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TOPOLOGICAL STRUCTURES II

PART 2

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CONTENTS

Part I

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NOWHERE DENSELY GENERATED TOPOLOGICAL PROPERTIES

.F. Mills & E. Wattel

0. INTRODUCTION

In [7] it is shown that a topological space without isolated points is compact if and only if each closed nowhere dense subset of the space is compact. We could express this fact by merely saying that the property compactness in spaces without isolated points is generated by the nowhere dense closed subsets of those spaces. Direct applications of this theorem can be found in [7] and [6]. In our Theorem 2 we show that for all cardinals κ and λ in a perfect space the property of (κ,λ)-compactness is generated by the closed nowhere dense subsets of the space. A direct consequence of this fact is that in a perfect space in which every nowhere dense subset is Lindelöf is itself hereditarily Lindelöf, which is related to an observation in [3].

Next we investigate normality, perfect normality and paracompactness which are nowhere densely generated properties in a relative sense. Then we state some examples which show that "Hausdorff" is not nowhere densely generated in the class of T_1 spaces, that normality, paracompactness, discreteness and metrizability are not nowhere densely generated in Tychonoff spaces. Finally we show that it is consistent with the usual axioms of set theory that separability is not nowhere densely generated in compact spaces and that metrizability is not nowhere densely generated in separable spaces.

In this paper all spaces are assumed to be T_1.

We are indebted to J. Vermeer who was the first to bring this type of problems to our attention.

DEFINITION 1. Let κ and λ be cardinal numbers such that $\lambda < \kappa$. A topological space X is called (κ,λ)-*compact* if and only if each open cover of X with cardinality less than κ has a subcover of cardinality less than λ.

REMARK. Compact = $(-,\omega)$-compact; Lindelöf = $(-,\omega_1)$-compact; countably compact = (ω_1,ω)-compact.

THEOREM 2. *Let* X *be a topological space such that every nowhere dense closed subset of* X *is* (κ,λ)*-compact. Then* X *itself is the union of a* (κ,λ)*-compact closed subset and a set consisting of isolated points.*

PROOF. We use transfinite induction. Suppose that we have already shown the theorem for (κ,ν)-compactness for all $\nu < \lambda$. Let D be the set of all isolated points of X and let E be X\D. Let $\mathcal{U}$ be a cover of E with open subsets of X and with cardinality less than κ. Let $\mathcal{V}$ be a subcover of $\mathcal{U}$ of minimal cardinality and well-order $\mathcal{V}$ in the most economical way. Fix a cardinal μ with $\lambda < \mu < \kappa$, and assume that every cover of E with less than μ open sets has a subcover of E of cardinality less than λ. Assume moreover that $\mathcal{V}$ has μ members.

CASE 1. μ is a regular cardinal.

Define $\mathcal{W}$ as the collection of the unions over initial segments in the well-ordering of $\mathcal{V}$. Let $N_0 = W_0$. We choose a point p_0 from $E \cap N_0$. We define N_α from the preceding N_β to be the first set $W \in \mathcal{W}$ such that

$$E \cap W \backslash c\ell_X(\cup N_\beta \mid \beta < \alpha) \neq \emptyset.$$

This is possible because otherwise, since the boundary of $\cup_{\beta<\alpha} N_\beta$ is nowhere dense, we would be able to construct a subcover of $\mathcal{V}$ for E of cardinality less than $\lambda < \mu$. We choose a point p_α from each set

$$E \cap N_\alpha \backslash c\ell_X(\cup N_\beta \mid \beta < \alpha).$$

We obtain a collection $\{p_\alpha\}_\alpha$ with the same cardinality as $\mathcal{V}$. Let

$$P = c\ell_X(\{p_\alpha \mid \alpha < \mu\}).$$

Now the set P cannot be covered by any subcollection of $\mathcal{V}$ of cardinality less than μ. Moreover, P is nowhere dense and therefore $\mu \leq \lambda$. Contradiction.

CASE 2a. μ is a singular cardinal larger than λ.

Assume that α is a regular cardinal with $\lambda \leq \alpha < \mu$. We define

$$V_\alpha^* = \cup\{V_\beta \mid \beta < \alpha\}.$$

Let us denote the boundary of V_α^* by D_α^* and let us denote $X\setminus(V_\alpha^* \cup D_\alpha^*)$ by E_α^*. There is an open cover $\mathcal{V}_\alpha$ of E which consists of $\mathcal{V}$ together with E_α^*. Let $\mathcal{W}_\alpha$ be a minimal subcover of this cover. The cardinality of this minimal subcover is less than $\alpha + \lambda < \mu$ and hence by induction we may assume that the cardinality is at most λ.

In this way we obtain an open cover for E of cardinality less than λ for each collection V_α^* and the union of all V_α^* contains E. Let Γ be a sequence of regular cardinals with limit μ and with $\#(\Gamma) = \text{cf}(\mu)$. Then $\cup\{\mathcal{W}_\gamma\setminus\{E_\gamma^*\} \mid \gamma \in \Gamma\}$ is a cover of E and a subcover of $\mathcal{V}$. Its cardinality is not larger than $\lambda * \text{cf}(\mu)$ and for singular $\mu > \lambda$ this amounts to less than μ. This means that the cardinality of $\mathcal{V}$ was not minimal when $\mu > \lambda$. The only thing which is left to prove is the case in which λ itself is singular.

CASE 2b. μ is singular and equal to λ.

Define a set of regular cardinals Λ which is cofinal with λ and such that $\text{cf}(\lambda) = \#(\Lambda)$. For every $\nu \in \Lambda$ let $\Psi(\nu)$ be the minimal cardinal such that $\mathcal{C}\ell_X \cup (\mathcal{V}_\alpha \mid \alpha < \Psi(\nu))$ cannot be covered by any subset of $\mathcal{V}$ of cardinality less than ν. Define

$$\Lambda' = \{\nu \mid \nu \in \Lambda;\ \Psi(\nu) > \Psi(\delta) \text{ for all } \delta \in \Lambda \text{ with } \delta < \nu\}.$$

Since $\mathcal{V}$ is a minimal cover of E it follows that $\#(\Lambda') = \#(\Lambda) = \text{cf}(\lambda)$.

Next we define W_ν and C_ν by induction for every $\nu \in \Lambda'$.

$$W_\nu = \cup(\mathcal{V}_\beta \mid \beta < \Psi(\nu))\setminus\mathcal{C}\ell_X(\cup(W_\delta \mid \delta < \nu)) \quad \text{and} \quad C_\nu = \mathcal{C}\ell_X(W_\nu).$$

Moreover, let

$$G_\nu = \mathcal{C}\ell_X(\cup(W_\mu \mid \mu \in \Lambda';\ \mu < \nu))\setminus\cup(W_\mu \mid \mu \in \Lambda';\ \mu < \nu)$$

for every $\nu \in \Lambda'$. Let $G = \cup(G_\nu \mid \nu \in \Lambda')$. Clearly G is nowhere dense and closed, and according to our assumptions it can be covered by some subcover of $\mathcal{V}$ of cardinality $o < \lambda$. Let H be a fixed open set containing G which is the union of o members of $\mathcal{V}$. The set Λ'' of all cardinals in Λ' which are larger than o has again the properties $\#(\Lambda'') = \#(\Lambda) = \text{cf}(\lambda)$ and λ is the limit of Λ''. Let $\nu \in \Lambda''$ and let V^*_ν be the union over a minimal subcollection of $\mathcal{V}$ which covers $\bigcup(C_\delta \mid \delta \in \Lambda''; \delta < \nu)$. Then the collection $C_\nu \setminus (V^*_\nu \cup H)$ is a closed non-empty subset of X, because we assumed that $\nu \in \Lambda'' \subset \Lambda$ was a regular cardinal and hence $\nu > \bigcup(\delta \mid \delta < \nu)$. Moreover, it follows that this particular collection cannot be covered by any subcollection of $\mathcal{V}$ of cardinality less than ν. Therefore it contains a nowhere dense closed subcollection N_ν which has a cover $\mathcal{O}_\nu$ which does not contain a subcover of cardinality less than ν.

We claim that $N = \bigcup(N_\nu)$ is a closed and nowhere dense subcollection of X which is not (κ,λ)-compact. First of all N is closed, since every point of N which would be in the closure of an infinite collection is also in G and hence in H but no point of N is in H. N is nowhere dense since every open set containing points of N intersects some $W_\nu \setminus N_\nu$ which is open and disjoint from N. Finally we take a cover $\mathcal{O}$ of X which is derived in the following way: For each $\nu \in \Lambda''$ we consider the collection $\mathcal{O}_\nu$ and intersect each member of it with $V^*_\nu \setminus G$. This is an open cover and for each N_ν we need at least ν members of $\mathcal{O}$ to cover it and therefore there exists no subcover of N chosen from $\mathcal{O}$ with less than λ members. This shows our theorem. □

COROLLARY 3.

(a) [7] *Let* X *be a space without isolated points and assume that every closed nowhere dense subspace of* X *is compact. Then* X *is itself compact.*

(b) [3] *Let* X *be a* CCC *space with the property that every nowhere dense set is Lindelöf, then* X *is itself Lindelöf.*

PROPOSITION 4.

(a) *Let* X *be a* T_1*-space with the property that each pair of disjoint closed nowhere dense subsets of* X *can be separated by a pair of disjoint open sets. Then* X *is normal.*

(b) *Let* X *be a* T_1*-space in which each closed nowhere dense set is a zeroset. Then* X *is perfectly normal.*

PROOF.

(a) Let A and B be two closed disjoint sets in X. Then their boundaries $\delta(A)$ and $\delta(B)$ are closed and nowhere dense in X and we can find open sets U_A and U_B in X such that $\delta(A) \subset U_A$; $\delta(B) \subset U_B$ and $U_A \cap U_B = \emptyset$. The set $U_A \setminus B$ is also a neighbourhood of $\delta(A)$ and so is $U_B \setminus A$ an open set containing $\delta(B)$. Moreover, $A \cup (U_A \setminus B)$ and $B \cup (U_B \setminus A)$ are disjoint neighbourhoods of A and B.

(b) According to part (a) the space X is normal because a nowhere dense closed subset is a zeroset, and any two disjoint zero-sets are always contained in disjoint open co-zerosets. Moreover, a closed set A is a G_δ because its boundary $\delta(A)$ is a G_δ and $A = \mathrm{Int}(A) \cup \delta(A)$. This proves the proposition. □

In the same way it can be shown that the property "Hausdorff" in T_1-spaces and the property "regular" in T_2-spaces are relatively nowhere densely generated. This means that a space is Hausdorff iff it is T_1 and each pair of points in a closed nowhere dense subspace can be separated by a pair of open subsets in the space itself. It is moreover easy to see that the property T_1 is nowhere densely generated in the absolute sense of Theorem 2.

THEOREM 5. *Let* X *be a regular space, such that for every closed nowhere dense subset* D *of* X, *and for every cover* $\mathcal{U}$ *of* D *with open subsets of* X, *there exists a refinement of* $\mathcal{U}$, *consisting of open sets in* X *which covers* D *and which is locally finite in* X. *Then* X *is paracompact.*

PROOF. First we show that X is normal. Let D be any nowhere dense closed subset of X, and let E be any neighbourhood of D. Then we can make an open neighbourhood 0_d for each point $d \in D$ such that $d \in 0_d \subset 0_d^- \subset E$. This is an open cover of D which has an open locally finite refinement $\mathcal{U}$ which covers D and which is locally finite in X. Let U be the union of $\mathcal{U}$; then $D \subset U \subset U^- \subset E$. According to Proposition 4(a) it follows that X is normal.

Let $\mathcal{U}$ be any open cover of X. Then we shall define an open locally finite refinement. Let $\mathcal{P}$ be a maximal collection of disjoint open sets which refines $\mathcal{U}$. Then $D = X\backslash \cup\mathcal{P}$ is closed and nowhere dense. Let $\mathcal{L}$ be a locally finite open refinement of $\mathcal{U}$ which covers D. Since $\cup\mathcal{L}$ is a neighbourhood of D there exists a closed neighbourhood of D inside $\cup\mathcal{L}$. Let us call it V. Define $\mathcal{P}^\#$ to be $\{P\backslash V \mid P \in \mathcal{P}\}$, then $\mathcal{P}^\# \cup \mathcal{L}$ is a locally finite cover which refines $\mathcal{U}$.

PROPOSITION 6. *Let* X *be a first countable Hausdorff space without isolated points and with the extra property that every closed nowhere dense subspace of* X *is regular. Then* X *is itself regular.*

PROOF. Let p be a point of X and let U be any open neighbourhood of p. Assume that no neighbourhood of p has a closure which is contained in U. Let $\mathcal{V} = \{V_i\}_i$ be a local base at p with the property that $V_{i+1} \subset V_i$ for each

$i \in \mathbb{N}$. Let W_1 be the first member of V which is contained in U. Then we choose a point q_1 in $(U^- \setminus U) \cap W_1^-$. Such a point exists because of our assumption. We can find a neighbourhood Q_1 of q_1 and a neighbourhood $W_2 \subset W_1$ of p in V such that $Q_1 \cap W_2 = \emptyset$. Next we choose a sequence $r_{i,1}$ which is contained in $Q_1 \cap W_1$ which converges to q_1. Since X has no isolated points, this is a nowhere dense subset of W_1. Next we choose a point q_2 in $W_2^- \cap (U^- \setminus U)$ and proceed in the same way by constructing a set W_3 and a set Q_2 such that $W_3 \cap Q_2 = \emptyset$. We next construct a sequence $r_{i,2}$ converging to q_2 which is contained in $W_2 \cap Q_2$. If we proceed by induction then we obtain a nowhere dense collection $\{p\} \cup \{r_{i,j}\} \cup \{q_j\}$. The closure D of this collection cannot be regular space, because $U \cap D$ is an open neighbourhood of p in D which has no closed subcollection with p in its interior. □

<u>EXAMPLE 7</u>. The following examples show that a T_1-space in which every closed nowhere dense subset is Hausdorff need not be T_2.

(a) Let X be the cofinite topology on a space with infinite cardinality.

(b) Let X be any set supplied with an "Ultrafilter topology", i.e. the nonvoid open subsets of X are precisely the members of a free ultrafilter.

<u>EXAMPLE 8</u>. The following example is meant to show that the properties discrete, metrizable, normal and paracompact are not nowhere densely generated. Our example is a *nodec* space, i.e. a space in which every nowhere dense subset is closed. This notion is due to van DOUWEN [1]. Suppose that $E(\mathbb{C})$ is the absolute of the circle.

A point p in $E(\mathbb{C})$ is called *remote* [2], [5] iff it is not in the closure of any subset of $\lim^{-1}[D]$ for any set D which is a nowhere dense subset of $\mathbb{C} \setminus \lim(p)$. Next we choose a remote point of $E(\mathbb{C})$ and take all its images under the circle group. This is our space X. It is clear that X is a nodec space. It contains precisely one member in each fiber of the mapping $\lim: E(\mathbb{C}) \to \mathbb{C}$. Moreover, it is clear that each nowhere dense subset is discrete and hence metrizable, paracompact and normal. However, X cannot be normal, because there exists an nowhere dense subset of $\mathbb{C}$ of cardinality 2^ω whose inverse image is discrete in X and X itself is separable.

<u>EXAMPLE 9</u>. [J. Vermeer]. The following example shows that a T_2-space without isolated points in which every closed nowhere dense subset is regular need not be regular.

Let X be any regular space without isolate points. Let $\kappa(X)$ be its KATĚTOV extension cf. [6]. Then $\kappa(X)$ is never regular but a nowhere dense subset of $\kappa(X)$ is the union of any nowhere dense subset of X together with a subset of the remainder. The closure of D is disjoint from the remainder whenever D is nowhere dense in X and the remainder itself is relatively discrete and hence every closed nowhere dense subset in $\kappa(X)$ is regular.

EXAMPLE 10. In a compact connected Souslin line each nowhere dense closed set is homeomorphic to a Cantor set. This shows that "Separable metric" is not nowhere densely generated in the ordinary sense. We obtain that it is consistent with the usual axioms of set theory, that there is a space in which every nowhere dense closed set is separable metric but the space itself is not.

