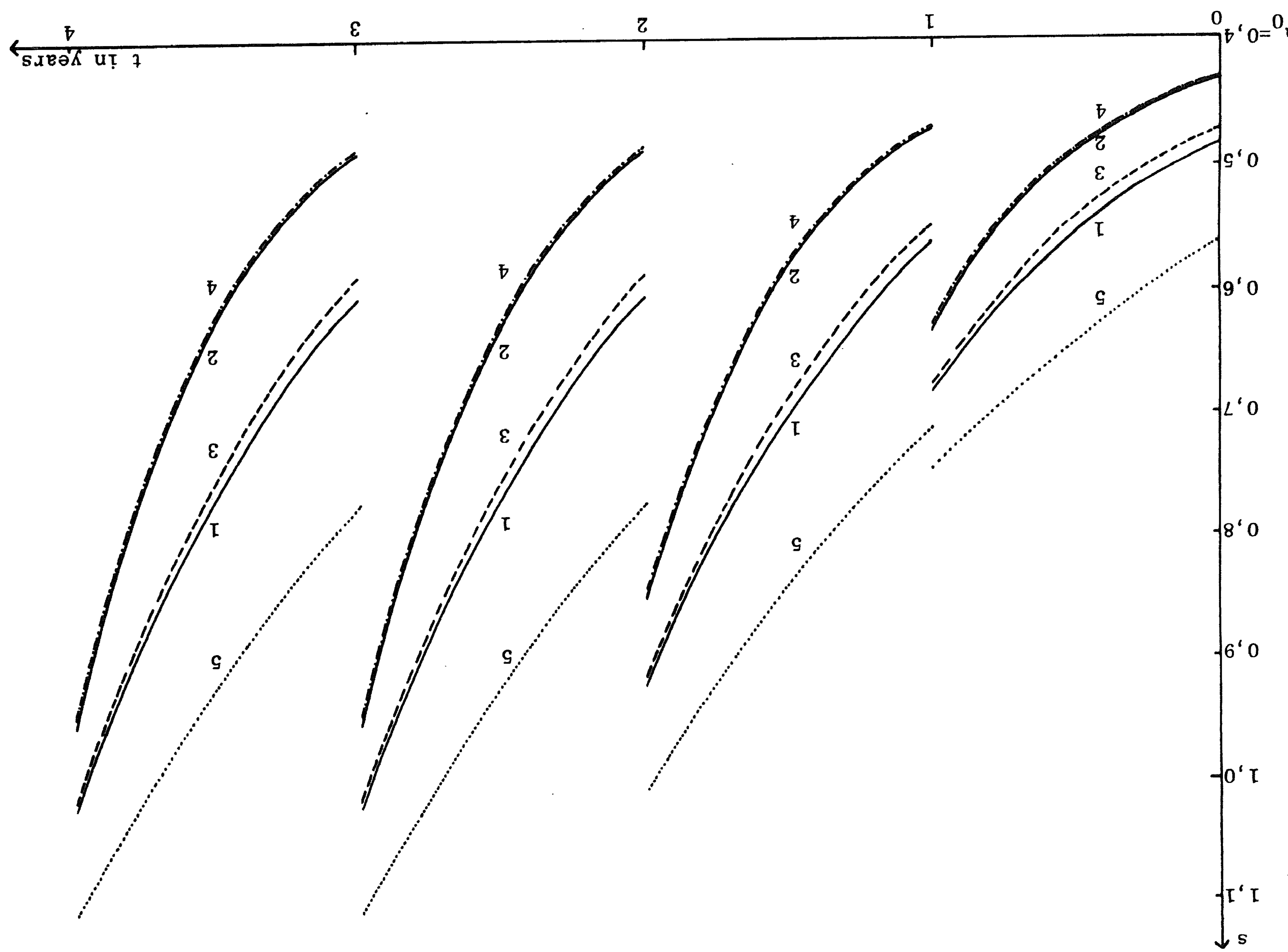

Figure 5.4 The optimal strategy for these distributions

6. A production problem with a non-denumerable state space

6.1 Introduction

The production of a continuous product can be realised on a finite number of production levels l_i, $i = 0, 1, \ldots, N$ with $l_0 = 0$. The product is kept in stock. The storage capacity is limited to a quantity M.

Orders arrive according to a Poisson-process with a mean of λ per unit of time. The order size $\underline{y}$ is a non-negative random variable with a given distribution function F(y) with finite mean and variance. The order size is assumed to be independent of the arrival process. Orders are fulfilled immediately by the available stock. If the size of an order exceeds the available stock then the supply is replenished by an emergency purchase.

The production can be controlled by switching over to another production level. There is no lead time needed to perform a change of production level. The following costs are involved in the operation of this system:

a) production costs $c_p(i)$ per unit of time for level l_i, $i = 0, 1, \ldots, N$ with $c_p(0) = 0$.

b) costs $c_q(i,j)$ of switching over from level l_i to level l_j, $i, j = 0, \ldots, N$.

c) costs c_r per unit of product of an emergency purchase.

d) stockholding costs c_s per unit of time per unit of product.

The decision maker wants to find the production strategy, which minimizes the (expected) average costs per unit of time.

In the following sections it will be demonstrated how the optimal strategy is obtained. In section 6.2 the strategy-independent notions such as the state space, the natural process, the set of feasible decisions in each state and the set A_0 are defined. In the same section functional equations are derived for the strategy-independent functions k(x;d) and t(x;d). Section 6.3 will be devoted to the solution of the functional equations for r(z;x) and c(z;x). It will be demonstrated that the solution can be obtained by solving a set of N linear equations making use of a finite Markov-chain imbedded in the Markov-process in A_z. In section 6.4

the computations which are involved in the method of section 6.3 are considered in more detail. In section 6.5 it will be demonstrated how the cutting mechanism is applied in this problem. Finally in section 6.6 numerical results will be given.

6.2 The strategy-independent notions

The state of the system x is specified by two state variables, an integer i for the productionlevel and a real variable s for the stocklevel. The stocklevel is restricted to the interval $0 \leq s \leq M$, so that the state space is given by

$$X = \{x = (i,s) \mid 0 \leq s \leq M,\ i = 0, 1, \ldots, N\}. \tag{6.1}$$

The natural process with initial state (0,s) with $0 < s \leq M$ implies the arrivals of orders and their supply. It also includes the first emergency purchase, which is done as soon as the initial stock s is exhausted. After the first emergency purchase the natural process remains in state (0,0) forever.

During the natural process with initial state (i,s), $i > 0$, the system continues to produce on the fixed productionlevel i. At each point of time when the available stock is exhausted by an order an emergency purchase is done. The natural process remains in state (i,M) forever as soon as this state is taken on.

A decision in this problem leads to a deterministic transformation. Hence a decision may be denoted by the state into which the corresponding transformation results. If a decision d in the state (i,s) is given by (j,s') then it follows that the set of feasible decisions in (i,s) is given by:

$$D(i,s) = \{(j,s') \mid j = 0, 1, \ldots, N,\ s' = s\}. \tag{6.2}$$

The strategy which supplies the demand only by emergency purchases is not considered, which is no restriction if the problem makes sense. According to this agreement and the definition of the natural process each strategy $z \in Z$ has to dictate both an intervention in state (0,0) as in the states (i,M), $i = 1, \ldots, N$. The intervention in the states (i,M),

$i = 1, \ldots, N$ results in a transformation to state (0,M). Hence the set A_0 is given by (c.f. (1.1)):

(6.3) $$A_0 = \{(0,0)\} \cup \{(i,M) \mid i = 1, \ldots, N\}.$$

In problems where no finite maximum stocklevel M is specified, we assume its existence. This is no restriction if the numerical value of M is chosen sufficiently large, so that the optimal strategy for the original problem is not excluded.

