MATHEMATICAL CENTRE TRACTS
4

GENERALIZED MARKOVIAN DECISION PROCESSES

PART II

PROBABILISTIC BACKGROUND

BY

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PART II

CHAPTER 1

The fundamental stochastic process

1. General properties

In this chapter we shall consider a class of stochastic processes with a common state space X^*.

The state space X^* with points x is an M-dimensional Borel set. Since X^* is also the parameter set of the class of stochastic processes considered, we denote the latter by $\{S^*_x ; x \in X^*\}$.

The stochastic processes S^*_x are defined by means of the following tools:

1) the state space X^* with points x;
2) a space Ω^* with points ω ;
3) a family of ω-functions $\{x^*_t(\omega) ; t \in [0,\infty)\}$, defined on Ω^*, such that for each $t \in [0,\infty)$ the ω-function $x^*_t(\omega)$ maps Ω^* into X^*;
4) the σ-field G^* of M-dimensional Borel sets in X^*;
5) the smallest σ-field H^* with respect to which the ω-functions $\{x^*_t(\omega) ; t \in [0,\infty)\}$ are measurable.
6) the function $P^*[K;x]$ of sets $K \in H^*$ and points $x \in X^*$, satisfying the properties:
   a) for each $x \in X^*$, the set function $P^*[K;x]$ assigns a probability measure to the sets $K \in H^*$;
   b) for each $K \in H^*$, the x-function $P^*[K;x]$ is measurable with respect to G^*.

A stochastic process $S^*_{x_o}$ is defined by a family of stochastic variables $\{\underline{x}^*_{t;x_o} ; t \in [0,\infty)\}$, the probability distributions of which are given by

$$\text{Prob } \{\underline{x}^*_{t;x_o} \in A\} = P^* [\Lambda_{t;A} ; x_o] , \tag{1.1}$$

where $\Lambda_{t;A}$ is defined by

$$\Lambda_{t;A} \overset{\text{def}}{=} \{\omega | x^*_t(\omega) \in A\} \tag{1.2}$$

and $A \varepsilon G^*$.

For each x the set function $P^*[K;x]$ represents a probability measure P^* defined on H^*. Consequently, for each $x \varepsilon X^*$ we have a triple $\{\Omega^*, H^*, P^*\}$. Such a triple is called a probability space. The stochastic processes S_x^* are defined by means of probability spaces with identical Ω^* and H^*, but with different probability measures P^*.

The points $x \varepsilon X^*$ and $\omega \varepsilon \Omega^*$ are called the states and the realizations of the stochastic processes respectively. The space Ω^* is called the sample space, while the functions $x_t^*(\omega)$ are named sample functions. Finally, the points $t \varepsilon [0,\infty)$ represent points of time.

Usually in the theory of stochastic processes the σ-field H^* is completed with all subsets of sets of probability measure 0. In this section, however, we consider various probability measures $P^*[K;x]$; one for each $x \varepsilon X^*$. So, if we want an "x-free" extension of H^*, we need to be more selective in completing the σ-field H^*.

Let Λ_o^* be an ω-set with the following properties:

1) for each $\omega \varepsilon \overline{\Lambda_o^*}$, the t-function $x_t^*(\omega)$ is continuous from the right;
2) in each bounded time interval in $[0,\infty)$ and for each $\omega \varepsilon \overline{\Lambda_o^*}$, the t-function $x_t^*(\omega)$ has only a finite number of discontinuities.

Assumption 1

For each $x \varepsilon X^*$, a set $K_x \varepsilon H^*$ can be found such that

a) $\Lambda_o^* \subset K_x$;

b) $P^*[K_x;x] = 0$.

The σ-field F^* is the smallest σ-field of ω-sets that contains H^* and includes all subsets of Λ_o^*.

The domain of definition of the set function $P^*[K;x]$ is from now on regarded as extended to F^*. This extension is unique (cf. [1] p.90).

Lemma 1.1

For each $K \varepsilon F^*$, the x-function $P^*[K;x]$ is measurable with respect to G^*.

Proof:

If $K \varepsilon H^*$, the x-function $P^*[K;x]$ is measurable with respect to G^* (cf. tool 6 of S_x^*). Further, if $K \subset \Lambda_o^*$, we have $P^*[K;x] = 0$ for all $x \varepsilon X^*$.

Let J be the class of ω-sets $K \varepsilon F^*$ for which the assertion is true.

We have now proved that

a) $J \supset H^*$;

b) $K \varepsilon J$ if $K \subset \Lambda_o^*$.

The following points can easily be verified:

c) $\overline{K} \varepsilon J$ if $K \varepsilon J$;

d) $\bigcup_{j=1}^{\infty} K_j \varepsilon J$ if $K_j \varepsilon J$ and if $K_j \subset K_{j+1}$ $(j=1,2,\ldots)$.

These properties of J imply that J is a σ-field, which contains H^* and includes all subsets of Λ_o^* ([2] p.599). Hence, $J=F^*$.

Now we are in a position to prove the following lemma:

Lemma 1.2.1

If I is any open time interval and if B is a closed set in X^*, then for

$$M_{I;B} \overset{\text{def}}{=} \{\omega \mid \forall_{t \varepsilon I}\ x_t^*(\omega) \varepsilon B\} \tag{1.3}$$

we have

$$M_{I;B} \varepsilon F^*. \tag{1.4}$$

Proof :

Let $\{t_j; j=1,2,\ldots\}$ be the set of all rational numbers in $[0,\infty)$. Obviously, we have

$$\bigcap_{t_j \varepsilon I} \Lambda_{t_j;B} \supset M_{I;B} \tag{1.5}$$

and thus

$$\overline{\Lambda_o^*} \cap \bigcap_{t_j \varepsilon I} \Lambda_{t_j;B} \supset \overline{\Lambda_o^*} \cap M_{I;B}\ . \tag{1.6}$$

The left hand sides of (1.5) and (1.6) belong to F^*. We now prove the converse of (1.6).

For each $t \varepsilon I$ a monotone decreasing subsequence $\{s_m; m=1,2,\ldots\}$ of $\{t_j; j=1,2,\ldots\}$ can be found such that

$$\lim_{m\to\infty} s_m = t \ . \tag{1.7}$$

Hence, if $\omega \varepsilon \overline{\Lambda^*_o} \cap \bigcap_{t_j \varepsilon I} \Lambda_{t_j;B}$, we find (B is closed)

$$x^*_t(\omega) = \lim_{s_m \downarrow t} x^*_{s_m}(\omega) \ \varepsilon B. \tag{1.8}$$

This result can be obtained for each $t \varepsilon I$ and therefore

$$\overline{\Lambda^*_o} \cap \bigcap_{t_j \varepsilon I} \Lambda_{t_j;B} \subset \overline{\Lambda^*_o} \cap M_{I;B}.^{1)} \tag{1.9}$$

From (1.6) and (1.9) it follows that

$$\overline{\Lambda^*_o} \cap M_{I;B} = \overline{\Lambda^*_o} \cap \bigcap_{t_j \varepsilon I} \Lambda_{t_j;B} \ \varepsilon \ F^*. \tag{1.10}$$

Since F^* includes all subsets of Λ^*_o we have

$$M_{I;B} \ \varepsilon F^*. \tag{1.4}$$

This terminates the proof.

Let i_1 and i_2 be the left and the right boundary point of an open interval I respectively. If B is a closed set in X^*, we have

1) $M_{\{i_1\}\cup I;B} = \Lambda_{i_1;B} \cap M_{I;B} \ \varepsilon F^*;$ (1.11)

2) $M_{I\cup\{i_2\};B} = M_{I;B} \cap \Lambda_{i_2;B} \ \varepsilon F^*;$ (1.12)

3) $M_{\{i_1\}\cup I\cup\{i_2\};B} = \Lambda_{i_1;B} \cap M_{I;B} \cap \Lambda_{i_2;B} \ \varepsilon F^*.$ (1.13)

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1) The identity (1.10) implies the separability of the stochastic processes $\{S^*_x; x \varepsilon X^*\}$ with respect to the class of all closed sets (cf. [2] p.51).

So we have proved the following lemma:

Lemma 1.2

If B is a closed set in X^* and if I is any interval in $[0,\infty)$, then

$$M_{I;B} \varepsilon F^*. \tag{1.14}$$

Lemma 1.3

If B is a closed set in X^*, there exists a sequence of open sets $\{B_n; n=1,2,\ldots\}$ and a sequence of closed sets $\{B^*_n ; n=1,2,\ldots\}$ satisfying :

1) $B_{n-1} \supset B^*_n \supset B_n \supset B^*_{n+1}$; (1.15)

2) $\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} B^*_n = B$. (1.16)

Proof :

Let B_n and B^*_n be defined by

$$B_n \overset{def}{=} \{x \mid \exists_{x_1} \varepsilon B;\ |x-x_1| < \tfrac{1}{n}\} \tag{1.17}$$

and

$$B^*_n \overset{def}{=} \{x \mid \exists_{x_1} \varepsilon B;\ |x-x_1| \leqq \tfrac{1}{n}\} \tag{1.18}$$

respectively.

The assertion will now be obvious.

Lemma 1.4.1

If B is any open set in X^* and if I is any interval in $[0,\infty)$, for

$$\Lambda_{I;B} \overset{def}{=} \{\omega \mid \exists_{t\,\varepsilon\, I}\ x^*_t(\omega)\ \varepsilon\ B\} \tag{1.19}$$

we have

$$\Lambda_{I;B}\ \varepsilon\ F^*. \tag{1.20}$$

Proof :

By lemma 1.2

$$\Lambda_{I;B} = \overline{M}_{I;\overline{B}}\ \varepsilon\ F^*. \tag{1.21}$$

Lemma 1.4.2

If B is any closed set in X^* and I is a bounded closed interval

$[i_1, i_2]$ in $[0,\infty)$, then

$$\Lambda_{I;B} \in F^{*}. \tag{1.22}$$

Proof :

We consider the sequence $T = \{t_j;\ j=1,2,\ldots\}$ consisting of

a) the points i_1 and i_2;

b) the rational points in I.

If

$$\omega \in \overline{\Lambda_o^*} \cap \Lambda_{I;B}\ , \tag{1.23}$$

then

$$\exists_{t\in I}\ x_t^*(\omega) \in B. \tag{1.24}$$

Since the t-function $x_t^*(\omega)$ is continuous from the right, the following statements are true:

$$\forall_k\ \exists_{t\in I}\ \exists_m\ \forall_{s\in[t,t+\frac{2}{m}]}\ x_t^*(\omega) \in B\ \&\ x_s^*(\omega) \in B_k^* \tag{1.25}$$

$$\forall_k\ \exists_m\ \forall_n\ \exists_{t_j\in T}\ \forall_{s\in[t_j;t_j+\frac{1}{m}]}\ x_{t_j}^*(\omega) \in B_n\ \&\ x_s^*(\omega) \in B_k^*\ . \tag{1.26}$$

Hence, (1.23) implies

$$\omega \in \overline{\Lambda_o^*} \cap \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{t_j\in T} \Lambda_{t_j;B_n} \cap M_{[t_j;t_j+\frac{1}{m}];B_k^*}. \tag{1.27}$$

Thus,

$$\overline{\Lambda_o^*} \cap \Lambda_{I;B} \subset \overline{\Lambda_o^*} \cap \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{t_j\in T} \Lambda_{t_j;B_n} \cap M_{[t_j,t_j+\frac{1}{m}];B_k^*}\ . \tag{1.28}$$

We shall now prove the converse of (1.28).

If (1.27) is true, then

$$\exists_{\{m_k;k=1,2,\ldots\}}^{m_k\geq 2m_{k-1}}\ \exists_{\{t_{kn};k=1,2,\ldots,n=1,2,\ldots\}}^{t_{kn}\in T}\ \forall_{s\in\left[t_{kn},t_{kn}+\frac{1}{2^{m_k-1}}\right]}$$

$$x_{t_{kn}}^*(\omega) \in B_n\ \&\ x_s^*(\omega) \in B_k^*. \tag{1.29}$$

For each k we consider the sequence of points $\{t_{kn}; n=1,2,\ldots\}$.

If $n \geqq n_o$, we find

$$x^*_{t_{kn}}(\omega) \in B_{n_o} \subset B^*_{n_o}. \tag{1.30}$$

Since I is closed and bounded, the points of accumulation $\{t_k^\alpha ; \alpha=1,2,\ldots\}$ of $\{t_{kn};n=1,2,\ldots\}$ belong to I. If (1.27) is true, one of the following cases will arise:

a) At least one of the points $\{t_k^\alpha; \alpha=1,2,\ldots\}$, say t_k^1, is a point of continuity of the t-function $x^*_t(\omega)$;

b) All points $\{t_k^\alpha; \alpha=1,2,\ldots\}$ are points of discontinuity of the t-function $x^*_t(\omega)$.

In case a) it follows from (1.30) that for the point ω considered we find

$$x^*_{t_k^1}(\omega) \in \bigcap_{n_o=1}^{\infty} B^*_{n_o} = B. \tag{1.31}$$

Hence,

$$\omega \in \Lambda_{I;B} \cap \overline{\Lambda^*_o}. \tag{1.32}$$

In case b), because of assumption 1, the number of accumulation points must be finite ($\omega \in \overline{\Lambda^*_o}$).

If s_k^α is defined by

$$s_k^\alpha = t_k^\alpha + \frac{1}{2^{m_k}} \quad ; k=1,2,\ldots , \tag{1.33}$$

then, since $\frac{1}{2^{m_k-1}} > 0$, for each α an integer n_α can be found such that

$$s_k^\alpha \in \left[t_{kn_\alpha}, t_{kn_\alpha} + \frac{1}{2^{m_k-1}}\right]. \tag{1.34}$$

Consequently,

$$x^*_{s_k^\alpha}(\omega) \in B^*_k; \; k=1,2,\ldots \tag{1.35}$$

If t'_k denotes the superior of $\{t_k^\alpha; \alpha=1,2,\ldots\}$ and if we consider the sequence $\{t'_k;k=1,2,\ldots\}$, then we can easily verify that this sequence runs through a finite number of points in I (points of discontinuity of $x^*_t(\omega)$). So a subsequence of $\{t'_k;k=1,2,\ldots\}$, say $\{t'_{k(h)};h=1,2,\ldots\}$, exists that satisfies

$$t'_{k(h)} = t' = \lim\inf \{t'_k; k=1,2,\ldots\}. \tag{1.36}$$

Now let s_k be defined by

$$s_k = t'_k + \frac{1}{2^{m_k}} \quad ; k=1,2,\ldots \tag{1.37}$$

Since $k(h) \geq h$ and thus $B^*_{k(h)} \subset B^*_h$, it follows from (1.35) that

$$x^*_{s_{k(h)}}(\omega) \in B^*_h \; ; h=1,2,3,\ldots \; . \tag{1.38}$$

Further, we can easily verify that $\lim_{h\to\infty} s_{k(h)} \downarrow t'$ and thus for each h_o $(\omega \in \overline{\Lambda^*_o})$

$$x^*_{t'}(\omega) = \lim_{h\to\infty} x^*_{s_{k(h)}}(\omega) \in B^*_{h_o} \; . \tag{1.39}$$

Consequently,

$$x^*_{t'}(\omega) \in \bigcap_{h_o=1}^{\infty} B^*_{h_o} = B. \tag{1.40}$$

Hence,

$$\omega \in \Lambda_{I;B} \cap \overline{\Lambda^*_o} \; . \tag{1.41}$$

We have now proved that both case a) and case b) lead to (1.41).

This implies that the converse of (1.28) is also true.

Thus,

$$\overline{\Lambda^*_o} \cap \Lambda_{I;B} = \overline{\Lambda^*_o} \cap \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{t_j \in I} \Lambda_{t_j;B_n} \cap M_{\left[t_j, t_j+\frac{1}{m}\right];B^*_k} \in F^*. \tag{1.42}$$

and therefore

$$\Lambda_{I;B} \in F^*. \tag{1.43}$$

This ends the proof.

The following lemma can easily be proved (cf. (1.11), (1.12) and (1.13)):

<u>Lemma 1.4.3</u>

If I is any interval in $[0,\infty)$ and if B is a closed set, then

$$\Lambda_{I;B} \in F^*. \tag{1.44}$$

Lemmas 1.2, 1.4.1 and 1.4.3 imply:

Lemma 1.4

If I is any interval in $[0,\infty)$ and if B is either closed or open, then

$$\Lambda_{I;B} \in F^{*} \tag{1.45}$$

and

$$M_{I;B} = \overline{\Lambda_{I;\overline{B}}} \in F^{*}. \tag{1.46}$$

If C is a closed set in X^{*} and if ω is a realization of a stochastic process S_x^{*}, let $t(\omega;C)$ be the moment that the system is for the first time in C. If the initial state of the stochastic process S_x^{*} belongs to C, then $t(\omega;C) = 0$.

This point of time can also be defined by

$$t(\omega;C) \overset{\text{def}}{=} \begin{cases} \inf\{t \mid x_t^{*}(\omega) \in C\}, & \text{if } x_t^{*}(\omega) \in C \text{ for some finite } t. \\ \infty, & \text{otherwise.} \end{cases} \tag{1.47}$$

Let the ω-set $\Xi_{I;C}$ be defined by

$$\Xi_{I;C} \overset{\text{def}}{=} \{\omega \mid t(\omega;C) \in I\}, \tag{1.48}$$

where I is an interval in $[0,\infty)$.

Lemma 1.5.1

For any interval I in $[0,\infty)$ and for each closed set C we have

$$\Xi_{I;C} \in F^{*}. \tag{1.49}$$

(Thus, $t(\omega;C)$ is measurable with respect to F^{*}.)

Proof:

Let us consider a closed interval $I = [i_1, i_2]$. It can easily be verified that for this choice of I the ω-set $\overline{\Lambda_o^{*}} \cap \Xi_{I;C}$ is given by

$$\overline{\Lambda_o^{*}} \cap \Xi_{I;C} = \overline{\Lambda_o^{*}} \cap M_{[0,i_1);\overline{C}} \cap \Lambda_{[i_1,i_2];C} \in F^{*}. \tag{1.50}$$

Hence, $\Xi_{I;C} \in F^{*}$.

The proofs for other types of intervals are obvious. This ends the proof.

Let us introduce the ω-functions $x^*(\omega;C)$, defined by

$$x^*(\omega;C) \overset{\text{def}}{=} \begin{cases} x^*_{t(\omega;C)}(\omega), & \text{if } t(\omega;C) < \infty. \\ x^*_o(\omega), & \text{if } t(\omega;C) = \infty. \end{cases} \tag{1.51}$$

Note that by this definition the state at the end of the period $[0,t(\omega;C)]$ is given by $x^*(\omega;C)$ if $t(\omega;C)<\infty$.

The function $x^*(\omega;C)$ is defined for each $\omega \in \Omega^*$.

Let the ω-set $\Delta_{B;C}$ be defined by

$$\Delta_{B;C} \overset{\text{def}}{=} \{\omega | x^*(\omega;C) \in B\} . \tag{1.52}$$

Lemma 1.5.2

For each $B \in G^*$ and for each closed set C we have

$$\Delta_{B;C} \in F^*. \tag{1.53}$$

(Thus, $x^*(\omega;C)$ is an ω-function which is measurable with respect to F^*.)

Proof:

Let for a fixed m the ω-function $x^*_{(m)}(\omega)$ be defined by (k=1,2,...)

$$x^*_{(m)}(\omega) \overset{\text{def}}{=} \begin{cases} x^*_o(\omega), & \text{if } \omega\in\Lambda^*_o \cup \overline{\Xi_{[0,\infty);C}} \\ x^*_{\frac{k}{2^m}}(\omega), & \text{if } \omega\in\Xi_{[\frac{k-1}{2^m},\frac{k}{2^m});C} \cap \overline{\Lambda^*_o} . \end{cases} \tag{1.54}$$

That is

$$x^*_{(m)}(\omega) = \sum_{k=0}^{\infty} \chi_{(m);k}(\omega)\, x^*_{\frac{k}{2^m}}(\omega), \tag{1.55}$$

where

$$\chi_{(m);0}(\omega) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } \omega\in\Lambda^*_o \cup \Xi_{[0,\infty);C} \\ 0, & \text{otherwise} \end{cases} \tag{1.56}$$

and for k=1,2,...

$$\chi_{(m);k}(\omega) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } \omega \in \Xi_{[\frac{k-1}{2^m}, \frac{k}{2^m});C} \cap \overline{\Lambda_o^*} \\ 0, & \text{otherwise} \end{cases} \tag{1.57}$$

Obviously, the ω-functions $\{\chi_{(m);k}(\omega);k=0,1,...\}$ are measurable with respect to F^*.

Consequently, the ω-functions $x^*_{(m)}(\omega)$ are measurable with respect to F^*.

It can easily be verified that for $m \to \infty$ the sequence of ω-functions $\{x^*_{(m)}(\omega);m=1,2,...\}$ converges everywhere to an ω-function, let us say $x^*_{(\infty)}(\omega)$. From this it follows that the ω-function $x^*_{(\infty)}(\omega)$ is measurable with respect to F^*.

Since for $\omega \in \overline{\Lambda_o^*}$ the t-function $x^*_t(\omega)$ is continuous from the right, we find for these points

$$x^*_{(\infty)}(\omega) = x^*(\omega;C). \tag{1.58}$$

Consequently, if $B \in G^*$,

$$\Delta_{B;C} \cap \overline{\Lambda_o^*} = \{\omega \mid x^*_{(\infty)}(\omega) \in B\} \cap \overline{\Lambda_o^*} \in F^*. \tag{1.59}$$

Thus,

$$\Delta_{B;C} \in F^*. \tag{1.60}$$

This ends the proof.

Let us assume that the set C has been chosen in such a way that for each $x \in X^*$ we have

$$P^*\left[\Xi_{[0,\infty);C};x\right] = 1. \tag{1.61}$$

Since each combination of a measurable ω-function and the probability space $\{\Omega^*;F^*;P^*\}$ generates a stochastic variable, the ω-functions $t(\omega;C)$ and $x^*(\omega;C)$ lead us to the stochastic variables $\underline{t}_{C;x}$ and $\underline{x}^*_{C;x}$. The probability distributions of these variables are given by

$$\text{Prob}\{\underline{t}_{C;x} \in I\} \overset{\text{def}}{=} P^*\left[\Xi_{I;C};x\right] \tag{1.62}$$

and

$$\text{Prob}\{\underline{x}^*_{C;x} \, \varepsilon \, B\} \overset{\text{def}}{=} P^* \left[\Delta_{B;C};x\right] \tag{1.63}$$

respectively.

The stochastic variable $\underline{t}_{C;x}$ represents the length of the time period preceding the moment at which the system first is in C, while $\underline{x}^*_{C;x}$ denotes the state at the end of this period if (1.61) is true.

Summarizing:

Lemma 1.5

If assumption 1 and condition (1.61) are satisfied, the probability distribution of the length $\underline{t}_{C;x}$ of the period preceding the moment at which the system first is in C and that of the state $\underline{x}^*_{C;x}$ at that point of time are defined. They are given by (1.62) and (1.63) respectively.

Let B be a closed set in X^* and let us define a family of ω-functions $\{x^*_t(\omega;B);\ t \, \varepsilon \left[0,\infty\right)\}$ by

$$x^*_t(\omega;B) \overset{\text{def}}{=} \begin{cases} x^*_{t(\omega;B)+t}(\omega), \text{ if } t(\omega;B) < \infty \\ x^*_t(\omega), \text{ if } t(\omega;B) = \infty. \end{cases} \tag{1.64}$$

Lemma 1.6

The ω-functions $\{x^*_t(\omega;B);\ t \, \varepsilon \left[0,\infty\right)\}$ are measurable with respect to F^*.

Proof:

Let us consider the ω-functions $x^*_{(m);t}(\omega)$, for k=1,2,..., defined by

$$x^*_{(m);t}(\omega) \overset{\text{def}}{=} \begin{cases} x^*_t(\omega), \text{ if } \omega \, \varepsilon \, \Lambda^*_o \cup \Xi_{\left[0,\infty\right);B} \\ x^*_{t+\frac{k}{2^m}}(\omega), \text{ if } \omega \, \varepsilon \, \overline{\Lambda^*_o} \cap \Xi_{\left[\frac{k-1}{2^m},\ \frac{k}{2^m}\right);B} \end{cases}. \tag{1.65}$$

Then

$$x^*_{(m);t}(\omega) = \sum_{k=0}^{\infty} \chi_k(\omega) x^*_{t+\frac{k}{2^m}}(\omega), \tag{1.66}$$

where

$$\chi_o(\omega) \overset{def}{=} \begin{cases} 1, \text{ if } \omega \in \Lambda_o^* \cup \overline{\Xi_{[0,\infty);B}} \\ 0, \text{ if } \omega \in \overline{\Lambda_o^*} \cap \Xi_{[0,\infty);B} \end{cases} \tag{1.67}$$

and for k=1,2,...

$$\chi_k(\omega) \overset{def}{=} \begin{cases} 1, \text{ if } \omega \in \overline{\Lambda_o^*} \cap \Xi_{[\frac{k-1}{2^m}, \frac{k}{2^m});B} \\ 0, \text{ otherwise.} \end{cases} \tag{1.68}$$

The remainder of the proof is identical with that of lemma 1.5.2 and is therefore omitted.

This ends the proof.

It follows from (1.64) that for each $\omega \in \overline{\Lambda_o^*}$

1) the t-function $x_t^*(\omega;B)$ is continuous from the right;
2) in each finite time interval in $[0,\infty)$ the t-function $x_t^*(\omega;B)$ has only a finite number of discontinuities.

If B and C are closed sets, let us introduce the ω-functions $t(\omega;B;C)$ and $x^*(\omega;B;C)$, defined by

$$t(\omega;B;C) \overset{def}{=} \begin{cases} \inf\{t \mid x_t^*(\omega;B) \in C\}, \text{ if } x_t^*(\omega;B) \in C \text{ for some finite } t \\ \infty, \text{ otherwise} \end{cases} \tag{1.69}$$

and

$$x^*(\omega;B;C) \overset{def}{=} \begin{cases} x^*_{t(\omega;B;C)}(\omega;B), \text{ if } t(\omega;B;C) < \infty \\ x_o^*(\omega;B), \text{ if } t(\omega;B;C) = \infty \end{cases} \tag{1.70}$$

respectively.

<u>Lemma 1.7</u>

If B and C are closed sets in X^*, the ω-functions $t(\omega;B;C)$ and $x^*(\omega;B;C)$, defined by (1.69) and (1.70), are measurable with respect to F^*.

Proof:

The function $x_t^*(\omega;B)$ has the same properties as the function $x_t^*(\omega)$. Therefore, lemma 1.7 is a direct consequence of lemmas 1.5.1 and 1.5.2.

If C is a closed set in X^*, a sequence of open sets $\{\bar{B}_n;n=1,2,...\}$ can be found such that (cf. lemma 1.3)

$$\bar{B}_n \supset \bar{B}_{n+1} \supset ..... \supset C \qquad (1.71)$$

and

$$\bigcap_{n=1}^{\infty} \bar{B}_n = C. \qquad (1.72)$$

Consequently, the sequence of closed sets $\{B_n;n=1,2,...\}$ satisfies

$$B_n \subset B_{n+1} \subset ..... \subset \bar{C} \qquad (1.73)$$

and

$$\bigcup_{n=1}^{\infty} B_n = \bar{C}. \qquad (1.74)$$

If C is a closed set in X^* and if ω is a realization of a stochastic process S_x^*, let $t(\omega;[C])$ be the moment that the system <u>enters</u> into C for the first time.

If the initial state of the stochastic process S_x^* belongs to $\bar{C}$, then we obviously have

$$t(\omega;[C]) = t(\omega;C). \qquad (1.75)$$

If the initial state is an element of C, then $t(\omega;C) = 0$ but the first entry in C does not occur before a state of $\bar{C}$ has been assumed.

Let us consider the sequence $\{t_n(\omega);n=1,2,...\}$, defined by

$$t_n(\omega) \overset{def}{=} t(\omega;B_n) + t(\omega;B_n;C). \qquad (1.76)$$

Obviously, the ω-functions $t_n(\omega)$ are measurable with respect to F^*. The function $t_n(\omega)$ represents the time needed for being first in B_n and then in C. Consequently, by (1.73)

$$t(\omega;[C]) \leqq t_n(\omega). \qquad (1.77)$$

Since

$$0 \leqq t_n(\omega) \leqq t_{n-1}(\omega), \tag{1.78}$$

we can define an ω-function $t_\infty(\omega)$ by

$$t_\infty(\omega) = \begin{cases} \lim\limits_{n \to \infty} t_n(\omega), \text{ if } t_n(\omega) < \infty \text{ for some n.} \\ \infty, \text{ otherwise.} \end{cases} \tag{1.79}$$

It can easily be verified that $t_\infty(\omega)$ is measurable with respect to F^*. It follows from the definition of $t(\omega; [C])$ that for each $\delta > 0$ and for some $t \in [0, t(\omega; [C]) + \delta)$ we have

$$x_t^*(\omega) \in \bar{C}. \tag{1.80}$$

Thus for some $t \in [0, t(\omega; [C]) + \delta)$ and a sufficient large n

$$x_t^*(\omega) \in B_n. \tag{1.81}$$

Hence, for each $\delta > 0$ and a sufficient large n

$$t(\omega; [C]) \overset{\leq}{=} t(\omega; B_n) + t(\omega; B_n; C) + \delta . \tag{1.82}$$

Thus, by (1.77) and (1.82)

$$t(\omega; [C]) = t_\infty(\omega). \tag{1.83}$$

So we have proved the following lemma:

Lemma 1.8.1

The ω-function $t(\omega; [C])$ is measurable with respect to F^*.