EXAMPLE 11. The continuum hypothesis implies that there exists a *Luzin* subspace of the real line, this is a subspace in which every nowhere dense subset is countable. [4]. If we take a Luzin set L in the reals $\mathbb{R}$ which has the additional property that for each point p of L also $-p \in L$ then we can make L into a topological space by considering it as a subspace of the Sorgenfrey line. Since every nowhere dense subset of this space is countable it follows that every closed nowhere dense subset of L is separable metric. However, under the continuum hypothesis the set L itself has cardinality 2^{ω} and the ordinary proof for the Sorgenfrey line also shows here that $L*L$ cannot be normal, and therefore L is not separable metric.

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NEW CHARACTERIZATIONS OF COVERING DIMENSION

J.Bruijning and J. Nagata

In the present paper we mean by dimension of a topological space X (denoted by dim X) the Katětov dimension of X. Namely, dim X = -1 iff X = $\emptyset$, and for a non-negative integer n, dim X $\leq$ n iff for every finite functionally open cover $\mathcal{U}$ of X (i.e. a cover consisting of cozerosets) there is a finite functionally open cover $\mathcal{V}$ such that $\mathcal{V}$ refines $\mathcal{U}$ and ord $\mathcal{V} \leq n+1$. It is well-known that dim X coincides with the ordinary Lebesgue covering dimension of X if X is a normal space and that dim X is equal to the Lebesgue dimension of βX if X is a Tychonoff space. Furthermore, dim X = dim $\tau(X)$ for every topological space X, where τ is the Tychonoff functor defined by MORITA [3]. Detailed proofs of the theorems discussed here will appear elsewhere, except for the proof of Lemma 2', which will be given in full detail (see [1] for Theorems 1 and 2, and [6] for Theorem 3. Also see [2], [4], [5] for general terminologies and notations of general topology and dimension theory).

$\mathbb{N}$ will be the set of positive integers and if $\mathcal{U}$ is a family of sets, $|\mathcal{U}|$ will be its cardinality.

Among various characterizations of dimension is the following classical theorem by Pontrjagin and Schnirelmann, which is interesting in the sense that it gives an analytical expression of dimension and also that it characterizes dimension in terms of global cardinality of covers while dimension is defined in terms of local cardinality of covers.

THEOREM (PONTRJAGIN and SCHNIRELMANN [7]). *Let* X *be a compact metrizable space. For a metric* d *for* X *and* $\varepsilon > 0$, *define* $N(\varepsilon,X,d) = \min\{n \in \mathbb{N} \mid$ *there is a cover* $\mathcal{U}$ *of* X *such that* $|\mathcal{U}| \leq n$ *and mesh* $\mathcal{U}$ (= $\sup\{\text{diameter } V \mid V \in \mathcal{U}\}$) $\leq \varepsilon\}$, *and define*

$$k(X,d) = \sup\{\inf\{-\frac{\log N(\varepsilon,X,d)}{\log \varepsilon} \mid 0 < \varepsilon < \varepsilon_0\} \mid \varepsilon_0 > 0\}.$$

Then $\dim X = \inf\{k(X,d) \mid d$ *is a metric for* $X\}$.

We are not going to give a proof of this theorem here, but instead we shall just give an example to explain the basic idea of the theorem. Let $J^2 = [0,1]\times[0,1]$ be the closed unit square. Then $\mathcal{U}_n = \{[\frac{k-1}{n}, \frac{k}{n}]\times[\frac{\ell-1}{n}, \frac{\ell}{n}] \mid k,\ell = 1,2,\ldots,n\}$ is a cover of I^2 with $N = |\mathcal{U}_n| = n^2$ and $\varepsilon = \text{mesh } \mathcal{U}_n = \frac{\sqrt{2}}{n}$. Thus

$$\lim_{n\to\infty} \frac{-\log N}{\log \varepsilon} = 2 = \dim I^2.$$

In fact the theorem is proved by use of an approximation of the n-dimensional space X by n-dimensional polyhedra. Pontrjagin-Schnirelmann's theorem can be generalized as follows.

THEOREM 1 (J. Bruijning). *Let* X *be a separable metrizable space; then* $\dim X = \inf\{k(X,d) \mid d$ *is a totally bounded metric for* $X\}$.

This theorem may not be very surprising if one recalls that the completion of a totally bounded metric space is a compact metric space, but perhaps more surprisingly the theorem can be extended to general topological spaces as follows.

Let $\mathcal{R}$ be the set of all continuous totally bounded pseudometrics on the topological space X. Then define a relation on $\mathcal{R}$ as follows: $d_1 > d_2$ for $d_1, d_2 \in \mathcal{R}$ iff for every $\varepsilon > 0$, there is a $\delta > 0$ such that $B(x,\delta,d_1) \subset B(x,\varepsilon,d_2)$ for all $x \in X$. Here, of course, $B(x,\delta,d_1) = \{y \in X \mid d_1(x,y) < \delta\}$. Then $\mathcal{R}$ turns out to be a directed set. Define $k(X,d)$ as in Pontrjagin-Schnirelmann's theorem, for all $d \in \mathcal{R}$. Then we have

THEOREM 2 (J. Bruijning). *Let* X *be an arbitrary topological space. Then*

$$\dim X = \sup\{\inf\{k(X,d) \mid d > d_0,\ d \in R\} \mid d_0 \in R\}.$$

A point of the proof of this theorem is to reduce the problem to compact metric spaces. To be a little more specific, let $d \in \mathcal{R}$. Then define an equivalence relation $\sim$ in X by $x \sim y$ iff $d(x,y) = 0$. Then we denote by $X/\!\sim$ and ϕ the quotient set of X with respect to $\sim$ and the quotient map, respectively. Define a metric d^* on $X/\!\sim$ by the formula $d^*(\phi(x),\phi(y)) = d(x,y)$. Then $\langle X/\!\sim, d^*\rangle$ is a totally bounded metric space, and thus its completion $\tilde{X}$ is a compact metric space, on which Pontrjagin-Schnirelmann's theorem holds.

This theorem 2 may be a good generalization, but it might look a bit too analytic for topologists. The formula on the right has four times "sup" or "inf" involved! In our attempt to find out a simpler characterization of dimension of non-metrizable spaces we have achieved the invention of a new function $\Delta_k(X)$ and, eventually, a new characterization of dimension by use of it. This will be discussed in the sequel.

DEFINITION 1. Let X be a topological space and $k \in \mathbb{N}$. Then $\Delta_k(X) = \min\{m \in \mathbb{N} \mid$ for every functionally open cover $\mathcal{U}$ of X with $|\mathcal{U}| \le k$, there exists a functionally open cover $\mathcal{V}$ of X with $|\mathcal{V}| \le m$ and $\mathcal{V}^\Delta < \mathcal{U}\}$. (Here $\mathcal{V}^\Delta = \{st(x,\mathcal{V}) \mid x \in X\}$.)

It is a simple observation that we may replace 'functionally open' by 'open' either or both times it occurs in the above formula if the space X is normal. While the definition of our function $\Delta_k(X)$ has a remote analogy with that of Pontrjugin-Schnirelmann's function $N(\varepsilon,X,d)$, a remarkable difference is that their function involves the metric d in its definition, while ours does not. This will give us an advantage in coping with non-metrizable spaces.

PROPOSITION 1. Let X be a topological space with $\dim X \le m$, where $m \ge 0$. Further let $k \in \mathbb{N}$. Then

$$\Delta_k(X) \le 2^k-1 \qquad \text{if } k \le n+1$$

$$\Delta_k(X) \le \binom{k}{1} + \ldots + \binom{k}{n+1} \quad \text{if } k \ge n+1.$$

(Here $\binom{k}{r} = \frac{k!}{r!(k-r)!}$).

Sketch of proof. We prove the case $k \ge n+1$. Let $\mathcal{U} = \{U_1,\ldots,U_k\}$ be a functionally open cover of X. Then we can shrink $\mathcal{U}$ to a functionally open cover $\mathcal{V} = \{V_1,\ldots,V_k\}$ (thus $V_i \subset U_i$ for all i) such that $\text{ord}\,\mathcal{V} \le n+1$. Shrink $\mathcal{V}$ further to a functionally closed cover $\mathcal{F} = \{F_1,\ldots,F_k\}$ of X. For every $A \subset \{1,\ldots,k\}$ define $W(A) = [\cap\{V_i \mid i \in A\}] \cap [\cap\{X\backslash F_i \mid i \notin A$. Then $\mathcal{W} = \{W(A) \mid A \subset \{1,\ldots,k\},\ W(A) \ne \emptyset\}$ is a functionally open cover of X such that $|\mathcal{W}| \le \binom{k}{1} + \ldots + \binom{1}{k+1}$ and $\mathcal{W}^\Delta < \mathcal{U}$. This implies the proposition.

Perhaps the following proposition is more interesting and substantial.

PROPOSITION 2. Let X be a normal space with dim X ≥ n, where either n ≥ 1 or n = 0 and X is infinite. Let $k \in \mathbb{N}$. Then

$$\Delta_k(X) \geq 2^k-1 \qquad \text{if } k \leq n+1$$

$$\Delta_k(X) \geq \binom{k}{1} +\ldots+ \binom{k}{n+1} \quad \text{if } k \geq n+1.$$

To prove the proposition, we need the following two lemmas.

LEMMA 1. *Let* X *be a normal space and* n ≥ 0. *Then* dim X ≤ n *iff every open cover* $\{W_1,\ldots,W_{n+2}\}$ *can be shrunk to a vanishing closed cover* $\{W_1',\ldots,W_{n+2}\}$, *i.e.* $W_i' \subset W_i$ *for* $1 \leq i \leq n+2$ *and* $\bigcap\{W_i' \mid 1 \leq i \leq n+2\} = \emptyset$. *(See e.g.* [2, p. 282]. *Obviously the lemma still holds even if* $\{W_1',\ldots,W_{n+2}'\}$ *is taken as a vanishing open cover.)*

LEMMA 2. *Let* X *be a normal space with* dim X ≥ n, *where either* n ≥ 1 *or* n = 0 *and* X *is infinite. Further, let* $k \in \mathbb{N}$. *Then there are* k *disjoint closed subsets of* X *with dimension* ≥ n.

In fact, lemma 2 is a special case of the following lemma, which in the present proof we don't need in its full generality.

LEMMA 2'. *Let* X *be a topological space with* dim X ≥ n ≥ 1. *Then there are infinitely many pairwise disjoint, functionally closed subsets of* X *with dimension* ≥ n.

A complete proof will be given of this lemma, because it has never been given before in its full generality. Since the proof is rather long, however, we postpone it until we have finished the main argument. For the moment, therefore, we proceed with a sketch of the proof of proposition 2. Let k ≥ n+1. Denote by C^k_{n+1} the family of all subsets of {1,...,k} which have n+1 elements. Then by Lemma 2 there are pairwise disjoint closed sets $C(\alpha)$, $\alpha \in C^k_{n+1}$, of X with dimension ≥ n. By lemma 1, for each $\alpha \in C^k_{n+1}$ there is an open cover $\mathcal{U}(\alpha) = \{U_i^\alpha \mid i \in \alpha\}$ of $C(\alpha)$ which cannot be shrunk to a vanishing closed cover of $C(\alpha)$. Define sets U_i, i = 1,...,k, of X by

$$U_i = [X\setminus\bigcup\{C(\alpha) \mid \alpha \in C^k_{n+1}\}] \cup [\bigcup\{U_i^\alpha \mid i \in \alpha\}].$$

Then $\mathcal{U} = \{U_i \mid 1 \leq i \leq k\}$ is an open cover of X.

Let $\mathcal{V}$ be a finite open cover of X such that $\mathcal{V}^\Delta < \mathcal{U}$; then we can prove that for every subset β of $\{1,\dots,k\}$ with $1 \le |\beta| \le n+1$, there is an element $V(\beta)$ of $\mathcal{V}$ such that $V(\beta) \subset U_i$ iff $i \in \beta$. Thus V contains at least $\binom{k}{1} + \dots + \binom{k}{n+1}$ distinct elements which proves that $\Delta_k(X) \ge \binom{k}{1} + \dots + \binom{k}{n+1}$.

We are not going to give here a detailed proof of the last claim, but we shall simply show the existence of such a $V(\beta)$ in the special case that $|\beta| = n+1$. For more details the reader is referred to [6]. So, let $\beta = \{i_1,\dots,i_{n+1}\} \in C^k_{n+1}$. Observe that every element of $\mathcal{V}$ which intersects $C(\beta)$ cannot be contained in U_i for $i \notin \beta$. In other words, such an element can be contained only in some of $U_{i_1},\dots,U_{i_{n+1}}$. Let $x \in C(\beta)$: $St(x,\mathcal{V})$ must be contained in at least one of $U_{i_1},\dots,U_{i_{n+1}}$, because $\mathcal{V}^\Delta < \mathcal{U}$. Put $F_{i_\ell} = \{x \in C(\beta) \mid St(x,V) \subset U_{i_\ell}\}$ for $\ell = 1,\dots,n+1$. Then $\{F_{i_1},\dots,F_{i_{n+1}}\}$ is a closed cover of $C(\beta)$ shrinking $\mathcal{U}(\beta)$. Recall the property we required of $U(\beta)$, and conclude that $F_{i_1} \cap \dots \cap F_{i_{n+1}} \ne \emptyset$. Select a point x_0 in this intersection. Denote by $V(\beta)$ any element of C containing x_0. Then $V(\beta) \subset U_{i_1} \cap \dots \cap U_{i_{n+1}}$, but $V(\beta) \not\subset U_i$ for $i \notin \{i_1,\dots,i_{n+1}\}$ as we observed earlier. This concludes the sketch of proof of Proposition 2.

One can extend Proposition 2 to Tychnoff spaces by use of βX and further to general topological spaces by use of the space $\tilde{X}$ to be defined in the following. Professor Morita kindly has informed us that the space $\tilde{X}$ is nothing but the result of applying his Tychonoff functor τ to X (see [3]).

Let $\mathcal{C}$ be the set of all continuous maps from X to $[0,1] = I$. Define $\phi\colon X \to \Pi\{I \mid f \in \mathcal{C}\}$ by $\phi(x) = F(x)_{f\in\mathcal{C}}$. Then $\tilde{X} = \phi(X)$ is a Tychonoff space, and both $\dim X = \dim \tilde{X}$ and $\Delta_k(X) = \Delta_k(\tilde{X})$ for all $k \in \mathbb{N}$. Now Proposition 2 for general spaces follows easily.

Finally we have then the following main theorem and its corollary.

THEOREM 3. (J. Nagata & J. Bruijning). *Let* X *be a topological space such that either* $1 \le \dim X \le \infty$ *or* $\dim X = 0$ *and* $\tilde{X}$ *is infinite. (This last condition is not too restrictive, e.g.* $\tilde{X} = X$ *if* X *is a Tychonoff space.) Let* $n = \dim X$ *and* $k \in \mathbb{N}$. *Then*

$$\Delta_k(X) = 2^k - 1, \qquad \text{if } k \le n+1,$$

$$\Delta_k(X) = \binom{k}{1} + \dots + \binom{k}{n+1}, \qquad \text{if } k \ge n+1.$$

COROLLARY. *If* X *is a topological space satisfying the condition of Theorem 3, then*

$$\dim X = \lim_{k\to\infty} \frac{\log \Delta_k(X)}{\log k} - 1.$$

We will conclude by giving the postponed proof of Lemma 2. Unless stated otherwise, in the sequel n will be an integer ≥0. We start with a definition.

DEFINITION 2. $A \subset X$ is called $\geq$ n-*dimensionally placed in* X if there exists a cozero cover $\mathcal{U} = \{U_1,\dots,U_{n+1}\}$ of X such that for every cozero shrinking $\mathcal{V}$ of $\mathcal{U}$, $\cap\mathcal{V} \cap A \neq \emptyset$.

We will abbreviate this by $\mathrm{pl}(X,A) \geq n$.