In this problem it is not possible to simplify the computation of the functions k(x;d) and t(x;d) by the choice of the subsets $A_{0,1}$ and $A_{0,2}$. Hence we choose:

(6.4) $$A_{0,1} = A_{0,2} = A_0$$

and consequently:

(6.5) $$\underline{w}_{0,1} = \underline{w}_{0,2}$$

(6.6) $$\underline{w}_{d,1} = \underline{w}_{d,2}.$$

To abbreviate the notation these walks are respectively denoted by $\underline{w}_0$ and $\underline{w}_d$.

Let the duration of a $\underline{w}_0$-walk with (i,s) as initial state be denoted by the random variable $\underline{t}_i(s)$ and the expected duration by $t_i(s)$. The costs incurred during this walk are denoted by $\underline{k}_i(s)$ and their expected value by $k_i(s)$. During the $\underline{w}_0$-walk the system is only subjected to the natural process. If the initial state is (0,s), $0 \leq s \leq M$ then the walk ends in state $(0,0) \in A_0$. If the initial state is (i,s), $i > 0$, $0 \leq s \leq M$ then the $\underline{w}_0$-walk ends in state (i,M).

Next the derivation of the functional equations for $t_i(s)$ and $k_i(s)$ will be discussed briefly. Consider primarily the case that $i = 0$. Let $\underline{\tau}$ denote the time interval elapsed between the start of the walk and the arrival of the first order. Let $\underline{y}$ denote the size of the first order. If $\underline{y}$ exceeds the available stock $s > 0$ an emergency purchase is done to replenish the order and the walk ends thereafter. If $\underline{y}$ does not exceed the available stock then the walk continues from state $(0,s-\underline{y})$ and from then on the duration will be $\underline{t}_0(s-\underline{y})$. For $\underline{t}_0(s)$, $s > 0$ it follows that:

(6.7) $$\underline{t}_0(s) = \underline{\tau} + \begin{cases} \underline{t}_0(s-\underline{y}) & \text{if } \underline{y} < s \\ 0 & \text{if } \underline{y} \geq s. \end{cases}$$

Recalling that $E\underline{\tau} = \frac{1}{\lambda}$, the following functional equation in the expected duration $t_0(s)$ is obtained:

(6.8) $$t_0(s) = \frac{1}{\lambda} + \int_0^s t_0(s-y)F(dy) \quad \text{for} \quad 0 < s \leq M.$$

If state (0,0) is the initial state then the walk ends immediately and its expected duration is given by:

(6.9) $$t_0(0) = 0.$$

The costs incurred during the time interval $\underline{\tau}$ are the stockholding costs which amount $c_s\, s\, \underline{\tau}$. If $\underline{y}$ exceeds the available stock s then the costs $c_r(s-\underline{y})$ of the necessary emergency purchase have to be added and if $\underline{y}$ does not exceed s then the costs $\underline{k}_0(s-\underline{y})$ are incurred in addition. For $\underline{k}_0(s)$ follows:

(6.10) $$\underline{k}_0(s) = c_s\, s\, \underline{\tau} + \begin{cases} k_0(s-\underline{y}) & \text{if } \underline{y} < s \\ c_r(\underline{y}-s) & \text{if } \underline{y} \geq s. \end{cases}$$

For the expected costs $k_0(s)$ the following functional equation is obtained:

(6.11) $$k_0(s) = \frac{c_s s}{\lambda} + c_r \int_s^\infty (y-s)F(dy) + \int_0^s k_0(s-y)F(dy) \text{ for } 0<s\leq M.$$

If state (0,0) is the initial state then it follows:

(6.12) $$k_0(0) = 0.$$

If in the initial state the system is producing on productionlevel $i > 0$ then the stocklevel increases linearly between the arrivals of orders. If the initial state is $(i,M) \in A_0$ then the walk ends immediately so we have:

(6.13) $\quad t_i(M) = 0 \quad$ for $i = 1, \ldots, N$

(6.14) $\quad k_i(M) = 0 \quad$ for $i = 1, \ldots, N$.

The duration $\underline{t}_i(s)$ follows from:

(6.15)
$$\underline{t}_i(s) = \begin{cases} \dfrac{M-s}{l_i} & \text{if } \underline{\tau} > \dfrac{M-s}{l_i} \\[2ex] \underline{\tau} + \underline{t}_i(0) & \text{if } \underline{\tau} \leq \dfrac{M-s}{l_i} \text{ and } \underline{y} > s + l_i \underline{\tau} \\[2ex] \underline{\tau} + \underline{t}_i(s + l_i \underline{\tau} - \underline{y}) & \text{if } \underline{\tau} \leq \dfrac{M-s}{l_i} \text{ and } \underline{y} \leq s + l_i \underline{\tau}. \end{cases}$$

From (6.15) the following functional equation is obtained for the expected duration $t_i(s)$:

(6.16)
$$t_i(s) =$$
$$+ \frac{M-s}{l_i} \int_{(M-s)/l_i}^{\infty} \lambda e^{-\lambda\tau} d\tau + \int_0^{(M-s)/l_i} \tau \, \lambda e^{-\lambda\tau} d\tau +$$
$$+ t_i(0) \int_0^{(M-s)/l_i} \lambda e^{-\lambda\tau} d\tau \int_{s+l_i\tau}^{\infty} F(dy) +$$
$$+ \int_0^{(M-s)/l_i} \lambda e^{-\lambda\tau} d\tau \int_0^{s+l_i\tau} t_i(s+l_i\tau-y) F(dy)$$

for $0 \leq s < M$ and $1 \leq i \leq N$.

By differentiation with respect to s this functional equation can be written in the form:

(6.17)
$$\frac{dt_i(s)}{ds} = \frac{\lambda}{l_i} t_i(s) - \frac{1}{l_i} - \frac{\lambda}{l_i} t_i(0)(1 - F(s)) +$$
$$- \frac{\lambda}{l_i} \int_0^s t_i(s - y) F(dy) \quad \text{for } 0 \leq s < M$$

and $1 \leq i \leq N$.

In a similar way the following functional equation is obtained in $k_i(s)$:

$$\frac{d\,k_i(s)}{ds} = \frac{\lambda}{l_i} k_i(s) - \frac{c_p(i)}{l_i} - \frac{c_s\,s}{l_i} - \frac{\lambda}{l_i} k_i(0)(1 - F(s)) + \tag{6.18}$$

$$- \frac{\lambda}{l_i} \int_0^s k_i(s - y)F(dy) +$$

$$- \frac{\lambda c_r}{l_i} \int_s^\infty (y - s)F(dy) \quad \text{for } 0 \leq s < M \text{ and } 1 \leq i \leq N.$$

From the functional equations (6.8), (6.11), (6.17) and (6.18) together with the conditions (6.9), (6.12), (6.13) and (6.14) the expected duration and the expected costs associated with the $\underline{w}_0$-walk with initial state $(i,s) \in X$ are obtained.

During the $\underline{w}_d$-walk with (i,s) as initial state the decision $(j,s) \in D(i,s)$ is taken. After the transformation to state (j,s) the walk is subjected to the natural process with (j,s) as initial state. The transformation does not take time and its costs amount $c_q(i,j)$. From the relations (1.3) and (1.4) it follows:

$$k(i,s;j,s) = c_q(i,j) + k_j(s) - k_i(s), \tag{6.19}$$

$$t(i,s;j,s) = t_j(s) - t_i(s), \tag{6.20}$$

for each state $(i,s) \in X$ and feasible decision $(j,s) \in D(i,s)$ with $j \neq i$. For $i = j$ the functions $k(i,s;j,s)$ and $t(i,s;j,s)$ are identical to zero.