Let us introduce the ω-function $x^*(\omega; [C])$, defined by

$$x^*(\omega; [C]) \overset{\text{def}}{=} \begin{cases} x^*_{t(\omega;[C])}(\omega), \text{ if } t(\omega; [C]) < \infty \\ x_0^*(\omega), \text{ if } t(\omega; [C]) = \infty. \end{cases} \tag{1.84}$$

Note, that by this definition the state at the end of the period $[0, t(\omega; [C])]$ is given by $x^*(\omega; [C])$ unless $t(\omega; [C]) = \infty$.

We shall now demonstrate that the ω-function $x^*(\omega; [C])$ is measurable with respect to F^*. To this end we introduce the sequence of ω-functions $\{x^*_{(n)}(\omega);n=1,2,...\}$, where

$$x^*_{(n)}(\omega) = \chi(\omega)\ x^*(\omega;B_n;C) + (1-\chi(\omega))\ x^*_o(\omega) \qquad (1.85)$$

with

$$\chi(\omega) = \begin{cases} 1, & \text{if } t(\omega;[C]) < \infty \text{ and } \omega \in \overline{\Lambda^*_o}\ . \\ 0, & \text{if } t(\omega;[C]) = \infty \text{ or if } \omega \in \Lambda^*_o\ . \end{cases} \qquad (1.86)$$

It can easily be verified that the ω-functions $\{x^*_n(\omega);n=1,2,...\}$ are measurable with respect to F^*.

Since

$$t(\omega;[C]) = \lim_{n\to\infty} (\ t(\omega;B_n) + t(\omega;B_n;C)\ )\ , \qquad (1.82)$$

we find for $\omega \in \overline{\Lambda^*_o}$

$$x^*(\omega;[C]) = x^*_{(n)}(\omega)\ . \qquad (1.87)$$

This implies that for all ω the sequences $\{x^*_{(n)}(\omega);n=1,2,...\}$ converge to a limit, say $x^*_\infty(\omega)$.

The ω-function $x^*_\infty(\omega)$ is measurable with respect to F^*.

Obviously, we have for $\omega \in \overline{\Lambda^*_o}$

$$x^*(\omega;[C]) = x^*_\infty(\omega)\ . \qquad (1.88)$$

So we have proved the following lemma:

<u>Lemma 1.8.2</u>

The ω-function $x^*(\omega;[C])$ is measurable with respect to F^*.

Let us introduce the ω-sets $\Xi_{I;[C]}$ and $\Delta_{B;[C]}$, defined by

$$\Xi_{I;[C]} \overset{def}{=} \{\omega|\,t(\omega;[C]) \in I\} \qquad (1.89)$$

and

$$\Delta_{B;[C]} \overset{def}{=} \{\omega|x^*(\omega;[C]) \in B\} \qquad (1.90)$$

respectively.

We now assume that the closed set C is chosen in such a way that for each x

$$P^{*}\left[\Xi_{[0,\infty);[C]};x\right] = 1. \qquad (1.91)$$

The ω-functions $t(\omega;[C])$ and $x(\omega;[C])$ together with the probability spaces $\{\Omega^{*};F^{*};P^{*}\}$ generate the stochastic variables $\underline{t}_{[C];x}$ and $\underline{x}^{*}_{[C];x}$; the corresponding probability distributions are given by

$$\text{Prob}\,\{\underline{t}_{[C];x}\ \varepsilon\ I\} \overset{\text{def}}{=} P^{*}\left[\Xi_{I;[C]};x\right] \qquad (1.92)$$

and

$$\text{Prob}\,\{\underline{x}^{*}_{[C];x}\ \varepsilon\ B\} \overset{\text{def}}{=} P^{*}\left[\Delta_{B;[C]};x\right] \qquad (1.93)$$

respectively.

The stochastic variable $\underline{t}_{[C];x}$ represents the length of the time period preceding the first entry in C, while $\underline{x}^{*}_{[C];x}$ denotes the state at the end of that period if (1.91) is true.

Summarizing:

Lemma 1.8

If assumption 1 and condition (1.86) are satisfied, the probability distribution of the length $\underline{t}_{[C];x}$ of the period preceding the first entry in C and that of the state $\underline{x}^{*}_{[C];x}$ at that point of time are defined. They are given by (1.92) and (1.93) respectively.

We now consider the ω-functions $x^{*}_{t}(\omega;[C])$, defined by

$$x^{*}_{t}(\omega;[C]) \overset{\text{def}}{=} \begin{cases} x^{*}_{t(\omega;[C])+t}(\omega), & \text{if } t(\omega;[C]) < \infty. \\ x^{*}_{t}(\omega), & \text{if } t(\omega;[C]) = \infty. \end{cases} \qquad (1.94)$$

Repeating the arguments made in the proof of lemma 1.6 we can prove:

Lemma 1.9

The ω-functions $\{x^{*}_{t}(\omega;[C]);t\ \varepsilon\ [0,\infty)\}$ are measurable with respect to F^{*}.

Our future discussions are based on the following assumption:

Assumption 2

If $x^{*}(t)$ is any mapping of the time axis $[0,\infty)$ into the state space X^{*}, one and only one point ω can be found such that

$$x_t^*(\omega) = x^*(t). \tag{1.95}$$

We introduce the following notation:

$$x_t^*(\omega;t_o) \overset{\text{def}}{=} x_{t+t_o}^*(\omega). \tag{1.96}$$

<u>Lemma 1.10</u>

For each $\omega \varepsilon \Omega^*$ and $t_o \varepsilon [0,\infty)$, one and only one point $\omega_1 \varepsilon \Omega^*$ can be found such that for $t \geq 0$

$$x_t^*(\omega_1) = x_t^*(\omega;t_o). \tag{1.97}$$

<u>Proof</u>:

If we write $x^*(t) = x_t^*(\omega;t_o)$ the assertion follows at once from assumption 2.

The point transformation, defined by (1.97), will be denoted by

$$\omega_1 = T_{t_o}(\omega). \tag{1.98}$$

The point transformation (1.98) also introduces a transformation of ω-sets.

The ω_1-set K_1 will be called the t_o-image of K if

$$K_1 = \{\omega_1 \mid \omega_1 = T_{t_o}(\omega),\ \omega \varepsilon K\} . \tag{1.99}$$

We write:

$$K_1 = T_{t_o}(K). \tag{1.100}$$

Conversely, we can define a set transformation $K = T_{t_o}^{-1}(K_1)$ by

$$T_{t_o}^{-1}(K_1) \overset{\text{def}}{=} \{\omega \mid \omega_1 = T_{t_o}(\omega),\ \ \omega_1 \varepsilon K_1\} . \tag{1.101}$$

If $K_1 \varepsilon F^*$, let us introduce the class

$$F_{t_o}^* \overset{\text{def}}{=} \{T_{t_o}^{-1}(K_1) \mid K_1 \varepsilon F^*\} . \tag{1.102}$$

Lemma 1.11

The set transformation $K = T_{t_o}^{-1}(K_1)$ generates an isomorphism of F^* with $F^*_{t_o}$.

Proof:

We first prove that $F^*_{t_o}$ is a σ-field. This can easily be done by verifying the following properties:

a) $\Omega^* \varepsilon F^*_{t_o}$

b) if $K = T_{t_o}^{-1}(K_1) \varepsilon F^*_{t_o}$, then

$$\overline{K} = \Omega^* - T_{t_o}^{-1}(K_1) = T_{t_o}^{-1}(\Omega^* - K_1) \varepsilon F^*_{t_o};$$

c) if $K_i = T_{t_o}^{-1}(K_{i;1}) \varepsilon F^*_{t_o}$ $(i=1,2,\ldots)$, we also have

$$\bigcup_{i=1}^{\infty} K_i = \bigcup_{i=1}^{\infty} T_{t_o}^{-1}(K_{i;1}) = T_{t_o}^{-1}\left(\bigcup_{i=1}^{\infty} K_{i;1}\right) \varepsilon F^*_{t_o}.$$

Consequently, $F^*_{t_o}$ is a σ-field.

Since

$$T_{t_o}(T_{t_o}^{-1}(K_1)) = \{\omega_1 | \omega_1 = T_{t_o}(\omega);\ \omega \varepsilon T_{t_o}^{-1}(K_1)\} = \\ = \{\omega_1 | \omega_1 \varepsilon K_1\} = K_1, \qquad (1.103)$$

the set transformation $K = T_{t_o}^{-1}(K_1)$ generates an isomorphism of F^* with $F^*_{t_o}$.

This proves the lemma completely.

Lemma 1.12

The σ-field $F^*_{t_o}$ satisfies $F^*_{t_o} \subset F^*$.

Proof:

Let J be the class of sets K belonging to both F^* and $F^*_{t_o}$. Obviously, J is a σ-field.

Let J_1 be the class of sets $K_1 \varepsilon F^*$ satisfying $T_{t_o}^{-1}(K_1) \varepsilon J$.

The following properties of J_1 can easily be verified:

a) If $K_1 \varepsilon J_1$, then $T_{t_o}^{-1}(\overline{K}_1) = \overline{T_{t_o}^{-1}(K_1)} \varepsilon J$. Thus, $\overline{K}_1 \varepsilon J_1$;

b) If $K_{1;i} \varepsilon J_1$, then $T_{t_o}^{-1}(\bigcup_{i=1}^{\infty} K_{1;i}) = \bigcup_{i=1}^{\infty} T_{t_o}^{-1}(K_{1;i}) \varepsilon J$. Thus, $\bigcup_{i=1}^{\infty} K_{1;i} \varepsilon J_1$.

Hence, J_1 is a σ-field.

c) If $K_1 \subset \Lambda_o^*$, then $T_{t_o}^{-1}(K_1) \subset \Lambda_o^*$ and thus $T_{t_o}^{-1}(K_1) \varepsilon J$. Consequently, $K_1 \varepsilon J_1^o$;

d) If $K_1 = \Lambda_{t;B}$ with $B \varepsilon G^*$, then $T_{t_o}^{-1}(K_1) = \Lambda_{t+t_o;B} \varepsilon J$. Thus, $\Lambda_{t;B} \varepsilon J_1$.

Hence, J_1 is a σ-field that contains the sets $\Lambda_{t;B}$ and the subsets of Λ_o^*. Thus, $J_1 = F^*$. Consequently, $J = F_{t_o}^* \subset F^*$.

This ends the proof.

<u>Lemma 1.13</u>

For each $\omega \varepsilon \Omega^*$ and for each closed set $C \varepsilon G^*$ one and only one point $\omega_1 \varepsilon \Omega^*$ can be found such that

$$x_t^*(\omega_1) = x_t^*(\omega; [C]); \quad t \geqq 0. \tag{1.104}$$

<u>Proof</u> :

If we write $x^*(t) = x_t^*(\omega; [C])$, the assertion follows at once from assumption 2.

The point transformation, defined by (1.104) will be denoted by

$$\omega_1 = T_{[C]}(\omega). \tag{1.105}$$

This point transformation also introduces a trnasformation of ω-sets in Ω^*. The ω_1-set K_1 will be called the $[C]$-image of K if

$$K_1 = \{\omega_1 | \omega_1 = T_{[C]}(\omega); \ \omega \varepsilon K\} . \tag{1.106}$$

Lemma 1.14

If $\Omega^{*}_{[C]}$ is defined by

$$\Omega^{*}_{[C]} \overset{\text{def}}{=} T_{[C]}(\Omega^{*}), \tag{1.107}$$

then

$$\Omega^{*}_{[C]} = \overline{\Xi}_{(0,\infty);[C]} \cup \Xi_{0;C} . \tag{1.108}$$

Proof:

If we have either $t(\omega;[C]) = 0$ or $t(\omega;[C]) = \infty$, by (1.94) and (1.104) we find $\omega_1 = T_{[C]}(\omega) = \omega$. Consequently, $t(\omega_1;[C]) = t(\omega;[C])$. Hence, $\omega_1 \varepsilon \overline{\Xi}_{(0,\infty);[C]}$.

If $0 < t(\omega;[C]) < \infty$, then $t(\omega_1;C) = 0$. Therefore, $\omega_1 \varepsilon \Xi_{0;C}$. So we have proved that

$$T_{[C]}(\Omega^{*}) \subset \overline{\Xi}_{(0,\infty);[C]} \cup \Xi_{0;C}. \tag{1.109}$$

We shall now demonstrate that the converse of (1.109) is also true.

If $\omega' \varepsilon \overline{\Xi}_{(0,\infty);[C]}$, then

$$\omega' = T_{[C]}(\omega') \tag{1.110}$$

and thus

$$\omega' \varepsilon T_{[C]}(\Omega^{*}). \tag{1.111}$$

If $\omega' \varepsilon \Xi_{0;C}$, if ω'' satisfies

$$t(\omega'';[C]) > 0 \tag{1.112}$$

and if ω''' is given by

$$x^{*}_{t}(\omega''') = \begin{cases} x^{*}_{t}(\omega''), \text{ if } t < t(\omega'';[C]) \\ x^{*}_{t-t(\omega'';[C])}(\omega'), \text{ if } t \geq t(\omega'';[C]) \end{cases} \tag{1.113}$$

then

$$\omega' = T_{[C]}(\omega'''). \tag{1.114}$$

Consequently, if $\omega' \in \Xi_{O;C}$, we also have

$$\omega' \in T_{[C]}(\Omega^*) . \tag{1.115}$$

Summarizing, if $\omega' \in \overline{\Xi}_{(0,\infty);[C]} \cup \Xi_{O;C}$, we find

$$\omega' \in T_{[C]}(\Omega^*) \tag{1.116}$$

and thus

$$T_{[C]}(\Omega^*) \supset \overline{\Xi}_{(0,\infty);[C]} \cup \Xi_{O;C} . \tag{1.117}$$

The relations (1.109) and (1.117) imply

$$T_{[C]}(\Omega^*) = \overline{\Xi}_{(0,\infty);[C]} \cup \Xi_{O;C} . \tag{1.118}$$

This ends the proof.

Conversely, we can define a set transformation $K = T^{-1}_{[C]}(K_1)$ by

$$T^{-1}_{[C]}(K_1) \stackrel{def}{=} \{\omega | \omega_1 = T_{[C]}(\omega);\ \omega_1 \in K_1\} . \tag{1.119}$$

Let us consider the following classes:

$$F^{*[C]} \stackrel{def}{=} \{K_1 \cap \Omega^*_{[C]} | K_1 \in F^*\} \tag{1.120}$$

$$F^*_{[C]} \stackrel{def}{=} \{T^{-1}_{[C]}(K_1) \mid K_1 \in F^{*[C]}\} . \tag{1.121}$$

Repeating the arguments made in the proofs of lemmas 1.11 and 1.12 we find:

<u>Lemma 1.15</u>

The set transformation $K = T^{-1}_{[C]}(K_1)$ generates an isomorphism of $F^{*[C]}$ with $F^*_{[C]}$.

<u>Lemma 1.16</u>

The σ-field $F^*_{[C]}$ satisfies $F^*_{[C]} \subset F^*$.

In future we shall use the point transformations $\omega_1 = T^j_{[C]}(\omega)$, defined by

$$T^{1}_{[C]}(\omega) = T_{[C]}(\omega) \qquad (1.122)$$

and

$$T^{j}_{[C]}(\omega) = T_{[C]}(T^{j-1}_{[C]}(\omega)); \; j=2,3,\dots \; . \qquad (1.123)$$

The point transformation $\omega_1 = T^{j}_{[C]}(\omega)$ introduces a transformation of ω-sets.

We write

$$K_1 = T^{j}_{[C]}(K) \qquad (1.124)$$

if

$$K_1 = \{\omega_1 \mid \omega_1 = T^{j}_{[C]}(\omega), \; \omega \in K\}. \qquad (1.125)$$

We obviously have

$$T^{1}_{[C]}(K) = T_{[C]}(K). \qquad (1.126)$$

Conversely, we can define a set transformation $K = T^{-j}_{[C]}(K_1)$ by

$$T^{-j}_{[C]}(K_1) = T^{-1}_{[C]}(T^{-j+1}_{[C]}(K_1)); \; j=1,2,\dots \qquad (1.127)$$

with

$$T^{0}_{[C]}(K_1) = K_1. \qquad (1.128)$$

By means of lemmas 1.15 and 1.16 we can easily verify:

<u>Lemma 1.17</u>

If $K \in F^{*}$, then $T^{j}_{[C]}(K) \in F^{*}$. (1.129)

If $K_1 \in F^{*}$, then $T^{-j}_{[C]}(K_1) \in F^{*}$. (1.130)

Consider a closed set C in X^{*}, satisfying the following assumption:

For each $j \geq 1$ and for each $x \in X^{*}$ we have

$$P^{*}\left[T^{-j+1}_{[C]}(\Xi_{[0,\infty);[C]});x\right] = 1. \qquad (1.131)$$

Let us define the ω-function $t_j(\omega;[C])$ by

$$t_j(\omega;[C]) = t(T^{j-1}_{[C]}(\omega);[C]). \qquad (1.132)$$

By lemma 1.17 the ω-function $t_j(\omega;[C])$ is measurable with respect to F^*.

By (1.131) the length of the period between the $(j\text{-}1)^{st}$ and the j^{th} entry in C is almost surely defined and equal to $t_j(\omega;[C])$.

Let us define the ω-function $x_j^*(\omega;[C])$ by

$$x_j^*(\omega;[C]) = x^*(T_{[C]}^{j-1}(\omega);[C]). \tag{1.133}$$

By lemma 1.17 the ω-function $x_j(\omega;[C])$ is measurable with respect to F^*.

By (1.131) the state at the j^{th} entry in C is almost surely defined and equal to $x_j(\omega;[C])$.

Summarizing:

Lemma 1.18

The ω-functions $t_j(\omega;[C])$ and $x_j^*(\omega;[C])$ $(j=1,2,\ldots)$, defined by (1.132) and (1.333) respectively, are measurable with respect to F^*.

The ω-functions $t_j(\omega;[C])$ and $x_j^*(\omega;[C])$ together with the probability spaces $\{\Omega^*;F^*;P^*\}$ generate the stochastic variables $\underline{t}_{[C];x;j}$ and $\underline{x}^*_{[C];x;j}$; the corresponding probability distributions are given by

$$\text{Prob}\ \{\underline{t}_{[C];x;j}\ \varepsilon\ I\} \overset{\text{def}}{=} P^*[T_{[C]}^{-j+1}(\Xi_I;[C]);x] \tag{1.134}$$

and

$$\text{Prob}\ \{\underline{x}^*_{[C];x;j}\ \varepsilon\ B\} \overset{\text{def}}{=} P^*[T_{[C]}^{-j+1}(\Delta_B;[C]);x]\ . \tag{1.135}$$

The stochastic variable $\underline{t}_{[C];x;j}$ represents almost surely the length of the period between the $(j\text{-}1)^{st}$ and the j^{th} entry in C, while $\underline{x}^*_{[C];x;j}$ denotes the state at the j^{th} entry.

So we have proved the following lemma:

Lemma 1.19

If the assumptions 1 and 2 and the condition (1.131) are satisfied, the probability distributions of the lengths $\underline{t}_{[C];x;j}$ of the periods between successive entries in C and those of the entry states $\underline{x}^*_{[C];x;j}$ are defined. They are given by (1.134) and (1.135).

2. Random losses

The considerations in this section do not longer start from the assumption that almost all t-functions $x^*_t(\omega)$ are continuous from the right. On the other hand we still assume that almost all t-functions $x^*_t(\omega)$ have only a finite number of discontinuities in a finite interval. Moreover, the assertions, stated in lemmas 1.5 ff., are supposed to be true. In chapter 2 of this part we shall show that in a special case these lemmas can be proved without the continuity assumption.

A stochastic process S^*_x is also called a random walk in X^*. Let us assume that losses are incurred during walks in X^*. We distinguish the following types of losses:

a) The "first type" loss is defined by means of a closed set A and a bounded real valued function $\gamma_{disc}(x)$, which is measurable with respect to G^*. If the initial state $x^*_o(\omega)$ of the random walk belongs to A, a loss $\gamma_{disc}(x^*_o(\omega))$ is incurred at the start. Moreover, each entry $x^*_j(\omega;[A])$ in A costs $\gamma_{disc}(x^*_j(\omega;[A]))$. In our future discussions we shall make use of a constant γ_d, that satisfies for each $x \in X^*$

$$|\gamma_{disc}(x)| \leqq \gamma_d < \infty; \qquad (1.136)$$

b) The "second type" loss is defined by means of a bounded continuous function $\gamma_{cont}(x)$. The "second type" loss incurred during the period $[s_1, s_2)$ is then given by the Riemann integral

$$\int_{s_1}^{s_2} \gamma_{cont}(x^*_t(\omega))dt. \qquad (1.137)$$

In our future discussions we shall make use of a constant γ_c that satisfies for each $x \in X^*$

$$|\gamma_{cont}(x)| \leqq \gamma_c < \infty. \tag{1.138}$$

In this section we consider random losses, which will be incurred in the periods $[0,t_o)$, $[0,t(\omega;B)]$ and $[0,t(\omega;[C]))$.

Let the ω-functions $\{\hat{t}_n(\omega;[A]);\ n=1,2,\ldots\}$ be defined by

$$\hat{t}_n(\omega;[A]) \overset{def}{=} \sum_{j=1}^{n} t_j(\omega;[A]). \tag{1.139}$$

Note that set A has been used in the definition of the "first type" loss.

We now assume that the closed set A satisfies for each x

$$\lim_{n\to\infty} P^{*}\left[\Xi_{t_o;[A];n};x\right] = 0, \tag{1.140}$$

where

$$\Xi_{t_o;[A];n} \overset{def}{=} \{\omega \mid \hat{t}_n(\omega;[A]) < t_o\}.$$

Let $n(\omega;t_o;[A])$ be the number of entries in A during the period $[0,t_o)$.

According to this definition

$$n(\omega;t_o;[A]) = n, \text{ if } \hat{t}_n(\omega;[A]) < t_o \leqq \hat{t}_{n+1}(\omega;[A]). \tag{1.141}$$

Obviously, the following lemma is true:

Lemma 1.20

The ω-function $n(\omega;t_o;[A])$ is measurable with respect to F^{*}.

We now start our discussion with the losses of the first type.

A real valued ω-function $k_{disc}(\omega;t_o)$ is defined by

$$k_{disc}(\omega;t_o) = \begin{cases} \sum\limits_{j=1}^{n(\omega;t_o;[A])} \gamma_{disc}(x_j^{*}(\omega;[A])), & \text{if } \omega\in\Xi_{0;\bar{A}} \text{ and } n(\omega;t_o;[A]) < \infty \\ \gamma_{disc}(x_o^{*}(\omega)) + \sum\limits_{j=1}^{n(\omega;t_o;[A])} \gamma_{disc}(x_j^{*}(\omega;[A])), & \text{if } \omega\in\Xi_{0;A} \text{ and } n(\omega;t_o;[A])<\infty. \\ 0, & \text{otherwise.} \end{cases} \tag{1.142}$$

Lemma 1.21

The ω-function $k_{disc}(\omega;t_o)$ is measurable with respect to F^*.

Proof:

Since

a) $\gamma_{disc}(x)$ is Borel measurable with respect to G^*,

b) $x_j^*(\omega;[A])$ are measurable with respect to F^* $(j=1,2,\ldots)$,

we find that both $(i=1,2,\ldots)$

$$\sum_{j=1}^{i} \gamma_{disc}(x_j^*(\omega;[A])) \text{ and } \gamma_{disc}(x_o^*(\omega))$$

are measurable with respect to F^*.

Let us introduce the ω-functions $\chi_o(\omega)$ and $\chi_i(\omega;t_o)$, defined by

$$\chi_o(\omega) \overset{def}{=} \begin{cases} 1, & \text{if } \omega\varepsilon\Xi_{O;A} \\ 0, & \text{otherwise} \end{cases} \tag{1.143}$$

and

$$\chi_i(\omega;t_o) \overset{def}{=} \begin{cases} 1, & \text{if } \hat{t}_i(\omega;[A]) < t_o \leq \hat{t}_{i+1}(\omega;[A]) \\ 0, & \text{otherwise} \end{cases} \tag{1.144}$$

respectively.

It can easily be verified that the ω-functions

$$\{\sum_{i=1}^{m} \chi_i(\omega;t_o)\ [\chi_o(\omega)\ \gamma_{disc}(x_o^*(\omega)) + \sum_{j=1}^{i} \gamma_{disc}(x_j^*(\omega;[A]))]; \quad ;m=1,2,\ldots\} \tag{1.145}$$

are measurable with respect to F^*.

Since the sequence (1.145) converges everywhere to $k_{disc}(\omega;t_o)$, this ω-function is measurable with respect to F^*.

This ends the proof.

By (1.140) and (1.142) $k_{disc}(\omega;t_o)$ represents almost surely the "first type" loss incurred in the period $[0,t_o)$.

We now consider a closed set B, satisfying

$$\lim_{n \to \infty} P^{*} \left[\Xi_{B;[A];n};x \right] = 0, \tag{1.146}$$

where

$$\Xi_{B;[A];n} \overset{\text{def}}{=} \{\omega \mid \hat{t}_n(\omega;[A]) \leq t(\omega;B)\} . \tag{1.147}$$

Let $n(\omega;B;[A])$ be the number of entries in A during the period $[0,t(\omega;B)]$.

According to this definition

$$n(\omega;B;[A]) = n, \text{ if } \hat{t}_n(\omega;[A]) \leq t(\omega;B) < \hat{t}_{n+1}(\omega;[A]). \tag{1.148}$$

The proof of the following lemma is obvious.

Lemma 1.22

The ω-function $n(\omega;B;[A])$ is measurable with respect to F^{*}.

A real valued ω-function $k_{disc}(\omega;B)$ is defined by

$$k_{disc}(\omega;B) = \begin{cases} \sum_{j=1}^{n(\omega;B;[A])} \gamma_{disc}(x_j^{*}(\omega;[A])), \text{ if } \omega \in \Xi_{0;\bar{A}} \text{ and } n(\omega;B;[A]) < \infty. \\ \sum_{j=1}^{n(\omega;B;[A])} \gamma_{disc}(x_j^{*}(\omega;[A])) + \gamma_{disc}(x_o^{*}(\omega)), \text{ if } \omega \in \Xi_{0;A} \text{ and } n(\omega;B;[A]) < \infty. \\ 0, \text{ otherwise.} \end{cases} \tag{1.149}$$

The following lemma can easily be proved (cf. lemma 1.21):

Lemma 1.23

The ω-function $k_{disc}(\omega;B)$ is measurable with respect to F^{*}.

By (1.146) and (1.149) $k_{disc}(\omega;B)$ represents almost surely the "first type" loss incurred in the period $[0,t(\omega;B)]$.

Next we consider a closed set C, satisfying

$$\lim_{n \to \infty} P^{*}\left[\Xi_{[C];[A];n};x\right] = 0, \tag{1.150}$$

where

$$\Xi_{[C];[A];n} \overset{\text{def}}{=} \{\omega \mid \hat{t}_n(\omega);[A]) < t(\omega;[C])\}. \qquad (1.151)$$

Let $n(\omega;[C];[A])$ be the number of entries in A during the period $[0,t(\omega;[C]))$.

According to this definition

$$n(\omega;[C];[A]) = n, \text{ if } \hat{t}_n(\omega;[A]) < t(\omega;[C]) \leqq \hat{t}_{n+1}(\omega;[A]). \qquad (1.152)$$

The proof of the following lemma is obvious.

<u>Lemma 1.24</u>

The ω-function $n(\omega;[C];[A])$ is measurable with respect to F^*.

A real valued ω-function $k_{disc}(\omega;[C])$ is now defined by

$$k_{disc}(\omega;[C]) \overset{\text{def}}{=} \begin{cases} \sum_{j=1}^{n(\omega;[C];[A])} \gamma_{disc}(x_j^*(\omega;[A])), \text{ if } n(\omega;[C];[A]) < \infty \text{ and } \omega \in \Xi_{0;\bar{A}}. \\ \sum_{j=1}^{n(\omega;[C];[A])} \gamma_{disc}(x_j^*(\omega;[A])) + \gamma_{disc}(x_0^*(\omega)), \text{ if } \\ \qquad n(\omega;[C];[A]) < \infty \text{ and } \omega \in \Xi_{0;A}. \\ 0, \text{ otherwise.} \end{cases} \qquad (1.153)$$

The following lemma can easily be proved (cf. lemma 1.21):

<u>Lemma 1.25</u>

The ω-function $k_{disc}(\omega;[C])$ is measurable with respect to F^*.

By (1.150) and (1.153) $k_{disc}(\omega;[C])$ represents almost surely the "first type" loss incurred in the period $[0,t(\omega;[C]))$.

Let us introduce the ω-function $x_t^{**}(\omega)$, defined by

$$x_t^{**}(\omega) \overset{\text{def}}{=} \begin{cases} \lim_{n\to\infty} x^*_{t+\frac{1}{n}}(\omega), \text{ if } \omega \in \overline{\Lambda_0^*}. \\ x_0^*(\omega), \text{ if } \omega \in \Lambda_0^*. \end{cases} \qquad (1.154)$$

<u>Lemma 1.26</u>

The ω-functions $\{x_t^{**}(\omega);t \in [0,\infty)\}$ are measurable with respect to F^*.

The t-functions $\{x_t^{**}(\omega); \omega \in \Omega^*\}$ are continuous from the right and have almost surely in each finite interval only a finite number of discontinuities.

Proof:

Consider the sequence of ω-functions $\{x_{n;t}^{**}(\omega);\ n=1,2,\ldots\}$, defined by

$$x_{n;t}^{**}(\omega) \overset{\text{def}}{=} \begin{cases} x^*_{t+\frac{1}{n}}(\omega), & \text{if } \omega \in \overline{\Lambda}{}_o^*. \\ x_o^*(\omega), & \text{if } \omega \in \Lambda_o^*. \end{cases} \tag{1.155}$$

The ω-functions $x_{n;t}^{**}(\omega)$ are measurable with respect to F^*. It can easily be verified that the sequence converges everywhere to $x_t^{**}(\omega)$. Consequently, the ω-function $x_t^{**}(\omega)$ is measurable with respect to F^*. Since $x_t^*(\omega)$ has only a finite number of discontinuities in a finite interval, the second part of the assertion is obvious.