EXAMPLE 1. If $\mathcal{U} = \{U_1,\dots,U_{n+1}\}$ is a cozero cover such that for every cozero shrinking $\mathcal{V}$, $\cap\mathcal{V} \neq \emptyset$ holds, and if $\cap\mathcal{U} \subset \bar{A} \subset X$, then $\mathrm{pl}(X,A) \geq n$. Indeed, if $\mathcal{V}$ is a cozero shrinking of $\mathcal{U}$, then $\cap\mathcal{V} \cap A \neq \emptyset$ since $\cap\mathcal{V}$ is open and is contained in $\cap\mathcal{U}$.

EXAMPLE 2. $\mathrm{pl}(X,A) \geq 0$ iff $A \neq \emptyset$.

LEMMA 3. *If* $A \subset B \subset X$ *and* $\mathrm{pl}(X,A) \geq n$, *then* $\mathrm{pl}(X,B) \geq n$.

PROOF. Obvious.

LEMMA 4. $\dim X \geq n$ *iff for some* $A \subset X$, $\mathrm{pl}(X,A) \geq n$.

PROOF. Obvious.

LEMMA 5. *If* U *is a cozeroset in* X, *and* V *is a cozeroset in* U, *then* V *is a cozeroset in* X.

PROOF. Let $f\colon X \to [0,1]$ be continuous, such that $f^{-1}((0,1]) = U$. Let $g\colon U \to [0,1]$ be continuous such that $g^{-1}((0,1]) = V$.

Define $h\colon X \to [0,1]$ by the formula

$$h(x) = \begin{cases} f(x)\cdot g(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

It is easily proved that h is continuous and that $h^{-1}((0,1]) = V$.

LEMMA 6. *Let* $A \subset U \subset Y \subset X$. *A a zeroset, U a cozeroset. Let* $pl(X,A) \geq n$. *Then* $pl(Y,A) \geq n$.

PROOF. *Let* $\mathcal{U} = \{U_1,\ldots,U_{n+1}\}$ *be a cozero cover of X such that for every cozero shrinking* $\mathcal{V}$ *of* $\mathcal{U}$, $\cap\mathcal{V} \cap A \neq \emptyset$ *holds. Put* $\mathcal{U}' = \{U_1 \cap Y,\ldots,U_{n+1} \cap Y\}$ *and* $\mathcal{U}'' = \{U_1 \cap U,\ldots,U_{n+1} \cap U\}$. *Let* $\mathcal{V}'$ *be a cozero shrinking of* $\mathcal{U}'$ *(this* $\mathcal{V}'$ *is a cover of Y), say* $\mathcal{V}' = \{V_1',\ldots,V_{n+1}'\}$. *Then* $\mathcal{V}'' = \{V_1' \cap U,\ldots,V_{n+1}' \cap U\}$ *is a cozero shrinking of* U". *Define* $V_i = V_i' \cup (U_i \backslash A)$ *for* $1 \leq i \leq n+1$, *and put* $\mathcal{V} = \{V_1,\ldots,V_{n+1}\}$. *Since* $U_i \backslash A$ *is a cozeroset,* $\mathcal{V}$ *consists of cozerosets; it is easy to see that* $\mathcal{V}$ *shrinks* $\mathcal{U}$. *By hypothesis,* $\cap\mathcal{V} \cap A \neq \emptyset$. *But this implies* $\cap\mathcal{V}' \cap A \neq \emptyset$. *From this it follows that* $pl(Y,A) \geq n$.

REMARK. The converse of the lemma also holds, i.e. in this situation $pl(U,A) \geq n$ implies $pl(X,A) \geq n$. The proof is omitted since it is of no relevance to our main argument.

COROLLARY. *In the situation of the lemma,* dim $Y \geq n$ *(see Lemma 4)*.

LEMMA. *If* $B_1,\ldots,B_m$ *are subsets of X such that* $pl(X,\cup\{B_i \mid 1 \leq i \leq m\}) \geq n$, *then for some j,* $pl(X,B_j) \geq n$.

PROOF. Suppose the conclusion does not hold; let $\mathcal{U} = \{U_1,\ldots,U_{n+1}\}$ be a cozero cover of X. It is easy to construct m consecutive shrinkings of $\mathcal{U}$ so as to obtain a cozero shrinking $\mathcal{V}$ such that $\cap\mathcal{V} \cap B_i = \emptyset$ for all i. This contradicts the fact that $pl(X,\cup\{B_i \mid 1 \leq i \leq m\}) \geq n$.

LEMMA 8. *Let* $A \subset X$ *be a zeroset, such that* $pl(X,A) \geq n \geq 1$. *Then there are zerosets* $A_1,A_2 \subset X$ *such that* $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 \subset A$, *and* $pl(X,A_1) \geq n$ *and* $pl(X,A_2) \geq n$.

PROOF. Let $\mathcal{U} = \{U_1,\ldots,U_{n+1}\}$ be a cozero cover of X such that for every cozero shrinking $\mathcal{V}$, $\cap\mathcal{V} \cap A \neq \emptyset$ holds. Let $\mathcal{F} = \{F_1,\ldots,F_{n+1}\}$ be a shrinking of $\mathcal{U}$ by zerosets. Since $A = \cup\{F_i \cap A \mid 1 \leq i \leq n+1\}$ and $pl(X,A) \geq n$, it follows from Lemma 7 that for some i, $pl(X,F_i \cap A) \geq n$; let us say that $pl(X,F_1 \cap A) \geq n$. Put $A_1 = F_1 \cap A$ and note that A_1 is a zeroset.

Now let V be a cozeroset and G a zeroset such that $F_1 \subset V \subset G \subset U_1$. Put $\mathcal{V} = \{U_1,U_2\backslash G,\ldots,U_{n+1}\backslash G\}$. Clearly, $\mathcal{V}$ is a cozero cover of X which shrinks $\mathcal{U}$. Let $\mathcal{W}$ be any cozeroshrinking of $\mathcal{V}$. Since $\mathcal{W}$ also shrinks $\mathcal{U}$, $\cap\mathcal{W} \cap A \neq \emptyset$. On the other hand, $\cap\mathcal{W} \cap V = \emptyset$. Therefore $\cap\mathcal{W} \cap (A\backslash V) \neq \emptyset$. This proves that $pl(X,A\backslash V) \geq n$. Put $A_2 = A\backslash V$, and note that A_2 is a zeroset in X. Observe also that $A_1 \cup A_2 \subset A$. This proves the lemma.

LEMMA 9. *Let* dim $X \geq n \geq 1$. *Then there exist collections* $\{B_i \mid i \in \mathbb{N}\}$ *and* $\{0_i \mid i \in \mathbb{N}\}$ *of zerosets and cozerosets, respectively, such that*

(i) $B_i \subset 0_i$ *for every* i

(ii) $\mathrm{pl}(X,B_i) \geq n$

(iii) $0_i \cap 0_j = \emptyset$ *if* $i \neq j$.

PROOF. We proceed inductively. Start by applying Lemma 8 with $A = X$. This gives us two disjoint zerosets A_1^1 and A_2^1, each of which is $\geq$ n-dimensionally placed. Let U_1^1 and U_2^1 be disjoint cozerosets, containing A_1^1 and A_2^1, respectively. Put $B_1 = A_1^1$ and $0_1 = U_1^1$. Now apply Lemma 8 with $A = A_2^1$. This gives us disjoint zerosets A_1^2 and A_2^2, contained in A_2^1 and both $\geq$ n-dimensionally placed.

Let U_1^2 and U_2^2 be disjoint cozerosets containing A_1^2 and A_2^2, respectively and both contained in U_2^1. Now put $B_2 = A_1^2$ and $0_2 = U_1^2$. It is clear that we can proceed inductively and thus obtain the required collections $\{B_i \mid i \in \mathbb{N}\}$ and $\{0_i \mid i \in \mathbb{N}\}$.

We now finally arrive at the proof of Lemma 2 which is repeated here.

LEMMA 2. *Let* dim $X \geq n \geq 1$. *Then there exists in* X *a collection* $\{F_i \mid i \in \mathbb{N}\}$ *of disjoint zerosets, all of dimension* $\geq$ n.

PROOF. Let $\{B_i \mid i \in \mathbb{N}\}$ and $\{0_i \mid i \in \mathbb{N}\}$ be as in the conclusion of Lemma 9. For every i, let G_i be a cozeroset and F_i a zeroset such that $B_i \subset G_i \subset F_i \subset 0_i$. Now it is easily seen that $\{F_i \mid i \in \mathbb{N}\}$ fullfills all requirements, the fact that dim $F_i \geq n$ following from the corollary to Lemma 6. This proves the theorem.

REMARK. It is easily seen that $\{F_i \mid i \in \mathbb{N}\}$ also satisfies the following:

(i) There exist disjoint cozerosets $\{0_i \mid i \in \mathbb{N}\}$ containing the F_i's.

(ii) For every i, and every set Y containing F_i, both dim $Y \geq n$ and $\mathrm{pl}(X,Y) \geq n$. In particular, $\mathrm{pl}(X,F_i) \geq n$ for all i.

REMARK. What can be said if dim $X = 0$ or dim $X = \infty$? We will briefly discuss these questions.

First, let dim $X = 0$. Let $F = \{f \mid f\colon X \to [0,1]$ is continuous$\}$ and define $\phi\colon X \to \Pi\{[0,1]_f \mid f \in F\}$ by the formula $(\phi(x))_f = f(x)$. Define $\tilde{X} = \phi(X)$. Now it appears that dim $\tilde{X}$ = dim $X = 0$ and that A_1, A_2 are disjoint zerosets in X if and only if $\phi(A_1)$ and $\phi(A_2)$ are disjoint zerosets in $\tilde{X}$. (See [2] for some details). Thus the problem how many disjoint zerosets which are ≥ 0 - dimensionally placed (i.e. which are non-empty) exist in X, is equivalent to the same problem in $\tilde{X}$. It follows that if $|\tilde{X}| = \infty$ one can find infinitely many such sets, while if $|\tilde{X}| = k < \infty$ exactly k can exist. As for the case dim $X = \infty$, note that the theorem implies that for every n, infinitely many disjoint zerosets of dimension $\geq n$ can be found. However, the following example shows that in the above statement "n" cannot be replaced by "∞". Let, for $n \in \mathbb{N}$, I^n be the n-cube and let Y be the topological sum of all I^n's. Let $X = Y \cup \{p\}$ be the one-point compactification of Y. Then it is clear that a closed set of X not containing p has finite dimension. Thus not even two disjoint zerosets of infinite dimension exist in X.

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TOWARDS A UNIFIED THEORY OF SEMI-METRIC AND METRIC SPACES

(S-METRIZABLE SPACES)

H.C. Reichel

> *"Metrization is the heart and*
> *the soul of General Topology"*
> (M.E. Rudin; [RU], p. 44)

0. INTRODUCTION

Distance functions obviously play a central role in topology. - Therefore, the concept of real-valued metrics and the "classical" theory of metric spaces have been generalized in many ways, and each of the assigned theories has its typical and, in most cases, useful applications. Roughly speaking, there are *essentially five main directions* of *generalizing* the "classical" concept of *metrics* and *metrizability*. (Of course, these fields are not completely "separated"; combinations of (some of) them yield also interesting theories.) We shortly describe some main ideas of these research fields:

(i) Modifying or cancelling some of the *axioms* defining a metric. Well-known examples of this research field are the theories of pseudo-, quasi- and semimetrics as well as Δ-metrics, symmetrics, and others.

(ii) Modifying the *range* of a metric, in other words: considering metrics d which, instead of $\mathbb{R}$, have some more special or more general range. Well-known examples are various kinds of so-called *"non-numerical" distance functions* d on a set or a space X. Such functions arise if - in the concept of a "classical" metric d - we replace the reals $\mathbb{R}$ by some other range of d, for example by ordered groups, semi-groups or even ordered sets. The respective distance functions are then defined by more or less obvious analogues of the "classical" axioms. Here, we should mention also the so-called probabilistic and statistical metrics d, where the basic idea is to use certain sets of distribution functions as the range of d, or to consider a "distance" to be a random variable, respectively. (Of course, in working with non-numerical distance functions d, we often have to modify also the basic axioms defining d.) One of the more interesting subjects in this field is the theory of ω_μ-metrizable spaces which – among others – plays an important role in this paper. As is well-known, a space X is ω_μ-metrizable if the topology of X can be generated by a "metric" d taking its values in a linearly ordered abelian group $(G,\leq)$ where ω_μ is the least cardinal α such that there is a strictly decreasing α-sequence in G $\{s_i | i < \alpha;\ s_i \neq 0 \in G\}$ converging to 0 w.r.t. the order topology of G. Equivalently, X is ω_μ-metrizable iff its topology can be induced by a uniform structure $\mathcal{U}$ with a totally ordered base $\mathcal{B}$ with cofinality ω_μ. (For more details, see below.) – On the other hand, the most general frame for studying distance-functions d without being forced to alter the axioms defining d (especially the triangle inequality) seems to be the concept of *semigroup*-valued metrics. It turns out that the resulting theory is much richer than the theory of ω -metrizable spaces; moreover we shall show that this concept does not only certain most of the other concepts of real-valued and non-numerical distance functions but also "unifies" them in a certain sense; see below. (This may also be thought of as a justification of the title of this paper.)

(iii) Studying metrization and metrizability problems in the realm of *cardinal functions*. By a definition of R.E. Hodel, we can assign to any topological space X a cardinal m(X), the *metrizability degree* of X, such that a regular space X is metrizable if and only if $m(X) = \omega_0$. The resulting cardinal function m reflects in some sense how metrizable a space X is. Using this cardinal function we can extend metrizability theorems to higher cardinals in a different way than was indicated in (ii). On the other hand, one of the major tasks is to compute m(X) by using some other cardinal functions and to compare them with m. This could give us another possibility to clarify and generalize connections between metrizability and other topological properties of X; ([HO_1], [NR_3], [RE_4].) – This research field seems to be the most recent one in this area.

(iv) Another, and especially famous field in this area stems from *weakening topological properties* of a space X necessary or *equivalent with metrizability* of X. Doing so we obtain new classes of spaces which contain the metrizable ones and have some more general and in most cases convenient properties and features. (E.g.: paracompact spaces, stratifiable spaces, Nagata-spaces, M-spaces, p-spaces, and many others, see e.g. [BL].) The different classes of topological spaces that arise in this way are usually joined together under the name of *"generalized metric spaces"*. In many important cases those new classes and theories arise while studying special problems of metrization theory.

(v) Modifying and exploring various *topological structures* associated with, and generated by given distance functions d on a set X. - To metric spaces (X,d) we usually assign a *topology* generated by the family of all balls $B_n(x) = \{y | d(x,y) < \frac{1}{n}\}$, $x \in X$, $n = 1,2,\ldots$. On the other hand, given a "general" distance function d on X the d-balls need not form a base for a topology on X; then, for example, we could use the family of all d-balls as a network (instead of a base) for a topology τ assigned to d (compare the theory of o-metrizable, symmetrizable and Δ-metrizable spaces; see §4). Another, and perhaps more natural possibility is to use the d-balls centered at $x \in X$, as a *local* base for a topology τ on X in which these balls then need not be open (see below). But for many purposes it is necessary (and even more natural in some sense) to study other topological structures generated by a distance function d which are more special or more general than topologies (uniformities, proximities, several kinds of "neighbourhood structures" and, last but not least, nearness structures). Recently, it seems to turn out that nearness structures as defined by H. HERRLICH seem to be an especially appropriate and natural frame for studying general distance functions. (See a forthcoming paper of the author.)