6.3 The solution of the functional equations for $c(z;x)$

For a given strategy z the functions $r(z;x)$ and $c(z;x)$ are obtained from the relations (1.13), (1.14), (1.16) and (1.18). Primarily the possibility of two or more simple ergodic sets of states is investigated. If the distribution function $F(y)$ is continuous almost everywhere and $F(M) < 1$ then two arbitrary states (i,s_1) and (i,s_2) on the same production level cannot belong to two disjunct simple ergodic sets. Some reflection shows that this is also impossible for two arbitrary states (i_1,s_1) and (i_2,s_2) without violating the definition of the problem. Hence it is allowed to put:

$$(6.21) \qquad r(z;x) = r(d\cdot z;x) = r(z) \qquad \text{for } x \in X$$

$$(6.22) \qquad D_z(x) = D(x) \qquad \text{for } x \in X.$$

In this problem the optimal strategy is obtained by the iterative approach (c.f. chapter 1). The first step is to solve the functional equations (1.14) and (1.18) for $c(z;x)$. Thereafter the strategy is improved by means of steps 2 and 3. The specific form the functional equations (1.14) and (1.18) take on in a particular problem depends strongly on the structure of the set A_z. Especially the structure of A_z is important in problems with a non-denumerable state space because then the functional equation (1.14) does not take in general the form of a set of linear equations. Because of this observation it is advantageous to consider primarily the structure of A_z for the strategies that emerge from the strategy-improvement steps.

Suppose that for a given strategy z, the function $c(z;x)$ is obtained in a computationally useful form. This may be either in polynomial or in tabular form. In the latter case intermediate values can be obtained by interpolation. The next step in the iterative approach is then to find for each state $(i,s) \in X$ the decision $d \in D(i,s)$ minimizing $c(d\cdot z;i,s)$. For each interval of states (i,s) with fixed i and $0 \leq s \leq M$ there are N+1 functions $c(d_k\cdot z;i,s)$ each corresponding to one feasible decision $d_k = (k,s)$, $k = 0, \ldots, N$. These N+1 functions will have a finite number of intersections on the interval $0 \leq s \leq M$. Let the locations of these intersections be denoted by the numbers r_j, $j = 1, \ldots, m_i-1$ ordered in increasing magnitude. Let $r_0 = 0$ and $r_{m_i} = M$. For each interval between two neighbouring intersections r_j and r_{j+1} one of the N+1 functions $c(d_k\cdot z;i,s)$ will minimize $c(d\cdot z;i,s)$ on the interval $r_j < s < r_{j+1}$. An intersection located in (i,r_j) will be called a separation state of z_1 if the decisions minimizing $c(d\cdot z;i,s)$ respectively on the adjacent intervals $r_{j-1} < s < r_j$ and $r_j < s < r_{j+1}$ are not equal.

Let the separation states be denoted by the numbers s_{ik}, $i = 0, \dots, N$ and $k = 0, \dots, n_i$ with $n_i \leq m_i$, $s_{i0} = 0$ and $s_{i,n_i} = M$. Let the production level, to which the decision minimizing $c(d\cdot z;i,s)$ on the interval $s_{i,k-1} < s < s_{ik}$ with respect to $d \in D(i,s)$ transforms the system, be denoted by j_{ik} with $i = 0, \dots, N$ and $k = 1, \dots, n_i$. The strategy z_1 is completely specified by the numbers s_{ik} and j_{ik}. To a separation state s_{ik} which is not a boundary state of A_{z_1} one of the decisions (j_{ik}, s_{ik}) and $(j_{i,k+1}, s_{ik})$ is assigned. The decision assigned to a separation state being also a boundary state of A_{z_1} is uniquely determined and should be the one which is also an intervention.

By step 3 of the iterative approach the strategy z_1 is reduced to strategy z_2. Some reflection shows that the basic structure of z_2 is the same as strategy z_1 so we conclude that all strategies generated by the strategy-improvement steps have the structure discussed above.

The method to solve $c(z;x)$ from (1.14) is demonstrated on a strategy with only one interval for each production level in which null-decisions are dictated. The method used can be extended to the general structure in which there are two or more disjoint intervals with null-decisions on some production levels. For illustration a strategy with only one interval with null-decisions for each production level is presented in figure 6.1.

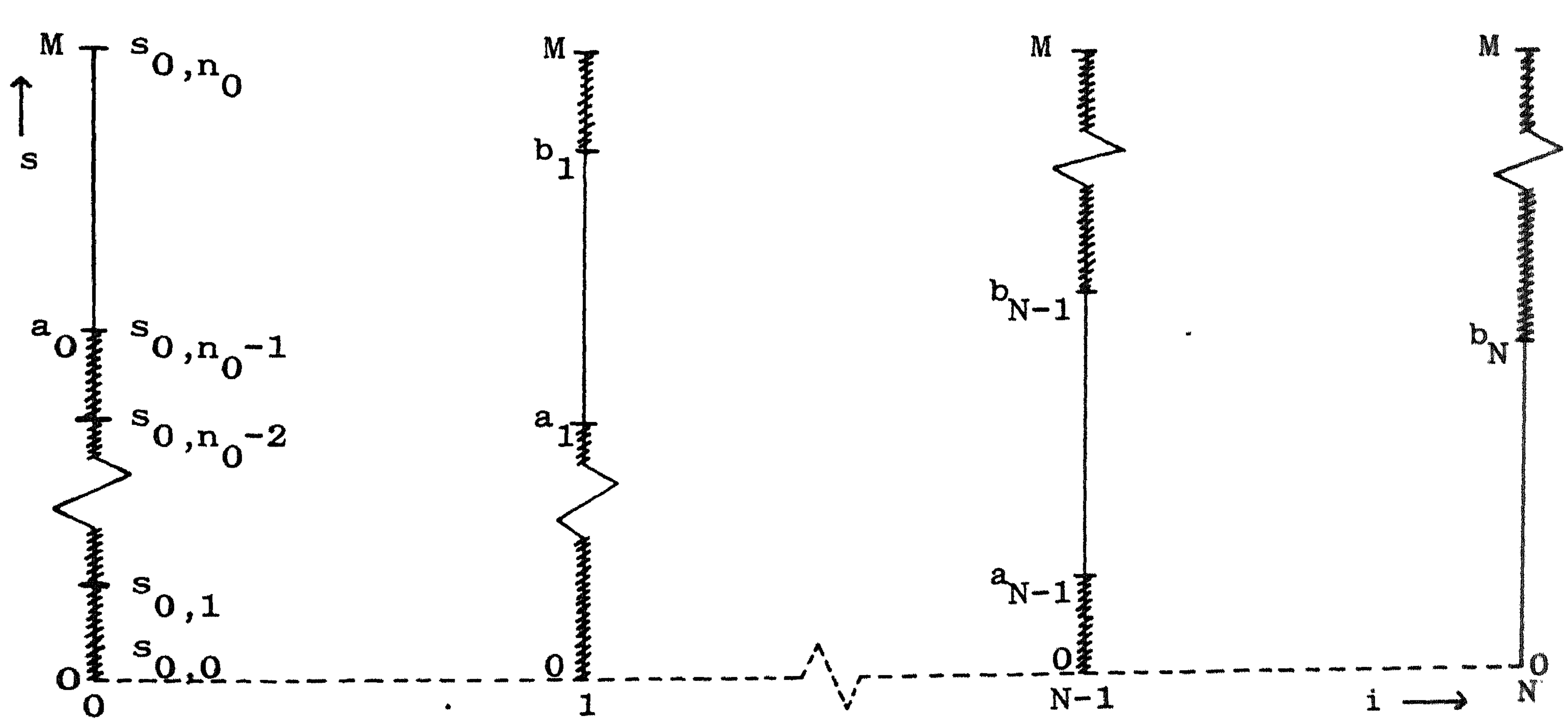

Figure 6.1 A strategy

The set A_z is indicated in figure 6.1 by the shaded intervals. The boundary states of A_z on production level i, $1 \leq i \leq N-1$ are denoted by a_i and b_i. For i = 0 the only boundary state is a_0 and for i = N the only boundary state is b_N. The separation states s_{0j}, j = 0, ..., n_0 are only indicated on production level i = 0.