This ends the proof.

Lemma 1.27

The ω-functions $\gamma_{cont}(x_t^{**}(\omega))$ are measurable with respect to F^*.

The t-functions $\gamma_{cont}(x_t^{**}(\omega))$ are continuous from the right and have almost surely only a finite number of discontinuities in a finite interval.

Proof:

Since $\gamma_{cont}(x)$ is a continuous function, the assertions are immediate.

Lemma 1.28

The Riemann integral

$$k_{cont}(\omega;s) \overset{\text{def}}{=} \int_0^s \gamma_{cont}(x_t^{**}(\omega))dt \tag{1.156}$$

exists for each $s < \infty$ and represents an ω-function which is measurable with respect to F^*.

Proof:

By lemma 1.27 we obviously have

$$\int_0^s \gamma_{cont}(x_t^{**}(\omega))dt = \lim_{n\to\infty} \frac{1}{2^n} \sum_{j=1}^{2^n} \gamma_{cont}(x_{\frac{js}{2^n}}^{**}(\omega)) \leqq s\gamma_c < \infty . \tag{1.157}$$

Consequently, $k_{cont}(\omega;s)$ exists and is measurable with respect to F^*.

This ends the proof.

The ω-function $k_{cont}(\omega;t_o)$ represents almost surely the "second type" costs incurred in the period $[0,t_o)$.

Next we introduce the ω-functions $k_{cont}(\omega;B)$ and $k_{cont}(\omega;[C])$, defined by

$$k_{cont}(\omega;B) \overset{def}{=} \begin{cases} k(\omega;t(\omega;B)), \text{ if } t(\omega;B) < \infty \\ 0, \text{ otherwise} \end{cases} \tag{1.158}$$

and

$$k_{cont}(\omega;[C]) \overset{def}{=} \begin{cases} k(\omega;t(\omega;[C])), \text{ if } t(\omega;[C]) < \infty \\ 0, \text{ otherwise} \end{cases} \tag{1.159}$$

respectively.

<u>Lemma 1.29</u>

The ω-functions $k_{cont}(\omega;B)$ and $k_{cont}(\omega;[C])$ are measurable with respect to F^*.

<u>Proof</u>:

Let us introduce the ω-functions $\{w_n(\omega);\ n=1,2,\dots\}$, defined for each j by

$$w_n(\omega) \overset{def}{=} \begin{cases} k(\omega;\frac{j}{n}), \text{ if } \frac{j-1}{n} \leqq t(\omega;B) < \frac{j}{n} . \\ 0, \text{ otherwise} \end{cases} \tag{1.160}$$

Obviously, the ω-functions $\{w_n(\omega);\ n=1,2,\dots\}$ are measurable with respect to F^*.

The sequence $\{w_n(\omega);\ n=1,2,\dots\}$ converges everywhere to the ω-function $k_{cont}(\omega;B)$. Consequently, the ω-function $k_{cont}(\omega;B)$ is measurable with respect to F^*.

The proof for $k_{cont}(\omega;[C])$ goes along the same lines and is therefore omitted.

This ends the proof.

If the sets B and C satisfy for each x

$$P^{*}\left[\Xi_{[0,\infty);B};x\right] = 1 \tag{1.161}$$

and

$$P^{*}\left[\Xi_{[0,\infty);[C]};x\right] = 1 \tag{1.162}$$

the ω-function $k_{cont}(\omega;B)$ represents almost surely the "second type" loss incurred in the period $[0,t(\omega;B)]$, while with regard to the period $[0,t(\omega;[C]))$ this loss almost surely is given by $k_{cont}(\omega;[C])$. Under (1.140), (1.146), (1.150), (1.161) and (1.162) the total costs incurred in the period $[0,t_o)$, $[0,t(\omega;B)]$ and $[0,t(\omega;[C]))$ are almost surely given by

$$k(\omega;t_o) \overset{def}{=} k_{disc}(\omega;t_o) + k_{cont}(\omega;t_o), \tag{1.163}$$

$$k(\omega;B) \overset{def}{=} k_{disc}(\omega;B) + k_{cont}(\omega;B) \tag{1.164}$$

and

$$k(\omega;[C]) \overset{def}{=} k_{disc}(\omega;[C]) + k_{cont}(\omega;[C]) \tag{1.165}$$

respectively.

Remark that

a) $|k(\omega;t_o)| \leqq t_o\ \gamma_c + n(\omega;t_o;[A])\ \gamma_d;$ (1.166)

b) $|k(\omega;B)| \leqq t(\omega;B)\ \gamma_c + n(\omega;B;[A])\ \gamma_d;$ (1.167)

c) $|k(\omega;[C])| \leqq t(\omega;[C])\ \gamma_c + n(\omega;[C];[A])\ \gamma_d\ .$ (1.168)

Obviously, we have:

Lemma 1.30

The ω-functions $k(\omega;t_o)$, $k(\omega;B)$ and $k(\omega;[C])$ are measurable with respect to F^{*}.

The ω-functions $k(\omega;t_o)$, $k(\omega;B)$, $k(\omega;[C])$, $n(\omega;t_o;[A])$, $n(\omega;B;[A])$ and $n(\omega;[C];[A])$ together with the probability spaces $\{\Omega^{*};F^{*};P^{*}\}$ generate the stochastic variables $\underline{k}_{t_o;x}$, $\underline{k}_{B;x}$, $\underline{k}_{[C];x}$,

$\underline{n}_{t_o;x}, \underline{n}_{B;x}$ and $\underline{n}_{[C];x}$; the corresponding probability distributions are given by

$$\text{Prob}\ \{\underline{k}_{t_o;x} \in I\} \overset{\text{def}}{=} P^* \left[K_{I;t_o};x\right], \tag{1.169}$$

$$\text{Prob}\ \{\underline{k}_{B;x} \in I\} \overset{\text{def}}{=} P^* \left[K_{I;B};x\right], \tag{1.170}$$

$$\text{Prob}\ \{\underline{k}_{[C];x} \in I\} \overset{\text{def}}{=} P^* \left[K_{I;[C]};x\right], \tag{1.171}$$

$$\text{Prob}\ \{\underline{n}_{t_o;x} = n\} \overset{\text{def}}{=} P^* \left[N_{n;t_o};x\right], \tag{1.172}$$

$$\text{Prob}\ \{\underline{n}_{B;x} = n\} \overset{\text{def}}{=} P^* \left[N_{n;B};x\right], \tag{1.173}$$

and

$$\text{Prob}\ \{\underline{n}_{[C];x} = n\} \overset{\text{def}}{=} P^* \left[N_{n;[C]};x\right], \tag{1.174}$$

where

$$K_{I;t_o} \overset{\text{def}}{=} \{\omega \mid k(\omega;t_o) \in I\}, \tag{1.175}$$

$$K_{I;B} \overset{\text{def}}{=} \{\omega \mid k(\omega;B) \in I\}, \tag{1.176}$$

$$K_{I;[C]} \overset{\text{def}}{=} \{\omega \mid k(\omega;[C]) \in I\}, \tag{1.177}$$

$$N_{n;t_o} \overset{\text{def}}{=} \{\omega \mid n(\omega;t_o;[A]) = n\}, \tag{1.178}$$

$$N_{n;B} \overset{\text{def}}{=} \{\omega \mid n(\omega;B;[A]) = n\}, \tag{1.179}$$

$$N_{n;[C]} \overset{\text{def}}{=} \{\omega \mid n(\omega;[C];[A]) = n\}, \tag{1.180}$$

and I is any interval in $(-\infty,+\infty)$.

So we have proved:

Lemma 1.31

Under (1.140), (1.146), (1.150), (1.161) and (1.162) the probability distributions of the random losses $\underline{k}_{t_o;x}, \underline{k}_{B;x}$ and $\underline{k}_{[C];x}$ incurred in the periods $[0,t_o)$, $[0,\underline{t}_{B;x}]$ and $[0,\underline{t}_{[C];x})$ respectively as well as those of the number of entries $\underline{n}_{t_o;x}, \underline{n}_{B;x}$ and $\underline{n}_{[C];x}$ in A during the same periods are defined. They are given by (1.169) through (1.174).

Finally, let us define the ω-functions $n_j(\omega;t_o;[A])$, $n_j(\omega;[C];[A])$, $k_j(\omega;t_o)$ and $k_j(\omega;[C])$ by

$$n_j(\omega;t_o;[A]) = n(T_{(j-1)t_o}(\omega);t_o;[A]) \tag{1.181}$$

$$n_j(\omega;[C];[A]) = n(T^{j-1}_{[C]}(\omega);[C];[A]) \tag{1.182}$$

$$k_j(\omega;t_o) = k(T_{(j-1)t_o}(\omega);t_o) \tag{1.183}$$

and

$$k_j(\omega;[C]) = k(T^{j-1}_{[C]}(\omega);[C]). \tag{1.184}$$

By means of lemmas 1.11 and 1.17 we can easily verify that the ω-functions $n_j(\omega;t_o;[A])$, $n_j(\omega;[C];[A])$, $k_j(\omega;t_o)$ and $k_j(\omega;[C])$ are measurable with respect to F^*.

We now assume that for each $j \geqq 1$ and for each x we have

$$\lim_{n\to\infty} P^*[T^{-1}_{(j-1)t_o}(\Xi_{t_o;[A];n});x] = 0 \tag{1.185}$$

$$\lim_{n\to\infty} P^*[T^{-j+1}_{[C]}(\Xi_{[C];[A];n});x] = 0 \tag{1.186}$$

and

$$P^*[T^{-j+1}_{[C]}(\Xi_{[0,\infty);[C]});x] = 1. \tag{1.187}$$

The ω-functions $n_j(\omega;t_o;[A])$ and $n_j(\omega;[C];[A])$ represent the number of entries in A during the periods $[(j-1)t_o,jt_o)$ and $[\hat{t}_{(j-1)}(\omega;[C]),\hat{t}_j(\omega;[C]))$ respectively.
By (1.185) the costs incurred in the period $[(j-1)t_o,jt_o)$ are almost surely given by $k_j(\omega;t_o)$. By (1.186) and (1.187) the costs incurred between the $(j-1)^{st}$ and the j^{th} entry in C are almost surely given by $k_j(\omega;[C])$.

The ω-functions $n_j(\omega;t_o;[A])$, $n_j(\omega;[C];[A])$, $k_j(\omega;t_o)$ and $k_j(\omega;[C])$ together with the probability spaces $\{\Omega^*;F^*;P^*\}$ generate the stochastic variables $\underline{n}_{t_o;x;j}$, $\underline{n}_{[C];x;j}$, $\underline{k}_{t_o;x;j}$ and $\underline{k}_{[C];x;j}$; the corresponding probability distributions are given by

$$\text{Prob}\ \{\underline{n}_{t_o;x;j} = n\} \stackrel{\text{def}}{=} P^*\left[T^{-1}_{(j-1)t_o}(N_{n;t_o});x\right] \qquad (1.188)$$

$$\text{Prob}\ \{\underline{n}_{[C];x;j} = n\} \stackrel{\text{def}}{=} P^*\left[T^{-j+1}_{[C]}(N_{n;[C]});x\right] \qquad (1.189)$$

$$\text{Prob}\ \{\underline{k}_{t_o;x;j} \in I\} \stackrel{\text{def}}{=} P^*\left[T^{-1}_{(j-1)t_o}(K_{I;t_o});x\right] \qquad (1.190)$$

and

$$\text{Prob}\ \{\underline{k}_{[C];x;j} \in I\} \stackrel{\text{def}}{=} P^*\left[T^{-j+1}_{[C]}(k_{I;[C]});x\right]. \qquad (1.191)$$

Let the stochastic variables $\{\hat{\underline{t}}_{[C];x;n}\ ;\ n=1,2,\ldots\}$ be defined by

$$\hat{\underline{t}}_{[C];x;n} = \sum_{j=1}^{n} \underline{t}_{[C];x;j}\ ;\ j=1,2,\ldots \qquad (1.192)$$

The stochastic variables $\underline{n}_{t_o;x;j}$ and $\underline{n}_{[C];x;j}$ represent the number of entries in A during the periods $[(j-1)t_o, jt_o)$ and $[\hat{\underline{t}}_{[C];x;j-1}, \hat{\underline{t}}_{[C];x;j})$ respectively.

The stochastic variables $\underline{k}_{t_o;x;j}$ and $k_{[C];x;j}$ represent almost surely the costs incurred in the periods $[(j-1)t_o, jt_o)$ and $[\hat{\underline{t}}_{[C];x;j-1}, \hat{\underline{t}}_{[C];x;j})$ respectively.

So we have proved the following lemma:

<u>Lemma 1.32</u>

Under (1.185), (1.186) and (1.187) the probability distributions of $\underline{n}_{t_o;x;j}$, $\underline{n}_{[C];x;j}$, $\underline{k}_{t_o;x;j}$ and $\underline{k}_{[C];x;j}$ are defined; they are given by (1.188) through (1.191).

3. Stationary strong Markov processes

Let us consider the ω-functions $\hat{x}^*_t(\omega;t_o)$ and $\hat{x}^*_t(\omega;[C])$ defined by

$$\hat{x}^*_t(\omega;t_o) \stackrel{\text{def}}{=} \begin{cases} x^*_t(\omega), & \text{if } t < t_o \\ x^*_{t_o}(\omega), & \text{if } t \geq t_o \end{cases} \qquad (1.193)$$

and

$$\hat{x}^*_t(\omega;[C]) \stackrel{\text{def}}{=} \begin{cases} x^*_t(\omega), & \text{if } t < t(\omega;[C]) \\ x^*(\omega;[C]), & \text{if } t \geq t(\omega;[C]) \end{cases} \qquad (1.194)$$

We can easily prove the following lemma;

Lemma 1.33

If C is a closed set in X^*, the ω-functions $\hat{x}^*_t(\omega;t_o)$ and $\hat{x}^*_t(\omega;[C])$ are measurable with respect to F^*.

We now introduce the following notation:

The class of ω-sets $\hat{H}^*_t$ is the smallest σ-field with respect to which the ω-functions $\{\hat{x}^*_t(\omega;\overset{\circ}{t}_o);t\,\varepsilon[0,\infty)\}$ are measurable.

The class of ω-sets $\hat{H}^*_{[C]}$ is the smallest σ-field with respect to which the ω-functions $\{\hat{x}^*_t(\omega;[C]);t\,\varepsilon\,[0,\infty)\}$ are measurable.

The class of ω-sets H^*_t is the smallest σ-field with respect to which the ω-functions $\{x^*_t(\omega;\overset{\circ}{t}_o);t\,\varepsilon\,[0,\infty)\}$ are measurable.

The class of ω-sets $H^*_{[C]}$ is the smallest σ-field with respect to which the ω-functions $\{x^*_t(\omega;[C]);t\,\varepsilon\,[0,\infty)\}$ are measurable.

Let F^*_1 be a σ-field of ω-sets in Ω^* that satisfies

$$F^*_1 \subset F^*. \tag{1.195}$$

Let $\hat{y}(\omega)$ be a measurable (F^*) and integrable ω-function satisfying for some $K\,\varepsilon\,F^*_1$ and for each $\Lambda\,\varepsilon\,F^*_1$

$$P^*[K \cap \Lambda;x] = \int_\Lambda P^*[d\omega;x]\;\hat{y}(\omega). \tag{1.196}$$

Then the conditional probability of K relative to F^*, denoted by

$$P^*[K;x \mid F^*_1], \tag{1.197}$$

is defined as any ω-function $y(\omega)$ which is almost surely equal to $\hat{y}(\omega)$.

By the Radon-Nicodym theorem ([1],p.132) a family of such ω-functions exists of which

a) each one is measurable with respect to F^*_1;

b) each two are identical except for an ω-set of probability measure 0.

Note that the expression (1.197) is an ω-function which is measurable with respect to F_1^*.

The ω-function $P^*[K;x \mid F_1^*]$ is called a regular conditional probability measure, if

1) for each $\omega \varepsilon \Omega^*$ the set function $P^*[K;x|F_1^*]$ is a probability measure defined on F^*;
2) for each $K \varepsilon F^*$ the ω-function $P^*[K;x|F_1^*]$ is measurable with respect to F_1^*.

In this book the probability space $\{\Omega^*;F^*;P^*\}$ will be called strongly Markovian if and only if

1) for each $t_o \varepsilon [0,\infty)$, for each $K \varepsilon H_{t_o}^*$ and for each $x \varepsilon X^*$ we have
$$P^*[K;x|\hat{H}_{t_o}^*] = P^*[T_{t_o}(K);x_{t_o}^*(\omega)]; \tag{1.198}$$
2) for each $x \varepsilon X^*$, for each closed set C in X^* satisfying
$$P^*[\Xi_{[0,\infty);[C]};x] = 1, \tag{1.199}$$
for each $K \varepsilon H_{[C]}^*$ we have
$$P^*[K;x \mid \hat{H}_{[C]}^*] = P^*[T_{[C]}(K);x^*(\omega;[C])] . \tag{1.200}$$

If $\{\Omega^*;F^*;P^*\}$ is a strongly Markovian probability space, the basic stochastic process S_x^* is called a "stationary strong Markov process".

In [1] on p.577 and in [5] on p.91 condition (1.200) is replaced by a more stringent one.

The equations (1.198) and (1.200) are equivalent to
$$P^*[K_1 \cap \Lambda_1;x] = \int_{\Lambda_1} P^*[d\omega;x]\; P^*[T_{t_o}(K_1);x_{t_o}^*(\omega)] \tag{1.201}$$
and
$$P^*[K_2 \cap \Lambda_2;x] = \int_{\Lambda_2} P^*[d\omega;x]\; P^*[T_{[C]}(K_2);x^*(\omega;[C])], \tag{1.202}$$

where $\Lambda_1 \varepsilon \hat{H}^*_{t_o}$, $K_1 \varepsilon H^*_{t_o}$, $\Lambda_2 \varepsilon \hat{H}^*_{[C]}$ and $K_2 \varepsilon H^*_{[C]}$.

Let the class of ω-sets $\hat{F}^*_{t_o}$ be the smallest σ-field of ω-sets containing $\hat{H}^*_{t_o}$ and including all subsets of Λ^*_o.

Let the class of ω-sets $\hat{F}^*_{[C]}$ be the smallest σ-field of ω-wets containing $\hat{H}^*_{[C]}$ and including all subsets of Λ^*_o.

The following lemma can easily be proved:

Lemma 1.34

If the probability space is strongly Markovian, then for each $x \varepsilon X^*, t_o \varepsilon [0,\infty)$ and closed set C satisfying (1.199), we have

$$P^* [K_1;x|\hat{F}^*_{t_o}] = P^*[T_{t_o}(K_1);x^*_{t_o}(\omega)] \qquad (1.203)$$

and

$$P^* [K_2;x|\hat{F}^*_{[C]}] = P^*[T_{[C]}(K_2);x^*(\omega;[C])], \qquad (1.204)$$

where $K_1 \varepsilon F^*_{t_o}$ and $K_2 \varepsilon F^*_{[C]}$.

Let $y_{t_o}(\omega_1)$ and $y(\omega)$ be two ω-functions, satisfying

a) $0 \leq y_{t_o}(\omega_1) \leq 1$ (1.205)

b) $y(\omega) = y_{t_o}(T_{t_o}(\omega))$. (1.206)

Lemma 1.35

1) If $y(\omega)$ is measurable with respect to $F^*_{t_o}$, then $y_{t_o}(\omega_1)$ is measurable with respect to F^*.
2) If $y_{t_o}(\omega_1)$ is measurable with respect to F^*, then $y(\omega)$ is measurable with respect to $F^*_{t_o}$.
3) If $\Lambda \varepsilon \hat{F}_{t_o}$, if $y(\omega)$ is measurable with respect to $F^*_{t_o}$ and if $\{\Omega^*;F^*;P^*\}$ is strongly Markovian, then

$$\int_\Lambda P^* [d\omega;x] \; y(\omega) = \int_\Lambda P^* [d\omega;x] \int_{\Omega^*} P^* [d\omega_1;x^*_{t_o}(\omega)] \; y_{t_o}(\omega_1). \qquad (1.207)$$

Proof:

We first consider the cases 1) and 2). If M_r and M'_r are defined by

$$M_r \overset{def}{=} \{\omega | y(\omega) \leqq r\} \tag{1.208}$$

and

$$M'_r \overset{def}{=} \{\omega | y_{t_o}(\omega_1) \leqq r\} \tag{1.209}$$

respectively, then we can easily verify that

$$M'_r = T_{t_o}(M_r) \tag{1.210}$$

and

$$M_r = T_{t_o}^{-1}(M'_r). \tag{1.211}$$

The assertions are now a simple consequence of lemma 1.11.

We consider the third case.

Let the sets $M_{k;m}$ and $M'_{k;m}$ be defined by

$$M_{k;m} \overset{def}{=} \{\omega \mid \frac{k-1}{2^m} \leqq y(\omega) < \frac{k}{2^m}\} \tag{1.212}$$

and

$$M'_{k;m} \overset{def}{=} \{\omega_1 \mid \frac{k-1}{2^m} \leqq y_{t_o}(\omega_1) < \frac{k}{2^m}\} \tag{1.213}$$

respectively.

We can easily verify that

$$M'_{k;m} = T_{t_o}(M_{k;m}) \tag{1.214}$$

and

$$M_{k;m} = T_{t_o}^{-1}(M'_{k;m}). \tag{1.215}$$

Thus,

$$M'_{k;m} \varepsilon F^{*}. \tag{1.216}$$

Moreover, by lemma 1.34

$$\int_\Lambda P^{*}[d\omega;x]\, y(\omega) = \lim_{n\to\infty} \sum_{k=1}^{2^m} \frac{k-1}{2^m} P^{*}[\Lambda \cap M_{k;m};x] =$$

$$= \lim_{m\to\infty} \sum_{k=1}^{2^m} \frac{k-1}{2^m} \int_\Lambda P^{*}[d\omega;x]\, P^{*}[M_{k;m};x \mid \hat{F}^{*}_{t_o}] =$$

$$= \lim_{m\to\infty} \sum_{k=1}^{2^m} \frac{k-1}{2^m} \int_\Lambda P^* [d\omega;x]\, P^* [M'_{k;m};x^*_{t_o}(\omega)] =$$

$$= \int_\Lambda P^* [d\omega;x] \{ \lim_{m\to\infty} \sum_{k=1}^{2^m} \frac{k-1}{2^m} P^* [M'_{k;m};x^*_{t_o}(\omega)] \} =$$

$$= \int_\Lambda P^* [d\omega;x] \int_{\Omega^*} P^* [d\omega_1;x^*_{t_o}(\omega)]\, y_{t_o}(\omega_1). \tag{1.217}$$

This ends the proof.

Finally, let $y_{[C]}(\omega_1)$ and $y(\omega)$ be two ω-functions, satisfying

a) $0 \leqq y_{[C]}(\omega_1) \leqq 1$; (1.218)

b) $y(\omega) = y_{[C]}(T_{[C]}(\omega))$. (1.219)

Using similar arguments as in the proof of lemma 1.35, we can prove:

Lemma 1.36

1) If $y(\omega)$ is measurable with respect to $F^*_{[C]}$, then $y_{[C]}(\omega_1)$ is measurable with respect to F^*.

2) If $y_{[C]}(\omega)$ is measurable with respect to F^*, then $y(\omega)$ is measurable with respect to $F^*_{[C]}$.

3) If $\Lambda \varepsilon \hat{F}_{|C|}$, if $y(\omega)$ is measurable with respect to $F^*_{[C]}$ and if $\{\Omega^*;F^*;P^*\}$ is strongly Markovian, then

$$\int_\Lambda P^* [d\omega;x]\, y(\omega) = \int_\Lambda P^* [d\omega;x] \int_{\Omega^*} P^* [d\omega_1;x^*(\omega;[C])]\, y_{[C]}(\omega_1). \tag{1.220}$$

4. Stationary Markov processes and random losses

In this section our discussions are based on the following assumption:

Assumption 3

For each $x \varepsilon X^*$ we have

$$P^* [\Lambda_{0;x};x] = 1 . \tag{1.221}$$

If ω represents a realization of the basic stochastic process, let $k_T(\omega)$ denote the costs incurred during the period $[0,T)$.

We shall show that, under conditions to be mentioned below, the limit

$$\lim_{T \to \infty} \frac{k_T(\omega)}{T} \tag{1.222}$$

almost surely exists.

Let us define the random variables $\{\underline{x}^*_{t_o;x;j};\ j=1,2,\dots\}$ by

$$\underline{x}^*_{t_o;x;j} = \underline{x}^*_{jt_o;x} . \tag{1.223}$$

If the functions $\{p^j_{t_o}(B;x);\ j=1,2,\dots\}$ are defined by

$$p^j_{t_o}(B;x) \stackrel{def}{=} P^* \left[\Lambda_{jt_o;B};x\right] , \tag{1.224}$$

then

$$\mathrm{Prob}\ \{\underline{x}^*_{t_o;x;j} \in B\} = p^j_{t_o}(B;x). \tag{1.225}$$

Let us assume that for each $j \geq 1$, $x \in X^*$, $\Lambda \in \hat{F}^*_{t_o}$ and $K \in F^*_{t_o}$ the Markov property

$$P^* \left[\Lambda \cap K;x\right] = \int_\Lambda P^* \left[d\omega;x\right] \ P^* \left[T_{t_o}(K);x^*_1(\omega;t_o)\right] \tag{1.226}$$

is true.

This property implies for each j, x and $B \in G^*$

$$p^j_{t_o}(B;x) = \int_{X^*} p^1_{t_o}(dx_1;x)\, p^{j-1}_{t_o}(B;x_1);\ j=1,2,\dots . \tag{1.227}$$

Since the functions $p^j_{t_o}(B;x)$ are

a) for each $B \in G^{*o}$ and for each $j \geq 1$ measurable with respect to G^*,

b) for each $x \in X^*$ and for each $j \geq 1$ a probability measure defined on G^*,

the relations (1.227) imply that the sequence of states $\{\underline{x}^*_{t_o;x;j};\ j=1,2,\dots\}$ constitutes a stationary Markov process with a discrete time parameter (cf. [2] p.190 ff.). So we have proved:

Lemma 1.37

Under (1.226) the sequence of states $\{\underline{x}^*_{t_o;x;j};\ j=1,2,\ldots\}$ constitutes a stationary Markov process with a discrete time parameter.

Let us make the following assumption:
There is a finite valued measure Q(C) of sets $C \in G^*$ with $Q(X^*)$ O, an integer $k \geqq 1$ and a positive η, such that for each $x \in X^*$ (cf. [2] , p. 192)

$$p^k_{t_o}(C;x) \leqq 1-\eta \text{ if } Q(C) \leqq \eta. \tag{1.228}$$

This assumption is called the "Doeblin condition". The following lemma can be proved (cf. [2] , p.214):

Lemma 1.38

Under (1.226) and (1.228) the function $p_{t_o}(C;x)$, given by

$$p_{t_o}(C;x) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} p^j_{t_o}(C;x), \tag{1.229}$$

defines for each $x \in X^*$ a stationary absolute probability distribution.

For the meaning of the concepts: "ergodic sets", "cyclically moving subsets" etc., the reader is referred to [2] . In this book, however, we prefer the name "simple ergodic set" to the term "ergodic set". The latter can be mixed up with the set of all ergodic states.

In [2] on p.207 ff. the following lemma is proved:

Lemma 1.39

If the initial state x belongs to a simple ergodic set E_i and if the number of cyclically moving subsets of E_i is c_i, then, under (1.226) and (1.228), the sequences $\{p^{nc_i+j}_{t_o}(C;x);\ n=1,2,\ldots\}$ $(j=1,2,\ldots c_i)$ converge for $n\to\infty$ exponentially fast and uniformly in C and x to a limit, denoted by $p^{\infty c_i+j}_{t_o}(C;x)$.

We now introduce an (M+1)-dimensional Cartesian space, which is the product space of X^* and the line $\Gamma = (-\infty,+\infty)$.

Let us consider the ω-functions $y_j(\omega;t_o)$, given by

$$y_j(\omega;t_o) \overset{\text{def}}{=} (x_j(\omega;t_o),\ n_j(\omega;t_o;[A]));j=1,2,\dots . \tag{1.230}$$

As we know the ω-function $n_j(\omega;t_o;\ [A]\ )$ represents the number of entries in A during the period $[(j-1)t_o, jt_o)$.

The ω-functions $y_j(\omega;t_o)$ map Ω^* into the product space $X^* \times \Gamma$.

<u>Lemma 1.40</u>

The ω-functions $\{y_j(\omega;t_o);\ j=1,2,\dots\}$ are measurable with respect to F^*.

<u>Proof</u>:

Let L be a linear Borel field of sets in Γ . If $U_1 \varepsilon G^*$ and if $U_2 \varepsilon L$ we have for each j

$$\{\omega \mid y_j(\omega;t_o) \varepsilon U_1 \times U_2\} = \tag{1.231}$$
$$= \{\omega | x_j(\omega;t_o) \varepsilon U_1\} \cap \{\omega | n_j(\omega;t_o;[A]) \varepsilon U_2\} .$$

Let J_k be the class of (M+1)-dimensional Borel sets U, satisfying

$$\{\omega \mid y_k(\omega;t_o) \varepsilon U\} \varepsilon F^*. \tag{1.232}$$

So, J_k contains all (M+1)-dimensional intervals. In addition, we have

a) if $U \varepsilon J_k$, then $\bar{U} \varepsilon J_k$;

b) if $U^i \varepsilon J_k$ (i=1,2,...), then $\bigcup_{i=1}^{\infty} U^i \varepsilon J_k$.

Consequently, J_k is a σ-field that includes the (M+1)-dimensional intervals.

Hence, J_k is the class of (M+1)-dimensional Borel sets.

This ends the proof.