In this paper, we are concerned with problems belonging to (i) and (ii), and we shall show how a unifying viewpoint could be found.*) Specifically, we study T_1spaces the topology of which can be generated by distance functions d on X taking their values in totally ordered *semigroups* S, and, more specifically, in 0^+-*semigroups*. This concept allows us to subsume in a natural way the concepts of metric and semimetric spaces as well as their different generalizations to higher cardinals (ω_μ-metric spaces; spaces which admit an inverse m-semimetric, recently introduced by P. NYIKOS; and some other theories of spaces metrizable by non-numerical distance functions). Hereby, all distance functions d: $X^2 \to S$ satisfy (analogues of) *all* usual "metric" axioms, the triangle inequality always included. As we shall see,

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*) Together with some other papers, say [KU], [HO_2], [$RE_{1,2,3,4}$], [RR], [NR_1], [$NY_{1,3}$], [NC]. [STE], and others, this could also partially serve as a survey on general (symmetric) distance functions different from probabilistic and statistical ones. - The author plans to write down a more detailed bibliography and/or survey on general distance functions in the near future.

the typical differences between (m-) metric and (m-) semi metric spaces are then not caused by the triangle inequality or by lacking it, respectively, but by algebraic properties of the value-semigroups S. Consequently, we shall study relations between topological properties of a T_1-space X which is metrizable "over" a linearly ordered semigroup S and structural properties of these value-semigroup; (§1). - Similarly, it will be of some interest to study relations between topological properties of such a space X and analytic properties of the distance function $d: X^2 \to S$ which generates the topology of X; (§§1 and 2). - Among others, this yields a useful characterization of m-*developable* and m-*Moore spaces*, respectively. At the same time we get generalizations as well as new and more direct (and perhaps more appropriate) proofs of two theorems of M. GAGRAT and S. NAIMPALLY (which themselves are improvements of two well-known theorems of H. COOK and S. NEDEV, respectively). The results of §§1 and 2 are also strongly related to results of R.W. HEATH ([HE_1], §3) and to questions posed by J. NAGATA and C.J. BORGES [BO]. -Many theorems of "classical" metrization theory carry over to the theory of S-metrizable spaces; however, it is typical for the case where the cofinality of S is uncountable that, in most cases, we then have to develop special proving-methods. Then there are also interesting theorems which do not have analogues for the countable case (see also [$NR_{1,3}$], [RE_3], [VA], [STH], [SI] and others cited there).

Whereas in §§1 and 2 we are concerned with metrizability of spaces over 0^+-*semigroups,* §3 studies metrizability over *totally ordered abelian semigroups* $(S;+,\leq)$ *in general*. In some sense, this section is the "heart" of the paper. It is this section which shows the *unifying concept* indicated in the title. Moreover, we see that in the theory of S-metrizable spaces we can restrict ourselves to only *three types* of semigroups: $\mathbb{R}_0^+$, $[\Pi\mathbb{Z}_i\ (i < \omega_\mu)]_0^+$, and the 0^+-semigroups S_α (α an infinite cardinal) described in §1 (Theorem 1.3). Those S_α are studied in more details in an earlier paper of H.C. REICHEL and W. RUPPERT ([RR]). - §3 also includes several examples. By one of these examples we see that not every S-valued distance function d on a set X gives rise to a topology on X as real-valued and "group-valued" metrics always do. We show that continuity of the semigroup operation (= "addition") in S at $(0,0) \in S^2$ is a sufficient condition which, however, is not necessary. On the other hand, we show that *no weaker condition* on S can guarantee that a distance-function $d: X^2 \to S$ induces a topology on an arbitrary set X.

Finally, §5 is concerned with paracompactness and similar properties of S-metrizable spaces. Every S-metrizable space X is ω_μ-*paracompact* if ω_μ is the cofinality of S_0^+ at 0. If, moreover, addition in S is *continuous* at 0, then X is *paracompact*.

REMARK. Non-numerical distance functions have been studied by many authors; some of them used ordered *sets* as the range of d, and consequently, obtain relatively general results. The tradition of studying such distance-functions starts in the early twenties by papers of K. MENGER, O. BLUMENTAL and M. FRÉCHET; it was continued by D. KUREPA, Z. MAMUZIĆ, APPERT and KY FAN, R. DE MARR and I. FLEISCHER as well as L. COLLATZ among many others; see e.g. [KU], [MAM]. However, if the range is a totally ordered *set*, Definition 1.1 and the construction of the 0^+-semigroups S_σ in §1, Theorem 1.3, may show how these concepts can also be *subsumed* under the somewhat "richer", more comprehensive and perhaps more appropriate theory of S-*metrizable spaces* where S is some 0^+-*semigroup*. Concerning symmetric as well as non-symmetric distance-functions d with (only) *partially ordered* groups $(G,\leq)$ as their range, see. e.g. [$RE_{1,2}$] and other papers cited there, especially an older paper of G. KALISCH. - Finally, let us say a few words or more on ω_μ-*metrizable spaces*. As we said before, these are T_1-spaces which are metrizable over linearly ordered abelian *groups* $(G,\leq)$; a theory which - as we shall see - is much more specific than the theory of S-metrizable spaces developed in this paper. The tradition of studying ω_μ-metrizable spaces had been started by F. HAUSDORFF and R. SIKORSKI and was continued e.g. by I. JUHÁSZ, D. HARRIS, F.W. STEVENSON and W.J. THRON, WANG SHU TANG, R.E. HODEL, P. NYIKOS and H.C. REICHEL to name only a few. (Compare the more detailed comments in §1 and the bibliographies of the papers mentioned there. See also an informative review of [NR_1] by J. VAUGHAN in Zbl. f. Math. 356 (1978) #54030.)

Another type of generalized distance functions has been studied by S. NEDEV, M.M. ČOBAN and others, under the name of "o-*metrics*", "Δ-*metrics*" and "*symmetric spaces*" (see e.g. [NE], [NC], [KO]). At the end of §4 we compare this concept with the concepts studied and developed in this paper. (The differences of these concepts are shown explicitly by *Example* 4 in §3; for more details on the theory of symmetrizable spaces, see. e.g. [SIW], [JA], [HS] and others).

Finally, let us shortly mention the old extended research field of *quasi metric spaces* and more generally, non-symmetric distance functions.

As a reference concerning *not necessarily symmetric* S-*metrics* let us cite only [RR] and [RE_1]. (These papers fit into the concept of the present paper which - as we said before - is mainly concerned with symmetric distances).

ADDED IN PROOF. By combining very recent results of M. HUŠEK, P. NYIKOS and H.C. REICHEL (see [RE_4], [NR_3] and a forthcoming paper of M. HUŠEK and H.C. REICHEL) we get a rather surprising characterization of ω_μ-metric spaces: for $\omega_\mu > \omega_0$, a space X is ω_μ-metrizable iff it is a ω_μ-additive, almost orderable space such that the diagonal ΔX of X is the intersection of not more than ω_μ many open sets in X. (Hereby, X is ω_μ-additive, if every intersection of fewer than ω_μ many open sets is again.)

This theorem is another instance showing that the class of ω_μ-metrizable spaces is a very interesting, but rather "small" generalization of metric spaces.

This fact may provide another argument convincing us that for studying general distance functions we should study semigroup-valued metrics rather than group valued ones.

As we said before, although our main topic in this paper are non-numerical distance functions we do not touch the theory of probabilistic - and statistical metrics here. Recently B. MORREL and J. NAGATA wrote a paper [MN] which throws a new light on the connections between statistical metric spaces and semimetric ones. Among others, the authors showed that any topological Menger space is semimetrizable and therefore (by our Theorem 1.3) metrizable over a certain semigroup S_0. Combining their further results with ours (say §4) shows that there are also other interesting and comparatively "narrow" connections between statistical metrics and general distance functions as discussed in this paper.

1. SPACES METRIZABLE OVER 0^+-SEMIGROUPS

A totally ordered abelian semigroup (S,+) with zero element 0 is an 0^+-*semigroup* if (i) $0 = \min S$, (ii) $a < b \Rightarrow a+c < b+c$, for all $a,b,c \in S$, and (iii) for each $s > 0$, there exists $t > 0$ with $0 < t < s$. The *cofinality of* S (cof S) is the smallest cardinal ω_μ such that there exists a strictly decreasing ω_μ-sequence $(s_i | i < \omega_\mu)$ converging to $0 \in S$ with respect to the order topology of S. In other words, cof S is the cofinality of $S \backslash \{0\}$ with the reverse order; hence cof S is always a regular cardinal. - 0^+-semigroups are especially interesting in general metrization theory; examples will be given later on. More details about (the algebraic theory of) 0^+-semigroups can be found e.g. in [RR] and the literature cited there.

By abuse of language, we call S a *continuous* 0^+*-semigroup* iff $s \to s+s$ defines a function $f: S \to S$ which is continuous at $0 \in S$, (i.e. "iff addition is continuous at $0 \in S$"). - It is easy to show that if S is the positive cone of a totally ordered abelian group G, S is always a continuous semigroup. - It turns out that continuity of S (in this weak sense) is the crucial point in studying topologies generated by S-valued distance functions.

DEFINITION 1.1. Let (X,τ) be a topological space, and S a totally ordered abelian semigroup $(S;+,<)$ with zero element 0, then X is *metrizable over* S iff there is a function $d: X^2 \to S$ satisfying

(i) $d(x,y) > 0$ and $d(x,y) = 0$ iff $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y,z \in X$,

and such that, for each $x \in X$, the system of all *balls* $B(x,s) = B_s(x) = \{y \in X \mid d(x,y) < s\}$, $s \in S\setminus\{0\}$, is a (not necessarily open) *neighbourhood-base* at x. - d is then often called an "S-*metric*" generating the topology of X; and X is also said to be "S-*metrizable*". - Obviously any such space X is a T_1-space, but X need not be a T_2-space (see §3, Example 2).

More generally, H.C. REICHEL and W. RUPPERT proved in [RR]:

PROPOSITION 1.2. ([RR]). *A* T_1*-space* X *is* ω_μ*-metrizable if and only if* X *is metrizable over a continuous* 0^+*-semigroup* S *with* $\operatorname{cof} S = \omega_\mu$.

As we shall see, the concept of metrizability over 0^+-semigroups provides a "natural" and unifying frame for studying problems connected with (non-numerical) distance functions in topology, and perhaps - together with other papers on the subject - it can give better insights showing what it "really" means to be *(semi-) metrizable,* or m-*developable*, respectively (compare §3). For more details in this respect, see also the papers [RR] and [RE_1] where spaces are studied which are metrizable over 0^+-semigroups by not necessarily symmetric distance functions. Note further that the order topology on S gives us the possibility to speak of *upper* and *lower semi-continuous (S-valued) distance functions* d, and - properly defined - functions $d(x,A)$, where A is a fixed closed set in X. Thus we can study relations between topological properties of X and analytic properties of these functions, (formally) analogous to the classical case of (semi-)metric spaces. (See §§2, and 3.) The results of these sections are related to questions posed and discussed by JUN-ITI NAGATA and C.J.R. BORGES [1]. First, we prove:

THEOREM 1.3. *A* T_1*-space* X *is semi-metrizable**) *if and only if* X *is metrizable over an* 0^+*-semigroup* S *with countable cofinality* **).

REMARK. In [HE_2], HEATH gave an example of a semimetric space in which no ball is an open set. This - among others - justifies also our Definition 1.1.

PROOF OF THEOREM 1.3. Let $d\colon X^2 \to S$ be an S-metric generating the topology of X and let $\{s_i \mid i = 1,2,\ldots\}$, $s_i \in S$, be a monotonically decreasing sequence converging to $0 \in S$ with respect to the order-topology of S.

For any pair $x,y \in X$, define $\rho(x,y) = n^{-1}$ where $n = \min\{i \mid d(x,y) \geq s_i\}$. ***) Then $\rho\colon X^2 \to \mathbb{R}$ is a semimetric compatible with the topology on X. We only have to show that, for any set $A \subset X$, $x \in cl(A)$ iff $\rho(x,A) = 0$. But this is obvious: if $x \in cl(A)$ we have $A \cap B(x,s) \neq \emptyset$ for every ball $B(x,s) = \{y \mid d(x,y) < s\}$, and therefore also $\inf\{\rho(x,y) \mid y \in A\} = \rho(x,A) = 0$. Conversely, $\rho(x,A) = 0$ implies that for every $n \in \mathbb{N}$ there exists $y \in A$ such that $\rho(x,y) \leq n^{-1}$, equivalently: $d(x,y) < s_{n-1}$. Thus every neighbourhood $B(x,s)$, $s \in S$, $s > 0$, intersects A, i.e. $x \in cl(A)$.

Now, conversely, let $\rho\colon X^2 \to \mathbb{R}$ be a *semimetric* for X; we shall construct an 0^+*-semigroup* S_0 of countable cofinality and a metric $d\colon X^2 \to S_0$ in

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*) As is well-known, X is a semimetric space if the topology of X is induced by a "semimetric" d on X, i.e. a $\mathbb{R}_0^+$-valued function on X^2 such that $d(x,y) = d(y,x)$; $d(x,y) = 0$ iff $x = y$; and for any set $A \subset X$, $x \in \bar{A}$ iff $d(x,A) = 0$. Semimetric spaces have been studied extensively, they play a great role e.g. in the theory of developable spaces and other generalizations of metric spaces (see e.g. [BL]).

**) This means that any semimetric space X can be "remetrized" by a suitable S-metric which satisfies *all* metric axioms, the *triangle inequality included*!

***) And $\rho(x,y) = 0$ if $x = y$

the sense of Definition 1.1 which generates the topology of X.*) For an arbitrary regular cardinal α fix a set $M = \{s_i \mid i < \alpha\}$ inversely well ordered by $s_i < s_j \Longleftrightarrow i > j$ and "add" an element $0 < s_i$ $(i < \alpha)$. Let $S_\alpha(M) =: S_\alpha$ denote the free abelian semigroup over M, i.e. the set of all finite "formal sums" $\Sigma\lambda_i s_i$, $s_i \in M$, λ_i a non-negative integer, with the "usual" addition. Identify the empty "word" with $0 \in M$. For different elements $s = \Sigma\lambda_i s_i$ and $t = \Sigma\mu_i s_i$, where $\Sigma\lambda_i \neq \Sigma\mu_i$, let $s < t$ if and only if $\Sigma\lambda_i < \Sigma\mu_i$; otherwise, if $\Sigma\lambda_i = \Sigma\mu_i$, take $j = \min\{i \mid \lambda_i \neq \mu_i\}$ and let $s < t$ if and only if $\lambda_j < \mu_j$. It is now easy to prove that S_α is an 0^+-semigroup which certainly is non-continuous and cof S_α = cof α **). Moreover denote $S_0 := S_{\omega_0}$.

Now we construct an S_0-*metric on* X compatible with the topology on X: for any pair $x,y \in X$, $x \neq y$, let $n = \min\{k \mid k^{-1} \leq \rho(x,y)\}$, and define $d(x,y) = s_n$. Moreover, let $d(x,x) = 0 \in S_0$, for all $x \in X$.

Then $d: X^2 \to S_0$ is an S_0-metric on X (it is easy to see that d is symmetric and satisfies the triangle inequality; further $d(x,y) = 0 \Leftrightarrow \rho(x,y) = 0$). We only have to show that (i): for every $x \in X$ and any open set $W \subset X$, $x \in W$, W contains a ball $B(x,s) = \{y \mid d(x,y) < s\}$, $s > 0$; and (ii): $x \in \text{int}\, B(x,s)$, for every ball with $s > 0$. But (i) holds because $x \in W$ and therefore $\rho(x,X\backslash W) > 0$ since $X\backslash W$ is a closed set. (Note that S_0 is an 0^+-semigroup and $\{s_i \mid i = 1,2,...\}$ is cofinal at $0 \in S_0$.) Further, (ii) holds since $x \notin cl(X\backslash B(x,s))$. Otherwise, for every $0 < r < s$, we could find $z \in X\backslash B(x,s)$ with $d(x,z) < r < s$ which is an obvious contradiction. (Note however, that B(x,s) *need not be open*.) □

As ω_μ-metrizable spaces generalize the "classical" theory of *metrizable* spaces to higher cardinals ω_μ, our Theorem 1.3 now shows a "natural" way of generalizing the concept of *semimetric* spaces to higher cardinals. We can also use the construction of the 0^+-semigroups S_m in the proof above to show how to *subsume* P. NYIKOS' concept of *inverse* m-*semimetrics* ([NY_2]) under the theory of spaces *metrizable over* 0^+-*semigroups*, too. An inverse m-semimetric for a space X is a symmetric function $q: X^2 \to m+1$ such that

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*) For this special purpose, we could give a simple construction for S_0; however, we shall present a general but perspicuous construction for semigroups S_α, where α is an arbitrary cardinal, and which also will be used in the following sections. S_{ω_0} will be denoted by S_0. - This construction as well as the proof itself will also show that any distance function which takes its values in a totally ordered *set* only, can be considered as a distance-function with an 0^+-*semigroup* as its range.