For $1 \leq i \leq N-1$ there are two disjoint intervals of intervention states (i,s) given by $0 \leq s \leq a_i$ and $b_i \leq s \leq M$. For i = 0 and i = N the only intervals of intervention states are $0 \leq s \leq a_0$ and $b_N \leq s \leq M$ respectively. From the definition of the problem it follows that if the system starts a walk in a non-intervention state then the first future intervention state $\underline{I}_1$ will be located on the same production level. For $1 \leq i \leq N-1$ the state $\underline{I}_1$ will be either state b_i or a state $(i,\underline{u})$ in the interval $0 \leq \underline{u} \leq a_i$. For i = 0 the state $\underline{I}_1$ will be located in the interval $0 \leq \underline{u} \leq a_0$ and for i = N the state $\underline{I}_1$ will be identical to b_N. Note that the intervention states (i,s) with $b_i < s \leq M$ and $1 \leq i \leq N$ are transient states.

The boundary states b_i, i = 1, ..., N play an important role in the method developed to solve the functional equation for c(z;x). The finite set constituted by these states is denoted by B_z. Note that $B_z \subset A_z$. The Markov-chain in B_z is imbedded in the Markov-process in A_z.

In the following discussion walks of the system will be considered with initial state (i,s) and ending in the first future state in B_z taken on. Denote the first future state in B_z by $\underline{b}$. Suppose that the $\underline{m}^{th}$ state in the sequence of future intervention states $\underline{I}_n$, n = 1, 2, ... of the Markov-process in A_z is identical to $\underline{b}$. Let the function ct(z;i,s) denote the expectation of the sum of the contributions t(i,s;z(i,s)) in the initial state (i,s) and $t(\underline{I}_n;z(\underline{I}_n))$ in the future intervention states $\underline{I}_n$, n = 1, ..., $\underline{m}$-1. In mathematical terms its definition is given by:

$$(6.23) \qquad ct(z;i,s) = t(i,s;z(i,s)) + E\left\{\sum_{n=1}^{\underline{m}-1} t(\underline{I}_n;z(\underline{I}_n))\right\},$$

where E denotes the expectation with respect to the joint probability distribution of $\underline{m}$ and $\underline{I}_n$, n = 1, ..., $\underline{m}$-1 given initial state (i,s) and decision z(i,s) dictated by strategy z in state (i,s). The function ck(z;i,s) is defined similarly by:

(6.24) $$ck(z;i,s) = k(i,s;z(i,s)) + E\{\sum_{n=1}^{m-1} k(\underline{I}_n;z(\underline{I}_n))\}.$$

Consider now the functional equation (1.14) for $x = (i,s) \in X$:

(6.25) $$c(z;i,s) = k(i,s;z(i,s)) - r(z)t(i,s;z(i,s)) + \\ + Ec(z;\underline{I}_1).$$

By repeated application of (6.25) it follows:

(6.26) $$c(z;i,s) = ck(z;i,s) - r(z)ct(z;i,s) + \\ + Ec(z;\underline{b}) \qquad \text{for } (i,s) \in X,$$

where the expectation is taken with respect to the probability distribution of $\underline{b}$ given (i,s) and strategy z. For the states b_i, i = 1, ..., N we have particularly:

(6.27) $$c(z;b_i) = ck(z;b_i) - r(z)ct(z;b_i) + \\ + Ec(z;\underline{b}) \qquad \text{for } i = 1, \ldots, N.$$

In accordance with (1.18) we define:

(6.28) $$c(z;b_N) = 0.$$

If $ck(z;b_i)$, $ct(z;b_i)$ and the transition probabilities of the Markov-chain in B_z are known in advance then the N+1 unknowns, $c(z;b_i)$ for i = 1, ..., N and r(z), can be solved uniquely from the N linear equations (6.27) and (6.28).

For the remaining states c(z;i,s) can be obtained from (6.26) if the functions ck(z;i,s), ct(z;i,s) and the probability distribution of $\underline{b}$ are known for a given strategy z and each state $(i,s) \in X$. Their computation will be considered in section 6.4.

Summarizing the necessary steps to solve the functional equation (6.25) for a given strategy z:

1. Compute the functions ck(z;i,s), ct(z;i,s) and the probability distribution of $\underline{b}$ for all $(i,s) \in X$.
2. Obtain $ck(z;b_i)$, $ct(z;b_i)$ and the transition probabilities of the Markov-chain in B_z from the results of step 1.
3. Solve $c(z;b_i)$, $i = 1, \ldots, N$ and r(z) from (6.27) and (6.28).
4. Compute c(z;i,s) from (6.26) for the remaining states.

6.4 Some computational aspects

In this section the computation of the functions ck(z;i,s), ct(z;i,s) and the probability distribution of $\underline{b}$ for given $(i,s) \in X$ is discussed.

Primarily the probability distribution of the first future intervention state $\underline{I}_1$ taken on during a walk of the system with initial state (i,s) will be determined. The following probabilities are defined for $(i,s) \notin A_z$:

(6.33) $$P(b_i;z;i,s) = P\{\underline{I}_1 = b_i \mid z;(i,s)\} \quad \text{for } 1 \leq i \leq N$$

(6.34) $$G(v;z;i,s) = P\{\underline{I}_1 = (i,\underline{u}),\ a_i - v \leq \underline{u} \leq a_i \mid z;(i,s)\}$$
$$\text{for } 0 \leq i \leq N-1 \text{ and } 0 \leq v < \infty.$$

In a similar way as done in section 6.2 for the functions $t_i(s)$ and $k_i(s)$, functional equations are derived for $P(b_i;z;i,s)$ and G(v;z;i,s).

For a state (N,s) with $0 \leq s < b_N$, the first future intervention state will be b_N with probability one, so we have:

(6.35) $$P(b_N;z;N,s) = 1 \qquad \text{for } 0 \leq s \leq b_N.$$

If state $(i,s) \notin A_z$ is located on production level $i = 0$, then $\underline{I}_1$ is located in the interval $0 \leq s \leq a_0$. For G(v;z;0,s) the following functional equation can be derived for $a_0 < s \leq M$

(6.36) $$G(v;z;0,s) = \begin{cases} F(s-a_0+v) - F(s-a_0) + \int_0^{s-a_0} G(v;z;0,s-y)F(dy) & \text{for } 0 \leq v < a_0 \\ 1 & \text{for } v \geq a_0. \end{cases}$$

Note that $G(v;z;0,s)$ is in fact a function of two independent variables $u = s - a_0$ and v. Both variables u and v do not depend on the strategy, hence $G(v;z;0,s)$ can be computed before the iterative part of the method is entered.