If U is an (M+1)-dimensional Borel set, let the ω-set $O_{k;U}$ be defined by

$$O_{k;U} \overset{\text{def}}{=} \{\omega \mid y_k(\omega;t_o)\ \varepsilon\ U\} . \tag{1.233}$$

Obviously,

$$O_{k;U} \in F_{(k-1)t_o}. \tag{1.234}$$

We now define the set functions $\{'p^k_{t_o}(U;x);\ k=1,2,\dots\}$ by

$$'p^1_{t_o}(U;x) \overset{def}{=} P^*[O_1;U;x]. \tag{1.235}$$

$$'p^k_{t_o}(U;x) \overset{def}{=} \int_{X^*} p^{k-1}_{t_o}(dx_1;x)\ 'p^1_{t_o}(U;x_1);\ k=2,3,\dots \tag{1.236}$$

By means of (1.226) we can easily verify that

$$\begin{aligned} P^*[O_{k;U};x] &= \int_{\Omega^*} P^*[d\omega;x]\ P^*[O_{1;U};x^*_{k-1}(\omega;t_o)] = \\ &= \int_{X^*} p^{k-1}_{t_o}(dx_1;x)\ P^*[O_{1;U};x^*_{k-1}(\omega;t_o)] = \\ &= \int_{X^*} p^{k-1}_{t_o}(dx_1;x)\ 'p^1_{t_o}(U;x_1) = 'p^k_{t_o}(U;x). \end{aligned} \tag{1.237}$$

If $a \in \Gamma$ and if $y = (x,a)$, let $''p^k_{t_o}(U;y)$ be defined by

$$''p^k_{t_o}(U;y) \overset{def}{=} 'p^k_{t_o}(U;x). \tag{1.238}$$

We can easily verify, that for $U_1 \in G^*$ we have

$$''p^k_{t_o}(U_1 \times \Gamma;y) = 'p^k_{t_o}(U_1 \times \Gamma;x) = p^k_{t_o}(U_1;x). \tag{1.239}$$

Consequently, (1.237) can be rewritten as follows:

$$''p^k_{t_o}(U;y) = \int_{X^*} {''p}^{k-1}_{t_o}(dy_1;y)\ ''p^1(U;y_1). \tag{1.240}$$

It can easily be proved that

a) for a given (M+1)-dimensional Borel set U the y-functions $''p^k_{t_o}(U;y)$ are measurable with respect to the class of all (M+1)-dimensional Borel sets;

b) for a given y the set function $''p^k_{t_o}(U;y)$ is a probability measure defined on the class of all (M+1)-dimensional Borel sets.

We now consider the stochastic variables $\{\underline{y}_{t_o;x;k};\ k=1,2,\dots\}$, generated by the ω-functions $\{y_k(\omega;t_o);\ k=1,2,\dots\}$ and the probability spaces $\{\Omega^*;F^*;P^*\}$.

Obviously, for each $k \geq 1$ and $x \in X^*$

$$\text{Prob}\left[\underline{y}_{t_o;x;k} \in U\right] = P^*\left[O_{k;U};x\right] = {}'p^k_{t_o}(U;x). \qquad (1.241)$$

The relations (1.241), (1.238) and (1.240) imply the following lemma:

Lemma 1.41

Under (1.226) the sequence of stochastic variables $\{\underline{y}_{t_o;x;k};\ k=1,2,\dots\}$ constitutes a stationary Markov process with a discrete time parameter.

Let us return to the Markov process $\{\underline{x}^*_{t_o;x;k};\ k=1,2,\dots\}$. If x belongs to a simple ergodic set E_i and if c_i is the number of cyclically moving sets of E_i, then, according to lemma 1.39, the limits for $n \to \infty$ of the sequences $\{p^{nc_i+j}_{t_o}(U;x);\ n=1,2,\dots\}$ converge to $p^{\infty c_i+j}_{t_o}(U;x)$ exponentially fast and uniformly in $U \in G^*$ and $x \in E_i$.
It follows from (1.239) and (1.240) that

$$\lim_{n\to\infty} {}''p^{nc_i+j}_{t_o}(U;y) =$$

$$= \lim_{n\to\infty} \int_{X^*} p^{nc_i+j-1}_{t_o}(dx_1;x)\ {}'p^1_{t_o}(U;x_1) =$$

$$= \int_{X^*} p^{\infty c_i+j-1}_{t_o}(dx_1;x)\ {}'p^1_{t_o}(U;x_1). \qquad (1.242)$$

Consequently, the limit exists.
Now let $''p^{\infty c_i+j}_{t_o}(U,y)$ be defined by

$$''p^{\infty c_i+j}_{t_o}(U;y) \overset{\text{def}}{=} \lim_{n\to\infty} {}''p^{nc_i+j}_{t_o}(U;y). \qquad (1.243)$$

By means of (1.240) we can easily verify that

$$''p_{t_o}^{\infty c_i+j}(U;y) = \int_{X^*} ''p_{t_o}^{\infty c_i+j-1}(dy_1;y)\ ''p_{t_o}^{1}(U;y_1). \qquad (1.244)$$

<u>Lemma 1.42</u>

Under (1.226) and (1.228) the sequences $\{''p_{t_o}^{nc_i+j}(U;y);\ n=1,2,\dots\}$ converge uniformly in $y \in E_i \times \Gamma$ and $U \in G^*$ $(j=1,2,\dots,c_i)$.

<u>Proof</u>:

Let the x-set B_{hr}^U be given by

$$B_{hr}^U \overset{def}{=} \{x \mid \frac{r-1}{2^h} < 'p_{t_o}^1(U;x) \leq \frac{r}{2^h}\}. \qquad (1.245)$$

It follows from (1.242), (1.240) and (1.245) that

$$|''p_{t_o}^{\infty c_i+j}(U;y) - \sum_{r=1}^{2^h} \frac{r}{2^h} p_{t_o}^{\infty c_i+j-1}(B_{hr}^U;x)| \leq 2^{-h} \qquad (1.246)$$

and

$$|''p_{t_o}^{nc_i+j}(U;y) - \sum_{r=1}^{2^h} \frac{r}{2^h} p_{t_o}^{nc_i+j-1}(B_{hr}^U;x)| \leq 2^{-h}. \qquad (1.247)$$

Consequently,

$$|''p_{t_o}^{\infty c_i+j}(U;y) - ''p_{t_o}^{nc_i+j}(U;y)| \leq$$

$$\leq \sum_{r=1}^{2^h} \frac{r}{2^h} |p_{t_o}^{\infty c_i+j-1}(B_{hr}^U;y) - p_{t_o}^{nc_i+j-1}(B_{hr}^U;y)| + 2^{-h+1}. \qquad (1.248)$$

For each $\eta > 0$ we can find an integer h_o such that for $h \geq h_o$ we have $2^{-h+1} < \frac{\eta}{2}$. Since the sequences $\{p_{t_o}^{nc_i+j-1}(B_{hr}^U;x);\ n=1,2,\dots\}$ converge uniformly in B_{hr}^U and x, an integer N_{ij} can be found such that for $n \geq N_{ij}$

$$\sum_{r=1}^{2^h} \frac{r}{2^h} |p_{t_o}^{\infty c_i+j-1}(B_{hr}^U;x) - p_{t_o}^{nc_i+j-1}(B_{hr}^U;x)| \leq \frac{\eta}{2}. \qquad (1.249)$$

Thus, for each $\eta > 0$ an integer N_{ij} can be found such that uniformly in U and x we have for $n \geqq N_{ij}$

$$|{}''p_{t_o}^{\infty c_i + j}(U;x) - {}''p_{t_o}^{nc_i + j}(U;x)| \leqq \eta. \qquad (1.250)$$

This ends the proof.

Lemma 1.43

If (1.226) and (1.228) hold, there is a finite valued measure $Q^*(U)$ of (M+1)-dimensional Borel sets U with $Q^*(X^* \times \Gamma) > 0$, satisfying for each $\eta > 0$, each simple ergodic set E_i, each $y \in E_i \times \Gamma$, each j and some $k_{ij} < \infty$

$$''p_{t_o}^{k_{ij} c_i + j}(U;y) \leqq Q^*(U) + \eta . \qquad (1.251)$$

Proof :

Suppose that the stochastic process $\{\underline{x}^*_{t_o;x;k};\ k=1,2,\ldots\}$ has m simple ergodic sets E_i. For each pair (i,j) the set functions $\{''p_{t_o}^{\infty c_i + j}(U;y);\ y \in E_i \times \Gamma\}$ are identical.

We now define $Q^*(U)$ by

$$Q^*(U) \overset{\text{def}}{=} \sum_{i=1}^{m} \sum_{j=1}^{c_i} {}''p_{t_o}^{\infty c_i + j}(U;y_i), \qquad (1.252)$$

where y_i is some point of $E_i \times \Gamma$.
The assertion is an immediate consequence of lemma 1.42.

This ends the proof.

The following lemma can be proved (cf. [2] , p.207):

Lemma 1.44

If the stochastic process $\{\underline{x}^*_{t_o;x;k};\ k=1,2,\ldots\}$ has m simple ergodic sets E_i and if (1.226) and (1.228) hold, then for some $\rho < 1$

$$1 - p_{t_o}^{n}(\bigcup_{i=1}^{m} E_i;x) \leqq \text{const}\cdot \rho^n;\ n=1,2,\ldots \qquad (1.253)$$

Now we shall prove that the stochastic process $\{\underline{y}_{t_o;x;k};\ k=1,2,\dots\}$ satisfies the Doeblin condition.

Lemma 1.45

If (1.226) and (1.228) hold, there is a finite valued measure $Q^*(U)$ of (M+1)-dimensional Borel sets U, with $Q^*(X^* \times \Gamma) > 0$, an integer $k \geqq 1$ and an $\eta' > 0$ such that for each $y \in X^* \times \Gamma$

$$''p_{t_o}^{k}(U;y) \leqq 1 - \eta', \text{ if } Q^*(U) \leqq \eta'. \qquad (1.254)$$

Proof:

Let the stochastic process $\{\underline{x}^*_{t_o;x;k};\ k=1,2,\dots\}$ have m simple ergodic sets E_i and let $Q^*(U)$ be given by

$$Q^*(U) = \sum_{i=1}^{m} \sum_{j=1}^{c_i} {''p_{t_o}^{\infty c_i + j}}(U;y_i), \qquad (1.255)$$

where y_i is some point of $E_i \times \Gamma$.

Let k_1 and k_2 be two integers, such that for some positive $\eta < \frac{1}{2}$:

a) for each x

$$1 - p_{t_o}^{k_1}\left(\bigcup_{i=1}^{m} E_i;x\right) \leqq \eta; \qquad (1.256)$$

b) $k_2 = \max\limits_{i,j} k_{ij}(c_i+1)$ (cf. lemma 1.43). (1.257)

Obviously, by (1.251), (1.256) and (1.257) we have for each Borel set U and y

$$''p_{t_o}^{k_1+k_2}(U;y) \leqq \int_{\bigcup_{i=1}^{m} E_i} p_{t_o}^{k_1}(dx_1;x)\ {'p_{t_o}^{k_2}}(U;x_1) + \eta \leqq Q^*(U) + 2\eta . \qquad (1.258)$$

For sets U, with $Q^*(U) \leqq \frac{1}{2} - \eta$, we find by means of (1.258)

$$''p_{t_o}^{k_1+k_2}(U;y) \leqq \tfrac{1}{2} + \eta = 1 - (\tfrac{1}{2} - \eta). \qquad (1.259)$$

Consequently, the triple (Q^*, k, η'), given by

$$Q^*(U) = \sum_{i=1}^{m} \sum_{j=1}^{c_i} {}''p_{t_o}^{\infty c_i+j}(U;y), \tag{1.260}$$

$$k = k_1 + k_2 \tag{1.261}$$

$$\text{and} \quad \eta' = \tfrac{1}{2} - \eta, \tag{1.262}$$

satisfies.

This ends the proof.

We now introduce for each x the assumption:

$$\sum_{n=1}^{\infty} \int_{X^*} p_{t_o}(dx_1;x) \; P^*\left[\Xi_{t_o};[A];n;x_1\right]<\infty \tag{1.263}$$

If x is the initial state of the stochastic process $\{\underline{x}_{t_o;x;k};k=1,2,\ldots\}$, the expected number of entries $\bar{n}_{t_o;x}$ in A during a period of length t_o in the steady state is given by

$$\begin{aligned}
\bar{n}_{t_o;x} &= \sum_{j=1}^{\infty} \int_{X^*} p_{t_o}(dx_1;x)j \; P^*\left[N_{j;t_o};x_1\right] = \\
&= \sum_{j=1}^{\infty} \int_{X^*} p_{t_o}(dx_1;x)j\{P^*\left[\Xi_{t_o};[A];j;x_1\right] + \\
&\qquad -P^*\left[\Xi_{t_o};[A];j+1;x_1\right]\} = \\
&= \sum_{j=1}^{\infty} \int_{X^*} p_{t_o}(dx_1;x) \; P^*\left[\Xi_{t_o};[A];j;x_1\right]<\infty .
\end{aligned} \tag{1.264}$$

Let a function f(y) be real valued and measurable with respect to the class of all (M+1)-dimensional Borel sets and let the ω-set F_I be defined by

$$F_I \overset{\text{def}}{=} \{\omega | f(y(\omega;t_o)) \in I\} . \tag{1.265}$$

The following lemma can be proved (cf. [2], p.220):

<u>Lemma 1.46</u>

If (1.226) and (1.228) hold and if for each initial state x

$$\int_{X^*} p_{t_o}(dx_1;x) \int_{-\infty}^{+\infty} |f| \; P^*[F_{df};x_1] < \infty, \qquad (1.266)$$

then for almost all ω the limit

$$\lim_{r\to\infty} \frac{1}{r} \sum_{j=1}^{r} f(y_j(\omega;t_o)) \qquad (1.267)$$

exists and is equal to

$$\int_{X^*} p_{t_o}(dx_1;x) \int_{-\infty}^{+\infty} fP^*[F_{df};x_1] \qquad (1.268)$$

if x belongs to a simple ergodic set of the stochastic process $\{\underline{x}^*_{t_o;x;j};\ j=1,2,\ldots\}$.

It follows from (1.230) that $n_j(\omega;t_o;[A])$ is a measurable function of $y_j(\omega;t_o)$. By lemma 1.46 and (1.264) we find:

<u>Lemma 1.47</u>

Under (1.226), (1.228) and (1.263), for almost all ω the limit

$$\lim_{r\to\infty} \frac{1}{r} \sum_{j=1}^{r} n_j(\omega;t_o;[A]) \qquad (1.269)$$

exists and is equal to

$$\sum_{j=1}^{n} \int_{X^*} p_{t_o}(dx_1;x) P^*[\Xi_{t_o;[A];j};x_1] \qquad (1.270)$$

if x belongs to a simple ergodic set of the stochastic process $\{\underline{x}^*_{t_o;x;j};\ j=1,2,\ldots\}$.

We now consider the sequence of ω-functions $\{k_j(\omega;t_o);j=1,2,\ldots\}$. If (1.185) holds, the ω-function $k_j(\omega;t_o)$ represents the losses incurred in the period $[(j-1)t_o, jt_o)$.

Obviously, we have

$$\int_{X^*} p_{t_o}(dx_1;x) \int_{-\infty}^{+\infty} |k| \; P^*[K_{dk;t_o};x_1] \leqq t_o\gamma_c + \bar{n}_{t_o;x}\,\gamma_d < \infty. \qquad (1.271)$$

Using similar arguments as above, we can prove the following lemma:

Lemma 1.48

Under (1.226), (1.228) and (1.263), for almost all ω the limit

$$\lim_{r \to \infty} \frac{1}{r} \sum_{j=1}^{r} k_j(\omega;t_o) \tag{1.272}$$

exists and is equal to

$$\int_{X^*} p_{t_o}(dx_1;x) \int_{-\infty}^{+\infty} kP^* \left[K_{dk;t_o};x_1 \right] \tag{1.273}$$

if x belongs to a simple ergodic set of the stochastic process $\{\underline{x}^*_{t_o};x;j;\ j=1,2,\ldots\}$.

As we know the (ω,T)-function $k_T(\omega)$ represents the losses incurred in the period $[0,T)$. We now prove:

Lemma 1.49

Under (1.226), (1.228) and (1.263), for almost all ω the limit

$$\lim_{T \to \infty} \frac{k_T(\omega)}{T} \tag{1.274}$$

exists and is equal to

$$\frac{1}{t_o} \int_{X^*} p_{t_o}(dx_1;x) \int_{-\infty}^{+\infty} kP^* \left[K_{dk;t_o};x_1 \right] \tag{1.275}$$

if x belongs to a simple ergodic set of the stochastic process $\{\underline{x}^*_{t_o};x;j;\ j=1,2,\ldots\}$.

Proof:

It follows from lemma 1.47 that almost surely

$$\lim_{r \to \infty} \frac{n_r(\omega;t_o;[A])}{r} = 0. \tag{1.276}$$

Let the numbers $n(T;t_o)$ be given by

$$n(T;t_o) = \left[\frac{T}{t_o}\right]^- . \tag{1.277}$$

Obviously, we have almost surely

$$\frac{\sum_{j=1}^{n(T;t_o)} k_j(\omega;t_o) - t_o\cdot\gamma_c - n_{n(T;t_o)+1}(\omega;t_o;[A])\gamma_d}{(n(T;t_o)+1)\cdot t_o} \leq$$

$$\leq \frac{k_T(\omega)}{T} \leq \frac{\sum_{j=1}^{n(T;t_o)} k_j(\omega;t_o) + t_o\cdot\gamma_c + n_{n(T;t_o)+1}(\omega;t_o;[A])\cdot\gamma_d}{n(T;t_o)\cdot t_o} \tag{1.278}$$

and thus

$$\frac{\frac{1}{n(T;t_o)}\sum_{j=1}^{n(T;t_o)} k_j(\omega;t_o) - \frac{t_o\cdot\gamma_c}{n(T;t_o)} - \frac{n_{n(T;t_o)+1}(\omega;t_o;[A])\cdot\gamma_d}{n(T;t_o)}}{t_o\cdot\left(1+\frac{1}{n(T;t_o)}\right)} \leq$$

$$\leq \frac{k_T(\omega)}{T} \leq$$

$$\leq \frac{\frac{1}{n(T;t_o)}\sum_{j=1}^{n(T;t_o)} k_j(\omega;t_o) + \frac{t_o\cdot\gamma_c}{n(T;t_o)} + \frac{n_{n(T;t_o)+1}(\omega;t_o;[A])\cdot\gamma_d}{n(T;t_o)}}{t_o} \tag{1.279}$$

If $T \to \infty$, then the assertion is an immediate consequence of lemma 1.48 and the relations (1.276) and (1.279).

This ends the proof.

We shall now show that under certain conditions the limit

$$\lim_{T\to\infty} \frac{k_T(\omega)}{T} \tag{1.274}$$

can also be expressed in a different form.

To this end we consider the random variables $\{\underline{x}^*_{[C];x;j};\ j=1,2,\ldots\}$ again.

If the functions $\{p^{j}_{[C]}(B;x);\ j=1,2,\ldots\}$ are defined by

$$p^{j}_{[C]}(B;x) \stackrel{def}{=} P^{*}\left[T^{-j+1}_{[C]}(\Delta_{B;[C]});x\right] \tag{1.280}$$

and if for $j=1,2,\ldots$

$$P^{*}\left[T^{-j+1}_{[C]}(\Xi_{[0,\infty);[C]});x\right] = 1, \tag{1.281}$$

then

$$\text{Prob}\ \{\underline{x}^{*}_{[C];x;j} \in B\} = p^{j}_{[C]}(B;x). \tag{1.282}$$

Let us assume that for each $j \geq 1$, $x \in X^{*}$, $\Lambda \in \hat{F}^{*}_{[C]}$ and $K \in F^{*}_{[C]}$ the Markov property

$$P^{*}\left[\Lambda \cap K;x\right] = \int_{\Lambda} P^{*}\left[d\omega;x\right]\ P^{*}\left[T_{[C]}(K)\ ;x^{*}_{1}(\omega;t_{o})\right] \tag{1.283}$$

is true.

Since

$$T_{[C]}\ \left[T^{-j+1}_{[C]}(\Delta_{B;[C]})\right] = T^{-j+2}_{[C]}(\Delta_{B;[C]}), \tag{1.284}$$

it follows from (1.280) and (1.283) that for $j=2,3,\ldots$

$$p^{j}_{[C]}(B;x) = \int_{X^{*}} p^{1}_{[C]}(dx_{1};x)\,p^{j-1}_{[C]}(B;x_{1}). \tag{1.285}$$

Since the function $p^{j}_{[C]}(B;x)$ is

a) for each $B \in G^{*}$ and for each $j \geq 1$ measurable with respect to G^{*}

b) for each $x \in X^{*}$ and for each $j \geq 1$ a probability measure defined on G^{*},

the equations (1.285) imply that the sequence of states $\{\underline{x}^{*}_{[C];x;j};\ j=1,2,\ldots\}$ constitutes a stationary Markov process with a discrete time parameter.

<u>Lemma 1.50</u>

Under (1.283), the sequence of states $\{\underline{x}^{*}_{[C];x;j};\ j=1,2,\ldots\}$ constitutes a stationary Markov process with a discrete time parameter.

Let us make the following assumption:
There is a finite valued measure $Q(B)$ of sets $B \in G^{*}$ with $Q(X^{*}) > 0$, an integer $k \geq 1$ and a positive η, such that for each $x \in X^{*}$ (cf. [2], p. 192)

$$p^{k}_{[C]}(B;x) \leq 1-\eta \ , \ \text{if } Q(B) \leq \eta. \qquad (1.286)$$

The following lemma can be proved (cf. [2] , p.214):

Lemma 1.51

Under (1.283) and (1.286) the function $p_{[C]}(B;x)$, given by

$$p_{[C]}(B;x) \stackrel{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} p^{j}_{[C]}(B;x), \qquad (1.287)$$

defines for each $x \in X^{*}$ a stationary absolute probability distribution.

We now introduce an (M+1)-dimensional Cartesian space, which is the product space of X^{*} and the line $\Gamma = [0,\infty)$.

Let us consider the ω-functions $y_j(\omega;[C])$, given by

$$y_j(\omega;[C]) \stackrel{\text{def}}{=} (x^{*}_j(\omega;[C]),\ t_j(\omega;[C]))\ ;\ j=1,2,\ldots\ . \qquad (1.288)$$

If for each x

$$P^{*}\,[T^{-j+1}_{[C]}(\Xi_{[0,\infty)\,;\,[C]});x] = 1, \qquad (1.289)$$

the ω-function $t_j(\omega;[C])$ represents the length of the period between the $(j-1)^{st}$ and the j^{th} entry in C.

The ω-functions $y_k(\omega;[C])$ $(k=1,2,\ldots)$ together with the probability spaces $\{\Omega^{*};F^{*};P^{*}\}$ generate the stochastic variables $\{\underline{y}_{[C];x;k};\ k=1,2,\ldots\}$. Using similar arguments as above, we can prove the following lemma:

Lemma 1.52

Under (1.283), (1.286) and (1.289) the stochastic process $\{\underline{y}_{[C];x;k};\ k=1,2,\ldots\}$

a) is a stationary Markov process with a discrete time parameter;
b) satisfies the Doeblin condition.

Let us assume that for each $x \varepsilon X^*$

$$0 < \int_{X^*} p_{[C]}(dx_1;x) \int_0^\infty tP^* [\Xi_{dt;[C]};x_1] < \infty. \qquad (1.290)$$

The following lemma can be proved (cf. lemma 1.48):

Lemma 1.53

Under (1.283), (1.286) and (1.290) for almost all ω the limit

$$\lim_{r\to\infty} \frac{1}{r} \sum_{j=1}^{r} t_j(\omega;[C]) \qquad (1.291)$$

exists and is equal to

$$\int_{X^*} p_{[C]}(dx_1;x) \int_0^\infty tP^* [\Xi_{dt;[C]};x_1] \qquad (1.292)$$

if x_1 belongs to a simple ergodic set.

Lemma 1.54

We now consider the ω-functions $\{n_j(\omega;[C];[A]);\ j=1,2,\dots\}$. The ω-function $n_j(\omega;[C];[A])$ represents the number of entries in A during the period $[\hat{t}_{j-1}(\omega;[C]),\ \hat{t}_j(\omega;[C]))$.

Next we assume that

$$\sum_{j=1}^{\infty} \int_{X^*} p_{[C]}(dx_1;x)\ P^* [\Xi_{[C];[A];j};x_1] < \infty. \qquad (1.293)$$

If x is the initial state of the stochastic process $\{\underline{x}_{[C];x;j};\ j=1,2,\dots\}$, the expected number of entries $\bar{n}_{[C];x}$ in A between two successive entries in C in the steady state is then given by

$$\bar{n}_{[C];x} = \sum_{j=1}^{\infty} \int_{X^*} p_{[C]}(dx_1;x) j\ P^* [N_{j;[C]};x_1] =$$

$$= \sum_{j=1}^{\infty} \int_{X^*} p_{[C]}(dx_1;x) j\ \{P^* [\Xi_{[C];[A];j};x_1] +$$

$$- P^{*}\left[\Xi_{[C];[A];j+1};x_1\right]\} =$$

$$= \sum_{j=1}^{\infty} \int_{X^{*}} p_{[C]}(dx_1;x)\; P^{*}\left[\Xi_{[C];[A];j};x_1\right] < \infty. \tag{1.294}$$

We can prove the following lemma (cf. lemma 1.47):

Lemma 1.55

Under (1.283), (1.286) and (1.293) for almost all ω the limit

$$\lim_{r\to\infty} \frac{1}{r} \sum_{j=1}^{r} n_j(\omega;[C];[A]) \tag{1.295}$$

exists and is equal to

$$\sum_{j=1}^{n} \int_{X^{*}} p_{[C]}(dx_1;x)\; P^{*}\left[\Xi_{[C];[A];j};x_1\right] \tag{1.296}$$

if x belongs to a simple ergodic set of the stochastic process $\{\underline{x}^{*}_{[C];x;j};\ j=1,2,\dots\}$.

We now consider the sequence of ω-functions $\{k_j(\omega;[C]);\ j=1,2,\dots\}$. If (1.290) and (1.293) hold, the ω-function $k_j(\omega;[C])$ represents the losses incurred in the period $[\hat{t}_{j-1}(\omega;[C]),\ \hat{t}_j(\omega;[C]))$.

Obviously, we have

$$\int_{X^{*}} p_{[C]}(dx_1;x) \int_{-\infty}^{+\infty} |k|\; P^{*}\left[K_{dk;[C]};x_1\right] \leq$$

$$\leq \gamma_c \int_{X^{*}} p_{[C]}(dx_1;x) \int_{0}^{\infty} t P^{*}\left[\Xi_{dt;[C]};x_1\right] +$$

$$+ \bar{n}_{[C];x}\, \gamma_d < \infty. \tag{1.297}$$

The following lemma can be proved (cf. lemma 1.48):

Lemma 1.56

Under (1.283), (1.286), (1.290) and (1.293) for almost all ω the limit

$$\lim_{r\to\infty} \frac{1}{r} \sum_{j=1}^{r} k_j(\omega;[C]) \tag{1.298}$$

exists and is equal to

$$\int_{X^*} p_{[C]}(dx_1;x)\int_{-\infty}^{+\infty} kP^*[K_{dk;[C]};x_1] \tag{1.299}$$

if x belongs to a simple ergodic set of the stochastic process $\{\underline{x}^*_{[C];x;j};\ j=1,2,\ldots\}$.

Finally, we prove:

<u>Lemma 1.57</u>

Under (1.283), (1.286), (1.290) and (1.293), for almost all ω the limit

$$\lim_{T\to\infty} \frac{k_T(\omega)}{T} \tag{1.300}$$

exists and is equal to

$$\frac{\int_{X^*} p_{[C]}(dx_1;x)\int_{-\infty}^{+\infty} kP^*[K_{dk;[C]};x_1]}{\int_{X^*} p_{[C]}(dx_1;x)\int_{-\infty}^{+\infty} tP^*[\Xi_{dt;[C]};x_1]} \tag{1.301}$$

if x belongs to a simple ergodic set of the stochastic process $\{\underline{x}^*_{[C];x;j};\ j=1,2,\ldots\}$.

<u>Proof</u> :

It follows from lemmas 1.53 and 1.55 that almost surely

$$\lim_{r\to\infty} \frac{t_r(\omega;[C])}{r} = 0 \tag{1.302}$$

and

$$\lim_{r\to\infty} \frac{n_r(\omega;[C];[A])}{r} = 0. \tag{1.303}$$

Let $n(T;\omega)$ be the number of entries in C during the period $[0,T)$. Obviously, we have almost surely

$$\frac{\sum_{j=1}^{n(T;\omega)} k_j(\omega;[C]) - t_{n(T;\omega)+1}(\omega;[C])\cdot\gamma_c - n_{n(T;\omega)+1}(\omega;[C];[A])\cdot\gamma_d}{\sum_{j=1}^{n(T;\omega)+1} t_j(\omega;[C])} \leqq$$

$$\leqq \frac{k_T(\omega)}{T} \leqq$$

$$\leqq \frac{\sum_{j=1}^{n(T;\omega)} k_j(\omega;[C]) + t_{n(T;\omega)+1}(\omega;[C])\cdot\gamma_c + n_{n(T;\omega)+1}(\omega;[C];[A])\cdot\gamma_d}{\sum_{j=1}^{n(T;\omega)} t_j(\omega;[C])} \tag{1.304}$$

and thus

$$\frac{\frac{1}{n(T;\omega)}\sum_{j=1}^{n(T;\omega)} k_j(\omega;[C]) - \frac{t_{n(T;\omega)+1}(\omega;[C])\cdot\gamma_c}{n(T;\omega)} - \frac{n_{n(T;\omega)+1}(\omega;[C];[A])\cdot\gamma_d}{n(T;\omega)}}{\frac{1}{n(T;\omega)}\sum_{j=1}^{n(T;\omega)} t_j(\omega;[C]) + \frac{t_{n(T;\omega)+1}(\omega;[C])}{n(T;\omega)}} \leqq$$

$$\leqq \frac{k_T(\omega)}{T} \leqq$$

$$\leqq \frac{\frac{1}{n(T;\omega)}\sum_{j=1}^{n(T;\omega)} k_j(\omega;[C]) + \frac{t_{n(T;\omega)+1}(\omega;[C])\cdot\gamma_c}{n(T;\omega)} + \frac{n_{n(T;\omega)+1}(\omega;[C];[A])\cdot\gamma_d}{n(T;\omega)}}{\frac{1}{n(T;\omega)}\sum_{j=1}^{n(T;\omega)} t_j(\omega;[C])} \tag{1.305}$$

If $T \to \infty$, then the assertion is an immediate consequence of the lemmas 1.56 and 1.53 and the relations (1.302) and (1.303).