**) i.e.: = α iff α is a regular cardinal

$q(x,A) = \sup\{q(x,y) \mid y \in A\} = m$ is equivalent to $x \in cl(A)$. A space admitting an inverse m-semimetric is called m-*semimetrizable*. (For an interesting study of these spaces, see [NY_2]).

THEOREM 1.4. *A* T_1*-space* (X,τ) *admits an inverse* m*-semimetric if and only if* X *is metrizable over an* 0^+*-semigroup* S *with* cof S = cof m.

PROOF. Let q: $X^2 \to m+1$ be an inverse m-semimetric for τ, and construct an 0^+-semigroup S_m as indicated in the proof of Theorem 1.3, modelled on a fixed monotonically decreasing m-sequence $\{s_i \mid i < m\}$ converging to $0 \in S_m$ w.r.t. the order topology on S_m. Further, for $x,y \in X$, $x \neq y$, define $d(x,y) = s_j$ where $j = q(x,y) < m$, and $d(x,x) = 0$ for every $x \in X$. Then d: $X^2 \to S_m$ is an S -metric on X, since d evidently satisfies the triangle inequality and the other axioms. (If m is singular, a similar construction will do: If cof m = ω_μ, construct S_{ω_μ} instead of S_m. Choose further an increasing well ordered ω_μ-sequence $\{k_i \mid k_i < \omega_\mu\}$ cofinal with m and let $d(x,y) = s_j$ where $j = \min\{k_i \mid q(x,y) < k_i\}$.) That the system of all balls $B(x,s)$, $s \in S_n$, $s > 0$, forms a neighbourhood base for every $x \in X$ is shown analogously to the proof of Theorem 1.3, too. (As in the case of countable cofinality, no ball $B(x,s)$ need be open. For regular m, every intersection of fewer than m neighbourhoods of $x \in X$ is a neighbourhood of x again.) Conversely, if S is an 0^+-semigroup with cof S = α, and d: $X^2 \to S$ is an S-metric for τ, fix a monotonically decreasing α-sequence $\{s_i \mid i < \alpha\}$ converging to $0 \in S$ w.r.t. the order topology on S. Then, for $x,y \in X$, $q(x,x) = m$ and $q(x,y) = i$, $i = \min\{j < \alpha \mid s_j \leq d(x,y)\}$, defines an inverse m-semimetric q for τ. We only have to show that for an arbitrary set $A \subset X$, $x \in cl(A)$ is equivalent to sup $\{q(x,y) \mid y \in A\} = m$. But this is obvious since the latter condition is equivalent with $A \cap B(x,s) \neq \emptyset$ for every ball $B(x,s)$, $s < S$, $s \neq 0$. (Note that $d(x,y) < s_i \Longleftrightarrow q(x,y) > i$.) □

One of the crucial points in the theory of S-metrizable spaces X is the *question* whether X can be metrized over a *continuous* 0^+-semigroup S or not. The following provides necessary and sufficient conditions for a space X metrizable over a 0^+-semigroup S to be metrizable over a *continuous* 0^+-semigroup S' (with cof S' = cof S). By Theorem 1.4 and Proposition 1.2 this is equivalent to finding (ω_μ-)*metrizability conditions for* (ω_μ-)*semimetric spaces*. (It is remarkable that even conditions as strong as paracompactness e.g., do not guarantee metrizability of a semimetric space; for counterexamples see the well-known papers of BORGES [BO] and HEATH [HE_1].) For more general results, compare §5.

In [NE], S. NEDEV showed that a semimetric space (X,d) is metrizable if d(x,A) defines a *continuous* function of x for each closed $A \subset X$. In [GN] GAGRAT and NAIMPALLY improved this theorem by showing that a semimetric space (X,d) is metrizable if d(x,A) is a *lower semicontinuous* function of x for each closed $A \subset X$. To prove this fact, the authors used a metrization theorem of A.V. ARHANGELSKIǏ which cannot be generalized directly to higher cardinals. We give a proof of their theorem which is more direct and appropriate; moreover, it allows a generalization for spaces which are metrizable over *arbitrary* 0^+-semigroups.

First we must define d(x,A) in the general case (note that in an arbitrary 0^+-semigroup, subsets need not have greatest lower bounds, therefore, we *cannot* take over the "classical" definition $d(x,A) = \inf\{d(x,y) \mid y \in A\}$):

DEFINITION 1.5. Let (X,d) be a space metrized over an 0^+-semigroup S (with cof S = m) and $A \subset X$. For $x \in X$, let $\underline{d(x,A)} = \inf\{d(x,y) \mid y \in A\}$ if every subset $T \subset S$ has an infimum, otherwise, take a monotonically decreasing m-sequence $\sigma = \{s_i \mid i < m\}$, $s_i \in S$, $s_i > 0$, converging to $0 \in S$ w.r.t. the order topology on S, and let $j = \min\{i \mid s_i \leq d(x,y)$ for every $y \in A\}$; then define $d(x,A;\sigma) = d(x,A) = s_j$. - Then it follows from Definition 1.1 that $x \in \bar{A}$ iff $d(x,A) = 0$.

In the following, every 0^+-semigroup S will be equipped with the order topology. Thus we can speak of *lower* (and *upper*) *semicontinuity* of d(x,A) as an S-valued function of $x \in X$ and of d(x,y) as an S-valued function on X^2.

If X is a metric space and A is a closed subset of X then d(x,A) is a *continuous* real-valued function on X. If, more generally, X is metrizable over a *continuous* 0^+-semigroup S the situation is similar: by Proposition 1.2, X could then equivalently be metrized over a linearly ordered abelian *group* G. Now, since a linearly ordered group is always a *topological* group w.r.t. its order topology, the "classical" proof showing continuity of d(x,A) for real-valued metrics works also for G-valued distance functions. (Note the linearly ordered *semigroups* S need *not* be topological semigroups w.r.t. their order topology; addition even need not be continuous at $0 \in S$. This is one of the crucial points in the theory of S-metrizable spaces!) So, for spaces X which are metrizable over *continuous* 0^+-semigroups S, we can assume without loss of generality that all functions d(x,A), $x \in X$ are (lower semi-)continuous functions.

The following lemma presents a converse assertion:

LEMMA 1.6. *Let* S *be an arbitrary* 0^+*-semigroup of cofinality* ω_μ, *and* $\{s_i \mid i \in \omega_\mu\}$ *be a monotonically decreasing* ω_μ*-sequence,* $s_i > 0$, *cofinal at* $0 \in S$*; let* (X,τ) *be a* T_1*-space whose topology is generated by an* S*-metric* $d\colon X^2 \to S$. *Then* X *is metrizable over a continuous* 0^+*-semigroup* S' *with* $\operatorname{cof} S' = \operatorname{cof} S$ *if, and only if, for every closed set* $A \subset X$, $d(x,A)$ *is a lower semicontinuous function of* x.

PROOF. By the preceding remarks, we only have to prove sufficiency. Let $\operatorname{cof} S = \omega_\mu \geq \omega_0$. We shall show ω_μ-metrizability of X by using a generalization of a well-known metrization theorem of A.H. FRINK proved by P. NYIKOS and H.C. REICHEL in [NR_1]: A T_1-space X is ω_μ-metrizable iff, for each $x \in X$, there exists a local base $\{W_i(x) \mid i < \omega_\mu\}$ such that:

(i) for all $\nu < \omega_\mu$, $\bigcap\{W_i(x) \mid i \leq \nu\}$ is a neighbourhood of x; and

(ii) for every $i < \omega_\mu$ and $x \in X$, there exists $k(i,x) < \omega_\mu$ such that $W_k(x) \cap W_k(y) \neq \emptyset$ implies $W_k(y) \subset W_i(x)$.

(Obviously, for $\omega_\mu = \omega_0$, condition (i) is trivially satisfied.)

Remember first that the not necessarily open balls in X, $B(x,s) = \{y \mid d(x,y) < s\}$, $s > 0$, form a neighbourhood base for every $x \in X$ which satisfies condition (i) of this theorem. To show (ii), fix a monotonically decreasing ω_μ-sequence $\{s_i > 0 \mid i < \omega_\mu\}$ cofinal at $0 \in S$. Now, for every ball $B(x,s)$, $s \neq 0$, denote $C(x,s) := \operatorname{cl}(X \setminus B(x,s))$. Then $x \in \operatorname{int} B(x,s_i) = X \setminus C(x,s_i)$ for all $i < \omega_\mu$; ergo $d(x,C(x,s_i)) = t_i > 0$ and $B(x,t_i) \subset \operatorname{int} B(x,s_i)$. By the lower semicontinuity of $d(x,C(x,s_i))$, we can find an $s_{i_1} < t_{i+1}$ such that $d(y,C(x,s_i)) \geq t_{i+1}$, for all $y \in B(x,s_{i_1})$. (Note that $t_{i+1} < t_i$.) Therefore $B(y,s_{i_1}) \subset B(y,t_{i+1}) \subset \operatorname{int} B(x,s_i)$ for all $y \in B(x,s_{i_1})$. Further, by the same argument, using s_{i_1} instead of s_i, and t_{i_1} instead of t_i, respectively, we can find an $s_{i_2} < t_{i_1+1} < s_{i_1}$ such that $B(z,s_{i_2}) \subset B(z,t_{i_1+1}) \subset \operatorname{int} B(x,s_{i_1})$ for every $z \in B(x,s_{i_2})$. Now let $z \in B(x,s_{i_2}) \cap B(p,s_{i_2})$ for an arbitrary point $p \in X$, then $p \in B(z,s_{i_2}) \subset \operatorname{int} B(x,s_{i_1})$, and therefore, $B(p,s_{i_2}) \subset B(p,s_{i_1}) \subset \operatorname{int} B(x,s_i)$. Summarizing, the system of all $W_i(x) = \operatorname{int} B(x,s_i)$ forms a *local base* of each $x \in X$ satisfying *condition* (ii) of the generalized Frink theorem cited above, and therefore, X is ω_μ*-metrizable* over a *continuous* semigroup (in fact, over a linearly ordered abelian group). □

If (X,d) is a semimetric space, d: $X^2 \to \mathbb{R}$, the same proof as above yields the theorem of GAGRAT and NAIMPALLY cited above as a corollary. (However, our proof is more direct and shorter and, moreover seems to be more appropriate; compare [GN].)

COROLLARY 1.7. ([GN] Theorem 4.2). *A semimetric space* (X,d) *is metrizable iff for all closed sets* $A \subset X$, d(x,A) *is a lower semicontinuous function of* x.

Now, Lemma 1.6, together with the above results, yields the following ω_μ-metrization theorem:

THEOREM 1.8. *For a* T_1*-space* X *the following are equivalent:*

(i) X *is* ω_μ*-metrizable;*

(ii) X *is metrizable over a* 0^+*-semigroup* S *with* cof $S = \omega_\mu$ *such that for every closed set* $A \subset X$, d(x,A) *is a lower semicontinuous (S-valued) function on* X.

SUMMARIZING DIAGRAM:

$$\begin{tikzcd}
\text{ω_μ-semimetrizable} \arrow[r, Leftrightarrow] & \text{metrizable over an 0^+-\textit{semigroup} S with cof $S = \omega_\mu$} \\
\text{ω_μ-metrizable} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & \text{metrizable over a \textit{continuous} 0^+-\textit{semigroup} S with cof $S = \omega_\mu$} \arrow[u, Rightarrow] \arrow[d, Leftrightarrow] \\
 & \text{metrizable over an 0^+-\textit{semigroup} S such that, for every closed $A \subset X$, $\underline{d(x,A)}$ is a \textit{lower semicontinuous} S-valued function of x. (cof m = cof S)}
\end{tikzcd}$$

It is typical for the theory of S-metrizable spaces, that for *uncountable* cardinals m = cof S, we often can prove theorems which do not have ω_0 analogues, i.e. analogues for metrizable or semimetrizable spaces. As an example, we now prove another (and rather "intrinsic") characterization of those S-metrizable spaces which are metrizable over *continuous* 0^+*-semigroups* S' with cof S' = cof S = m. In [NR], P. NYIKOS and H.C. REICHEL have shown that, if $m > \omega_0$, every m-metrizable space X has a compatible *non-archimedean* m-metric, i.e. there is a totally ordered abelian group G with cof G = cof m and a compatible metric d: $X^2 \to G$ satisfying $\underline{d(x,z) \le \max\{d(x,y), d(y,z)\}}$ for all $x,y,z \in X$, and generating the topology of X. (Compare also [STH].) -Let $\{g_i \mid i < m\}$ be a monotonically decreasing m-sequence in G converging to $0 \in G$, then - as an easy consequence of the strong triangle inequality every covering $\mathcal{B}_i$, $i < m$, consisting of all balls $B(x,g_i)$, $x \in X$, is in fact a partition $\mathcal{C}_i$ of X consisting of clopen sets C_α, $\alpha \in A_i$, where A_i is a suitable index set. Trivially, for every set C^i_α and $j < i$ there is exactly one $C^j_\gamma \in \mathcal{C}_j$ with $C^i_\alpha \subset C^j_\gamma$. Therefore, for every $i < m$, a transfinite induction argument will yield a total order $<_i$ on $\mathcal{C}_i$ such that for every $j < i$, $C^i_\alpha < C^i_\beta$ implies $C^j_\gamma < C^j_\delta$ if $C^i_\alpha \subset C^j_\gamma$ and $C^i_\beta \subset C^j_\delta$. Moreover, for every $x \in X$ and $i < m$, there is a uniquely defined set $C^i_x \in \mathcal{C}_i$ containing x. Now, for $x,y \in X$, let $x < y$ if and only if for every $i < m$, $C^i_x < C^i_y$. Since X is a T_2-space and the sets C^i_x, $i < m$, form a local basis at every $x \in X$, this procedure defines a *total order* < on X such that the given topology τ_d (induced by d on X) is *finer* than the *order topology* induced by < on X. Moreover, every C^i_x, $i < m$, is a (possibly degenerate) interval which follows from the construction of <. In other words, $(X,\tau_d,<)$ is a *generalized ordered space* and therefore - by a well-known result of ČECH (see e.g. LUTZER [LU]) - X is *suborderable,* i.e. can be embedded as a subspace of an orderable space.

The converse is also true which follows from a remarkable result of P. NYIKOS in [NY_2] and our Propositions 1.2 and 1.4. In this paper, P. NYIKOS shows that if the topology τ of a suborderable space X can be generated by an inverse m-semimetric q, then τ can also be generated by an uniformity which has a totally ordered base $\mathcal{B}$ with cof $\mathcal{B}$ = m, and, as is well-known and was indicated before, every such space is m-*metrizable* (see [STH]).