If $(i,s) \notin A_z$ is located on production level i with $1 \leq i \leq N-1$ then $\underline{I}_1$ will be either state b_i or a state $(i,\underline{u})$ with $0 \leq \underline{u} \leq a_i$. Then the following functional equation follows for $P(b_i;z;i,s)$, $1 \leq i \leq N-1$, $a_i < s < b_i$:

(6.37) $P(b_i;z;i,s) =$

$$\int_{(b_i-s)/l_i}^{\infty} \lambda e^{-\lambda\tau} d\tau + \int_0^{(b_i-s)/l_i} \lambda e^{-\lambda\tau} d\tau \int_0^{s+l_i\tau-a_i} P(b_i;z;i,s+l_i\tau-y)F(dy)$$

and for $G(v;z;i,s)$ with $1 \leq i \leq N-1$ and $a_i < s < b_i$:

(6.38a) $G(v;z;i,s) =$

$$\int_0^{(b_i-s)/l_i} \lambda e^{-\lambda\tau}\{F(s + l_i\tau - a_i + v) - F(s + l_i\tau - a_i)\}d\tau +$$

$$+ \int_0^{(b_i-s)/l_i} \lambda e^{-\lambda\tau} d\tau \int_0^{s+l_i\tau-a_i} G(v;z;i,s+l_i\tau-y)F(dy)$$

$$\text{for } 0 \leq v < a_i$$

(6.38b) $$G(v;z;i,s) = 1 - P(b_i;z;i,s) \qquad \text{for } a_i \leq v < \infty.$$

It will now be shown that both probabilities can be expressed in functions, which are independent of the strategy and from which they can be obtained quite easily. This fact reduces considerably the amount of computation involved in the iterative solution of this problem.

By differentiation of (6.37) with respect to s it follows:

$$(6.39)\qquad \frac{dP(b_i;z;i,s)}{ds} =$$

$$\frac{\lambda}{l_i} P(b_i;z;i,s) - \frac{\lambda}{l_i}\int_0^{s-a_i} P(b_i;z;i,s-y)F(dy).$$

By integration of (6.36) over $a_i < s < u$ and performing the obvious integration by parts, (6.39) can be written as follows:

$$(6.40)\qquad P(b_i;z;i,u) =$$

$$P(b_i;z;i,a_i) + \frac{\lambda}{l_i}\int_0^{u-a_i} P(b_i;z;i,u-y)(1-F(dy))dy.$$

Note that (6.40) can only be solved relative to a multiplicative constant. This constant is chosen in such a way that:

$$(6.41)\qquad P(b_i;z;b_i) = 1 \qquad \text{for } 1 \leq i \leq N-1.$$

Let $Q_i(w)$ be defined by the unique solution of the functional equation:

$$(6.42)\qquad Q_i(w) = 1 + \frac{\lambda}{l_i}\int_0^{w} Q_i(w-y)(1-F(y))dy \quad \text{for } 0 \leq w < \infty.$$

Then the solution of (6.40) and (6.41) in terms of the function $Q_i(w)$ is as follows:

$$(6.43)\qquad P(b_i;z;i,s) = \frac{Q_i(s - a_i)}{Q_i(b_i - a_i)} \qquad \text{for } a_i < s \leq b_i.$$

which can be verified by substitution of (6.43) in (6.40).

Note that $Q_i(w)$ is independent of the strategy. If a_i and b_i are specified by the strategy then $P(b_i;z;i,s)$ is easily obtained from (6.43).

It can be shown that the solution of (6.38) can be expressed in $Q_i(w)$. Then we obtain for $G(v;z;i,s)$:

(6.44a) $G(v;z;i,s) =$

$$+ \frac{Q_i(s-a_i)}{Q_i(b_i-a_i)} \frac{\lambda}{l_i} \int_0^{b_i-a_i} \{F(b_i-a_i+v-w) - F(b_i-a_i-w)\} Q_i(w)\,dw$$

$$- \frac{\lambda}{l_i} \int_0^{s-a_i} \{F(s-a_i+v-w) - F(s-a_i-w)\} Q_i(w)\,dw \qquad \text{for } 0 \leq v < a_i,$$

(6.44b) $$G(v;z;i,s) = 1 - \frac{Q_i(s - a_i)}{Q_i(b_i - a_i)} \qquad \text{for } a_i \leq v < \infty,$$

using the condition:

(6.45) $$G(v;z;i,b_i) = 0 \qquad \text{for } 0 \leq v < \infty.$$

Let the function $H_i(v;u)$ be defined by

(6.46) $$H_i(v;u) = \frac{\lambda}{l_i} \int_0^u \{F(u+v-w) - F(u-w)\} Q_i(w)\,dw$$

for $0 \leq v < \infty$, $0 \leq u < \infty$ and $1 \leq i \leq N-1$, then (6.44) can be written:

(6.47) $G(v;z;i,s) =$

$$\begin{cases} \dfrac{Q_i(s - a_i)}{Q_i(b_i - a_i)} H_i(v;b_i-a_i) - H_i(v;s-a_i) & \text{for } 0 \leq v < a_i \\[2ex] 1 - \dfrac{Q_i(s - a_i)}{Q_i(b_i - a_i)} & \text{for } a_i \leq v < \infty. \end{cases}$$

Also the function $H_i(u;v)$ is independent of the strategy and can be computed before the iterative part of the method is entered. The probability $G(v;z;i,s)$ can be easily obtained from (6.47) if the numerical values of a_i and b_i are specified by the strategy.

The probability distribution of $\underline{b}$ for a given state $(i,s) \in X$ if strategy z is applied will be denoted by:

(6.48) $$P(b_j;z;i,s) = P\{\underline{b} = b_j \mid z;i,s\} \qquad \text{for } b_j \in B_z.$$

Now the computation of the functions ck(z;i,s) and ct(z;i,s) and the probabilities $P(b_j;z;i,s)$ will be discussed. Note that the following relations are true:

$$\lim_{s \uparrow b_i} ct(z;i,s) = 0 \quad \text{for } i = 1, \ldots, N \tag{6.49}$$

$$\lim_{s \uparrow b_i} ck(z;i,s) = 0 \quad \text{for } i = 1, \ldots, N. \tag{6.50}$$

Because the computation of ct(z;i,s) is similar to the computation of ck(z;i,s), the discussion will be restricted to ck(z;i,s) and $P(b_j;z;i,s)$.

For states (N,s) with $0 \leq s \leq b_N$ we have according to (6.50):

$$ck(z;N,s) = 0. \tag{6.51}$$

For the probability $P(b_j;z;N,s)$ it follows for $0 \leq s \leq b_N$:

$$P(b_j;z;N,s) = \begin{cases} 1 & \text{for } j = N \\ 0 & \text{for } j = 1, \ldots, N-1. \end{cases} \tag{6.52}$$

For intervention states (i,s) with $0 \leq s \leq a_i$ and $0 \leq i \leq N-1$ it follows according to (6.24):

$$ck(z;i,s) = k(i,s;j,s) + ck(z;j,s) \tag{6.53}$$

where (j,s) denotes the intervention dictated by z in state (i,s). Also we have for these states:

$$P(b_k;z;i,s) = P(b_k;z;j,s). \tag{6.54}$$

For non-intervention states (i,s) with $a_i < s < b_i$ we have either $\underline{I}_1 = b_i$ or $\underline{I}_1 = (i,\underline{u})$ with $0 \leq \underline{u} \leq a_i$. If $\underline{I}_1 = b_i$ there is no contribution to the expectation in (6.24), so it follows for ck(z;i,s):

$$(6.55)\qquad ck(z;i,s) = \int_0^{a_i} ck(z;i,a_i-v)G(dv;z;i,s) + $$

$$+ \left\{1 - \frac{Q_i(s - a_i)}{Q_i(b_i - a_i)} - G(a_i^-;z;i,s)\right\}ck(z;i,0)$$

and for $P(b_j;z;i,s)$ we obtain for $j \neq i$

$$(6.56)\qquad P(b_j;z;i,s) = \int_0^{a_i} P(b_j;z;i,a_i-v)G(dv;z;i,s) +$$

$$+ \left\{1 - \frac{Q_i(s-a_i)}{Q_i(b_i-a_i)} - G(a_i^-;z;i,s)\right\}P(b_j;z;i,0).$$

The probability $P(b_i;z;i,s)$ follows straightforwardly from (6.39).