This ends the proof.

CHAPTER 2

The decision process

1. The basic probability space

In this section we consider a family of stochastic processes $\{S_x^o;\ x \in X\}$. For the definition of these stochastic processes the reader is referred to chapter 1 ($*$ =0, M=N).[1]

Let Λ_o be an ω^o-set with the following properties:

1) For each $\omega^o \in \bar{\Lambda}_o$, the t-function $x_t^o(\omega^o)$ is continuous from the right;
2) In each bounded time interval in $[0,\infty)$ and for each $\omega^o \in \bar{\Lambda}_o$, the t-function $x_t^o(\omega^o)$ has only a finite number of discontinuities.

Assumption 1

For each $x \in X$, a set $K_x \in H^o$ can be found such that

a) $\Lambda_o \subset K_x$;

b) $P^o[K_x;x] = 0$.

Assumption 2

If x(t) is any mapping of the time axis $[0,\infty)$ into the state space X, one and only one point ω^o can be found such that

$$x_t^o(\omega^o) = x(t), \tag{2.1}$$

These assumptions imply that the probability spaces $\{\Omega^o;F^o;P^o\}$ have the properties of $\{\Omega^*;F^*;P^*\}$. These properties have been considered in chapter 1 of this part.

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1) It is convenient to index the points ω and to suppress the index 0 in X^o and G^o; $*$=0 means read 0 where we wrote $*$ in chapter 1.

Moreover, we assume the following:

Assumption 3

For each $x \in X$ we have (cf. p.1)

$$P^{o}\left[\Lambda_{o;x};x\right] = 1. \tag{2.2}$$

This assumption implies that the initial state of the stochastic process S_x^o is x with probability 1.

In part I a strategy z is given by means of a function $z(B;x)$ of sets $B \in G$ and points $x \in X$ with the following properties:

1) For each $x \in X$ the set function $z(B;x)$ is a probability measure defined on G;
2) For each $B \in G$ the x-function $z(B;x)$ is measurable with respect to G;
3) A closed set A_z can be found such that (cf.p.9)
   a) $z(A_z;x) = 0$ and thus $z(\bar{A}_z;x) = 1$ if $x \in A_z$;
   b) $z(\{x\};x) = 1$ if $x \in \bar{A}_z$ and if the set $\{x\}$ only contains the single point x.
   c) for each $x \in X$ we have

$$P^{o}\left[\Xi_{[0,\infty);A_z};x\right] = 1 \tag{2.3}$$

and

$$\int_0^\infty tP^{o}\left[\Xi_{dt;A_z};x\right] < \infty. \tag{2.4}$$

The application of a strategy z involves extra transitions. As soon as a state of A_z, say x_1, is assumed, an instantaneous transition will happen with transition probabilities $z(B;x_1)$.

In order to be able to describe the resulting random walk in X, the extra transitions have to be incorporated in the original stochastic process S_x^o. To this end we assume that, if the transition in question leads to a state x_2, the process goes on like a $S_{x_2}^o$-process. Note that by point 3a) of $z(B;x)$ the state $\underline{x}_2$ almost surely belongs to $\bar{A}_z$.

Let us now consider the set functions

$$\{p^j(B;x;z);\ j=0,1,\ldots\}, \tag{2.5}$$

defined by (cf.p.10)

$$p^{o}(B;x;z) \overset{\text{def}}{=} P^{o}\left[\Delta_{B;A_z};x\right], \tag{2.6}$$

$$p^{1}(B;x;z) \overset{\text{def}}{=} \begin{cases} \int_X z(dy;x)P^{o}\left[\Delta_{B;A_z};y\right], & \text{if } x \in A_z \\ \int_{A_z} p^{o}(dI_1;x;z) \int_X z(dy;I_1)P^{o}\left[\Delta_{B;A_z};y\right], & \text{if } x \in \bar{A}_z. \end{cases} \tag{2.7}$$

and

$$p^{k}(B;x;z) \overset{\text{def}}{=} \int_{A_z} p^{k-1}(dI_k;x;z)p^{1}(B;I_k;z);\ k=2,3,\ldots \tag{2.8}$$

recursively.[2)]

The following lemma can easily be proved:

<u>Lemma 2.1</u>

The functions $\{p^{k}(B;x;z);\ k=0,1,\ldots\}$ satisfy:

1) For each $x \in X$ the set function $p^{k}(B;x;z)$ is a probability measure defined on G with

$$p^{k}(A_z;x;z) = 1; \tag{2.9}$$

2) For each set $B \in G$, the x-function $p^{k}(B;x;z)$ is measurable with respect to G.

It follows from the construction of the set function $p^{k}(B;x;z)$, that it represents the probability distribution of the initial point $\underline{I}_{k+1}$ of the $(k+1)^{st}$ added transition.

We now consider a sequence of spaces $\{\Omega^{k};\ k=1,2,\ldots\}$. These spaces are isomorphic with Ω^{o}. Consequently, there exist 1-1 point transformations

$$\omega^{k} = T_{kh}(\omega^{h});\ k,h=1,2,\ldots \qquad \text{(in (2.10))}$$

from Ω^{h} onto Ω^{k} satisfying:

---

2) States in A_z are denoted by $I, I_1, \ldots$ etc.

a) $\omega^k = T_{kk}(\omega^k)$, $k=0,1,\ldots;$ (2.11)

b) $\omega^k = T_{kh}(T_{hj}(\omega^j))$; $k,h,j=0,1,\ldots$. (2.12)

These point transformations induce set transformations, denoted by

$$K^k = T_{kh}(K^h);\ k,h=0,1,\ldots \qquad (2.13)$$

and defined by

$$T_{kh}(K^h) \stackrel{\text{def}}{=} \{\omega^k | \omega^k = T_{kh}(\omega^h);\ \omega^h \in K^h\} . \qquad (2.14)$$

The set transformation (2.13) has the following properties:

a) $K^k = T_{kk}(K)$, $k=0,1,\ldots;$ (2.15)

b) $K^k = T_{kh}(T_{hj}(K^j))$, if $K^k = T_{kj}(K^j)$

$k,h,j=0,1,2,\ldots$. (2.16)

Let the class F^k of sets K^k be defined by

$$F^k \stackrel{\text{def}}{=} \{K^k | K^k = T_{ko}(K^o);\ K^o \in F^o\} . \qquad (2.17)$$

Obviously, the class F^k is isomorphic with F^o. Consequently, the following lemma is true:

<u>Lemma 2.2</u>

The class F^k is a σ-field of ω^k-sets.

In order to simplify the notation, from now on we drop the space-index k in the notation of the sets K^k. Corresponding sets in different spaces will be denoted by the same symbol.

Next we introduce the ω^k-functions $\{x_t^k(\omega^k);\ t \in [0,\infty)\}$, defined by

$$x_t^k(\omega^k) = x_t^o(T_{ok}(\omega^k));\ k=0,1,\ldots . \qquad (2.18)$$

If $B \in G$ and if $\Lambda_{t;B}$ is defined by

$$\Lambda_{t;B} \overset{\text{def}}{=} \{\omega^o \mid x_t^o(\omega^o) \in B\} , \tag{2.19}$$

then $\Lambda_{t;B} \in F^o$ and thus

$$T_{ko}(\Lambda_{t;B}) = \{\omega^k \mid x_t^k(\omega^k) \in B\} \in F^k. \tag{2.20}$$

So we have proved:

Lemma 2.3

The ω^k-functions $\{x_t^k(\omega^k);\ t \in [0,\infty)\}$ are measurable with respect to F^k $(k=1,2,\ldots)$.

Finally, we introduce the set functions $\{P^k[K;x;z];k=1,2,\ldots\}$, defined on F^k by

$$P^k[K;x;z] \overset{\text{def}}{=}$$

$$\overset{\text{def}}{=} \int_{A_z} p^{k-1}(dI_k;x;z) \int_X z(dy;I_k)P^o[K;y] . \tag{2.21}$$

The proof of the following lemma is left to the reader:

Lemma 2.4

For each x and k the set function $P^k[K;x;z]$ is a probability measure. For each set K the x-function $P^k[K;x;z]$ is measurable with respect to G.

The ω-functions $\{x_t^k(\omega^k);t \in [0,\infty)\}$ together with the probability space $\{\Omega^k;F^k;P^k\}$ generate the stochastic process S_x^k $(k=0,1,\ldots)$.

The initial state of this process is not x (cf. assumption 3), but obeys the probability distribution

$$q(B;x;z) = P^k[\Lambda_{o;B};x;z] =$$

$$= \int_{A_z} p^{k-1}(dI_k;x;z)z(B;I_k). \tag{2.22}$$

It can easily be verified that the set function q(B;x;z) represents the probability distribution of the state into which the system is transferred by the k^{th} added transition. Note that apart from the initial distribution the process S_x^k does not depend on the strategy applied.

By (2.18) and (2.21) the set $\Lambda_o \in F^k$ and has the following properties:

1) For each $\omega^k \in \bar{\Lambda}_o$, the t-function $x_t^k(\omega^k)$ is continuous from the right;
2) In each bounded time interval in $[0,\infty)$ and for each $\omega^k \in \bar{\Lambda}_o$, the t-function $x_t^k(\omega^k)$ has only a finite number of discontinuities;
3) For each $x \in X$, we have

$$P^k\,[\Lambda_o;x;z] = 0. \tag{2.23}$$

If $x(t)$ is any mapping of the time axis $[0,\infty)$ into the state space X, then it follows from assumption 2 and (2.18) that one and only one point ω^k can be found such that

$$x_t^k(\omega^k) = x(t). \tag{2.24}$$

So we have proved the following lemma:

<u>Lemma 2.5</u>

The stochastic processes S_x^k (k=1,2,...) satisfy the assumptions 1 and 2 of the S_x^* process ($*$=k; M=N) and have initial probability distributions.

Up to now the probability spaces $\{\Omega^k;F^k;P^k\}$ have been considered eparately. In the remainder of this section, however, we shall contruct one single probability space $\{\Omega;F;P\}$ which is in fact the Carsian product of the probability spaces $\{\Omega^k;F^k;P^k\}$.

Let Ω be the product space of the spaces Ω^k (k=0,1,...) and let H the smallest σ-field of sets K that contains the cylinder sets

$$K = K_o \times K_1 \times \ldots = \prod_{i=0}^{\infty} K_i, \tag{2.25}$$

iere

a) $K_i \in F^i$, $i=0,1,\ldots$;

b) only a finite number of ω^i-sets K_i are different from Ω^i.

The points of Ω are denoted by $\omega = (\omega^o, \omega^1, \ldots)$. We now consider the point transformation

$$\omega_1 = T_{(k)}(\omega), \tag{2.26}$$

defined by

$$\omega_1^j = \omega^{k+j} \; ; \quad j=0,1,\ldots \; . \tag{2.27}$$

The definition of the point transformation $T_{(k)}(\omega)$ implies:

<u>Lemma 2.6</u>

For each $\omega\epsilon\Omega$ one and only one point $\omega_1 \epsilon \Omega$ can be found such that

$$\omega_1 = T_{(k)}(\omega). \tag{2.28}$$

By means of the point transformation (2.26) we can define a set transformation $'K = T_{(k)}(K)$, given by

$$T_{(k)}(K) \overset{\text{def}}{=} \{\omega_1 \mid \omega_1 = T_{(k)}(\omega);\ \omega \epsilon K\} \; . \tag{2.29}$$

Next we consider a set transformation of sets $K \epsilon H$, denoted by

$$''K = T_{(k);\omega^o\ldots\omega^{k-1}}(K) \tag{2.30}$$

and defined by

$$T_{(k);\omega^o\ldots\omega^{k-1}}(K) \overset{\text{def}}{=} T_{(k)}(K \cap \{\omega^o\} \times \ldots \times \{\omega^{k-1}\} \times \prod_{i=k}^{\infty} \Omega_i), \tag{2.31}$$

where $\{\omega^j\}$ is the point set in Ω^j containing the single point ω^j.

We now prove the following lemma:

<u>Lemma 2.7</u>

For each $\omega \epsilon \Omega$ the set transformation $T_{(k);\omega^o\ldots\omega^{k-1}}(K)$ induces a σ-homomorphism of H onto H.

Proof:

If K is a product set of the type

$$K = \prod_{j=0}^{r} K_j \times \prod_{j=r+1}^{\infty} \Omega^j, \tag{2.32}$$

with $K_j \varepsilon F^j$ and $r \geqq k$, then

$$T_{(k);\omega^o\ldots\omega^{k-1}}(K) = \begin{cases} \prod_{j=k}^{r} K_j \times \prod_{j=r+1}^{\infty} \Omega^j, \text{ if } \omega^j \varepsilon K_j;\ j \leqq k-1 \\ \emptyset, \text{ if } \exists_{j \leqq k-1}\ \omega^j \varepsilon \overline{K}_j. \end{cases} \tag{2.33}$$

Consequently, $T_{(k);\omega^o\ldots\omega^{k-1}}(K) \varepsilon H$.

From the definition of $T_{(k);\omega^o\ldots\omega^{k-1}}(K)$ it follows that $\omega_1 \varepsilon T_{(k);\omega^o\ldots\omega^{k-1}}(K)$ implies $(\omega^o,\ldots,\omega^{k-1},\omega_1) \varepsilon K$ and conversely.

Hence,

$$T_{(k);\omega^o\ldots\omega^{k-1}}(\overline{K}) = \overline{T_{(k);\omega^o\ldots\omega^{k-1}}(K)} \tag{2.34}$$

and

$$T_{(k);\omega^o\ldots\omega^{k-1}}(\bigcup_{i=1}^{\infty} K^i) = \bigcup_{i=1}^{\infty} T_{(k);\omega^o\ldots\omega^{k-1}}(K^i). \tag{2.35}$$

Let J be the class of ω-sets $K \varepsilon H$ which satisfy

$$T_{(k);\omega^o\ldots\omega^{k-1}}(K) \varepsilon H.$$

Then, by (2.33) the class J contains the product sets (2.32). Because of (2.34) and (2.35) J is a σ-field and therefore, $J = H$.
Let J_1 be the class of ω_1-sets "K, defined by

$$J_1 \overset{def}{=} \{"K \mid "K = T_{(k);\omega^o\ldots\omega^{k-1}}(K);\ K \varepsilon H\}. \tag{2.36}$$

have just proved that $J_1 \subset H$.
the other hand, if $"K \varepsilon H$, then

$$"K = T_{(k);\omega^o\ldots\omega^{k-1}}(\prod_{i=0}^{k-1} \Omega_i \times "K) \tag{2.37}$$

and thus

$$''K \in J_1 .$$

Hence,

$$J_1 = H. \tag{2.38}$$

This proves the lemma completely.

If $K_k \in F^k$, let us introduce the ω-functions $P^k(K_k;\omega^o \ldots \omega^{k-1})$, defined by (cf.p.10)

$$P^k \left[K_k;\omega^o \ldots \omega^{k-1}\right] \overset{\text{def}}{=} \int_X z(dy\,;x^{k-1}(\omega^{k-1};A_z))\; P^o\left[K_k;y\right] . \tag{2.39}$$

The following lemma can easily be proved:

<u>Lemma 2.8</u>

For each $K_k \in F^k$ the ω-function $P^k\left[K_k;\omega^o \ldots \omega^{k-1}\right]$ is measurable with respect to H. For each ω the set function $P^k\left[K_k;\omega^o \ldots \omega^{k-1}\right]$ is a probability measure, defined on F^k.

Next we prove:

<u>Lemma 2.9</u>

If $K_k \in F^k$, we have

$$P^k\left[K_k;x;z\right] = \int_{\Omega^o} P^o\left[d\omega^o;x\right] \int_{\Omega^1} P^1\left[d\omega^1;\omega^o\right] \ldots \int_{\Omega^k} P^k\left[K_k;\omega^o \ldots \omega^{k-1}\right] ; \quad k=1,2,\ldots . \tag{2.40}$$

<u>Proof</u>:

This lemma is proved by induction.

If k=1, then according to (2.6), (2.21) and (2.39) we find

$$P^1\left[K_1;x;z\right] = \int_{A_z} P^o(dI_1;x;z) \int_X z(dy\,;I_1) P^o\left[K_1;y\right] =$$

$$= \int_{\Omega^o} P^o\left[d\omega^o;x\right]\; P^1\left[K_1;\omega^o\right] . \tag{2.41}$$

Thus, the assertion is true for k=1.

Let us now assume that the assertion is also true for k=n-1 and let

$M_{j;m}$ be defined by

$$M_{j;m} \overset{\text{def}}{=} \{\omega^{n-1} \mid \frac{j-1}{2^m} < \int_X z(dy;x^{n-1}(\omega^{n-1};A_z))P^o \left[K_n;y\right] \leqq \frac{j}{2^m}\} =$$

$$= \{\omega^{n-1} \mid \frac{j-1}{2^m} < P^n \left[K_n;\omega^o\ldots\omega^{n-1}\right] \leqq \frac{j}{2^m}\} . \qquad (2.42)$$

According to (2.21)

$$P^n \left[K_n;x;z\right] = \int_{A_z} P^{n-1}(dI_n;x;z) \int_X z(dy;I_n)P^o \left[K_n;y\right] =$$

$$= \int_{\Omega^{n-1}} P^{n-1} \left[d\omega^{n-1};x;z\right] \int_X z(dy;x^{n-1}(\omega^{n-1};A_z))P^o \left[K_n;y\right] =$$

$$= \lim_{m\to\infty} \sum_{j=1}^{2^m} \frac{j}{2^m} P^{n-1} \left[M_{j;m};x;z\right] =$$

$$= \lim_{m\to\infty} \sum_{j=1}^{2^m} \frac{j}{2^m} \int_{\Omega^o} P^o\left[d\omega^o;x\right] \ldots \int_{\Omega^{n-1}} P^{n-1} \left[M_{j;m};\omega^o\ldots\omega^{n-2}\right] =$$

$$= \int_{\Omega^o} P^o \left[d\omega^o;x\right] \ldots \{\lim_{m\to\infty} \sum_{j=1}^{2^m} \frac{j}{2^m} P^{n-1} \left[M_{j;m};\omega^o\ldots\omega^{n-2}\right]\} =$$

$$= \int_{\Omega^o} P^o \left[d\omega^o;x\right] \ldots \int_{\Omega^{n-1}} P^{n-1} \left[d\omega^{n-1};\omega^o\ldots\omega^{n-2}\right] \cdot$$

$$\cdot P^n \left[K_n;\omega^o\ldots\omega^{n-1}\right] . \qquad (2.43)$$

Hence the assertion is also true for k=n.

This ends the proof.

We now consider the cylinder set $K \in H$, given by

$$K = \prod_{i=0}^{\infty} K_i \qquad (2.44)$$

a) $K_i \in F^i$;

) only a finite number of ω^i-sets K_i different from Ω^i.

›r each cylinder set K of the type (2.44) we can define

.) a number m_K by

$$m_K \overset{\text{def}}{=} \inf\{i \mid \forall_{j \geq i+1}\ K_j = \Omega^j\}\ , \qquad (2.45)$$

2) functions $I_K(\omega^o \ldots \omega^{m_K})$ by

$$I_K(\omega^o \ldots \omega^{m_K}) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } (\omega^o \ldots \omega^{m_K}) \in \prod_{i=0}^{m_K} K_i . \\ 0, & \text{otherwise.} \end{cases} \qquad (2.46)$$

Now we are in a position to define a set function $P[K;x;z]$ on the class of all sets of the type (2.44).

Let $P[K;x;z]$ be defined by

$$P[K;x;z] =$$
$$= \int_{\Omega^o} P^o[d\omega^o;x] \int_{\Omega^1} P^1[d\omega^1;\omega^o] \ldots \int_{\Omega^{m_K}} P^{m_K}[d\omega^{m_K};\omega^o \ldots \omega^{m_K-1}] \cdot$$
$$\cdot I_K(\omega^o \ldots \omega^{m_K}) . \qquad (2.47)$$

It can easily be verified that the right hand side of (2.47) exists.

<u>Lemma 2.10</u>

a) The domain of definition of the set function $P[K;x;z]$, can be extended to H. For each $x \in X$ the set function $P[K;x;z]$ is a probability measure defined on H. For each $K \in F$ the x-function $P[K;x;z]$ is measurable with respect to G.

b) If $K_k \in F^k$ and if K_k^c is given by

$$K_k^c = \prod_{j=0}^{k-1} \Omega^j \times K_k \times \prod_{j=k+1}^{\infty} \Omega^j , \qquad (2.48)$$

then

$$P^k[K^k;x;z] = P[K_k^c;x;z] . \qquad (2.49)$$

<u>Proof</u>:

Point a) has been proved by I. Tulcea (cf. [1] p.137).

We now consider point b). The proof runs as follows (cf. (2.40)):

$$P[K_k^c;x;z] =$$
$$= \int_{\Omega^o} P^o[d\omega^o;x] \ldots \int_{\Omega^k} P^k[d\omega^k;\omega^o \ldots \omega^{k-1}]\ I_{K_k^c}(\omega^o \ldots \omega^k) =$$

$$= \int_{\Omega^o} P^o \left[d\omega^o;x\right] \ldots \int_{\Omega^k} P^k \left[K_k;\omega^o \ldots \omega^{k-1}\right] =$$

$$= P^k \left[K_k;x;z\right] . \qquad (2.50)$$

So we have obtained a probability space $\{\Omega;H;P\}$ with $\{\Omega^i;F^i;P^i\}$ as factor probability spaces.

Since ω^k is the k^{th} component of ω, the ω^k-functions $\{x_t^k(\omega^k);t \varepsilon [0,\infty)\}$ can also be defined on Ω . We define the ω-functions $\{x_t^k(\omega);t \varepsilon [0,\infty)\}$ by

$$x_t^k(\omega) \overset{def}{=} x_t^k(\omega^k). \qquad (2.51)$$

It follows from (2.50) and (2.51), that the stochastic processes $\{S_x^k;k=0,1,\ldots\}$ can also be defined by means of the ω-functions $\{x_t^k(\omega);t \varepsilon [0,\infty)\}$ and the probability spaces $\{\Omega;H;P\}$.

In the final part of this section we shall discuss some properties of the probability measure $P\left[K;x;z\right]$.

<u>Lemma 2.11</u>

If $K \varepsilon H$, the ω-function

$$\int_X z(dy;x^{j-1}(\omega;A_z))P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(K);y;z\right] \ ; \ j=1,2,\ldots \qquad (2.52)$$

: measurable with respect to H.

<u>ɔof</u>:

Let K be the product set

$$K = \prod_{h=0}^{\infty} K_j, \qquad (2.53)$$

$K_h \varepsilon F^h$.

we have

$$\int_X z(dy;x^{j-1}(\omega;A_z))P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(K);y;z\right] =$$

$$= \begin{cases} \int_X z(dy;x^{j-1}(\omega;A_z))P\left[\prod_{h=j}^{\infty} K_h;y;z\right] , \text{ if } \omega^i \varepsilon K_i ; \\ \qquad\qquad i=0,\ldots,j-1. \\ 0, \text{ otherwise.} \end{cases} \qquad (2.54)$$

Hence,

$$\int_X z(dx;x^{j-1}(\omega;A_z))P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(K);x;z\right] =$$

$$= \chi(\omega)\int_X z(dx;x^{j-1}(\omega;A_z))P\left[\prod_{i=j}^{\infty} K_i;x;z\right], \qquad (2.55)$$

where

$$\chi(\omega) \overset{\text{def}}{=} \begin{cases} 1, \text{ if } \omega^i \in K_i \text{ for } i=0,1,\ldots,j-1 \\ 0, \text{ otherwise.} \end{cases} \qquad (2.56)$$

Thus, if K is of the form (2.53), the ω-function (2.52) is measurable with respect to H.

Let J be the class of ω-sets $K \in H$ which satisfy the assertion. So J includes the product sets (2.53). It can easily be verified that J also contains:

a) the complements of J-sets;

b) the limit of any monotone sequence of J-sets.

Consequently, J=H.

This ends the proof.

Lemma 2.12

If $K \in H$, we have for each $x \in X$ and $j \geq 1$

$$P[K;x;z] = \int_\Omega P[d\omega;x;z]\int_X z(dy;x^{j-1}(\omega;A_z))\cdot$$

$$\cdot P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(K);y;z\right] =$$

$$= \int_\Omega P[d\omega;x;z]\; P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(K);x_o^j(\omega);z\right]. \qquad (2.57)$$

Proof:

Let us consider the cylinder set $K \in H$, given by

$$K = \prod_{i=0}^{\infty} K_i \qquad (2.58)$$

with

a) $K_i \in F^i$;

b) only a finite number of sets K_i different from Ω^i.

If m_K is defined by

$$m_K \overset{def}{=} \inf \{i \mid \forall_{h > i} \; K_h = \Omega^h\} \tag{2.59}$$

and if $\chi(\omega)$ is defined by

$$\chi(\omega) = \begin{cases} 1, & \text{if } \omega^i \in K_i; \; i=0,1,\dots,j-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.60}$$

then for each j we have

$$\begin{aligned} P\left[K;x;z\right] &= \\ &= \int_{\Omega^o} P^o\left[d\omega^o;x\right] \dots \int_{\Omega^{j-1}} P^{j-1}\left[d\omega^{j-1};\omega^o\dots\omega^{j-2}\right] \cdot \chi(\omega) \cdot \\ &\quad \cdot \int_X z(dx_1;x^{j-1}(\omega^{j-1};A_z)) \cdot P\left[\prod_{h=j}^{\infty} K_h;x_1;z\right] = \\ &= \int_{\Omega} P\left[d\omega;x;z\right] \int_X z(dx_1;x^{j-1}(\omega^{j-1};A_z)) \cdot \\ &\quad \cdot P\left[T_{(j);\omega^o\dots\omega^{j-1}}(K);x_1;z\right]. \end{aligned} \tag{2.61}$$

From (2.61) we deduce that the product sets K satisfy the assertion. By using similar arguments as in the proof of lemma 2.11 we can complete the proof of this lemma.

We now consider the ω-set M_o, defined by

$$M_o \overset{def}{=} \bigcup_{j=0}^{\infty} \prod_{i=0}^{j-1} \Omega^i \times \Lambda_o \times \prod_{i=j+1}^{\infty} \Omega^i. \tag{2.62}$$

Obviously, we have for each $x \in X$

$$P\left[M^o;x;z\right] = 0. \tag{2.63}$$

By completing the measure, the domain of definition of $P\left[K;x;z\right]$ is from now on extended to the σ-field F, the smallest σ-field including H and containing all subsets of M_o.

The proof of the following lemma is left to the reader (cf. lemma 1.1):

Lemma 2.13

For each set $K \varepsilon F$, the x-function $P[K;x;z]$ is measurable with respect to G.

2. The probabilistic foundation of the decision process

In section 1 we have stipulated that the application of a strategy involves extra transitions. We assumed that an instantaneous transition with transition probabilities $z(B;x_1)$ occurs if a state, say x_1, of a closed set A_z is reached. If such a transition leads to a state x_2, then the process goes on like a $S^o_{x_2}$-process. The resulting random walk is called the decision process and is denoted by $S_{x;z}$ if x is the initial state.

Since the initial distribution of the S^k_x-process represents the probability distribution of the state into which the system is transferred by the k^{th} added transition (cf. p. 60), the stochastic process S^k_x can be used for the description of that part of the decision process which will take place between the k^{th} and the $(k+1)^{st}$ added transition. Hereafter this part is called the $(k+1)^{st}$ stretch of the decision process.

In such a presentation the points $\omega^k \varepsilon \Omega^k$ determine realizations of the $(k+1)^{st}$ stretch. Hence the points $\omega \varepsilon \Omega$ determine realizations of the whole decision process.

In this section we shall demonstrate that decision processes can also be defined by means of probability spaces $\{\Omega;F;P\}$.

Obviously, the successive states in A_z, reached by the system, can for almost all ω be represented by $\{x^j(\omega;A_z);\ j=0,1,\dots\}$. The lengths $\{t^j(\omega;A_z);\ j=0,1,\dots\}$ of the time intervals between the added transitions are defined and measurable with respect to F^j (cf. lemmas 1.5.1 and 1.5.2 with $*=j$).

The sequence of ω-functions $\{x^j(\omega;A_z);\ j=0,1,\dots\}$ together with a probability space $\{\Omega;F;P\}$ generate a sequence of stochastic variables, denoted by $\{\underline{I}_{j+1};\ j=0,1,\dots\}$.

We already know that the applied strategy z effects an extra transition in the random states $\{\underline{I}_j;\ j=1,2,\dots\}$.

Obviously, we have for j=1,2,... (cf. (2.6), (2.7) and (2.8))

$$\text{Prob}\ \{\underline{I}_{j+1} \varepsilon B \mid \underline{I}_1 = I_1,\dots,\underline{I}_j = I_j\} =$$

$$= \int_X z(dy;I_j)P^o\left[\Delta_{B;A_z};y\right] = p^1(B;I_j;z) \qquad (2.64)$$

and

$$\text{Prob}\ \{\underline{I}_{j+1} \varepsilon B\} = P^j\left[\Delta_{B;A_z};x;z\right] = p^j(B;x;z) \qquad (2.65)$$

with

$$p^j(B;x;z) = \int_{A_z} p^{j-1}(dI_j;x;z)p^1(B;I_j;z). \qquad (2.66)$$

The following theorem is an immediate consequence of lemma 2.1 and the equations (2.64) through (2.66) (cf. [2] p.190 ff.):

Theorem 1

The stochastic variables $\{\underline{I}_k;\ k=1,2,\dots\}$ constitute a stationary Markov process with a discrete time parameter.