Summarizing the arguments and results above, and applying our Propositions 1.2 and 1.4, we obtain:

THEOREM 1.9. *Let* S *be an* 0^+*-semigroup of uncountable cofinality. A space* X *metrizable over* S, *is metrizable over a continuous* 0^+*-semigroup* S' *with* cof S' = cof S *if and only if* X *is a suborderable space.*

For *uncountable* cardinals m, there is yet *another* "intrinsic" necessary and sufficient condition for S-metrizable spaces to be S'-metrizable over a *continuous* 0^+-semigroup S' (and hence being m-metrizable by Proposition 1.2): we have already mentioned (in the proof above) that, if X is S-metrizable over a *continuous* 0^+-semigroups and cof S = $m > \omega_0$, we can assume without loss of generality that X is a *non-archimedeanly* G-metrizable over some linearly ordered abelian group G. (In fact we can assume that G is the lexicographically ordered product $\Pi\, \mathbb{Z}_i$ $(i < m)$ where $\mathbb{Z}_i$ denotes the additive group of the integers; see §4 and [STH], [RE_2].) Then it is easily seen that two balls B(x,s) and B(y,t) in such a space either have empty intersection or one of them contains the other. (In the proof of Theorem 1.9 we already used the equivalent fact that the system of all balls with a fixed radius forms a clopen partition of the whole space.) Therefore, every space which is metrizable over a *continuous* 0^+-semigroup S with uncountable cofinality m (i.e. every m-metrizable space) is a *non-archimedean topological space.* (As is well-known, a T_1-space X is called a *non-archimedean space* iff X has a base $\mathcal{B}$ such that any two basis sets either have empty intersection or one contains the other; for more details about the such spaces, see e.g. [NR_2], [NY_1], [RE_3], and the literature cited there.) - Interestingly enough, there is also a converse of this fact: let d: $X^2 \to S$ be a compatible distance function on an arbitrary *non-archimedean* topological space X, and let cof S = m, then we can show m-metrizability of X by using any of the m-metrizability theorems proved e.g. in [NR_1], but this result follows also from Theorem 1.9, since any non-archimedean topological space is *suborderable*. Thus we conclude:

THEOREM 1.10. *A space* X *metrizable over an* 0^+*-semigroup* S *with uncountable cofinality* cof S *is metrizable over a continuous* 0^+*-semigroup* S' *with* cof S' = cof S *if and only if* X *is a non-archimedean topological space.*

Concluding this section, the construction of the semigroups S_α in the proof of Theorem 1.3 allows us to give an "internal" topological *characterization of spaces* X *metrizable over* 0^+*-semigroups* S by special properties of local bases. This result generalizes a theorem of J. CEDER [CE], who characterizes semimetrizable spaces in his Theorem 6.1.*)

THEOREM 1.11. *Let* S *be an* 0^+*-semigroup with* cof $S \leq \alpha$, *then a* T_1*-space* X *is metrizable over* S *if and only if for each point* $x \in X$, *there are two nested neighbourhood-bases* $\mathcal{U}(x) = \{U_i(x) \mid i < \alpha\}$ *and* $\mathcal{B}(x) = \{V_i(x) \mid i < \alpha\}$ *such that* $V_i(x) \subset U_i(x)$ *for every* $i < \alpha$, *and* $y \in V_i(x)$ *implies* $x \in U_i(y)$ *for* $y \in X$.

PROOF. For the necessity, let X be metrizable over S, choose a minimal strictly decreasing α-sequence $\{s_i \mid i < \alpha;\ s_i < s_j \Leftrightarrow i > j\}$ in S, converging to zero w.r.t. the order topology of S, and for each $x \in X$, put $U_i(x) = V_i(x) = B_i(x) = \{y \in X \mid d(x,y) < s_i\}$.

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*) Compare our Corollary 1.12, which collects necessary and sufficient (local) conditions for a space X to be metrizable by general semigroup-valued distance functions.

To prove the sufficiency, fix any pair x, y, $x \neq y$, of points in X, and let $j(x,y) = j = \min\{i \mid x \notin U_i(y) \text{ or } y \notin U_i(x)\}$*), and define $d(x,y) = s_{j-1}$ if j is a successor ordinal (i.e. s_j has an immediate "predecessor" $s_{j-1} > s_j$) and let $d(x,y) = s_j$ if j is limit ordinal. Moreover, let $d(x,y) = 0 \in S$ if $y \in U_i(x)$ for all $i < \alpha$ (i.e. if $x = y$; note that X is a T_1-space). Then the family of all $B_i(x) = \{y \in X \mid d(x,y) < s_i\}$, $i < \alpha$, is a local base at x:

(i) for any $i < \alpha$ and $j \geq i$, we have $B_j(x) \subset U_i(x)$, since $z \in B_i(x)$ means $d(z,x) < s_j \leq s_i$ and therefore, $z \in U_i(x)$ as well as $x \in U_i(z)$ by definition of d;

(ii) if $j > i$, $V_j(x) \subset B_i(x)$, since for any $z \in V_j(x)$, we have $x \in U_j(z)$ and $z \in V_j(x) \subset U_j(x)$, by the special property of the given local bases $\mathcal{U}(x)$ and $\mathcal{B}(x)$, and therefore $d(z,x) \leq s_j < s_i$. Thus $z \in B_i(x)$, and we are done. □

If we combine Theorem 1.11 with our characterization of ω_μ-metrizable spaces in [NR_1], p. 6, and with our (slightly modified) characterization of ω_μ-quasimetrizable**) spaces in [RR], p. 238, (see also [RE_1], [RE_2] and [RI]) we obtain the following:

<u>COROLLARY 1.12</u>. *A* T_1*-space* X *is*

(i) *metrizable over an* 0^+*-semigroup* S *with* cof $S \leq \alpha$,

(ii) *metrizable over a continuous* 0^+*-semigroup* S *with* cof $S \leq \alpha$,

(iii) *quasimetrizable over an* 0^+*-semigroup* S *with* cof $S \leq \alpha$,

(iv) *quasimetrizable over a continuous* 0^+*-semigroup* S *with*

(1) cof $S = \omega_0$, or (2) $\omega_0 <$ cof $S \leq \alpha$.

if and only if to every point $x \in X$*, we can assign two nested neighbourhood-bases* $\mathcal{U}(x) = \{U_i(x) \mid i < \alpha\}$ *and* $\mathcal{B}(x) = \{V_i(x) \mid i < \alpha\}$ *such that* $V_i(x) \subset U_i(x)$ *for every* $i < \alpha$*, and such that for* $y \in X$*, the following conditions hold respectively:*

(i) $y \in V_i(x)$ *implies* $x \in U_i(y)$; ([Th. 1.11])

(ii) $y \in V_i(x)$ *implies* $V_i(y) \subset U_i(x)$ *and* $y \notin U_i(x)$ *implies* $V_i(x) \cap V_i(y) = \emptyset$; ([$NR_1$])

(iii) $U_i(x) = V_i(x)$ *for every* $i < \alpha$ *and every* $x \in X$ *(i.e.: if every point has a nested nbhd-base of cofinality* α*)*; ([RR]; *and* §3, *Example* 2a).

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*) We can assume that $U_1(x) = X$ for every $x \in X$.

**) X is ω_μ-quasimetrizable iff X is quasimetrizable over a totally ordered abelian group G with cof $G = \omega_\mu$.

(iv) (1) $\alpha = \omega_0$, *and* $V_i(x) = U_i(x)$ *for all* $i < \alpha$ *and every* $x \in X$, *and* $y \in U_{i+1}(x)$ *implies* $U_{i+1}(y) \subset U_i(x)$; ([RI], p. 35)

(2) *for* $\alpha > \omega_0$, *we have a stronger condition:* $V_i(x) = U_i(x)$ *for all* $i < \alpha$ *and every* $x \in X$, *and* $y \in U_i(x)$ *implies* $U_i(y) \subset U_i(x)$; ([RR], [RE_1]).

-Compare condition (ii)!

2. m-MOORE SPACES AND S-METRIZABILITY

This section is concerned with some aspects of the general relationship between m-*developability* and *metrizability* of T_1-spaces X *over arbitrary* 0^+-*semigroups*. Specializing our main result to countable cofinality, we shall obtain a well-known theorem on *developability* of *semimetric* spaces proved by GAGRAT and NAIMPALLY.

Let m be a regular cardinal. By a definition of P. NYIKOS [NY_2], a space X is *developable over* m if there exists a set of open covers of X: $\{\mathcal{B}_i \mid i < m\}$ such that

(i) $\mathcal{B}_i$ refines $\mathcal{B}_j$ whenever $i > j$, and

(ii) For each point $x \in X$, the collection $\{St(x,\mathcal{B}_i) \mid i < m\}$ (where $St(x,\mathcal{B}_i) = \cup\{B \mid x \in B \in \mathcal{B}_i\}$) is a local base at x.

Regular spaces which are developable over m, are called m-*Moore spaces;* obviously ω_0-Moore spaces are Moore spaces in the "classical" sense and vice-versa.

Spaces metrizable over *continuous* 0^+-semigroups S are m-*Moore* spaces for m = cof S because of Proposition 1.2 and the above proof of 1.9, respectively.**) This is *not true* for spaces metrizable over *arbitrary* 0^+-semigroups S in general. It is well-known that there are semimetric spaces, hence by (1.3) S-metrizable spaces with cof S = ω_0, which are not developable.

However, we can prove:

<u>THEOREM 2.1</u>. *A* T_1-*space* X *is developable over* $m \geq \omega_0$ *if and only if its topology can be generated by an upper semicontinuous metric* d *over an* 0^+-*semigroup* S *with* cof S = m.*)

---

*) It would also suffice to assume upper semicontinuity of d at all points of the diagonal ΔX of X.

**) Note that cof S is always a regular cardinal.

REMARK. Note that in Theorem 2.1 the 0^+-semigroup S need not be a continuous 0^+-semigroup. - Neither can "upper semicontinuous metric d over S" be replaced by "continuous metric d over S": as will be shown in §4, Example 2, there exists a *developable* space X which does *not* admit any compatible *lower* semicontinuous distance function on X.

PROOF OF THEOREM 2.1. (a) Let $d: X^2 \to S$ be a compatible S-metric on X, then d is *separately* (i.e. in each variable) *upper semicontinuous* if and only if every ball $B(x,s)$, $s > 0$, is *open*: let $d(z,x) < s$, then there is a ball $B(z,t)$ such that for every $y \in B(z,t)$, $d(x,y) < s$. Conversely, let $d(x,y) = s \in S$ and $t > s$, then there is a ball $B(y,r)$, $r \in S$, such that $B(y,r) \subset B(x,t)$, i.e. $d(x,z) < t$ for every $z \in B(y,r)$.

(b) Now fix a monotonically decreasing m-sequence $\{s_i \in S/i < m\}$ converging to $0 \in S$, and let us construct an m-development for X. $d: X^2 \to S$ is upper semicontinuous at $(x,x) \in S^2$ for every $x \in X$. Now since $d(x,x) = 0$, for every $s \in S$, $s > 0$, there are neighbourhoods $B(x,s_1)$ and $B(x,s_2)$ such that $d(z,y) < s$ if $z \in B(x,s_1)$ and $y \in B(x,s_2)$. Equivalently, for every $s_i \in S$, there is an $s_{i,x} \in S$, $s_{i,x} < s_i$, such that $d(y,z) < s_i$ for all $y,z \in B(x,s_{i,x})$. By a transfinite induction argument, we can suppose that $s_{i,x} > s_{j,x}$ if $i < j$, for every $x \in X$. Now we assert that the system of all $\mathcal{B}_i := \{B(x, s_{i,x}) \mid x \in X\}$, $i < m$ is am m-*development* for X. Since, by part (a), all balls are open, we have to show that for every $x \in X$, the family of all $St(x,\mathcal{B}_i)$, $i < m$, forms a local base at x. But this follows immediately from the construction of $\mathcal{B}_i$: for an arbitrary $i < m$, let $B(y,s_{i,y}) \in \mathcal{B}_i$ and $x \in B(y,s_{i,y})$, then $d(x,z) < s_i$ for every $z \in B(y,s_{i,y})$, i.e.: $St(x,\mathcal{B}_i) \subset B(x,s_i)$.

(c) Now let the system of open covers $\mathcal{B}_i = \{0_{i,\alpha} \mid \alpha \in A_i\}$ of X, $i < m$, be an m-development for X. Moreover, let S_m be the 0^+-semigroup constructed in the proof of Th. (1.3), and let $\{s_i \in S_m \mid i < m\}$ be a monotonically decreasing m-sequence of generators of S_m. - For $x,y \in X$, define $j = \min\{i < m \mid y \notin St(x,\mathcal{B}_i)\}$, and $d(x,y) = s_j$. Since X is a T_1-space, $d: X^2 \to S_m$ is an S_m-metric on X satisfying all axioms (the triangle inequality included) and generating the topology of X, because $d(x,y) < s_i$ is equivalent with $y \in St(x,\mathcal{B}_i)$. We only have to show that $d: X^2 \to S_m$ is an upper *semicontinuous* function:

First, let $d(x,y) = 0 \in S$, then $x = y$ because X is a T_1-space. For any $s_i \in S$, $s_i > 0$, x is contained in at least one open set $0_{i,\alpha} \in \mathcal{B}_i$.

So for every pair $z_1,z_2 \in O_{i,\alpha}$, we have $z_1 \in St(z_2,\mathcal{B}_i)$ and $z_2 \in St(z_1,\mathcal{B}_i)$ which is equivalent with $d(z_1,z_2) < s_i$. If, on the other hand $d(x,y) = s_i > 0$, then for each $s_j > s_i$, $y \in St(x,\mathcal{B}_j)$. So there is an open set $O_{j,\alpha} \in \mathcal{B}_j$ containing both x and y. Therefore for every pair $z_1 \in O_{j,\alpha}$ and $z_2 \in O_{j,\alpha}$, we have $z_1 \in St(z_2,\mathcal{B}_j)$ and $z_2 \in St(z_1,\mathcal{B}_j)$ which is equivalent to $d(z_1,z_2) < s_j$. - Summarizing, $d: X^2 \to S_m$ is an upper semicontinuous function. □

Specializing to $m = \omega_0$ and using Theorem 1.3 we obtain the following

COROLLARY 2.2. ([GN]). *A* T_1*-space* X *is developable if and only if it has a compatible upper semicontinuous semimetric.*

REMARK. Corollary 2.2 has been proved by M. GAGRAT and S.A. NAIMPALLY in [GN], Theorem 3.3. However, the proof given by Gagrat and Naimpally uses Lodato-proximities and Mozzochi-uniformities and leans heavily on a result of MOZZOCHI and a theorem of M. BROWN proved in [Summer Inst. Set Theoretic Topology, Madison, Wisconsin (1957) p. 65]. The proof given above is shorter and more direct, so much the more it shows how the theorem of Gagrat and Naimpally can be generalized to higher cardinalities in a natural way. Note that the theorem of Gagrat and Naimpally itself is a generalization, of a result of H. COOK ([CO]).

In [GI], GITTINGS studies semimetrizable spaces where every ball is open (so-called o-*semimetric spaces*). By combining part (a) of the proof above and Theorem 2.1 with a more or less obvious generalization of GITTINGS results to higher cardinals, we obtain the following theorem.

THEOREM 2.3. *For a* T_1*-space* X *the following are equivalent.*

(a) X *is metrizable over an* 0^+*-semigroup* S *such that* $d: X^2 \to S$ *is separately semicontinuous*

(b) X *is metrizable over an* 0^+*-semigroup* S *such that all balls* $B(x,s)$, $x \in X$, $s \in S$, $s > 0$, *are open.*

(c) X *is semi developable over a regular cardinal* m, *i.e.* X *has an* m*-sequence of not necessarily open covers* G_i, $i < m$, G_j *refining* G_i *iff* $j > i$ *and such that for every* $x \in X$, $\{St(x,G_i) \mid i < m\}$ *is an open neighbourhood-base at* x.

Since there are semimetrizable spaces which are not o-semimetrizable (see e.g. [HE_2], [RR]) and o-semimetrizable spaces which are not developable (see [BO]), the class σ of spaces described by Theorem 2.3 lies *between* the class of m-*developable* spaces and the class of spaces *metrizable* over 0^+-*semigroups* S with cof S = m. In analogy to [GI], spaces which are characterized by property (b) of the theorem above are called m-o-semimetrizable.

<u>Summarizing diagram</u>:

$$\begin{tikzcd}
\text{m-semimetric} \arrow[r, Leftrightarrow] & \text{metrizable over an } 0^+\text{-semigroup } S \\
\text{m-o-semimetrizable} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & \begin{array}{l}\text{metrizable over an } 0^+\text{-semigroup } S \text{ such that } d\colon X^2 \to S \\ \text{is } \textit{separately upper semicontinuous}\end{array} \arrow[u, Rightarrow] \\
\text{m-developable} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & \begin{array}{l}\text{metrizable over an } 0^+\text{-semigroup } S \text{ such that } d\colon X^2 \to S \\ \text{is } \textit{upper semicontinuous}\end{array} \arrow[u, Rightarrow] \\
\text{m-metrizable} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & \text{metrizable over a } \textit{continuous } 0^+\textit{-semigroup } S \arrow[u, Rightarrow]
\end{tikzcd}$$

Concerning the problem under which additional conditions the *vertical arrows* can be reversed, see §1 and the following

<u>REMARK</u>. In [NY_2], P. NYIKOS proved that for uncountable cardinals m, a *suborderable* space has a compatible inverse m-semimetric iff it admits an m-development. A reformulation of this result using our Theorem 1.3 yields the following

<u>THEOREM 2.4</u>. *For uncountable regular cardinals* m, *a suborderable space* X *is* m-*developable if and only if* X *is metrizable over an* 0^+*-semigroup* S *with cofinality* m.