The computation of $ck(z;i,s)$ can be performed in the order $i = N, N-1, \ldots, 0$ if for each state (i,s) with $0 \leq s \leq a_i$ the decision dictated by z is given by (j,s) with $j > i$. Primarily (6.53) is used to compute $ck(z;N-1,s)$ for $0 \leq s \leq a_{N-1}$. The right member of (6.53) is completely known because of (6.51). The computation of $ck(z;i,s)$ proceeds further using alternately (6.53) and (6.55) until finally $ck(z;0,s)$ is obtained. If the decisions (j,s) in the states (i,s) with $0 \leq s \leq a_i$, $0 \leq i \leq N-1$ do not involve the increase of the production level $(j > i)$ then the production levels can be rearranged in such an order that the same procedure can be used. The computation of $P(b_j;z;i,s)$ is performed in a similar way.

The functions $ck(z;i,s)$, $ct(z;i,s)$ and $P(b_j;z;i,s)$ were obtained by means of numerical integration. Their function values were obtained in states on a grid in the state space. For intermediate states the corresponding function values were obtained by interpolation.

6.5 The cutting mechanism

Let the strategy z_1 be obtained in the way described on page 84 and let A be a closed set satisfying

$$(6.57) \qquad A \supset A_0.$$

If the system is subjected to a mixed strategy $A.\hat{z}$ with initial state (i,s), let $(i,\underline{y})$ be the first state taken on in the set A. Note that $(i,\underline{y})$ is located on the same production level as (i,s), hence the cutting mechanism has to be applied to each production level separately. This mechanism will be used in order to make "holes" in the set A_{z_1}. Such a hole will be an open interval and since in practice the functions $c(\hat{z};i,s)$ are piecewise continuous only a finite number of holes can be expected. Consequently the set $A'_{\hat{z}}$ is the intersection of A_{z_1} with a finite number of complements of open intervals. These open intervals on a fixed production level will be denoted by (a_j,b_j) with $b_{j-1} \leq a_j < b_j$, $j = 1,2,\ldots$ and $b_0 = 0$ and are successively determined in this order.

To facilitate their construction the mixed strategies $\hat{z}_j$, $j = 1,2,\ldots$ are introduced, which are defined by

$$(6.58) \qquad \hat{z}_j(i,s) = \begin{cases} \text{null-decision} & \text{if } a_j < s < b_j \\ \hat{z}_{j-1}(i,s) & \text{otherwise} \end{cases}$$

with $\hat{z}_0(i,s) = \hat{z}(i,s)$. From this definition it follows that for each state (i,s) $a_j < s < b_j$, we have

$$(6.59) \qquad c(A'_{\hat{z}}.\hat{z};i,s) = c(\hat{z}_k;i,s) \qquad \text{for } k \geq j.$$

Primarily the construction of the j^{th} open interval will be considered for production level $i = 0$. Let the probability distribution of $(0,\underline{v})$, the first state taken on in the set I with $I = \{(0,y) \mid 0 \leq y \leq q\}$ during the natural process with initial state $(i,s) \notin I$, be denoted by $G_0(v;I;s)$ and defined by

$$(6.60) \qquad G_0(v;I;s) = P\{q-v \leq \underline{v} \leq q \mid I;0,s\}.$$

The function $c(I.\hat{z}_{j-1};0,s)$ is then given by

(6.60) $$c(I.\hat{z}_{j-1};0,s) = + \int_0^q c(\hat{z}_{j-1};0,q-v)G_0(dv;I;s) + + c(\hat{z}_{j-1};0,0)(1-G_0(q^-;I;s))$$

Let q_1 be the smallest number q satisfying

(6.61) $$c(I.\hat{z}_{j-1};0,s) < c(\hat{z}_{j-1};0,s) \qquad \text{for } q < s < q+\delta$$

where δ is some positive number. Obviously q_1 also satisfies

(6.62) $$\lim_{s\downarrow q_1} c(I_1.\hat{z}_{j-1};0,s) \leq \lim_{s\downarrow q_1} c(\hat{z}_{j-1};0,s)$$

where $I_1 = \{(0,y) \mid 0 \leq y \leq q_1\}$. Denoting $c(\hat{z}_{j-1};0,s) - c(\hat{z}_{j-1};0,0)$ by $\bar{c}(\hat{z}_{j-1};0,s)$ it follows from (6.60) and (6.62) that q_1 can also be obtained from the relation

(6.63) $$\lim_{s\downarrow q_1} \int_0^{q_1} \bar{c}(\hat{z}_{j-1};0,q_1-v)G_0(dv;I_1;s) \leq \lim_{s\downarrow q_1} \bar{c}(\hat{z}_{j-1};0,s).$$

Let $p_1 \leq M$ be defined by the largest value of p satisfying

(6.64) $$\int_0^{q_1} \bar{c}(\hat{z}_{j-1};0,q_1-v)G_0(dv;I_1;s) \leq \bar{c}(\hat{z}_{j-1};0,s) \qquad \text{for } q_1 < s \leq p.$$

By means of (6.64) p_1 can be determined. If two numbers q_1 and p_1 can be found satisfying (6.63) and (6.64) respectively then $a_j = q_1$ and $b_j = p_1$. Note that it follows from (6.61) that $a_j \geq b_{j-1}$, $j = 2,3,\ldots$. If a p_1 satisfying (6.64) cannot be found then the last hole on productionlevel $i = 0$ will be $(a_j, M]$.

Next the construction of the j^{th} hole on production level $i > 0$ will be considered. Consider closed sets I which are the complements of open intervals (q,p) with $b_{j-1} \leq q < p \leq M$ obeying the inequality

(6.65) $$c(I.\hat{z}_{j-1};i,s) \leq c(\hat{z}_{j-1};i,s) \qquad \text{for } q < s < p.$$

Let $P_i(p;I;s)$ denote the probability that the first state $(i,\underline{v})$ taken on in I is the state (i,p) or mathematically

(6.66) $$P_i(p;I;s) = P\{(i,\underline{v}) = (i,p) \mid I;i,s\}$$

for $q < s < p$. Let $G_i(v;I;s)$ denote the probability defined by

(6.67) $$G_i(v;I;s) = P\{(i,\underline{v}) = (i,y),\ q-v \leq y \leq q \mid I;i,s\}.$$

These probabilities can be expressed in the functions $Q_i(w)$ and $H_i(v;u)$ in a similar way as $P(b_i;z;i,s)$ and $G(v;z;i,s)$. The results are

(6.68) $$P_i(p;I;s) = \frac{Q_i(s-q)}{Q_i(p-q)}$$

and

(6.69) $$G_i(v;I;s) = \begin{cases} \dfrac{Q_i(s-q)}{Q_i(p-q)} H_i(v;p-q) - H_i(v;s-q) & \text{for } 0 \leq v < q \\[2ex] 1 - \dfrac{Q_i(s-q)}{Q_i(p-q)} & \text{for } q \leq v < \infty. \end{cases}$$

By means of (6.68) and (6.69) an inequality equivalent to (6.65) will be derived, which is more attractive from a computational point of view. For $c(I.\hat{z}_{j-1};i,s)$ we have (cf. (1.31)):

(6.70) $$c(I.\hat{z}_{j-1};i,s) = $$
$$+ P_i(p;I;s)\ c(\hat{z}_{j-1};i,p) + \int_0^q c(\hat{z}_{j-1};i,q-v) G_i(dv;I;s)$$
$$+ \{1 - P_i(p;I;s) - G_i(q^-;I;s)\}\ c(\hat{z}_{j-1};i,0).$$