This stochastic process is called the decision process on A_z.

Henceforth our considerations are based on the following assumption:

Assumption 4

There is a finite valued measure Q(U) of sets $U \varepsilon G$ with $Q(X) > 0$, an integer $k \geq 1$ and a positive η, such that for each $I \varepsilon A_z$ (cf. [2],). 192)

$$p^k(U;I;z) \leq 1-\eta\ ,\ \text{if}\ Q(U) \leq \eta\ . \qquad (2.67)$$

can now prove (cf. [2], p.214):

ι 2.14

Under (2.67), the function $p(U;I_1;z)$, given by

$$p(U;I_1;z) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} p^j(U;I_1;z), \qquad (2.68)$$

defines for each $I_1 \varepsilon A_z$ a stationary absolute probability distribution.

The sequence of ω-functions $\{t^j(\omega;A_z);\ j=0,1,\dots\}$ together with the probability space $\{\Omega;F;P\}$ also generate a sequence of stochastic variables. These stochastic variables are denoted by $\{\underline{t}_j;\ j=0,1,2,\dots\}$.

Using similar arguments as in the proof of lemma 1.47 we can prove:

Lemma 2.15

Under (2.4) and (2.67) the limit

$$\lim_{r\to\infty} \frac{1}{r}\sum_{j=1}^{r} t^j(\omega;A_z) \tag{2.69}$$

exists for almost all ω and is equal to

$$\int_{A_z} p(dI;I_1;z)\int_X z(dy;I)\int_0^\infty tP^o\left[\Xi_{dt;A_z};y\right] \tag{2.70}$$

if the initial state I_1 belongs to a simple ergodic set of the stochastic process $\{\underline{I}_k;\ k=1,2,\dots\}$.

Lemma 2.16

Under the assumptions 1,3 and 4 and by property 3[a]) of the function $z(B;x)$, we have for each $I_1 \varepsilon A_z$

$$\int_{A_z} p(dI;I_1;z)\int_X z(dx;I)\int_0^\infty tP^o\left[\Xi_{dt;A_z};x\right] > 0. \tag{2.71}$$

Proof:

Obviously, we have by assumption 3 for each x

$$P^o\left[\Xi_{[0,\frac{1}{n});A_z};x\right] = P^o\left[\Xi_{[0,\frac{1}{n});A_z}\cap\Lambda_{o;x};x\right]. \tag{2.72}$$

Let us consider the limit

$$\lim_{n\to\infty} P^o\left[\Xi_{[0,\frac{1}{n});A_z};x\right] = P^o\left[\bigcap_{n=1}^{\infty}\Xi_{[0,\frac{1}{n});A_z}\cap\Lambda_{o;x};x\right]. \tag{2.73}$$

If $x \varepsilon \bar{A}_z$, by the definition of Λ_o we find

$$\bigcap_{n=1}^{\infty} \Xi_{[0,\frac{1}{n});A_z} \cap \Lambda_{o;x} \subset \Lambda_o. \tag{2.74}$$

Hence, if $x \in \overline{A}_z$,

$$\lim_{n\to\infty} P^o \left[\Xi_{[0,\frac{1}{n});A_z};x\right] = 0 \tag{2.75}$$

and thus

$$\lim_{n\to\infty} P^o \left[\Xi_{[\frac{1}{n},\infty);A_z};x\right] = 1. \tag{2.76}$$

Therefore, if $x \in \overline{A}_z$,

$$\int_0^{\infty} tP^o \left[\Xi_{dt;A_z};x\right] > 0. \tag{2.77}$$

The assertion now is an immediate consequence of property 3^a) of the function $z(B;x)$.

This ends the proof.

Now we are in a position to prove the following theorem:

<u>Theorem 2</u>

Under the assumptions 1,3 and 4 and by property 3^a) of the function $z(B;x)$, the limit

$$\lim_{r\to\infty} \frac{1}{r} \sum_{j=1}^{r} t^j(\omega;A_z) \tag{2.78}$$

exists and is positive for almost all ω. In particular, if the initial state I_1 belongs to a simple ergodic set of the stochastic process $\{\underline{I}_k;\ k=1,2,\ldots\}$, the limit (2.78) is almost surely equal to

$$\int_{A_z} p(dI;I_1;z) \int_X z(dx;I) \int_0^{\infty} tP^o \left[\Xi_{dt;A_z};x\right]. \tag{2.79}$$

<u>Proof</u>:

If the initial state I_1 is an ergodic state, then the assertion is an immediate consequence of lemmas 2.15 and 2.16.

If the initial state is a transient state then for almost all ω the system will stay outside all simple ergodic sets only a finite number of times in its transitions (cf. [2] , p.207). Consequently, for almost all ω an integer $n(\omega)$ can be found such that for $n \geqq n(\omega)$ the states $x^n(\omega;A_z)$ are ergodic. Hence, for almost all ω

$$\lim_{r \to \infty} \frac{1}{r} \sum_{j=1}^{r} t^j(\omega;A_z) =$$

$$= \int_{A_z} p(dI;\hat{I};z) \int_X z(dx;I) \int_0^\infty tP^o\left[\Xi_{dt;A_z};x\right] > 0, \tag{2.80}$$

where $\hat{I}$ stands for $x^{n(\omega)}(\omega;A_z)$.

This completes the proof.

Let us now consider the ω-functions $\{u^k_{t;1}(\omega); t \in [0,\infty)\}$ and $\{u^k_{t;2}(\omega); t \in [0,\infty)\}$ for $k=0,1,\ldots,$ defined by

$$u^o_{t;1}(\omega) = u^o_{t;2}(\omega) \overset{\text{def}}{=} x^o_t(\omega), \tag{2.81}$$

$$u^k_{t;1}(\omega) \overset{\text{def}}{=} \begin{cases} u^{k-1}_{t;1}(\omega), \text{ if } t \leqq \hat{t}_k(\omega;A_z) \\ x^k_{t-\hat{t}_k(\omega;A_z)}(\omega), \text{ if } t > \hat{t}_k(\omega;A_z) \end{cases} ; \ k > 0 \tag{2.82}$$

and

$$u^k_{t;2}(\omega) \overset{\text{def}}{=} \begin{cases} u^{k-1}_{t;2}(\omega), \text{ if } t < \hat{t}_k(\omega;A_z) \\ x^k_{t-\hat{t}_k(\omega;A_z)}(\omega), \text{ if } t \geqq \hat{t}_k(\omega;A_z) \end{cases} ; \ k > 0 \tag{2.83}$$

where

$$\hat{t}_k(\omega;A_z) \overset{\text{def}}{=} \sum_{j=0}^{k-1} t^j(\omega;A_z). \tag{2.84}$$

Note that the t-functions $u^k_{t;1}(\omega)$ and $u^k_{t;2}(\omega)$ only differ if $t = \hat{t}_j(\omega;A_z)$ $(j=1,2,\ldots,k)$.

Lemma 2.17

The ω-functions $\{u^k_{t;1}(\omega); k=1,2,\ldots;\ t \in [0,\infty)\}$ are measurable with respect to H.

Proof:

This lemma will be proved by induction. It follows from (2.81), that the assertion is true for k=0.

Let us assume the assertion to be true for k=n-1 and let us consider the sequence of ω-functions $\{u^n_{m;t}(\omega);\ m=1,2,\ldots\}$, defined for $k=1,\ldots,2^m$ by

$$u^n_{m;t}(\omega) \overset{\text{def}}{=} \begin{cases} x^n_{t-\frac{(k-1)t}{2^m}}(\omega), & \text{if } \frac{(k-1)t}{2^m} \leqq \hat{t}_n(\omega;A_z) < \frac{kt}{2^m} \\ u^{n-1}_{t;1}(\omega), & \text{if } \hat{t}_n(\omega;A_z) \geqq t\ . \end{cases} \tag{2.85}$$

Let the ω-functions $\{\chi_k(\omega); k=0,\ldots,2^m\}$ be defined by

$$\chi_o(\omega) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } \hat{t}_n(\omega;A_z) \geqq t \quad \text{or } \omega^n \in \Lambda_o \\ 0, & \text{otherwise} \end{cases} \quad , \tag{2.86}$$

and

$$\chi_k(\omega) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } \frac{(k-1)t}{2^m} \leqq \hat{t}_n(\omega;A_z) < \frac{kt}{2^m} \text{ and } \omega^n \in \overline{\Lambda}_o \\ 0, & \text{otherwise} \end{cases} \quad , \tag{2.87}$$

where ω^n is the n^{th} component of ω.

It can easily be verified that the ω-functions $\{\chi_k(\omega); k=0,\ldots,2^m\}$ are measurable with respect to H.

Consequently, the ω-functions

$$u^n_{m;t}(\omega) = \chi_o(\omega)u^{n-1}_{t;1}(\omega) + \sum_{k=1}^{2^m} \chi_k(\omega)x^n_{t-\frac{(k-1)t}{2^m}}(\omega);\ m=1,2,\ldots \tag{2.88}$$

re measurable with respect to H.

It can easily be verified that for each ω the sequence $\{u^n_{m;t}(\omega);$ 2,...} converges to a limit, which is by consequence measurable respect to H. Since for $\omega \in \prod_{j=0}^{n-1} \Omega^j \times \overline{\Lambda}_o \times \prod_{j=n+1}^{\infty} \Omega^j$ this limit is l to $u^n_{t;1}(\omega)$, the latter is measurable with respect to H.

By a similar reasoning, we can prove:

Lemma 2.18

The ω-functions $\{u^k_{t;2}(\omega); k=0,1,\dots; t \in [0,\infty)\}$ are measurable with respect to H.

For each fixed k the ω-functions $\{u^k_{t;1}(\omega); t \in [0,\infty)\}$ together with the probability space $\{\Omega;F;P\}$ generate a stochastic process

$$\{\underline{u}^k_{t;x;1};\quad t \in [0,\infty)\} . \tag{2.89}$$

Let the stochastic variables $\{\hat{\underline{t}}_j;\ j=1,2,\dots\}$ be defined by

$$\hat{\underline{t}}_j = \sum_{k=0}^{j-1} \underline{t}_k . \tag{2.90}$$

The stochastic process (2.89) describes the state of the system in X if after the k^{th} extra transition no more extra transitions are added and if at $\{\hat{\underline{t}}_j;\ j=1,2,\dots,k\}$ only the initial point of the corresponding extra transition is recorded.
Similarly, we find that for each k the ω-functions $\{u^k_{t;2}(\omega); t \in [0,\infty)\}$ together with the probability space $\{\Omega;F;P\}$ generate a stochastic process

$$\{\underline{u}^k_{t;x;2};\quad t \in [0,\infty)\} . \tag{2.91}$$

The stochastic processes (2.89) and (2.91) are identical with the exception of the random points of time $\{\hat{\underline{t}}_j; j=1,2,\dots,k\}$. In (2.91) the state <u>after</u> the effectuation of the extra transition is presented at $\hat{\underline{t}}_j$.

In order to evade difficulties in determining the state at $\hat{\underline{t}}_j$ we introduce the product space X' of two N-dimensional Cartesian spaces X_1 and X_2. The σ-fields of all 2N-dimensional Borelsets in X' is denoted by G', while the corresponding σ-fields in X_1 and X_2 are called G_1 and G_2 respectively. Note that the spaces X, X_1 and X_2 are isomorphic.

Next let for each k and $t \in [0,\infty)$ the ω-functions $u^k_{t;1}(\omega)$ and $u^k_{t;2}(\omega)$ map Ω into X_1 and X_2 respectively.

From now on states are represented by points $x' \in X'$.

If $x' \in A_z \times X_2$, the x_1-component of the "state" x' determines the initial point of the extra transition, while the x_2-component describes the state just after the effectuation of the transition.

If $x' \in \bar{A}_z \times X_2$, then the x_1 and the x_2-components of x' are equal.

Obviously, the ω-functions $\{u_t^k(\omega);k=0,1,\ldots;t \in [0,\infty)\}$, defined by

$$u_t^k(\omega) \overset{\text{def}}{=} (u_{t;1}^k(\omega);u_{t;2}^k(\omega)), \tag{2.92}$$

map Ω into X'.

The proofs of the following lemmas are obvious.

Lemma 2.19

The ω-functions $\{u_t^k(\omega);k=0,1,\ldots;t \in [0,\infty)\}$ are measurable with respect to F.

Lemma 2.20

The ω-functions $\{u_t^k(\omega);k=0,1,\ldots;t \in [0,\infty)\}$ have the following properties:

a) For each $\omega \in \bar{M}_o$ the t-function $u_{t;2}^k(\omega)$ is continuous from the right;
b) For each $\omega \in \bar{M}_o$ the t-function $u_t^k(\omega)$ has only a finite number of discontinuities in each finite time interval.

We now consider the ω-functions $x_{t;1}(\omega)$ and $x_{t;2}(\omega)$ for $k=1,2,\ldots$, defined by

$$x_{t;1}(\omega) \overset{\text{def}}{=} u_{t;1}^{k-1}(\omega), \text{ if } t \leqq \hat{t}_k(\omega;A_z) \text{ and } \lim_{h\to\infty} \hat{t}_h(\omega;A_z) = \infty . \tag{2.93}$$

$$x_{t;2}(\omega) \overset{\text{def}}{=} u_{t;2}^{k-1}(\omega), \text{ if } t < \hat{t}_k(\omega;A_z) \text{ and } \lim_{h\to\infty} \hat{t}_h(\omega;A_z) = \infty; \tag{2.94}$$

$$x_{t;1}(\omega) = x_{t;2}(\omega) = x_t^o(\omega), \text{ if } \lim_{h\to\infty} \hat{t}_h(\omega;A_z) < \infty. \tag{2.95}$$

Note that the ω-functions $x_{t;1}(\omega)$ and $x_{t;2}(\omega)$ only differ at the points of time $\{\hat{t}_k(\omega;A_z);\ k=1,2,\ldots\}$.

The following lemmas can easily be verified:

<u>Lemma 2.21</u>

The ω-functions $\{x_{t;1}(\omega);t\ \varepsilon\ [0,\infty)\}$ and $\{x_{t;2}(\omega);t\ \varepsilon\ [0,\infty)\}$ are measurable with respect to H.

<u>Lemma 2.22</u>

The ω-functions $\{x_t(\omega);t\ \varepsilon\ [0,\infty)\}$, defined by

$$x_t(\omega) \overset{\text{def}}{=} (x_{t;1}(\omega),x_{t;2}(\omega)), \tag{2.96}$$

map Ω into X' and are measurable with respect to F.

<u>Lemma 2.23</u>

The ω-functions $\{x_t(\omega);t\ \varepsilon[0,\infty)\}$ have the following properties:

a) For each $\omega\ \varepsilon\,\overline{M}_o$ the t-function $x_{t;2}(\omega)$ is continuous from the right;

b) For each $\omega\ \varepsilon\,\overline{M}_o$ the t-function $x_t(\omega)$ has only a finite number of discontinuities in each finite time interval.

The ω-functions $\{x_t(\omega);t\ \varepsilon\ [0,\infty)\}$ together with a probability space $\{\Omega;F;P\}$ generate a stochastic process

$$\{\underline{x}_{t;x};t\ \varepsilon\ [0,\infty)\} \tag{2.97}$$

in X'.

Since

a) by theorem 2 we have almost surely

$$\lim_{h\to\infty} \hat{t}_h(\omega;A_z) = \infty\ ; \tag{2.98}$$

b) the stochastic processes $\{\underline{u}^k_{t;x;1};t\ \varepsilon\ [0,\infty)\}$ and $\{\underline{u}^k_{t;x;2};t\ \varepsilon\ [0,\infty)\}$ describe the evolution in the state of the system if only k extra transitions are added,

it follows from (2.93), (2.94) and (2.95) that the stochastic process $\{\underline{x}_{t;x};t\ \varepsilon\ [0,\infty)\}$ describes the whole decision process in X'.

If $\underline{x}_{t;x} \varepsilon A_z \times X_2$, the x_1-component of the state $\underline{x}_{t;x}$ determines the initial point of the extra transition, while the x_2-component describes the state just after the effectuation of this transition. The two components are equal if $\underline{x}_{t;x} \varepsilon \bar{A}_z \times X_2$.

Note that by virtue of assumption 3 the x_1-component of the initial state $\underline{x}_{o;x}$ is equal to x with probability 1.

If $x \varepsilon \bar{A}_z$, then the x_2-component of the initial state $\underline{x}_{o;x}$ also is equal to x with probability 1.

The x_2-component of $\underline{x}_{o;x}$ obeys the probability distribution $z(B;x)$ if $x \varepsilon A_z$. Consequently, for $x \varepsilon A_z$ the decision process has an initial distribution.

So we have proved:

Lemma 2.24

Under assumptions 1,3 and 4, the decision process in X' can be defined by means of a stochastic process.

3. Properties of the decision process

In this section we shall show that, notwithstanding the decision process does not satisfy assumption 1 (cf. chapter 1 of this part), the assertions stated in lemmas 1.5.1 through 1.9 can still be proved.

As we noted at the end of section 2 the decision process $\{\underline{x}_{t;x}; t \varepsilon [0,\infty)\}$ has an initial probability distribution if $x_1 \varepsilon A_z$. In the coming discussion we shall demonstrate that decision processes with given initial x'-states can also be defined.

For that purpose we have to define set functions $\{P[K;x';z]; x' \varepsilon X'\}$. Properties of these set functions are investigated at the end of this section.

Let us start with introducing the ω-functions $\{v_t(\omega); t \varepsilon [0,\infty)\}$, defined by

$$v_t(\omega) \overset{\text{def}}{=} (x_{t;2}(\omega); x_{t;2}(\omega)). \qquad (2.99)$$

The assertion stated in the following lemma can easily be proved.

Lemma 2.25

The ω-functions $\{v_t(\omega); t \in [0,\infty)\}$ have the following properties:

a) For each $t \in [0,\infty)$ the ω-function $v_t(\omega)$ is measurable with respect to F;

b) For each $\omega \in \overline{M}_o$ the t-function $v_t(\omega)$ is continuous from the right;

c) For each $\omega \in \overline{M}_o$ the t-function $v_t(\omega)$ has only a finite number of discontinuities in each finite interval.

Next we consider the ω-functions $\{x_j(\omega;A_z \times X_2); j=1,2,\ldots\}$, defined by

$$x_j(\omega;A_z \times X_2) \stackrel{\text{def}}{=} (x^{j-1}(\omega;A_z), x_o^j(\omega)). \qquad (2.100)$$

We easily verify:

Lemma 2.26

The ω-functions $\{x_j(\omega;A_z \times X_2); j=1,2,\ldots\}$ are measurable with respect to F.

It follows from (2.99) and (2.100) that

$$x_t(\omega) = \begin{cases} x_j(\omega;A_z \times X_2), \text{ if } t = \hat{t}_j(\omega;A_z) \\ v_t(\omega), \text{ if } t \neq \hat{t}_j(\omega;A_z); j=1,2,\ldots \end{cases} \qquad (2.101)$$

If C is a closed set in X', the ω-function $t(\omega;C)$ represents the moment that the system is for the first time in C.

In other words (cf. (1.47))

$$t(\omega;C) \stackrel{\text{def}}{=} \begin{cases} \inf\{t \mid x_t(\omega) \in C\}, \text{ if } x_t(\omega) \in C \text{ for some finite } t \\ \infty, \text{ otherwise.} \end{cases} \qquad (2.102)$$

If $'t(\omega;C)$ and $''t(\omega;C)$ are defined by

$$'t(\omega;C) \stackrel{\text{def}}{=} \begin{cases} \hat{t}_j(\omega;A_z), \text{ if } x_k(\omega;A_z \times X_2) \in \overline{C} \ (k=1,\ldots j-1) \text{ and } x_j(\omega;A_z \times X_2) \in C \\ \infty, \text{ otherwise} \end{cases} \qquad (2.103)$$

and

$$''t(\omega;C) \overset{\text{def}}{=} \begin{cases} \inf\{t \mid v_t(\omega) \in C\}, & \text{if } v_t(\omega) \in C \text{ for some finite } t \\ \infty, & \text{otherwise} \end{cases} \quad (2.104)$$

respectively, then we obviously have

$$t(\omega;C) = \min('t(\omega;C),\ ''t(\omega;C)). \quad (2.105)$$

Since the ω-functions $\hat{t}_j(\omega;A_z)$ and $x_k(\omega;A_z \times X_2)$ are measurable with respect to F, it follows from (2.103) that the ω-function $'t(\omega;C)$ is measurable with respect to F. Moreover, lemmas 2.25 and 1.5.1 imply that the ω-function $''t(\omega;C)$ is measurable with respect to F. ($M_o = \Lambda_o^*$!) Consequently, we find:

Lemma 2.27.1

If C is a closed set in X', then the ω-function $t(\omega;C)$, defined by (2.102), is measurable with respect to F.

We now consider the ω-function $x(\omega;C)$, defined by (cf. (1.51))

$$x(\omega;C) \overset{\text{def}}{=} \begin{cases} x_{t(\omega;C)}(\omega), & \text{if } t(\omega;C) < \infty \\ x_o(\omega), & \text{if } t(\omega;C) = \infty. \end{cases} \quad (2.106)$$

By (2.103) and (2.104)

$$x(\omega;C) = \begin{cases} x_j(\omega;A_z \times X_2), & \text{if } \hat{t}_{j-1}(\omega;A_z) < t(\omega;C) = \hat{t}_j(\omega;A_z) < \infty \\ v(\omega;C), & \text{if } t(\omega;C) = ''t(\omega;C) \end{cases}, \quad (2.107)$$

where

$$v(\omega;C) \overset{\text{def}}{=} \begin{cases} v_{''t(\omega;C)}(\omega), & \text{if } ''t(\omega;C) < \infty \\ x_o(\omega), & \text{if } ''t(\omega;C) = \infty. \end{cases} \quad (2.108)$$

Using similar arguments as in the proof of lemma 1.5.2 we can prove that the ω-function $v(\omega;C)$ is measurable with respect to F. Obviously, we have:

Lemma 2.27.2

If C is a closed set in X', then the ω-function $x(\omega;C)$, defined by (2.106), is measurable with respect to F.

If B is a 2N-dimensional Borelset, if C is a 2N-dimensional closed set and if I is an interval in $[0,\infty)$, the ω-sets $\Lambda_{t;B;z}$, $\Xi_{I;C;z}$ and $\Delta_{B;C;z}$ are defined by

$$\Lambda_{t;B;z} \stackrel{def}{=} \{\omega \mid x_t(\omega) \in B\} , \tag{2.109}$$

$$\Xi_{I;C;z} \stackrel{def}{=} \{\omega \mid t(\omega;C) \in I\} , \tag{2.110}$$

and

$$\Delta_{B;C;z} \stackrel{def}{=} \{\omega \mid x(\omega;C) \in B\} \tag{2.111}$$

respectively.

We now assume the set C to be chosen in such a way that for each $x_1 \in X_1$ we have

$$P\left[\Xi_{[0,\infty);C;z};x_1;z\right] = 1. \tag{2.112}$$

Since each combination of a measurable ω-function and the probability space $\{\Omega;F;P\}$ generates a stochastic variable, the ω-functions $t(\omega;C)$ and $x(\omega;C)$ lead us to the stochastic variables $\underline{t}_{C;x_1}$ and $\underline{x}_{C;x_1}$. The probability distributions of these stochastic variables are given by

$$\text{Prob } \{\underline{t}_{C;x_1} \in I\} \stackrel{def}{=} P\left[\Xi_{I;C;z};x_1;z\right] \tag{2.113}$$

and

$$\text{Prob } \{\underline{x}_{C;x_1} \in B\} \stackrel{def}{=} P\left[\Delta_{B;C;z};x_1;z\right] \tag{2.114}$$

respectively.

The stochastic variable $\underline{t}_{C;x_1}$ represents the length of the time period preceding the moment at which the system first is in C, while $\underline{x}_{C;x_1}$ denotes the state at the end of this period if (2.112) is true.

Summarizing:

<u>Lemma 2.27</u>

If the assumptions 1,2,3 and 4 and condition (2.112) are satisfied, the probability distribution of the length $\underline{t}_{C;x_1}$ of the period preceding the moment at which the system first is in C and that of the state $\underline{x}_{C;x_1}$ at that point of time are defined. They are given by (2.113) and (2.114) respectively.

Let B be a closed set in X' and let us define a family of ω-functions $\{x_t(\omega;B); t \in [0,\infty)\}$ by

$$x_t(\omega;B) \stackrel{def}{=} \begin{cases} x_{t(\omega;B)+t}(\omega), & \text{if } t(\omega;B) < \infty \\ x_t(\omega), & \text{if } t(\omega;B) = \infty. \end{cases} \quad (2.115)$$

Lemma 2.28

The ω-functions $\{x_t(\omega;B); t \in [0,\infty)\}$ are measurable with respect to F.

Proof.

We first consider the ω-functions $\{v_t(\omega;B); t \in [0,\infty)\}$, defined by

$$v_t(\omega;B) \stackrel{def}{=} \begin{cases} v_{t(\omega;B)+t}(\omega), & \text{if } t(\omega;B) < \infty \\ x_t(\omega), & \text{if } t(\omega;B) = \infty. \end{cases} \quad (2.116)$$

It follows from lemma 2.25 and 1.6 ($\Lambda_o^* = M_o$) that the ω-function $v_t(\omega;B)$ is measurable with respect to F.

We obviously have

$$x_t(\omega;B) = \begin{cases} v_t(\omega;B), & \text{if } t(\omega;B) = \infty \text{ or if for each } k \\ & \quad t(\omega;B) + t \neq \hat{t}_k(\omega;A_z) \\ x_k(\omega;A_z \times X_2), & \text{if } t(\omega;B) + t = \hat{t}_k(\omega;A_z) > \\ & \quad > \hat{t}_{k-1}(\omega;A_z). \end{cases} \quad (2.117)$$

We now define the ω-functions $\{\chi_k(\omega); k=0,1,\ldots\}$ by

$$\chi_o(\omega) \stackrel{def}{=} \begin{cases} 1, & \text{if } t(\omega;B) = \infty \text{ or if for each } k \\ & \quad t(\omega;B)+t \neq \hat{t}_k(\omega;A_z) \\ 0, & \text{otherwise} \end{cases} \quad (2.118)$$

and

$$\chi_k(\omega) \stackrel{def}{=} \begin{cases} 1, & \text{if } t(\omega;B) + t = \hat{t}_k(\omega;A_z) > \hat{t}_{k-1}(\omega;A_z) \\ 0, & \text{otherwise.} \end{cases} \quad (2.119)$$

Next we consider the sequence

$$\{\chi_o(\omega)\ v_t(\omega;B) + \sum_{k=1}^{n} \chi_k(\omega)\ x_k(\omega;A_z \times X_2); n=1,2,\ldots\}. \quad (2.120)$$

Since the ω-functions $X_k(\omega)$, $v_t(\omega;B)$ and $x_k(\omega;A_z \times X_2)$ are measurable with respect to F, all elements of (2.120) are measurable. Consequently, the limit $x_t(\omega;B)$ of this sequence is measurable with respect to F.

This ends the proof.

If B and C are closed sets, let us introduce the ω-functions $t(\omega;B;C)$ and $x(\omega;B;C)$, defined by

$$t(\omega;B;C) \overset{def}{=} \begin{cases} \inf \{t \mid x_t(\omega;B) \in C\}, & \text{if } x_t(\omega;B) \in C \text{ for some finite } t \\ \infty, & \text{otherwise} \end{cases} \qquad (2.121)$$

and

$$x(\omega;B;C) \overset{def}{=} \begin{cases} x_{t(\omega;B;C)}(\omega;B), & \text{if } t(\omega;B;C) < \infty \\ x_o(\omega;B), & \text{if } t(\omega;B;C) = \infty \end{cases} \qquad (2.122)$$

respectively.

Lemma 2.29

If B and C are closed sets in X', the ω-functions $t(\omega;B;C)$ and $x(\omega;B;C)$, defined by (2.121) and (2.122), are measurable with respect to F.

Proof:

The t-function $x_t(\omega;B)$ has the same properties as the function $x_t(\omega)$. Therefore, lemma 2.29 is a direct consequence of lemmas 2.27.1 and 2.27.2.

This ends the proof.

If C is a closed set in X' and if ω is a realization of a stochastic process $S_{z;x}$, let $t(\omega;[C])$ be the moment that the system enters into C for the first time.

Repeating the arguments used in the proof of lemma 1.8.1, we can prove the following lemma:

Lemma 2.30.1

The ω-function $t(\omega;[C])$ is measurable with respect to F.

Let us introduce the ω-function $x(\omega;[C])$, defined by

$$x(\omega;[C]) \overset{def}{=} \begin{cases} x_{t(\omega;[C])}(\omega), & \text{if } t(\omega;[C]) < \infty \\ x_o(\omega), & \text{if } t(\omega;[C]) = \infty \end{cases} \quad . \tag{2.123}$$

Note that by this definition the state at the end of the period $[0,t(\omega;[C])]$ is given by $x(\omega;[C])$ unless $t(\omega;[C]) = \infty$.

<u>Lemma 2.30.2</u>

The ω-function $x(\omega;[C])$ is measurable with respect to F.

<u>Proof</u>:

We first consider the function $v(\omega;[C])$, defined by

$$v(\omega;[C]) \overset{def}{=} \begin{cases} v_{t(\omega;[C])}(\omega), & \text{if } t(\omega;[C]) < \infty \\ x_o(\omega), & \text{if } t(\omega;[C]) = \infty \end{cases} \quad . \tag{2.124}$$

By using similar arguments as in the proof of lemma 1.82 we can prove that the ω-function $v(\omega;[C])$ is measurable with respect to F.