<u>PROBLEM</u>. Is there a topological property which, together with metrizability over 0^+-semigroups S, characterizes spaces developable over m and which holds also for countable cardinals m? (Compare [GI], Problem (2).)

3. METRIZABILITY OVER GENERAL LINEARLY-ORDERED SEMIGROUPS $(S;+,\leq)$:

0^+-semigroups as they have been used in the preceding sections are special cases of linearly ordered abelian semigroups. In general, an abelian semigroup (S,+) with zero element 0 is called a linearly (or totally) ordered semigroup if there is a linear order $\leq$ on S such that for all $a,b,c \in S$, $a < b$ implies $a+c \leq b+c$. For our purposes, we further assume that $0 \leq a$ for every $a \in S$.

In this section, we study metrizability of topological spaces X over *arbitrary* totally ordered abelian semigroups $(S,\leq)$ which seems to be a very convenient frame for studying general metrizability problems in connection with general distance functions. To that end, recall Definition 1.1, which we refer to when speaking of spaces "*metrizable over* $(S,+,\leq)$". As we shall see, it is this general concept which unifies the paper, and most of the other different ways of generalizing "classical" metrics, respectively.

Without loss of generality, we assume further that for each $s \in S$, $s > 0$, there exists $t \in S$ with $0 < t < s$. (Obviously, if S does not satisfy this property, any space metrizable over S is discrete; vice versa, any discrete space X is metrizable over the semigroup $\{0,1\}$ with $0+1 = 1+0 = 1$ and $1+1 = 1$, and such that $0 < 1$.)

First, we are concerned with some relations between topological properties of a space metrizable over a totally ordered abelian semigroup S and structural (=algebraical) properties of S.

(For other aspects and details of this scope, see also a paper of H.C. REICHEL and W. RUPPERT [RR].)

Among others, Theorem 3.1 and the examples thereafter may show to what extent countability inherent in classical (quasi-)metrization theorems can be replaced by order theoretic properties. For other aspects of this realm, compare e.g. a paper of J. VAUGHAN [VA] on linearly stratifiable spaces.

Let us consider *three properties* a totally ordered abelian semigroup $(S,\leq)$ may have:

(A) $r < s \Rightarrow r+t < s+t$ for all $r,s,t \in S$

(B) for each $s > 0$, there is a t, $0 < t < s$, such that $t+t < s$ (i.e.: addition, $s \to s+s$, is continuous at $0 \in S$ w.r.t. the order topology on S);

(C) there is an element $r \in S$, $r > 0$, such that every subset $T \subseteq S$ containing an element $t \leq r$ has a g.l.b.; i.e.: for subsets $T \subseteq S$ containing sufficiently small elements, inf T exists.

In §1 we have seen that the class Σ of T_1-spaces X *metrizable over* S coincides with the class of m-*semimetrizable* spaces *if* S *satisfies* (A) (i.e. if S is an 0^+-*semigroup*); and with the class of m-*metrizable* spaces *if* S *satisfies* (A) *and* (B). In the following, we show that Σ coincides with the class of *metric* spaces *if* S *satisfies* (A), (B) *and* (C). - At the same time, we present an analogue of this theorem for *quasimetric* spaces. Moreover, we shall present *three examples* of totally ordered abelian semigroups $\underline{S}$ which satisfy *two* of these *properties* and *lack* the *third one* respectively, as well as *non-metrizable* topological spaces which are *metrizable over* $\underline{S}$ alternately. (Roughly speaking, this shows that for developing the "classical" theory of real-valued (quasi-)metric spaces, we need exactly those three "algebraic" properties of the semigroup of the nonnegative reals $\mathbb{R}_0^+$; in other words, these and only these three properties of the semigroup $\mathbb{R}_0^+$ are "responsible" for the results of the "classical" theory of real-valued (quasi-)metrics and the topologies induced by them.)

THEOREM 3.1. *For a topological space* X *are equivalent:*

(i) X *is metrizable* [*quasimetrizable*] *over a totally ordered abelian semigroup* $(S,\le)$ *satisfying* (A), (B) *and* (C).

(ii) X *is metrizable* [*quasimetrizable*] *(over* $\mathbb{R}$*)*.

PROOF. Let the topology τ of X be induced by $d\colon X \times X \to S$. If $S\setminus\{0\}$ has a minimal element, τ is discrete, and we are done. Otherwise, choose an inversely well ordered cofinal ω_μ-sequence $\{s_i \in S \mid i < \omega_\mu\}$ with minimal cardinality converging to $0 \in S$ with respect to the order topology on S. Now by induction, construct a sequence $s_n \in S$, starting with (a sufficiently small) $s_1 \in S$ and such that $s_{i+1} + s_{i+1} < s_i$ for all $i = 1,2,\ldots$. This is possible because of property (B). Then, by hypothesis (C) $\inf\{s_i \mid i=1,2,\ldots\} = t$ exists in S. Moreover $t+t < s_j+s_j < s_i$, for all $i,j = 1,2,\ldots,i<j$. Thus $t+t \le \inf\{s_j \mid j \in \mathbb{N}\} = t$, and therefore: $t = 0$, as follows from property (A). (Note that (A), (B), (C) were used *explicitly*.) Now, since ω_μ was chosen to be the minimal cardinality of inversely well ordered net in S converging to $0 \in S$, we have $\omega_\mu = \omega_0$. Thus the topology τ on X induced by d is metrizable which, for example, follows by A.H. FRINK's metrization theorem (see e.g. [NA]): (X,τ) is a Hausdorff space as follows from property (B) and the triangle inequality; and $W_n(p) = \{y \mid d(p,y) < s_n \in S\}$, $n = 1,2,\ldots$, is a local base of any point $p \in X$, satisfying: $W_{i+2}(p) \cap W_{i+2}(q) \neq \emptyset \Rightarrow W_{i+2}(q) \subset W_i(p)$ by consrtuction of the net $\{s_i \mid i < \omega_0\}$. Clearly, let $z \in W_{i+2}(p) \cap W_{i+2}(q)$ the, by triangle inequality, $d(p,q) \le s_{i+2} + s_{i+2} < s_{i+1} < s_i$.

If X is quasimetrizable over S the arguments above hold word by word with one exception: instead of A.H. Frink's theorem we use the quasimetrization theorem of H. RIBEIRO (see e.g. RIBEIRO's paper [RI], p. 35, or a paper of M. SION and G. ZELMER [SZ]): (X,τ) is quasimetrizable iff every point $x \in X$ has a countable local base $\{U_n(x) \mid n = 1,2,\dots\}$ such that: $y \in U_{n+1}(x) \Rightarrow U_{n+1}(y) \subset U_n(x)$. (Compare our Corollary 1.12). And this property is satisfied automatically by the d-balls $B(x,s) = \{y \in X \mid d(x,y) < s\}$, $s \in S\backslash\{0\}$, as used before. - For a detailed study of spaces quasimetrizable over 0^+-semigroups and general quasimetrizability theory, see also a paper of H.C. REICHEL and W. RUPPERT [RR]. □

Summarizing diagram.

$$\begin{tikzcd}
\text{m-semimetrizable for some cardinal m} \arrow[r, Leftrightarrow] & \text{metrizable over a totally ordered abelian semigroup S satisfying (A)} \\
\text{m-metrizable for some cardinal m} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & \text{metrizable over a totally ordered abelian semigroup S satisfying (A) and (B)} \arrow[u, Rightarrow] \\
\text{metrizable} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & \text{metrizable over a totally ordered abelian semigroup S satisfying (A), (B) and (C)} \arrow[u, Rightarrow]
\end{tikzcd}$$

It may be interesting to compare this table with the diagrams in §§1 and 2. - Note moreover, that in the theory of S-metrizable spaces X we can restrict ourselves essentially to *three kinds of 0^+-semigroups* S.

(i) by our Theorems 1.3 and 1.4, and m-*semimetrizable* space is metrizable over some *semigroups* S_α (as they have been constructed and described in the proof of Theorem 1.3); and

(ii) any m-*metrizable* space can be remetrized by either a "classical" real-valued metric if $m = \omega_0$, or (iii) by a metric $d: X^2 \to G$ where G is the lexicographically ordered direct product of m copies of the additive group of integers $\mathbb{Z}$. (Compare example 1 below; for more details, see [STH], [RR] and [RE$_2$].)

EXAMPLES:

EXAMPLE 1. Let X be the set of all nets $\{a_i \in \mathbb{R} \mid i < \omega_\mu\}$ for some cardinal $\omega_\mu > \omega_0$, and let G be the direct product $\Pi \mathbb{Z}_i$ $(i < \omega_\mu)$ where every $\mathbb{Z}_i$ denotes a copy of the additive group of the integers, topologized by the lexicographic order on G. Then G is a topological group, and $S = G_0^+$ is a topological semigroup satisfying properties (A) *and* (B) *but not* (C). Now let

$d: X \times X \to S$ be such that, for $(a_i),(b_i) \in X$: $d((a_i),(b_i)) = (x_j) \in S$ where $x_j = 1$ iff $j := \min\{i \mid a_i \neq b_i\}$, and $x_k = 0$ for all other $k < \omega_\mu$; moreover, let $d((a_i),(b_i)) = 0 \in S$ iff $a_i = b_i$ for all $i < \omega_\mu$. Then d satisfies all metric axioms, and the system of all d-balls is a base for a T_1-topology τ on X which is *not metrizable,* since (X,τ) is not first countable.

EXAMPLE 2. Let $S_0 = S_{\omega_0}$ be the 0^+-semigroup constructed in the proof of Theorem 1.3. Then S_0 *satisfies* (A) *and* (C) *but not* (B) since by construction of S_{ω_0}, $s_i+s_i > s_1$, for all $i < \omega_0$. Now let $M = \{s_i \mid i < \omega_0\}$ and for $X = M$, consider the distance function $d: X \times X \to S_0$ with $d(x,y) = \min\{x,y\}$, $x \neq y \in X$; and $d(x,x) = 0$ for all $x \in X$. Then d satisfies all metric axioms and the topology τ_d on X induced by the system of all d-balls $B_i(x) = \{y \mid d(x,y) < s_i\}$, $i = 1,2,\ldots, x \in X$, is *not metrizable,* because (X,τ_d) is T_1 but not Hausdorff. By Theorem 1.3, (X,τ_d) is *semimetrizable*; moreover, we can show that (X,τ_d) is *quasimetrizable* (over $\mathbb{R}$): for each $x \in X$, let $U_i(x) = \{y \mid d(x,y) < s_i\}$ if $x > s_i$ and $U_i(x) = \{y \in X \mid y \leq s_i\}$ if $x \leq s_i$. Then for each set $U_i(x)$ and $y \in U_i(x)$, we have $U_i(y) \subset U_i(x)$. Therefore, by P. RIBEIRO's theorem we can conclude that (X,τ_d) is (in fact non-archimedeanly) quasimetrizable. Indeed, X is even a γ-space in the sense of [EL]. Referring to the remark after Theorem 2.1 we add that X is also developable (because d is upper semicontinuous) but X does *not* admit any topologically compatible distance function d' which is *lower semicontinuous*: if d' is any S-valued metric compatible with τ_d and S is any linearly ordered semigroup, then d' cannot be lower semicontinuous at any pair (x,y) of points $x \neq y$ in X, since any two neighbourhoods of x and y intersect and therefore, any neighbourhood of $(x,y) \in X^2$ contains a pair (z,z) so that $d(z,z) = 0 \in S$. (A development of X can be given also explicitly: let $\mathcal{B}_i$ consist of all balls $B_i(x)$, $x \in M$ and $x \geq s_i$, then $\{\mathcal{B}_i \mid i < \omega_0\}$ is a development of X.)

EXAMPLE 2a. In order to construct a space X which is *quasimetrizable over* S_0 but *not quasimetrizable* over $\mathbb{R}$, take any first countable T_1-space which is not quasimetrizable. For every point $p \in X$, fix a local base $\{B_i(p) \mid i = 1,2,\ldots\}$ totally ordered by inclusion, and define $d_{S_0}(p,q) = s_j \in S_0$ where $j = \min\{i \mid q \notin B_i(p)\}$, as well as $d_{S_0}(p,p) = 0$, for every $p \in X$. Then it is readily proved that the distance-function d_{S_0} on X satisfies the axioms of a *quasimetric over* S_0 and the topology τ of X is generated by d_{S_0}. -Compare our Corollary 1.12, (iii).

EXAMPLE 3. Let $S = \{x_i \mid i < \omega_\mu\}$ be an uncountable and inversely well ordered set such that $x_i < x_j$ iff $i > j$. Moreover, suppose that S contains an element 0 such that $0 < x_i$, for all $i < \omega_\mu$. If, for $s,t \in S$, we define $s+t = \max\{s,t\}$, S becomes a totally ordered abelian semigroup with properties (B) *and* (C) *but not* (A): we have $x_3 < x_2$, but $x_3 + x_1 = x_2 + x_1$. Now let X be the same set as used in Example 1 metrized by $d_S\colon X \times X \to S$, $d_S((a_i),(b_i)) = x_j$ iff $j = \min\{i \mid a_i \neq b_i\}$, and $d_S((a_i),(b_i)) = 0$ iff $a_i = b_i$ for all $i < \omega_\mu$. Then the *S-metric* d_S *generates the* same *topology* τ on X as the distance function described in Example 1. And obviously, (X,τ) is *not* (quasi-) *metrizable*.

REMARK 3.2. In several interesting papers, S. NEDEV, M.M. ČOBAN and others present a detailed study of so-called *o-metric spaces*, see e.g. [NE] or [NC], where a space X is called *o-metrizable* if there exists a real-valued function d on X^2 such that (i) $d(x,y) = 0$ if $x = y$, and (ii) a set $F \subset X$ is closed iff for every $x \notin F$, $\inf\{d(x,y) \mid y \in F\} > 0$. If d is a symmetric function the space X is called *symmetrizable*. If X is o-metrizable by a function satisfying the triangle inequality, X is called *Δ-metrizable*. (Compare e.g. [KO] and [NE].) Note that the concept of symmetric spaces is weaker than the concept of semimetric spaces, even if the additional axiom $d(x,y) = 0 \Longleftrightarrow x = y$ is satisfied. In a symmetrizable space, the balls $B(x,\varepsilon)$, $x \in X$, $\varepsilon > 0$, only form a *network* for the topology, i.e.: for every open set $0 \subset X$ and $x \in X$, there is a ball $B(x,\varepsilon)$ contained in 0. The balls need *not* be *neighbourhoods* of x! A symmetrizable space is *semimetric* iff $x \in \operatorname{int} B(x,\varepsilon)$ for every ball $B(x,\varepsilon)$, or equivalently, if $d(x,y)$ defines a continuous function of x at each point $y \in X$; see e.g. [NE], [SIW] and others cited there. To show the difference explicitly, let us give an example which is derived from Example 2.1 in REICHEL-RUPPERT [RR]:

EXAMPLE 4. Let $X = \{x \in \mathbb{R} \mid 0 < x < \frac{1}{2}\}$ and $\mathcal{B}_{\varepsilon,y} = \{x \mid 0 < x < \varepsilon\} \cup \{z \mid |z-y| < \varepsilon\}$, $\varepsilon > 0$, $y \in X$, be a base for a topology τ on X. Then τ is a T_1-topology on X. Let S be the 0^+-semigroup consisting of all real numbers $s > \frac{1}{2}$ and $0 \in \mathbb{R}$, with the usual addition. For $x \in X$, let $\phi(x) = \frac{1}{2}$ if x is irrational and $\phi(x) = x$ otherwise. Then $d(x,y) = \min\{|x-y|, \phi(x), \phi(y)\} + \frac{1}{2}$, and $d(x,x) = 0$ for each $x \in X$, defines an *S-valued symmetric distance function* d on X satisfying *all* metric axioms (the triangle inequality included!) and the system of all balls $B(x,s)$, $x \in X$, $s > 0$ is a *network* for the topology τ. However, it is easily seen that *for no* $x \in X$, $s > 0$, $B(x,s)$ is a *neighbourhood* of x. Therefore, (X,τ) cannot be metrized by $d\colon X^2 \to S$ in the sense of

Definition 1.1. Note that, for example, the closure of $A = \{\frac{\sqrt{2}}{n} \mid n = 1,2,\ldots\}$ is the whole space X, whereas $d(x,A) = \inf\{d(x,y) \mid y \in A\} > 0$, for every $x \notin A$; i.e.: $x \in cl(A)$ *need not* imply that $d(x,A) = 0$ as it would do if d were an S-*metric* generating the topology of X.