Using (6.68) and (6.69) relation (6.70) can be written

(6.71) $$c(I.\hat{z}_{j-1};i,s) =$$

$$+ \frac{Q_i(s-q)}{Q_i(p-q)} \{c(\hat{z}_{j-1};i,p) + \int_0^q c(\hat{z}_{j-1};i,q-v)\ H_i(dv;p-q)\}$$

$$- \int_0^q c(\hat{z}_{j-1};i,q-v)\ H_i(dv;s-q)$$

$$+ c(\hat{z}_{j-i};i,0)\ \{1 - \frac{Q_i(s-q)}{Q_i(p-q)} - (\frac{Q_i(s-q)}{Q_i(p-q)}\ H_i(q;p-q) - H_i(q;s-q)\}$$

By subtraction of $c(\hat{z}_{j-1};i,0)$ from both sides of (6.71) and denoting $c(I.\hat{z}_{j-1};i,s) - c(\hat{z}_{j-1};i,0)$ and $c(\hat{z}_{j-1};i,s) - c(\hat{z}_{j-1};i,0)$ respectively by $\bar{c}(I.\hat{z}_{j-1};i,s)$ and $\bar{c}(\hat{z}_{j-1};i,s)$ we obtain

(6.72) $$\bar{c}(I.\hat{z}_{j-1};i,s) =$$

$$+ \frac{Q_i(s-q)}{Q_i(p-q)} \{\bar{c}(\hat{z}_{j-1};i,p) + \int_0^q \bar{c}(\hat{z}_{j-1};i,q-v)\ H_i(dv;p-q)\}$$

$$- \int_0^q \bar{c}(\hat{z}_j;i,q-v)\ H_i(dv;s-q)$$

The inequality (6.65) is equivalent to

(6.73) $$\frac{Q_i(s-q)}{Q_i(p-q)} \{\bar{c}(\hat{z}_{j-1};i,p) + \int_0^q \bar{c}(\hat{z}_{j-1};i,q-v)\ H_i(dv;p-q)\}$$

$$- \int_0^q \bar{c}(\hat{z}_{j-1};i,q-v)\ H_i(dv;s-q) \le \bar{c}(\hat{z}_{j-1};i,s).$$

Let the function $f_i(s,t;\hat{z}_{j-1})$ be defined by

(6.74) $$f_i(s,t;\hat{z}_{j-1}) =$$

$$\frac{1}{Q_i(s-t)} \{\bar{c}(\hat{z}_{j-1};i,s) + \int_0^t \bar{c}(\hat{z}_{j-1};i,t-v)\ H_i(dv;s-t)\}$$

for $b_{j-1} \le t \le s \le M$.

For a fixed q let p_1 be the largest value of p, $q < p \leq M$, satisfying

$$\bar{c}(I.\hat{z}_{j-1};i,s) \leq \bar{c}(\hat{z}_{j-1};i,s) \tag{6.75}$$

or equivalently

$$f_i(p,q;\hat{z}_{j-1}) \leq f_i(s,q;\hat{z}_{j-1}) \qquad \text{for } q < s < p. \tag{6.76}$$

It will be clear from (6.76) that p_1 coincides with the largest value of p for which the absolute minimum of $f_i(p,q;\hat{z}_{j-1})$ on the interval $q < p \leq M$ is assumed. In this way to each q a number $p_1 = p(q)$ can be assigned. Let the closed set I_q be the complement of an open interval $q < s < p(q)$. Let q_1 be the smallest number q satisfying

$$\bar{c}(I_q.\hat{z}_{j-1};0,s) < \bar{c}(\hat{z}_{j-1};0,s) \qquad \text{for } q < s < q+\delta \tag{6.77}$$

where δ is a positive number. If the functions in (6.77) are piecewise continuous then q_1 is the smallest value of q satisfying

$$\lim_{s\downarrow q} \bar{c}(I_q.\hat{z}_{j-1};0,s) \leq \lim_{s\downarrow q} \bar{c}(\hat{z}_{j-1};0,s) \tag{6.78}$$

From (6.72) it follows because $Q(0) = 1$ that the lefthand side of (6.78) is equal to

$$\frac{1}{Q_i(p(q_1)-q_1)} \{\bar{c}(\hat{z}_{j-1};i,p(q_1)) + \int_0^{q_1} \bar{c}(\hat{z}_{j-1};i,q_1-v)H_i(dv;p(q_1)-q_1)\} \tag{6.79}$$

$$= f_i(p(q_1),q_1;\hat{z}_{j-1}).$$

From (6.74) it follows, because $Q(0) = 1$ and $H(v;0) = 0$ for all v, that we have

$$\bar{c}(\hat{z}_{j-1};0,q_1) = f_i(q_1,q_1;\hat{z}_{j-1}). \tag{6.80}$$

Hence (6.78) is equivalent to

$$f_i(p(q_1),q_1;\hat{z}_{j-1}) = f_i(q_1,q_1;\hat{z}_{j-1}). \tag{6.81}$$

Consequently a_j and b_j are equal to the numbers q_1 and $p(q_1)$ both satisfying (6.81) with $p(q_1)$ being the largest number p for which $f_i(p,q_1;\hat{z}_{j-1})$ assumes its absolute minimum on $q_1 < p \leq M$. If no two numbers q_1 and $p_1 = p(q_1)$ satisfying these two conditions cannot be found then (a_{j-1},b_{j-1}) is the last hole on that production level.

The physical interpretation of (6.81) is that it is indifferent in the states (i,a_j) and (i,b_j) $j = 1,2,...$ to intervene or not to intervene according to strategy $\hat{z}_{j-1}$.

If for some fixed $i > 0$ the state $(i,0) \notin A_{z_1}$ then $a_1 = 0$. To find b_1 the function $f_i(p,\hat{z}_1)$ is defined by

(6.81) $$f_i(p;\hat{z}) = \bar{c}(\hat{z};i,p).$$

The inequality (6.75) becomes

(6.82) $$f_i(p;\hat{z}) \leq f_i(s;\hat{z})$$

and b_1 will be the largest value of p for which the absolute minimum of $f_i(p;z)$ is attained on the interval $0 \leq p \leq M$.

6.6 Numerical results

In this section two numerical examples will be given of the iterative approach to the optimal strategy. The results were obtained by a computer program especially written for this problem.

Example 1:

The following numerical data are given:

Production levels: $l = (l_0, \dots, l_N) = (0, 4, 6, 8)$.

Production costs: $c_p = (0, 8, 12, 16)$.

Switching-over costs:

$$c_q = \begin{bmatrix} 0 & 5.5 & 7.5 & 9 \\ 2 & 0 & 3 & 5.5 \\ 3.5 & 2 & 0 & 3 \\ 4.5 & 3.5 & 2 & 0 \end{bmatrix}$$

Costs of emergency purchases: $c_r = 15.$

Stockholding costs: $c_s = 0.5.$

Maximum stocklevel: $M = 30.$

Mean number of orders per time unit: $\lambda = 1.$

Order size distribution: exponential with expectation 4.17.