Obviously, we have

$$x(\omega;[C]) = \begin{cases} v(\omega;[C]), & \text{if } t(\omega;[C]) \neq \hat{t}_k(\omega;A_z) \text{ for each } k \\ x_k(\omega;A_z \times X_2), & \text{if } t(\omega;[C]) = \hat{t}_k(\omega;A_z) > \\ & > \hat{t}_{k-1}(\omega;A_z). \end{cases} \tag{2.125}$$

The proof is immediate.

Let us introduce the ω-sets $\Xi_{I;[C];z}$ and $\Delta_{B;[C];z}$, defined by

$$\Xi_{I;[C];z} \overset{def}{=} \{\omega | t(\omega;[C]) \in I\} \tag{2.126}$$

and

$$\Delta_{B;[C];z} \overset{def}{=} \{\omega | x(\omega;[C]) \in B\} \tag{2.127}$$

respectively.

We now assume the closed set C to be chosen in such a way that for each $x_1 \in X_1$

$$P\left[\Xi_{[0,\infty);[C];z};x_1;z\right] = 1. \qquad (2.128)$$

The ω-functions $t(\omega;[C])$ and $x(\omega;[C])$ together with the probability space $\{\Omega;F;P\}$ generate the stochastic variables $\underline{t}_{[C];x_1}$ and $\underline{x}_{[C];x_1}$; the corresponding probability distributions are given by

$$\text{Prob}\ \{\underline{t}_{[C];x_1}\ \varepsilon\ I\} \overset{\text{def}}{=} P\left[\Xi_{I;[C];z};x_1;z\right] \qquad (2.129)$$

and

$$\text{Prob}\ \{\underline{x}_{[C];x_1}\ \varepsilon\ B\} \overset{\text{def}}{=} P\left[\Delta_{B;[C];z};x_1;z\right] \qquad (2.130)$$

respectively.

The stochastic variable $\underline{t}_{[C];x_1}$ represents the length of the time period preceding the first entry in C, while $\underline{x}_{[C];x_1}$ denotes the state at the end of that period if (2.128) is true.

Summarizing:

Lemma 2.30

If the assumptions 1,2,3 and 4 and condition (2.128) are satisfied, the probability distribution of the length $\underline{t}_{[C];x_1}$ of the period preceding the first entry in C and that of the state $\underline{x}_{[C];x_1}$ at that point of time are defined. They are given by (2.129) and (2.130) respectively.

We now consider the ω-functions

$\{x_t(\omega;[C];\ t\ \varepsilon\ [0,\infty)\}$, defined by

$$x_t(\omega;[C]) \overset{\text{def}}{=} \begin{cases} x_{t(\omega;[C])+t}(\omega), & \text{if } t(\omega;[C]) < \infty \\ x_t(\omega), & \text{if } t(\omega;[C]) = \infty . \end{cases} \qquad (2.131)$$

Repeating the arguments used in the proof of lemma 2.28, we can prove:

Lemma 2.31

The ω-functions $\{x_t(\omega;[C]);t\ \varepsilon\ [0,\infty)\}$ are measurable with respect to F.

We shall now demonstrate that decision processes with a given initial x'-state can also be defined. To this end we introduce the σ-field $\hat{H}_o$, the smallest σ-field of ω-sets with respect to which the ω-function $x_o(\omega)$ is measurable.

Lemma 2.32

A conditional probability measure $P[K;x_1;z \mid \hat{H}_o]$ can be defined on F.

Proof:

We first consider a set K of the following type:

$$K = K_o \times {}'K \tag{2.132}$$

with $K_o \in F^o$ and $'K \in F$. [3]

We can easily verify that the ω-function $\mu(K;\omega)$, defined by

$$\mu(K;\omega) \overset{def}{=} \begin{cases} P[K;x_{o;2}(\omega);z], & \text{if } x_{o;1}(\omega) = x_{o;2}(\omega) \in \bar{A}_z \\ P[K_o^c;x_{o;1}(\omega);z]\; P['K;x_{o;2}(\omega);z], & \text{otherwise} \end{cases} \tag{2.133}$$

with $K_o^c = K_o \times \Omega$, [3]
is measurable with respect to $\hat{H}_o$.

Since for $B_1 \in G_1$ and $B_2 \in G_2$ (cf. lemma 2.12)

$$\text{a) } P[K \cap \Lambda_{o;B_1 \times B_2;z};x_1;z] =$$

$$= P[K \cap \Lambda_{o;B_1 \times B_2;z} \cap \Lambda_{o;A_z \times X_2;z};x_1;z] +$$

$$+ P[K \cap \Lambda_{o;B_1 \times B_2;z} \cap \Lambda_{o;\bar{A}_z \times X_2;z};x_1;z]; \tag{2.134}$$

---

3) Really, F is a σ-field of sets in $\Omega = \prod_{j=0}^{\infty} \Omega^j$ and not in $'\Omega = \prod_{j=1}^{\infty} \Omega^j$. But, since the spaces Ω and 'Ω are isomorphic, isomorphic σ-fields, denoted with the same symbol, do not cause confusion.

b) $P\left[K \cap \Lambda_{o;B_1 \times B_2;z} \cap \Lambda_{o;A_z \times X_2;z};x_1;z\right] =$

$$= P\left[K \cap \Lambda_{o;(B_1 \cap A_z) \times B_2;z};x_1;z\right] =$$

$$= \int_{\Lambda_{o;(A_z \cap B_1) \times X_2;z}} P\left[d\omega;x_1;z\right] \int_{B_2} z(dx_2;x^o(\omega;A_z)) \quad P\left[T_{1;\omega^o}(K);x_2;z\right] =$$

$$= \int_{\Lambda_{o;(A_z \cap B_1) \times X_2;z}} P\left[d\omega;x_1;z\right] \int_{B_2} z(dx_2;x_1) \; P\left[T_{1;\omega^o}(K);x_2;z\right] =$$

$$= \int_{\Lambda_{o;(A_z \cap B_1) \times X_2;z} \cap K_o^c} P\left[d\omega;x_1;z\right] \int_{B_2} z(dx_2;x_1) \; P\left[{}'K;x_2;z\right] =$$

$$= P\left[\Lambda_{o;(A_z \cap B_1) \times X_2;z} \cap K_o^c;x_1;z\right] \int_{B_2} z(dx_2;x_1) \; P\left[{}'K;x_2;z\right] =$$

$$= \int_{\Lambda_{o;(A_z \cap B_1) \times B_2;z}} P\left[d\omega;x_1;z\right] \; P\left[K_o^c;x_{o;1}(\omega);z\right] \; P\left[{}'K;x_{o;2}(\omega);z\right]; \tag{2.135}$$

c) $P\left[K \cap \Lambda_{o;B_1 \times B_2;z} \cap \Lambda_{o;\bar{A}_z \times X_2;z};x_1;z\right] =$

$$= \int_{\Lambda_{o;(\bar{A}_z \cap B_1) \times B_2;z}} P\left[d\omega;x_1;z\right] \; P\left[K;x_{o;2}(\omega);z\right], \tag{2.136}$$

we find

$$P\left[K \cap \Lambda_{o;B_1 \times B_2;z};x_1;z\right] =$$

$$= \int_{\Lambda_{o;(A_z \cap B_1) \times B_2;z}} P\left[d\omega;x_1;z\right] \; P\left[K_o^c;x_{o;1}(\omega);z\right] \; P\left[{}'K;x_{o;2}(\omega);z\right] +$$

$$+ \int_{\Lambda_{o;(\bar{A}_z \cap B_1) \times B_2;z}} P\left[d\omega;x_1;z\right] \; P\left[K;x_{o;2}(\omega);z\right] = \int_{\Lambda_{o;B_1 \times B_2;z}} P\left[d\omega;x_1;z\right] \; \mu(K;\omega). \tag{2.137}$$

The class of all finite unions of disjunct product sets $B_1 \times B_2$, with $B_1 \varepsilon G_1$ and $B_2 \varepsilon G_2$, is a field of x'-sets. If $\bigcup_{i=1}^{n} B_{1i} \times B_{2i}$ is such a union, then we obviously have

$$P\left[K \cap \Lambda_{o;\bigcup_{i=1}^{n} B_{1i} \times B_{2i};z};x_1;z\right] =$$

$$= \sum_{i=1}^{n} \int_{\Lambda_{o;B_{1i} \times B_{2i};z}} P\left[d\omega;x_1;z\right] \mu(K;\omega) =$$

$$= \int_{\Lambda_{o;\bigcup_{i=1}^{n} B_{1i} \times B_{2i};z}} P\left[d\omega;x_1;z\right] \mu(K,\omega) \qquad (2.138)$$

Now let J_K be the class of sets $B \varepsilon G'$ with the following property:

$$P\left[K \cap \Lambda_{o;B;z};x_1;z\right] = \int_{\Lambda_{o;B;z}} P\left[d\omega;x_1;z\right] \mu(K;\omega). \qquad (2.139)$$

By (2.138) the class J_K includes the field of all finite unions of disjunct product sets. Moreover, we can easily prove that J_K contains all limits of monotone sequences of J_K-sets. Consequently, J_K includes the σ-field G'. Hence $J_K = G'$.
Since the σ-field $\hat{H}_o$ consists of sets of the form $\Lambda_{o;B;z}$ with $B \varepsilon G'$, we have now proved that

$$P\left[K \cap \Lambda;x_1;z\right] = \int_{\Lambda} P\left[d\omega;x_1;z\right] \quad \mu(K;\omega) \qquad (2.140)$$

if K satisfies (2.132) and $\Lambda \varepsilon \hat{H}_o$.

The set function $\mu(K;\omega)$ is for each ω a product probability measure, defined on the class of all product sets of the type (2.132), and therefore the domain of definition can be extended to H and F uniquely.

Henceforth the set function $\mu(K;\omega)$ is defined on F.

Now let J be the class of ω-sets $K \varepsilon F$ with the following properties:

a) the ω-function $\mu(K;\omega)$ is measurable with respect to $\hat{H}_o$;

b) for each $\Lambda \in \hat{H}_o$

$$P\left[K \cap \Lambda; x_1; z\right] = \int_{\Lambda} P\left[d\omega; x_1; z\right] \mu(K;\omega). \tag{2.141}$$

It can easily be verified that

a) $\Omega \in J$;

b) if $K \in J$, then $\overline{K} \in J$;

c) if $K_i \in J$ and if $K_i \subset K_{i+1}$ ... $(i=1,2,\ldots)$, then $\bigcup_{i=1}^{\infty} K_i \in J$.

Consequently, J is a σ-field.

According to (2.140) the class J includes product sets. Thus, $J \supset H$.

It follows from (2.133) that $K = \prod_{j=0}^{k-1} \Omega^j \times \Lambda_o \times \prod_{j=k+1}^{\infty} \Omega^j$ satisfies for each $\omega \in \Omega$

$$\mu(K;\omega) = 0. \tag{2.142}$$

Therefore, for each $\omega \in \Omega$ (cf. (2.62))

$$\mu(M_o;\omega) = 0. \tag{2.143}$$

This implies that all subsets of M_o belong to J. Hence, $J = F$.

Finally, let us define $P\left[K; x_1; z \mid \hat{H}_o\right]$ by

$$P\left[K; x_1; z \mid \hat{H}_o\right] = \mu(K;\omega). \tag{2.144}$$

The proof is complete.

The following properties of $P\left[K; x_1; z \mid \hat{H}_o\right]$ can easily be proved and are stated for later reference:

1) for each $\omega \in \Omega$ and $K \in F$

$$P\left[K; x_1; z \mid \hat{H}_o\right] = P\left[K; x_{o;2}(\omega); z\right], \text{ if } x_{o;1}(\omega) = x_{o;2}(\omega) \in \overline{A}_z; \tag{2.145}$$

2) for each $\omega \in \Omega$ and $x' \in X'$

$$P\left[\Lambda_{o;\{x_1\}\times\{x_2\};z};x_1;z \mid \hat{H}_o\right] = \begin{cases} 1, & \text{if } x_o(\omega) = (x_1,x_2) \\ 0, & \text{if } x_o(\omega) \neq (x_1,x_2) \end{cases} \tag{2.146}$$

where the product set $\{x_1\}\times\{x_2\}$ is the point set containing the single point $x' = (x_1,x_2)$.

Let us define a family of probability measures $\{P[K;x';z];x' \in X'\}$, with F as domain of definition, by

$$P[K;x';z] \overset{def}{=} P[K;x_1;z \mid \hat{H}_o], \tag{2.147}$$

where ω satisfies $x_o(\omega) = x' = (x_1,x_2)$.

The decision process $\{x_{t;x';z};t \in [0,\infty)\}$ with initial state x' is now defined by means of the ω-functions $\{x_t(\omega);t \in [0,\infty)\}$ and the probability space $\{\Omega;F;P\}$, where P is given by (2.147).

Let H_z be the smallest σ-field of ω-sets with respect to which the ω-functions $\{x_t(\omega);t \in [0,\infty)\}$ are measurable.

Presently, we need the following result:

Lemma 2.33

If $K \in H_z$, then

$$P[K;x_1;z \mid \hat{H}_o] = \begin{cases} P[K;x_{o;2}(\omega);z], & \text{if } x_{o;1}(\omega) = x_{o;2}(\omega) \in \bar{A}_z \\ P[T_{(1);\omega^o}(K);x_{o;2}(\omega);z], & \text{if } x_{o;1}(\omega) \in A_z. \end{cases} \tag{2.148}$$

Proof:

Let us first consider a set K, given by

$$K = \Lambda_{t;B_1\times B_2;z}, \tag{2.149}$$

with $B_1 \in G_1$ and $B_2 \in G_2$.

Obviously, we have

$$\Lambda_{t;B_1\times B_2;z} \cap \Lambda_{o;A_z\times X_2;z} = \begin{cases} \Lambda_{o;A_z} \times {}'\Lambda_{t;B_1\times B_2;z}, & \text{if } t>0 \\ \Lambda_{o;A_z\cap B_1} \times {}'\Lambda_{o;B_2\times X_2;z}, & \text{if } t=0 \end{cases}$$

(2.150)

where ${}'\Lambda_{t;B_1\times B_2;z}$ and ${}'\Lambda_{o;B_2\times X_2;z}$ are

a) sets in ${}'\Omega = \prod_{j=1}^{\infty} \Omega^j$;

b) isomorphic with $\Lambda_{t;B_1\times B_2;z}$ and $\Lambda_{o;B_2\times X_2;z}$ respectively.

Thus, if $t>0$,

$$\mu(\Lambda_{t;B_1\times B_2;z};\omega) = \mu(\Lambda_{t;B_1\times B_2;z}\cap\Lambda_{o;A_z\times X_2;z};\omega) +$$

$$+ \mu(\Lambda_{t;B_1\times B_2;z}\cap\Lambda_{o;\bar{A}_z\times X_2;z};\omega) =$$

$$= \mu(\Lambda_{o;A_z}\times{}'\Lambda_{t;B_1\times B_2;z};\omega) + \mu(\Lambda_{t;B_1\times B_2;z}\cap\Lambda_{o;\bar{A}_z\times X_2;z};\omega).$$

(2.151)

By (2.133) and (2.145) we find, if $t>0$,

$$\mu(\Lambda_{t;B_1\times B_2;z};\omega) =$$

$$= \begin{cases} P[\Lambda_{t;B_1\times B_2;z};x_{o;2}(\omega);z], & \text{if } x_{o;1}(\omega)\in\bar{A}_z \\ P[\Lambda^c_{o;A_z};x_{o;1}(\omega);z]\; P[\Lambda_{t;B_1\times B_2;z};x_{o;2}(\omega);z], & \text{if } x_{o;1}(\omega)\in A_z \end{cases}$$

(2.152)

and consequently,

$$\mu(\Lambda_{t;B_1\times B_2;z};\omega) = P[\Lambda_{t;B_1\times B_2;z};x_{o;2}(\omega);z]. \tag{2.153}$$

However, if t=0, we find by (2.150)

$$\mu(\Lambda_{t;B_1 \times B_2;z};\omega) = \mu(\Lambda_{o;A_z} \cap \Lambda_{o;B_1 \times B_2;z};\omega) +$$

$$+ \quad \mu(\Lambda_{o;\bar{A}_z} \cap \Lambda_{o;B_1 \times B_2;z};\omega) =$$

$$= \mu(\Lambda_{o;A_z \cap B_1} \times' \Lambda_{o;B_2 \times X_2;z};\omega) + \mu(\Lambda_{o;\bar{A}_z} \cap \Lambda_{o;B_1 \times B_2;z};\omega). \tag{2.154}$$

By (2.133) and (2.145)

$$\mu(\Lambda_{o;B_1 \times B_2;z};\omega) =$$

$$= \begin{cases} P\left[\Lambda_{o;B_1 \times B_2;z};x_{o;2}(\omega);z\right], & \text{if } x_{o;1}(\omega) \in \bar{A}_z \\ P\left[\Lambda^c_{o;A_z \cap B_1};x_{o;1}(\omega);z\right] \cdot P\left[\Lambda_{o;B_2 \times X_2;z};x_{o;2}(\omega);z\right], & \text{if } x_{o;1}(\omega) \in A_z \end{cases} \tag{2.155}$$

and consequently,

$$\mu(\Lambda_{o;B_1 \times B_2;z};\omega) = \begin{cases} P\left[\Lambda_{o;B_1 \times B_2;z};x_{o;2}(\omega);z\right], & \text{if } x_{o;1}(\omega) \in \bar{A}_z \\ 0, & \text{if } x_{o;1}(\omega) \in \bar{B}_1 \cap A_z \\ P\left[\Lambda_{o;B_2 \times X_2;z};x_{o;2}(\omega);z\right], & \text{if } x_{o;1}(\omega) \in B_1 \cap A_z. \end{cases} \tag{2.156}$$

From (2.153) and (2.156) it follows that for any t

$$\mu(\Lambda_{t;B_1 \times B_2;z};\omega) = \begin{cases} P\left[\Lambda_{t;B_1 \times B_2;z};x_{o;1}(\omega);z\right], & \text{if } x_{o;1}(\omega) \in \bar{A}_z \\ P\left[T_{(1);\omega^o}(\Lambda_{t;B_1 \times B_2;z});x_{o;2}(\omega);z\right], & \text{if } x_{o;1}(\omega) \in A_z. \end{cases} \tag{2.157}$$

Hence, the ω-sets of the type (2.149) satisfy the assertion.

Let J be the class of sets K with the following properties:

a) $J \subset H_z$;

b) the sets K satisfy (2.148).

The following points can easily be verified:

a) $\Lambda_{t;B_1 \times B_2;z} \in J$;

b) if $K \in J$, then $\bar{K} \in J$;

c) if $K_i \in J$ and if $K_i \subset K_{i+1}$ ... (i=1,2,...), then $\bigcup_{i=1}^{\infty} K_i \in J$.

Consequently, J is a σ-field that includes the sets $\Lambda_{t;B_1 \times B_2;z}$. Hence, $J = H_z$.

Lemma 2.34

If $K \in H$, we have for each $x' \in X'$ and $j \geq 1$

$$P[K;x';z] = \int_{\Omega} P[d\omega;x';z] \int_X z(dy;x^{j-1}(\omega;A_z)) \cdot$$

$$\cdot P[T_{(j);\omega^o\ldots\omega^{j-1}}(K);y;z] =$$

$$= \int_{\Omega} P[d\omega;x';z] \; P[T_{(j);\omega^o\ldots\omega^{j-1}}(K);x_o^j(\omega);z] . \tag{2.158}$$

Proof:

The proof of this lemma is similar to that of lemma 2.12.

We now make the following assumption:

Assumption 5

The basic probability space $\{\Omega^o;F^o;P^o\}$ is strongly Markovian.

In the final part of this section we shall prove two additional properties of the set function $P[K;x';z]$. In virtue of these properties the decision process is a stationary strong Markov process, as we shall see in section 4.

Let us consider the ω-functions $\{\hat{x}_t(\omega;t_o);t \in [0,\infty)\}$, defined by

$$\hat{x}_t(\omega;t_o) \overset{\text{def}}{=} \begin{cases} x_t(\omega), & \text{if } t \leq t_o \\ x_{t_o}(\omega), & \text{if } t > t_o. \end{cases} \tag{2.159}$$

The following lemma can easily be proved:

<u>Lemma 2.35</u>

The ω-functions $\{\hat{x}_t(\omega;t_o); t \in [0,\infty)\}$ are measurable with respect to F.

Let the class of ω-sets $\hat{H}_{t_o}$ be the smallest σ-field with respect to which the ω-functions $\{\hat{x}_t(\omega;t_o); t \in [0,\infty)\}$ are measurable.

Note that $\hat{H}_{t_o}$ also represents the smallest σ-field with respect to which the ω-functions $\{x_t(\omega); t \leq t_o\}$ are measurable.

Next we consider the ω-sets $\{\Xi^1_{j;t;z}; j=1,2,\ldots\}$ and $\{\Xi^2_{j;t;z}; j=0,1,\ldots\}$, defined by

$$\Xi^1_{j;t;z} \overset{\text{def}}{=} \{\omega | \hat{t}_j(\omega;A_z) = t;\ \hat{t}_k(\omega;A_z) \neq t \text{ if } k \neq j\} \tag{2.160}$$

$$\Xi^2_{j;t;z} \overset{\text{def}}{=} \{\omega | \hat{t}_j(\omega;A_z) < t < \hat{t}_{j+1}(\omega;A_z)\} \tag{2.161}$$

with $\hat{t}_o(\omega;A_z) = 0$.

Obviously, we have

$$\Xi^i_{j;t;z} \in H;\ i=1,2. \tag{2.162}$$

<u>Lemma 2.36</u>

If $\Lambda \in \hat{H}_{t_o}$, the ω-sets

$$\Lambda \cap \Xi^2_{o;t_o;z} \tag{2.163}$$

and

$$T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda \cap \Xi^2_{j;t_o;z});\ j=1,2,\ldots \tag{2.164}$$

are cylinder sets of the respective forms

$$(\Lambda^o \cap \Xi_{(t_o,\infty);A_z}) \times \prod_{h=1}^{\infty} \Omega^h \tag{2.165}$$

and

$$(\Lambda^j \cap \Xi_{(t_o^j,\infty);A_z}) \times \prod_{h=1}^{\infty} \Omega^h; \; j=1,2,\ldots, \tag{2.166}$$

where $\Lambda^j \in \hat{F}^o_{t_o^j}$ (cf. p.38 with $* = 0$) and $t_o^j = t_o - \hat{t}_j(\omega;A_z)$.

<u>Proof</u>:

Obviously, we have

$$\{\omega \mid \hat{x}_t(\omega;t_o) \in B\} = \begin{cases} \Lambda_{t;B;z}, & \text{if } t \leqq t_o \\ \Lambda_{t_o;B;z}, & \text{if } t > t_o \end{cases} . \tag{2.167}$$

We first consider the case

$$\Lambda = \Lambda_{t;B_1 \times B_2;z} . \tag{2.168}$$

It can easily be verified that for $t \leqq t_o$, we find

a) $$\Lambda_{t;B_1 \times B_2;z} \cap \Xi^2_{o;t_o;z} = (\Lambda_{t;B_1 \cap B_2} \cap \Xi_{(t_o,\infty);A_z}) \times \prod_{j=1}^{\infty} \Omega^j \tag{2.169}$$

Hence for $t \leqq t_o$ the set $\Lambda_{t;B_1 \times B_2;z}$ satisfies the assertion;

b) $$T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda_{t;B_1 \times B_2;z} \cap \Xi^2_{j;t_o;z}) =$$

$$= T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda_{t;B_1 \times B_2;z}) \cap T_{(j);\omega^o\ldots\omega^{j-1}}(\Xi^2_{j;t_o;z}) =$$

$$= \begin{cases} \emptyset, & \text{if } t < \hat{t}_j(\omega;A_z) \text{ and } \omega \in \Lambda_{t;\overline{B_1 \times B_2};z} \text{ or if } t = \hat{t}_j(\omega;A_z) \text{ and } \omega \in \Lambda_{t;\overline{B}_1 \times X_2;z} \\ \Xi_{(t_o-\hat{t}_j(\omega;A_z),\infty);A_z} \times \prod_{j=1}^{\infty} \Omega^j, & \text{if } t < \hat{t}_j(\omega;A_z) \text{ and } \omega \in \Lambda_{t;B_1 \times B_2;z} \\ (\Lambda_{o;B_2} \cap \Xi_{(t_o-\hat{t}_j(\omega;A_z),\infty);A_z}) \times \prod_{j=1}^{\infty} \Omega^j, & \text{if } t = \hat{t}_j(\omega;A_z) \text{ and } \omega \in \Lambda_{t;B_1 \times X_2;z} \\ (\Lambda_{t-\hat{t}_j(\omega;A_z);B_1 \cap B_2} \cap \Xi_{(t_o-\hat{t}_j(\omega;A_z),\infty);A_z}) \times \prod_{j=1}^{\infty} \Omega^j, & \text{if } t_o \geqq t > t_j(\omega;A_z). \end{cases} \tag{2.170}$$

Since the ω-function $\hat{t}_j(\omega;A_z)$ only depends on the components $(\omega^o,\ldots,\omega^{j-1})$, it follows from (2.170) that sets of the type (2.168) with $t \leq t_o$ satisfy the assertion.

Now let J be the class of ω-sets Λ with the following properties:

a) $\Lambda \varepsilon \hat{H}_{t_o}$;

b) the ω-sets Λ satisfy the assertion.

The following points can easily be verified:

a) $\Lambda_{t;B_1 \times B_2;z} \varepsilon J$ if $t \leq t_o$;

b) if $K \varepsilon J$, then $\bar{K} \varepsilon J$;

c) if $K_i \varepsilon J$ and if $K_i \subset K_{i+1} \ldots$ $(i=1,2,\ldots)$ then $\bigcup_{i=1}^{\infty} K_i \varepsilon J$.

Consequently, J is a σ-field that includes the sets $\Lambda_{t;B_1 \times B_2;z}$ with $t \leq t_o$. Hence $J=\hat{H}_{t_o}$

This ends the proof.

Lemma 2.37

If $\Lambda \varepsilon F^o_t$ and if $B \varepsilon G'$, then, under the assumptions 1 through 5, for each $s \varepsilon [0,\infty)$ and $x_1 \varepsilon X_1$ we have

$$\int_{\Lambda \cap \Xi_{(t_1,\infty);A_z}} P^o\left[d\omega^o;x_1\right] \int_{X_2} z(du;x^o(\omega^o;A_z)) \quad P\left[T_{(1);\omega^o}(\Lambda_{s+t_1;B;z});u;z\right]$$

$$=\int_{\Lambda \cap \Xi_{(t_1,\infty);A_z}} P^o\left[d\omega^o;x_1\right] P\left[\Lambda_{s;B;z};x^o_{t_1}(\omega^o);z\right]. \tag{2.171}$$

Proof:

Let us consider the functions $y_{t_1}(\omega^o_1)$ and $y(\omega^o)$, defined by (cf. (1.205) and (1.206))

$$y_{t_1}(\omega^o_1) \overset{def}{=} \int_{X_2} z(du;x^o(\omega^o_1;A_z)) \quad P\left[T_{(1);\omega^o_1}(\Lambda_{s;B;z});u;z\right] \tag{2.172}$$

and (cf. (1.98))

$$y(\omega^o) = y_{t_1}(T_{t_1}(\omega^o)). \tag{2.173}$$

By lemmas 1.35 and 2.12

$$\int_{\Lambda \cap \Xi_{(t_1,\infty);A_z}} P^o\left[d\omega^o;x_1\right] y(\omega^o) =$$

$$=\int_{\Lambda \cap \Xi_{(t_1,\infty);A_z}} P^o\left[d\omega^o;x_1\right]\int_{\Omega^o} P^o\left[d\omega_1^o;x_{t_1}^o(\omega^o)\right] y_{t_1}(\omega_1^o) =$$

$$=\int_{\Lambda \cap \Xi_{(t_1,\infty);A_z}} P^o\left[d\omega^o;x_1\right] P\left[\Lambda_{s;B;z};x_{t_1}^o(\omega^o);z\right] . \qquad (2.174)$$

If $\omega^o \varepsilon \Xi_{(t_1,\infty);A_z}$ and if $\omega_1^o = T_{t_1}(\omega^o)$, then we can easily verify that

$$x^o(\omega^o;A_z) = x^o(\omega_1^o;A_z). \qquad (2.175)$$

Since for $\omega^o \varepsilon \Xi_{(t_1,\infty);A_z}$ and $\omega_1^o = T_{t_1}(\omega^o)$ holds

$$t(\omega^o;A_z) = t(\omega_1^o;A_z) + t_1, \qquad (2.176)$$

we find

$$T_{(1);\omega_1^o}(\Lambda_{s;B;z}) = T_{(1);\omega^o}(\Lambda_{s+t_1;B;z}). \qquad (2.177)$$

Consequently, by (2.172), (2.173), (2.175) and (2.177),

$$y(\omega^o) = \int_{X_2} z(du;x^o(\omega^o;A_z))\ P\left[T_{(1);\omega^o}(\Lambda_{s+t_1;B;z});\ ;u;z\right]. \qquad (2.178)$$

Hence, by (2.174) and (2.178)

$$\int_{\Lambda\cap\Xi_{(t_1,\infty);A_z}} P^o\left[d\omega^o;x_1\right] \int_{X_2} z(du;x^o(\omega^o;A_z))\ P\left[T_{(1);\omega^o}(\Lambda_{s+t_1;B;z});u;z\right] =$$

$$\int_{\Lambda\cap\Xi_{(t_1,\infty);A_z}} P^o\left[d\omega^o;x_1\right] P\left[\Lambda_{s;B;z};x_{t_1}^o(\omega^o);z\right]. \qquad (2.179)$$

This ends the proof.