The results of the iteration cycle are presented in the tables 6.1 up to 6.9. The subsequent strategies are denoted by $z^{(n)}$, $n = 0, 1, 2.$ The strategies $z_1^{(n)}$, $n = 0, 1, \ldots$ are the strategies obtained by minimizing (1.23) with $z = z^{(n)}$. The numbers s_{ik}, $i = 0, \ldots, 3$, $k = 0, \ldots, n_i$ denote the separation states on production level i with $s_{i0} = 0$ and $s_{in_i} = M$. The production level assigned to interval $s_{ik-1} < s < s_{ik}$ by the strategy is denoted by the integer j_{ik}.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0		0		0		0	
	3		3		3		3
3		2		1		19	
	0		1		2		2
30		30		30		30	

Table 6.1 The initial strategy $z^{(0)}$, $r(z^{(0)}) = 22.770.$

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0		0		0		0	
	3		3		3		3
3.81		2.80		2.32		7.58	
	2		2		2		1
4.83		4.16		7.51		29.07	
	1		1		1		0
24.95		30		29.51		30	
	0				0		
30				30			

Table 6.2 Strategy $z_1^{(0)}$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0 3.81 4.83 13.61 30	3 2 1 0	0 2.80 3.77 30	3 2 1	0 2.25 16.82 29.50 30	3 2 1 0	0 16.61 29.06 30	3 1 0

Table 6.3 Strategy $z^{(1)}$, $r(z^{(1)}) = 21.362$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0 5.99 8.78 14.43 30	3 2 1 0	0 4.62 7.57 22.24 30	3 2 1 0	0 3.98 16.82 21.63 30	3 2 1 0	0 16.61 21.07 30	3 1 0

Table 6.4 Strategy $z_1^{(1)}$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0 5.99 8.78 14.08 30	3 2 1 0	0 4.62 6.55 26.27 30	3 2 1 0	0 3.82 24.92 30	3 2 0	0 24.15 30	3 0

Table 6.5 Strategy $z^{(2)}$, $r(z^{(2)}) = 21.218$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0		0		0		0	
	3		3		3		3
5.75		4.40		3.82		14.14	
	2		2		2		1
8.07		6.81		14.87		21.97	
	1		1		1		3
14.62		24.96		22.93		23.38	
	0		0		2		0
30		30		23.99		30	
					0		
				30			

Table 6.6 Strategy $z_1^{(2)}$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0		0		0		0	
	3		3		3		3
5.75		4.40		3.78		17.74	
	2		2		2		1
8.07		6.80		18.44		21.97	
	1		1		1		3
14.56		25.62		22.93		23.96	
	0		0		2		0
30		30		24.62		30	
					0		
				30			

Table 6.7 Strategy $z^{(3)}$, $r(z^{(3)}) = 21.153$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0		0		0		0	
	3		3		3		3
6.05		4.63		3.98		17.33	
	2		2		2		1
8.35		7.01		18.14		21.97	
	1		1		1		3
14.56		25.32		22.93		23.80	
	0		0		2		0
30		30		24.42		30	
					0		
				30			

Table 6.8 Strategy $z_1^{(3)}$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}
0		0		0		0	
	3		3		3		3
6.05		4.63		3.98		17.54	
	2		2		2		1
8.35		7.00		18.29		21.97	
	1		1		1		3
14.44		25.47		22.93		23.86	
	0		0		2		0
30		30		24.50		30	
					0		
				30			

Table 6.9 Strategy $z^{(4)}$, $r(z^{(4)}) = 21.152$.

For the strategy $z^{(4)}$ the numbers $c(z^{(4)};0,0) - c(z^{(4)};i,s)$ for $i = 0, 1, 2, 3$ and $s = 0(3)30$, being the values of these states relative to the most unfavourable state (0,0) are presented in table 6.10.

↓s \ i →	0	1	2	3
0	0.00	3.50	6.00	9.00
3	17.88	21.38	23.88	26.88
6	30.19	34.66	37.66	39.19
9	41.02	46.52	48.04	47.91
12	50.41	55.91	55.56	54.42
15	57.75	62.81	61.25	59.71
18	65.54	67.96	65.96	64.46
21	72.25	72.13	70.13	68.63
24	77.89	76.12	74.41	73.39
27	82.44	80.44	78.94	77.94
30	85.91	83.91	82.41	81.41

Table 6.10 The relative values of some states for strategy $z^{(4)}$.

Example 2:

The following numerical data are given:

Production levels: $l = (l_0, \ldots, l_N) = (0, 4, 5, 6, 7)$.

Production costs: $c_p = (0, 8, 10, 12, 14)$.

Switching-over costs:

$$c_q = \begin{bmatrix} 0 & 5.5 & 7.5 & 9 & 10 \\ 2 & 0 & 3 & 5.5 & 7.5 \\ 3.5 & 2 & 0 & 3 & 5.5 \\ 4.5 & 3.5 & 2 & 0 & 3 \\ 5.25 & 4.5 & 3.5 & 2 & 0 \end{bmatrix}$$

Costs of emergency purchases: $c_r = 15$.

Stockholding costs: $c_s = 0.5$.

Maximum stocklevel: $M = 30$.

Mean number of orders per unit time: $\lambda = 1$.

Order size distribution: gamma-distribution with 4 degrees of freedom and expectation 4.5.

The strategies $z^{(n)}$, n = 0, 1, 2, 3 are presented in the tables 6.11 up to 6.14.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}	s_{4k}	j_{4k}
0		0		0		0		0	
	4		4		4		4		4
4.5		3.5		2.5		1.5		17.5	
	3		3		3		3		0
6.5		5.5		4.5		18.5		30	
	2		2		2		0		
8.5		7.5		19.5		30			
	1		1		0				
10.5		20.5		30					
	0		0						
30		30							

Table 6.11 The initial strategy $z^{(0)}$, $r(z^{(0)}) = 20.283$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}	s_{4k}	j_{4k}
0		0		0		0		0	
	4		4		4		4		4
5.50		3.97		3.25		2.47		23.08	
	3		3		3		3		0
6.14		4.50		3.85		23.67		30	
	2		2		2		0		
7.50		6.05		24.47		30			
	1		1		0				
12.88		25.18		30					
	0		0						
30		30							

Table 6.12 Strategy $z^{(1)}$, $r(z^{(1)}) = 19.990$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}	s_{4k}	j_{4k}
0		0		0		0		0	
	4		4		4		4		4
6.44		4.65		3.67		2.90		17.69	
	2		2		3		3		1
7.64		6.05		3.94		18.41		20.68	
	1		1		2		1		4
13.04		24.26		19.44		20.55		22.52	
	0		0		1		3		0
30		30		20.77		22.98		30	
					2		0		
				23.79		30			
					0				
				30					

Table 6.13 Strategy $z^{(2)}$, $r(z^{(2)}) = 19.919$.

s_{0k}	j_{0k}	s_{1k}	j_{1k}	s_{2k}	j_{2k}	s_{3k}	j_{3k}	s_{4k}	j_{4k}
0		0		0		0		0	
	4		4		4		4		4
6.26		4.81		3.94		3.15		17.46	
	1		2		2		3		1
12.94		6.05		19.02		18.18		20.68	
	0		1		1		1		4
30		24.15		21.43		20.55		22.38	
			0		2		3		0
		30		23.65		22.84		30	
					0		0		
				30		30			

Table 6.14 Strategy $z^{(3)}$, $r(z^{(3)}) = 19.916$.

For the strategy $z^{(3)}$ the numbers $c(z^{(3)};0,0) - c(z^{(3)};i,s)$ for $i = 0, 1, 2, 3$ and $s = 0(3)30$, being the values of these states relative to the most unfavourable state (0,0), are presented in table 6.15.

s↓ \ i→	0	1	2	3	4
0	0.00	2.50	4.50	7.00	10.00
3	22.25	24.76	26.76	29.76	32.26
6	36.44	40.20	43.20	45.10	46.44
9	48.10	53.60	54.71	55.12	55.08
12	57.08	62.58	62.33	61.52	60.66
15	64.71	68.81	67.43	65.96	64.82
18	71.53	73.15	71.19	69.65	68.65
21	77.38	76.58	74.58	73.25	72.35
24	82.23	80.24	78.73	77.73	76.98
27	86.09	84.09	82.59	81.59	80.84
30	88.94	86.94	85.44	84.44	83.69

Table 6.15 The relative values of some states for strategy $z^{(3)}$.

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