Lemma 2.38

If $\Lambda \varepsilon \hat{H}'_{t_o}$, under the assumptions 1 through 5, for each $s \varepsilon [0,\infty)$, $x \varepsilon X'$ and $B \varepsilon G'$ we have

$$P[\Lambda \cap \Lambda_{s+t_o;B;z};x';z] = \tag{2.180}$$
$$= \int_{\Omega} P[d\omega;x';z]\ P[\Lambda_{s;B;z};x_{t_o}(\omega);z].$$

Proof:

It can easily be verified that for each $x' \varepsilon X'$ and $t_o > 0$ we have (cf. (2.76), (2.160) and (2.161))

$$P[\bigcup_{j=1}^{\infty} \Xi^1_{j;t_o;z} \cup \bigcup_{j=0}^{\infty} \Xi^2_{j;t_o;z};x';z] = 1. \tag{2.181}$$

So that, by lemma 2.34,

$$P[\Lambda \cap \Lambda_{s+t_o;B;z};x';z] =$$
$$= P[\Lambda \cap (\bigcup_{j=1}^{\infty} \Xi^1_{j;t_o;z} \cap \bigcup_{j=0}^{\infty} \Xi^2_{j;t_o;z}) \cap \Lambda_{s+t_o;B;z};x';z] =$$
$$= P[\Lambda \cap \Xi^2_{o;t_o;z} \cap \Lambda_{s+t_o;B;z};x';z] +$$
$$+ \sum_{j=1}^{\infty} \sum_{i=1}^{2} \int_{\Omega} P[d\omega;x';z] \int_{X_2} z(du;x^{j-1}(\omega;A_z)) \cdot$$
$$\cdot P[T_{(j);\omega o\ldots\omega j-1}(\Lambda \cap \Xi^i_{j;t_o;z} \cap \Lambda_{s+t_o;B;z});u;z]. \tag{2.182}$$

We first consider the term $P[\Lambda \cap \Xi^2_{o;t_o;z} \cap \Lambda_{s+t_o;B;z};x';z]$ of the right hand side of (2.182).

By lemmas 2.36 and 2.37 we find

$$P\left[\Lambda \cap \Xi^2_{o;t_o;z} \cap \Lambda_{s+t_o;B;z};x';z\right] =$$

$$= P\left[((\Lambda^o \cap \Xi_{(t_o,\infty);A_z}) \times \prod_{h=1}^{\infty} \Omega^h) \cap \Lambda_{s+t_o;B;z};x';z\right] =$$

$$= \int_{\Lambda^o \cap \Xi_{(t_o,\infty);A_z}} P^o\left[d\omega^o;x_1\right] \int_{X_2} z(du;x^o(\omega^o;A_z)) \cdot$$

$$\cdot P\left[T_{(1);\omega^o}(\Lambda_{s+t_o;B;z});u;z\right] =$$

$$= \int_{\Lambda^o \cap \Xi_{(t_o,\infty);A_z}} P^o\left[d\omega^o;x_1\right] P\left[\Lambda_{s;B;z};x^o_{t_o}(\omega^o);z\right] =$$

$$= \int_{\Lambda \cap \Xi^2_{o;t_o;z}} P\left[d\omega;x';z\right] P\left[\Lambda_{s;B;z};x_{t_o;2}(\omega);z\right]. \qquad (2.183)$$

Since for $\omega \in \Xi^2_{o;t_o;z}$ holds $x_{t_o;1}(\omega) = x_{t_o;2}(\omega) \in \bar{A}_z$, we find by means of (2.145) and (2.147)

$$P\left[\Lambda_{s;B;z};x_{t_o;2}(\omega);z\right] = P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right]. \qquad (2.184)$$

Hence, by (2.183) and (2.184)

$$P\left[\Lambda \cap \Xi^2_{o;t_o;z} \cap \Lambda_{s+t_o;B;z};x';z\right] = \int_{\Lambda \cap \Xi^2_{o;t_o;z}} P\left[d\omega;x';z\right] \cdot$$

$$\cdot P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right]. \qquad (2.185)$$

Next we consider the term

$$\int_{\Omega} P\left[d\omega;x';z\right] \int_{X_2} z(du;x^{j-1}(\omega;A_z)) \cdot$$

$$\cdot P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda \cap \Xi^2_{j;t_o;z} \cap \Lambda_{s+t_o;B;z});u;z\right] \qquad (2.186)$$

of the right hand side of (2.182).

By means of lemmas 2.34 and 2.36 the expression (2.186) can be rewritten in

$$\int_{\Omega} P\left[d\omega;x';z\right] \int_{X_2} z(du;x^{j-1}(\omega;A_z)) \cdot$$

$$\cdot \int_{\Lambda^j \cap \Xi_{(t_o^j;\infty);A_z}} P^o\left[d\omega_1^o;u\right] \int_{X_2} z(dv;x^o(\omega_1^o;A_z))\ P\left[T_{(1);\omega_1^o}(\Lambda_{s+t_o^j;B;z});\ ;v;z\right] . \qquad (2.187)$$

By means of lemma 2.37, we find for (2.187)

$$\int_{\Omega} P\left[d\omega;x';z\right] \int_{X_2} z(du;x^{j-1}(\omega;A_z)) \cdot$$

$$\cdot \int_{\Lambda^j \cap \Xi_{(t_o^j;\infty);A_z}} P^o\left[d\omega_1^o;u\right] \cdot P\left[\Lambda_{s;B;z};x^o_{t_o^j}(\omega_1^o);z\right] =$$

$$= \int_{\Lambda \cap \Xi^2_{j;t_o;z}} P\left[d\omega;x';z\right]\ P\left[\Lambda_{s;B;z};x_{t_o;2}(\omega);z\right] . \qquad (2.188)$$

Since, by (2.145) and (2.147),

$$P\ \left[\Lambda_{s;B;z};x_{t_o;2}(\omega);z\right] = P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right] \qquad (2.189)$$

for each $\omega \in \Xi^2_{j;t_o;z}$, (2.186) becomes

$$\int_{\Lambda \cap \Xi^2_{j;t_o;z}} P\left[d\omega;x';z\right]\ \ P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right] . \qquad (2.190)$$

Thus,

$$\int_{\Omega} P\left[d\omega;x';z\right] \int_{X_2} z(du;x^{j-1}(\omega;A_z)) \cdot$$

$$\cdot\ P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda \cap \Xi^2_{j;t_o;z} \cap \Lambda_{s+t_o;B;z});\ u;z\right] =$$

$$= \int_{\Lambda \cap \Xi^2_{j;t_o;z}} P\left[d\omega;x';z\right]\ P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right] . \qquad (2.191)$$

We now consider the term

$$\int_{\Omega} P\left[d\omega;x';z\right] \int_{X_2} z(du;x^{j-1}(\omega;A_z)) \cdot$$

$$\cdot\ P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda \cap \Xi^1_{j;t_o;z} \cap \Lambda_{s+t_o;B;z});\ ;u;z\right] . \quad (2.192)$$

Since for each $\omega \in \Xi^1_{j;t_o;z}$ we have $\hat{t}_j(\omega;A_z) = t_o$, (2.192) becomes

$$\int_{\Lambda \cap \Xi^1_{j;t_o;z}} P\left[d\omega;x';z\right] P\left[(T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda_{s+t_o;B;z});x^j_o(\omega^j);z\right] . \quad (2.193)$$

According to assumption 2 to each $\omega \in \Omega$ corresponds one and only one point $\omega_1 = (\omega_1^o,\omega_1^1,\ldots) \in \Omega$, given by

$$x^o_t(\omega_1) = x^{j-1}_{t+t_o-\hat{t}_{j-1}(\omega;A_z)}(\omega) \quad (2.194)$$

and

$$x^k_t(\omega_1) = x^{k+j-1}_t(\omega);\ k=1,2,\ldots . \quad (2.195)$$

We obviously have

$$x^1_o(\omega_1) = x^j_o(\omega) \quad (2.196)$$

and

$$T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda_{s+t_o;B;z}) = T_{(1);\omega_1^o}(\Lambda_{s+t_o-\hat{t}_j(\omega;A_z);B;z}) . \quad (2.197)$$

Thus, if $\hat{t}_j(\omega;A_z) = t_o$,

$$P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda_{s+t_o;B;z});x^j_o(\omega^j);z\right] =$$

$$= P\left[T_{(1);\omega_1^o}(\Lambda_{s;B;z});x^1_o(\omega^1_1);z\right] . \quad (2.198)$$

By (2.147) and (2.148)

$$P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda_{s+t_o;B;z});x^j_o(\omega^j);z\right] =$$

$$= P\left[T_{(1);\omega_1^o}(\Lambda_{s;B;z});x_o^1(\omega_1^1);z\right] = P\left[\Lambda_{s;B;z};x_o(\omega_1);z\right] =$$

$$= P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right]. \tag{2.199}$$

Consequently, the expression (2.193) turns out to be equal to

$$\int_{\Lambda\cap\Xi^1_{j;t_o;z}} P\left[d\omega;x';z\right] \; P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right]. \tag{2.200}$$

Hence, by (2.192), (2.193) and (2.200)

$$\int_\Omega P\left[d\omega;x';z\right] \int_{X_2} z(du;x^{j-1}(\omega;A_z)) \cdot$$

$$\cdot P\left[T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda\cap\Xi^1_{j;t_o;z}\cap\Lambda_{s+t_o;B;z});\right.$$

$$\left.;u;z\right] =$$

$$= \int_{\Lambda\cap\Xi^1_{j;t_o;z}} P\left[d\omega;x';z\right] \; P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right]. \tag{2.201}$$

Finally, it follows from (2.185), (2.191) and (2.201) that

$$P\left[\Lambda\cap\Lambda_{s+t_o;B;z};x';z\right] = \int_{\Lambda\cap\Xi^2_{o;t_o;z}} P\left[d\omega;x';z\right] \; P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right] +$$

$$+ \sum_{i=1}^{2}\sum_{j=1}^{\infty}\int_{\Lambda\cap\Xi^i_{j;t_o;z}} P\left[d\omega;x';z\right] \; P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right] =$$

$$= \int_\Lambda P\left[d\omega;x';z\right] \; P\left[\Lambda_{s;B;z};x_{t_o}(\omega);z\right]. \tag{2.202}$$

This ends the proof.

Let C be a closed set in X', satisfying for each $x'\varepsilon X'$

$$P\left[\Xi_{[0,\infty);[C];z};x';z\right] = 1. \tag{2.203}$$

We now introduce the functions $\{\hat{x}_t(\omega;[C]);t\in[0,\infty)\}$, defined by

$$\hat{x}_t(\omega;[C]) \overset{def}{=} \begin{cases} x_t(\omega), \text{ if } t \leq t(\omega;[C]) \\ x(\omega;[C]), \text{ if } t \geq t(\omega;[C]). \end{cases} \qquad (2.204)$$

The following lemma can easily be proved:

<u>Lemma 2.39</u>

The ω-functions $\{\hat{x}_t(\omega;[C]);t\in[0,\infty)\}$ are measurable with respect to F.

Let the class of ω-sets $\hat{H}_{[C]}$ be the smallest σ-field with respect to which the ω-functions $\{\hat{x}_t(\omega;[C]);t\in[0,\infty)\}$ are measurable.

If C is a closed set in X', let the x_1-set $\hat{C}$ be defined by

$$\hat{C} \overset{def}{=} \{x_1 \mid (x_1,x_1) \in C\} . \qquad (2.205)$$

The proof of the following lemma is left to the reader.

<u>Lemma 2.40</u>

If $C \in G'$, then $\hat{C} \in G_1$.

Let us consider the ω-sets $\{\Xi^1_{j;[C];z};j=1,2,\ldots\}$ and $\{\Xi^2_{j;[C];z};j=0,1,\ldots\}$, defined by

$$\Xi^1_{j;[C];z} \overset{def}{=} \{\omega \mid t(\omega;[C]) = \hat{t}_j(\omega;A_z); \\ \hat{t}_k(\omega;A_z) \neq t(\omega;[C]),k\neq j\} \qquad (2.206)$$

and

$$\Xi^2_{j;[C];z} \overset{def}{=} \{\omega \mid \hat{t}_j(\omega;A_z) < t(\omega;[C]) < \hat{t}_{j+1}(\omega;A_z)\} \qquad (2.207)$$

with $\hat{t}_o(\omega;A_z) = 0$.

Obviously, we have

$$\Xi^i_{j;[C];z} \in H;\ i=1,2 \qquad (2.208)$$

Finally, the ω^o-set $\Xi_{(t_{[\hat{C}]},\infty);A_z}$, defined by

$$\Xi_{(t_{[\hat{C}]},\infty);A_z} \overset{def}{=} \{\omega^o | t(\omega^o;A_z) > t(\omega^o;[C])\} \quad (2.209)$$

will be used in the coming discussion.

Clearly, (cf.p.38)

$$\Xi_{(t_{[\hat{C}]},\infty);A_z} \in \hat{F}^o_{[\hat{C}]} \quad . \quad (2.210)$$

Lemma 2.41

If $\Lambda \in \hat{H}_{[C]}$, the ω-sets

$$\Lambda \cap \Xi^2_{o;[C];z} \quad (2.111)$$

and

$$T_{(j);\omega^o\ldots\omega^{j-1}}(\Lambda \cap \Xi^2_{j;[C];z});j=1,2,\ldots \quad (2.112)$$

are cylinder sets of the respective forms

$$(\Lambda^o \cap \Xi_{(t_{[\tilde{C}]},\infty);A_z}) \times \prod_{h=1}^{\infty} \Omega^h \quad (2.213)$$

and

$$(\Lambda^j \cap \Xi_{(t_{[\hat{C}]},\infty);A_z}) \times \prod_{h=1}^{\infty} \Omega^h;j=1,2,\ldots \quad , \quad (2.214)$$

where $\Lambda^j \in \hat{F}^o_{[\hat{C}]}$.

Proof:

The proof of this lemma is similar to that of lemma 2.36.

Let us introduce the ω-set $\Lambda_{s;B;[C];z}$, defined by (cf.(2.131))

$$\Lambda_{s;B;[C];z} \overset{def}{=} \{\omega | x_s(\omega;[C]) \in B\} \quad . \quad (2.215)$$

Lemma 2.42

If $\Lambda \in \hat{F}^o_{[C]}$ and if $B \in G'$, then, under the assumptions 1 through 5 and (2.203), for each $s \in [0,\infty)$ and $x_1 \in X_1$ we have

$$\int_{\Lambda\cap\Xi_{(t_{[\hat{C}]},\infty);A_z}} P^o\left[d\omega^o;x_1\right] \int_{X_2} z(du;x^o(\omega;A_z)) \quad P\left[T_{1;\omega^o}(\Lambda_{s;B;[C];z});u;z\right] =$$

$$=\int_{\Lambda\cap\Xi_{(t_{[\hat{C}]},\infty);A_z}} P^o\left[d\omega^o;x_1\right] \quad P\left[\Lambda_{s;B;z};x^o(\omega^o;[\hat{C}]);z\right] . \qquad (2.216)$$

Proof:

The proof of this lemma is similar to that of lemma 2.37.

Lemma 2.43

If $\Lambda \in \hat{H}_{[C]}$, under the assumptions 1 through 5 and (2.203), for each $s \in [0,\infty)$, $x' \in X'$ and $B \in G'$ we have

$$P[\Lambda \cap \Lambda_{s;B;[C];z};x';z] =$$

$$= \int_{\Lambda} P\left[d\omega;x';z\right] \; P\left[\Lambda_{s;B;z};x(\omega;[C]);z\right] . \qquad (2.217)$$

Proof:

The proof is similar to that of lemma 2.38.

4. A new foundation of the decision process

In this section we shall give a formulation of the decision process which is similar to that of the fundamental stochastic process in chapter 1. Next we shall show that these two stochastic processes have nearly equal properties.

Let the class H_z be the smallest σ-field of ω-sets with respect to which the ω-functions $\{x_t(\omega); t \in [0,\infty)\}$ are measurable.

We now introduce the ω-set $M_{o;z}$, the smallest set with the following properties:

1) for each $\omega \in \overline{M}_{o;z}$, the t-function $x_{t;2}(\omega)$ is continuous from the right;
2) in each bounded time interval in $[0,\infty)$ and for each $\omega \in \overline{M}_{o;z}$ the t-function $x_t(\omega)$ has only a finite number of discontinuities.

Since $M_{o;z} \subset M_o$ we have for each x_1

$$P\left[M_{o;z};x_1;z\right] = 0. \qquad (2.218)$$

Let the class F_z be the smallest σ-field of ω-sets with the following properties:

1) $F_z \supset H_z$;
2) F_z contains all subsets of $M_{o;z}$;
3) the ω-functions $t(\omega;C)$, $t(\omega;[C])$, $x(\omega;C)$ and $x(\omega;[C])$ are measurable with respect to F_z if C is any closed set in X'.

We now consider

1) a space Ω^z with points ω^z;
2) a family of ω^z-functions $\{x_t^z(\omega^z);t \in [0,\infty)\}$, defined on Ω^z, such that
   a) for each $t \in [0,\infty)$ the ω^z-function $x_t^z(\omega^z)$ maps Ω^z into X';
   b) if x'(t) is any mapping of the time axis into the state space X', one and only one point ω^z can be found such that

$$x_t^z(\omega^z) = x'(t); \qquad t \in [0,\infty). \qquad (2.219)$$

Consequently, a 1-1 correspondence exists between realizations of the decision process and points $\omega^z \in \Omega$.

Similar to the ω-functions $t(\omega;C), t(\omega;[C])$, $x^{*}(\omega;C)$ and $x^{*}(\omega;[C])$ in chapter 1 of this part, we can define ω^z-functions $t^z(\omega^z;C)$, $t^z(\omega^z;[C])$, $x^z(\omega^z;C)$ and $x^z(\omega^z;[C])$. [4)]

Since each point $\omega \in \Omega$ corresponds to one and only one realization $\{x'(t);t \in [0,\infty)\}$ of the decision process, (2.219) also defines a point transformation

$$\omega^z = T_z(\omega) \qquad (2.220)$$

from Ω onto Ω^z.

If $\omega^z = T_z(\omega)$, then

---

4) In order to save confusion the ω^z-functions $t(\omega^z;C)$ and $t(\omega^z;[C])$ have been indexed.

$$x_t(\omega) = x_t^z(T_z(\omega)), \qquad (2.221)$$

$$t(\omega;C) = t^z(T_z(\omega);C), \qquad (2.222)$$

$$t(\omega;[C]) = t^z(T_z(\omega);[C]), \qquad (2.223)$$

$$x(\omega;C) = x^z(T_z(\omega);C), \qquad (2.224)$$

and

$$x(\omega;[C] = x^z(T_z(\omega);[C]). \qquad (2.225)$$

Let Λ_o^z be the smallest ω^z-set with the following properties:

1) for each $\omega^z \varepsilon \bar{\Lambda}_o^z$, the t-function $x_{t;2}^z(\omega^z)$ is continuous from the right;

2) in each bounded time interval in $[0,\infty)$ and for each $\omega^z \varepsilon \bar{\Lambda}_o^z$ the t-function $x_t^z(\omega^z)$ has only a finite number of discontinuities.

Next we define the ω^z-sets $\Lambda_{t;B}^z$, $\Xi_{I;C}^z$, $\Xi_{I;[C]}^z$, $\Delta_{B;C}^z$, $\Delta_{B;[C]}^z$ and $\Lambda_{s;B;[C]}^z$ by

$$\Lambda_{t;B}^z \overset{\text{def}}{=} \{\omega^z \mid x_t^z(\omega^z) \varepsilon B\}, \qquad (2.226)$$

$$\Xi_{I;C}^z \overset{\text{def}}{=} \{\omega^z \mid t^z(\omega^z;C) \varepsilon I\}, \qquad (2.227)$$

$$\Xi_{I;[C]}^z \overset{\text{def}}{=} \{\omega^z \mid t^z(\omega^z;[C]) \varepsilon I\}, \qquad (2.228)$$

$$\Delta_{B;C}^z \overset{\text{def}}{=} \{\omega^z \mid x^z(\omega^z;C) \varepsilon B\} \qquad (2.229)$$

$$\Delta_{B;[C]}^z \overset{\text{def}}{=} \{\omega^z \mid x^z(\omega^z;[C]) \varepsilon B\} \qquad (2.230)$$

and

$$\Lambda_{s;B;[C]}^z \overset{\text{def}}{=} \{\omega^z \mid x_s^z(\omega^z;[C]) \varepsilon B\}. \qquad (2.231)$$

Obviously, if we define the set transformations

$$K = T_z^{-1}(K_1) \qquad (2.232)$$

and

$$K_1 = T_z(K) \qquad (2.333)$$

by

$$T_z^{-1}(K_1) \overset{\text{def}}{=} \{\omega \mid \omega^z = T_z(\omega);\ \omega^z \in K_1\} \tag{2.234}$$

and

$$T_z(K) \overset{\text{def}}{=} \{\omega^z \mid \omega^z = T_z(\omega);\omega \in K\} \tag{2.235}$$

respectively,

then

$$\Lambda_o^z = T_z(M_{o;z}), \tag{2.236}$$

$$M_{o;z} = T_z^{-1}(\Lambda_o^z), \tag{2.237}$$

$$\Lambda_{t;B}^z = T_z(\Lambda_{t;B;z}), \tag{2.238}$$

$$\Lambda_{t;B;z} = T_z^{-1}(\Lambda_{t;B}^z), \tag{2.239}$$

$$\Lambda_{s;B;[C]}^z = T_z(\Lambda_{s;B;[C];z}) \tag{2.240}$$

and

$$\Lambda_{s;B;[C];z} = T_z^{-1}(\Lambda_{s;B;[C]}^z). \tag{2.241}$$

Let the class H^z be the smallest σ-field of ω^z-sets with respect to which the ω^z-functions $\{x_t^z(\omega);t \in [0,\infty)\}$ are measurable.

We now introduce F^z, the smallest σ-field with the following properties:

1) $F^z \supset H^z$;
2) F^z contains all subsets of Λ_o^z;
3) the ω-functions $t^z(\omega^z;C)$, $t^z(\omega^z;[C])$, $x^z(\omega^z;C)$ and $x^z(\omega^z;[C])$ are measurable with respect to F^z if C is any closed set in X'.

The following lemma can easily be proved:

Lemma 2.44

The set transformation $K = T_z^{-1}(K_1)$ generates an isomorphism of F^z with F_z.

Now we are in a position to define probability measures on F^z.

These set functions,

$$\{P^z\ [K_1;x_1]\ ;\ x_1 \in X_1\}\ , \tag{2.242}$$

are defined on F^z by

$$P^z \left[K_1;x_1\right] \stackrel{def}{=} P\left[T_z^{-1}(K_1);x_1;z\right] \quad . \tag{2.243}$$

Hence, the ω^z-functions $\{x_t^z(\omega^z);t\varepsilon\left[0,\infty\right)\}$ and the probability space $\{\Omega^z;F^z;P^z[\cdot\,;x_1]\}$ provide us with an alternative description of the decision process in X'.

Decision processes, defined in this way, are denoted by

$$S_{x_1}^z \equiv \{\underline{x}_{t;x_1}^z;t\,\varepsilon\left[0,\infty\right)\} \quad . \tag{2.244}$$

We already know that, if a decision process is described by means of a set function $P\left[K;x_1;z\right]$, the x_2-component of the initial state obeys an initial distribution. In section 3 we found a set function $P\left[K;x';z\right]$ that describes the decision process in case the initial x_2-state has also been given.

If on F^z the set functions

$$\{P^z\left[K_1;x'\right] \;;\; x'\,\varepsilon\,X'\} \tag{2.245}$$

are defined by

$$P^z\left[K_1;x'\right] \stackrel{def}{=} P\left[T_z^{-1}(K_1);\,x';z\right] \quad , \tag{2.246}$$

then the ω^z-functions $\{x_t^z(\omega^z);t\varepsilon\left[0,\infty\right)\}$ together with the probability space $\{\Omega^z;F^z;P^z[\cdot\,;x']\}$ generate the decision process with initial state x'.

Decision processes, defined in this way, are denoted by

$$S_{x'}^z \equiv \{\underline{x}_{t;x'}^z;t\,\varepsilon\left[0,\infty\right)\} \quad . \tag{2.247}$$

Finally, let us compare the fundamental stochastic processes $\{S_x^*;x\,\varepsilon\,X^*\}$, described in chapter 1, with the decision processes $\{S_{x'}^z;x'\,\varepsilon\,X'\}$.

It follows from lemma 2.23 and (2.221) that the decision processes $\{S_{x'}^z;x'\,\varepsilon\,X'\}$ do not satisfy assumption 1 completely. (Cf. chapter 1, p.2 ,*=z and M=2N). [5)]

In the points $\{\hat{t}_j(\omega^z;A_z);j=1,2,\dots\}$ almost all t-functions $\{x_t^z(\omega^z);\omega^z \varepsilon \Omega^z\}$ are not continuous from the right. Therefore the proofs of lemmas 1.1 through 1.9 do not apply to decision processes. However, in this chapter (lemma 2.22 ff.) we have demonstrated that the assertions stated in lemmas 1.1 through 1.9 remain true for these processes.

By the choice of Ω^z the decision processes $\{S^z_{x'};x' \varepsilon X'\}$ satisfy assumption 2. (Cf. chapter 1,p.17 , *=z and M=2N). According to (2.146), (2.147), (2.238) and (2.246) assumption 3 (cf. chapter 1,p.40 , *=z) is also fulfilled. This implies that the results obtained in chapter 1 of this part also apply to decision processes.

5. Stationary strong Markovian decision processes

In this section we shall show, that if the basic probability space $\{\Omega^o;F^o;P^o\}$ is strongly Markovian the decision processes $\{S^z_{x'};x' \varepsilon X'\}$ are stationary strong Markov processes.

It follows from lemma 2.38, (2.226) and (2.246) that for each pair of non-negative values (t_o,s), $B \varepsilon G'$, $x' \varepsilon X'$ and $\Lambda \varepsilon \hat{H}^z_{t_o}$ (cf. chapter 1,p.37 , *=z)

$$P^z\left[\Lambda^z_{t_o+s;B};x'\right] = \int_\Lambda P^z\left[d\omega^z;x'\right] P^z\left[\Lambda^z_{s;B};x^z_{t_o}(\omega^z)\right] . \tag{2.248}$$

Lemma 2.45.1

If $\Lambda \varepsilon \hat{H}^z_{t_o}$, $t_o \varepsilon [0,\infty)$, $x' \varepsilon X'$ and $B \varepsilon G'$ then, under the assumptions 1 through 5, we have for each $K \varepsilon H^z_{t_o}$ (cf. chapter 1,p. 37, *=z)

$$P^z\left[K \cap \Lambda;x'\right] = \int_\Lambda P^z\left[d\omega^z;x'\right] \quad P^z\left[T_{t_o}(K);x^z_{t_o}(\omega^z)\right] . \tag{2.249}$$

---

5) *=O means: "read O where we wrote *"

Proof:

Let J be the class of ω^z-sets K with the following properties:

a) $K \in H^z_{t_o}$;

b) the sets K satisfy (2.249).

Obviously, by (2.248)

$$\Lambda^z_{t_o+s;B} \in J; \quad s \geqq 0. \tag{2.250}$$

We can easily verify that

a) $\Omega^z \in J$;

b) if $K \in J$, then $\overline{K} \in J$;

c) if $K_i \in J$ $(i=1,2,\ldots)$ and if $K_i \subset K_{i+1} \subset \ldots$, then $\bigcup_{i=1}^{\infty} K_i \in J$.

Consequently, J is a σ-field that contains the sets $\Lambda^z_{t_o+s;B}$ with $s \geqq 0$. Hence, $J=\hat{H}^z_{t_o}$.

This ends the proof.

If follows from lemma 2.43, (2.231) and (2.246) that for each $s \in [0,\infty)$, $B \in G'$, $x' \in X'$, $\Lambda \in \hat{H}^z_{[C]}$ (cf. chapter 1 p. 36, $*$=z) and closed set C in X', satisfying

$$P^z \left[\Xi^z_{[0,\infty);[C]} ; x' \right] = 1, \tag{2.251}$$

we have

$$P^z \left[\Lambda^z_{s;B;[C]} \cap \Lambda ; x' \right] =$$

$$= \int_{\Lambda} P^z \left[d\omega^z ; x' \right] P^z \left[\Lambda^z_{s;B} ; x^z(\omega^z;[C]) \right] . \tag{2.252}$$

Lemma 2.45.2

If $\Lambda \in \hat{H}^z_{[C]}$, $x' \in X'$, $B \in G'$ and C is a closed set in X', then, under the assumptions 1 through 5 and (2.251), we have for $K \in H^z_{[C]}$ (cf. chapter 1, p.36, $*$=z)

$$P^z\,[K;x'] = \int_\Lambda P^z\,[d\omega^z;x']\quad P^z\,[T_{[C]}(K);x^z(\omega^z;[C])]\ . \tag{2.253}$$

Proof:

The proof is similar to that of lemma 2.45.1.

Lemma 2.45

Under the assumptions 1 through 5,

1) for each $t_o \varepsilon [0,\infty)$, $K \varepsilon H^z_{t_o}$ and $x' \varepsilon X'$ the conditional probability measure $P^z\,[K;x' \mid \hat{H}^z_{t_o}]$ can be defined by

$$P^z\,[K;x' \mid \hat{H}^z_{t_o}] = P^z\,[T_{t_o}(K);x^z_{t_o}(\omega^z)]\ ; \tag{2.254}$$

2) for each $x' \varepsilon X'$, closed set C in X', satisfying

$$P^z\,[\Xi^z_{[0,\infty);[C]};x'] = 1, \tag{2.255}$$

$K \varepsilon H^z_{[C]}$, the conditional probability measure $P^z[K;x' \mid \hat{H}^z_{[C]}]$ can be defined by

$$P^z\,[K;x' \mid \hat{H}^z_{[C]}] = P^z\,[T_{[C]}(K);x^z(\omega^z;[C])]. \tag{2.256}$$

Proof:

The assertions are immediate consequences of (2.249) and (2.253).

Finally, lemma 2.45 implies (cf. chapter 1 p.37):

Theorem 3

Under the assumptions 1 through 5, the decision processes $\{S^z_{x'};x' \varepsilon X'\}$ are stationary strong Markov processes.

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