4. S-METRICS WHICH DO NOT GENERATE TOPOLOGIES

Example 3.4 shows that a distance-function d on a set X with values in an 0^+-semigroup $(S,+)$ need not give raise to a topology on X; i.e.: the system of all balls $B_s(x) = \{y \in X \mid d(x,y) < s\}$, $s \in S\setminus\{0\}$, need not be a neighbourhood base at $x \in X$ with respect to any topology τ on X.

It is obvious that continuity of "addition" in S, $(s,t) \to s+t$, at $(0,0) \in S^2$ is a sufficient condition that a distance function $d\colon X^2 \to S$ induces a toplogy on X. Moreover, in §1 we saw that the following only formally weaker condition is sufficient, too:

(*) The function $f\colon S \to S$, defined by $f(s) = s+s$, $s \in S$, is continuous at $0 \in S$ w.r.t. the order topology on S.

In fact, Proposition 1.2 showed that this condition guarantees an ω_μ-metrizable topology on X. On the other hand, the results of the former sections showed that (*) is not necessary for a distance function $d\colon X^2 \to S$ to induce a toplogy on X. In fact, we proved that the class of T_1-spaces (X,τ) where τ is induced by a distance function $d\colon X^2 \to S$ with cof $S = \omega_\mu$ coincides with the class of ω_μ-semimetrizable spaces.

Now we shall show that *no* condition on S weaker than (*) can guarantee that a distance function $d\colon X^2 \to S$ on a set X gives raise to a topology on X:

THEROEM 4.1. *Let* $(S,+)$ *be an* 0^+-*semigroup such that the function* $s \to s+s$, $s \in S$, *is not continuous at* $0 \in S$. *Then there is always a set* X *and a distance function* $d\colon X^2 \to S$ *satisfying all metric axioms such that, for* $x \in X$, *the family of the balls* $B_s(x)$, $s \in S\setminus\{0\}$, *cannot be a local base for a topology* τ *on* X.

PROOF. First it is easy to see that if $s \to s+s$ is not continuous at $0 \in S$, then there must be an element $c \in S$, $c > 0$, such that $s+t > c$, for every pair $(s,t) \neq (0,0)$. Now let $Y = S\setminus\{0\}$ and define $d\colon Y^2 \to S$ by $d(p,q) = \min\{p,q,c\}$ if $p \neq q$, and $d(p,p) = 0$, for all $p,q \in Y$. Then d satisfies all metric axioms, the triangle inequality included, and the family $\mathcal{B}$ of balls $B_s(p)$, $p \in Y$, $s \in S\setminus\{0\}$, is an (open) base for a topology τ on Y (although

S is *not* a continuous 0^+-semigroup as defined in §1). Further, let $X = Y^2$, and define a distance function d': $X^2 \to S$ as follows

$$d'((p_1,q_1),(p_2,q_2)) = \begin{cases} d(p_1,p_2) & \text{if} \quad q_1 = q_2 \\ d(q_1,q_2) & \text{if} \quad p_1 = p_2 \\ c & \text{otherwise} \end{cases}$$

Then we can show by direct computation that d' satisfies all metric axioms including the triangle inequality.*) But for *no* point $x = (p,q)$ in X, the balls $B_s(x)$, $s \in S\backslash\{0\}$, can form a *neighbourhood base* for any topology τ on X, since for no $y \in B_s(x)$, $y \neq x$, we can find a ball $B_t(y)$ with $0 < t < s < c$ such that $B_t(y) \subset B_s(x)$: Clearly, for $x = (p,q)$ and $s < \min\{c,p,q\}$, $B_s(x) = \{(u,v) \in X \mid v = q \text{ and } u < s\} \cup \{(u,v) \mid u = p \text{ and } v < s\} \cup \{x\}$: but for any $y \in B_s(x)$, $y \neq x$, say $y = (p,w)$ with $w < s$, and for $0 < t < s$, there is a point $z = (m,w) \in X$, where $0 < m < t$, such that $z \in B_t(y)$ but $z \in B_s(x)$ since $d'(z,x) = c > s$.

Note however, that there are other S-valued matrics on X,d'': $X^2 \to S$, for example: $d''((p_1,q_1),(p_2,q_2)) = \max\{d(p_1,p_2),d(q_1,q_2)\}$, or 0 respectively, such that the d-balls $B_s(x)$, $x \in X$, $s \in S\backslash\{0\}$, do form an (open) base for a T_1-topology on X.

Comparing our Theorem 1.3 and 4.1 with the paper of MORREL and NAGATA [MN] will show interesting connections between semigroup-valued distances and statistical metrics!

5. PARACOMPACTNESS PROPERTIES OF SPACES METRIZABLE OVER 0^+-SEMIGROUPS

One of the most useful theorems in metrization theory states that every metrizable space is paracompact. Now, by our Proposition 1.2, every T_1-space X which is *metrizable* over a *continuous* 0^+*-semigroup* S is ω_μ-metrizable for some cardinal ω_μ and hence X is *paracompact*. (That every ω_μ-metrizable space is paracompact has been shown independently, and by different methods, by I. JUHÁSZ [JU], A. HAYES [HA] and P. NYIKOS and H.C. REICHEL [NR$_2$] (see also [RE$_3$]).) Now by generalizing and combining methods developed by R.H. BING, L. McAULEY, D.K. BURKE and R.A. STOLTENBERG, we can show that any T_1-space X *metrizable* over on *arbitrary* 0^+*-semigroup* S with cof $S \leq \omega_\mu$, is ω_μ*-paracompact*. Let us first define this concept which

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*) To show this, we have to use the fact that S is *not* a continuous 0^+-semigroup, and the special property of $c \in S$ described before.

generalizes *σ-paracompactness*, and hence subparacompactness, in a natural way:

DEFINITION. A T_1-space X is ω_μ*-paracompact* if for every open cover $\mathcal{O}$ of X there is a system $\{\mathcal{O}_i \mid i < \omega_\mu\}$ of open covers $\mathcal{O}_i$ of X such that for every $x \in X$, there is an index $j = j(x) < \omega_\mu$ such that $St(x,\mathcal{O}_j)$ is a subset of some set $0 \in \mathcal{O}$.

ω_0-paracompactness in the sense of this definition has been introduced by A.V. ARHANGELSKII in [AR] under the name "*σ-paracompactness*". In [BU], D.K. BURKE showed that σ-paracompactness is equivalent with "F_σ*-screenability*" of spaces X, defined by McAULEY in [MA] (i.e. every open cover has a σ-discrete closed refinement), and further, equivalent with *subparacompactness* defined by BURKE in [BU]. Obvious modifications of the main theorem in his paper will also yield *conditions equivalent* with ω_μ*-paracompactness*. (For more details on subparacompact spaces, see also [BS], [CR] and [BL].) For further investigations, note also that every space metrizable over an 0^+-semigroup S with cof S = ω_μ is ω_μ-additive, i.e.: any intersection of fewer than ω_μ open sets is open again.

REMARK. In [GFW] the authors define a space X to be m-*paracompact* iff every cover consisting of ≤m open sets has a locally finite open refinement. This concept obviously generalizes *countable paracompactness* and therefore, in analogy to countably compact and initially-m-compact spaces, should be better denoted by "*initially*-m-*paracompactness*". - Compare also R.E. HODEL's definition of the paracompactness degree of X; ([HO_1]).

In [MA] and [BS] L.F. McAULEY, D.K. BURKE and R.A. STOLTENBERG have shown that any semimetrizable space is σ-paracompact. By generalizing their methods, using our Theorems 1.3 and 1.4 and combining this with the above mentioned result of Juhász, Hayes and Nyikos-Reichel, we obtain:

THEOREM 5.1. *Any* T_1*-space* X *which is metrizable over an* 0^+*-semigroup* S *with* cof S ≤ ω_μ, *is* ω_μ*-paracompact. If* X *is metrizable over a continuous* 0^+*-semigroup* S, X *is paracompact.*

PROOF. Let X be metrizable over S and cof S ≤ ω_μ. A straight generalization of McAULEY's proof of Lemma 1 in [MA] shows that any open cover $\mathcal{O}$ of X has a closed ω_μ-discrete refinement $\mathcal{A}$, i.e.: $\mathcal{A}$ is the union of ω_μ many discrete collections $\mathcal{A}_i$, $i < \omega_\mu$, of closed sets. Indeed, let $\{s_i \mid i < \omega_\mu\}$ be a strictly decreasing ω_μ-sequence in S, converging to zero w.r.t. the order topology of S, and let the sets in $\mathcal{O}$ be well ordered. Then, for $0 \in \mathcal{O}$ and $i < \omega_\mu$, let $x(0,i) = \{p \in X \mid$ no element of $P \in \mathcal{O}$ with $p \in P$ precedes 0 w.r.t. the well ordering of $\mathcal{O}$; and the ball $B_i(p) = \{x \mid d(x,p) < s_i\}$ is contained in $0\}$. Now, if we let X_i be the collection of all such sets $x(0,i)$, $0 \in \mathcal{O}$, X_i will be a discrete collection of closed sets, and $\cup X_i$ $(i < \omega_\mu)$ covers X, hence $\{X_i \mid i < \omega_\mu\}$ is the ω_μ-discrete refinement of $\mathcal{O}$ we were looking for. (Compare [MA], p. 797 and [CR], Theorem 2.6.) Further, in [BS], BURKE and STOLTENBERG showed that any space with the property that every open cover of X has a σ-discrete refinement, is σ-paracompact. A straight generalization of their proof to arbitrary cardinals ω_μ yields the result cited in Theorem 5.1.*) - So, by the remarks preceding this theorem we are done. □

REMARK. Theorem 5.1 enables us to find reasonable conditions guaranteeing that an S-metrizable space is *paracompact*. For example, McAULEY has shown that a σ-*paracompact* (i.e. *subparacompact*) space X is *paracompact* iff X is *collectionwise normal*; see also [BU]. (By P. NYIKOS' paper [NY_4], one can even drop the word "collection wise" if PMEA, the product measure extension axiom, is assumed.) As before, this theorem of McAULEY can also be generalized. - On the other hand, note that a paracompact S-metric space need not be ω_μ-metrizable: the bowtie space, [STE], is a paracompact but non-metrizable semimetric and hence S_{ω_0}-metrizable space. (See our Theorem 1.3.)

PROBLEM. For uncountable cardinals ω_μ, is every paracompact S-metrizable space with cof $S = \omega_\mu$, ω_μ-metrizable?

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*) Remember that any S-metrizable space is ω_μ-additive, if cof $S = \omega_\mu$.

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THE NORMAL MOORE SPACE PROBLEM

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1. INTRODUCTION AND EARLY RESULTS

The normal Moore space problem has for forty years been one of the most outstanding problems in point-set topology. Still unsolved, its investigation has led to many beautiful results and new techniques. The purpose of this survey is to acquaint topologies and set-theorists with the problem, its current status, and the remaining difficulties. Although the offshoots of the problem form a delta system with many fascinating streams, for reasons of time and space we shall mainly confine ourselves to the central channel, to whit the attempt to prove (it consistent) that every normal Moore space is metrizable. Because of the problem's essentially set-theoretic nature, we do not need too many topological definitions. The following will suffice for now.

<u>DEFINITION 1</u>. A *development* of a space X is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that for any $x \in X$ and any open V containing x, for some n $\bigcup\{U \in \mathcal{U}_n : x \in U\} \subseteq V$. A space is *developable* if it has a development. A regular T_1 developable space is a *Moore space*. A collection $\mathcal{Y}$ of subsets of a space X is *discrete* if no $x \in X$ is in the closure of more than one element of $\mathcal{Y}$. A collection $\mathcal{Y}$ is *normalized* if for each $\mathcal{Z} \subseteq \mathcal{Y}$ there exist disjoint open sets U, V such that $\bigcup\{Y : Y \in \mathcal{Z}\} \subseteq U$, $\bigcup\{Y : Y \in \mathcal{Y} - \mathcal{Z}\} \subseteq V$. $\mathcal{Y}$ is *separated* if there exist pairwise disjoint open sets $U_Y \supset Y$, each $Y \in \mathcal{Y}$. X is (λ-) *collectionwise normal* if each discrete collection (of power $\leq\lambda$) is separated. X is (λ-) *collectionwise Hausdorff* if each discrete collection of ($\leq\lambda$) points is separated. The *character* of $x \in X$ is $\chi(x,X)$ = least cardinal of a neighbourhood base for x. $\chi(X)$, the character of X, is the

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sup of the characters of points in X. A *Q-set* is a set B of reals such that in the subspace topology on B, every subset of B is an F_σ.

REMARKS. Spaces with countable character are often called *first countable* spaces. Developable spaces are first countable. Discrete collections of points are the same as closed subspaces which inherit the discrete topology. In a normal space, every discrete collection is normalized. Collectionwise normal spaces are normal.

Moore spaces were studied by Moore [M] in his program of topologically axiomatizing Euclidean spaces. Non-metrizable Moore spaces are well-known; the most familiar example is variously known as the *bubble space, tangent disk space,* or *Niemytzki plane.*

EXAMPLE 1. Let B be a set of real numbers. M(B), *the Moore space derived from* B, is obtained by giving the usual Euclidean neighbourhoods to points in the upper half-plane above the x-axis, and letting neighbourhoods of points in B (considered as a subset of the x-axis) be the point together with the interior of a disk tangent to the axis from above at that point. When B is the entire x-axis, M(B) is the bubble space.

It is easy to see that the bubble space is a Moore space. It is not metrizable since it has a countable dense set but does not have a countable base.

The *general metrization problem* asks for a topological characterization of metrizability. Several early results showed that the existence of a development satisfying additional conditions was equivalent to metrizability. However these were not regarded as satisfactory solutions since they were too close conceptually to actually having a metric. In [JO_1], F.B. JONES conjectured that every *normal* Moore space is metrizable, and this has become known as *the normal Moore space conjecture.* In that paper he also proved what to a set-theoretically naive topologist must surely be a most startling result:

THEOREM 1. $2^{\aleph_0} < 2^{\aleph_1}$ *implies every separable normal Moore space is metrizable.*

Jones told me that he tried for several years to prove the hypothesis, and that when he told Moore of his efforts, the latter - who was acquainted with Sierpiński's work - was highly amused.

The general metrization problem was solved by Nagata, Smirnov, and Bing around 1950. BING's formulation in [B_1] included the following result.

THEOREM 2. *A space is metrizable if and only if it is a collectionwise normal Moore space.*

Having spotlighted the importance of the question of when normality implies collectionwise normality, Bing provided in the same paper what was for twenty years the only example of a normal non-collectionwise normal space. Bing's example is indeed a normal space of character $2^{\aleph_1}$ with a discrete unseparated collection of $\aleph_1$ points.

EXAMPLE 2. Let X be a set of power $\aleph_1$. Let $\mathcal{S}$ be the collection of all subsets of X. For each $S \in \mathcal{S}$, let $D_S = \{0,1\}$ with the discrete topology and let $Y = \Pi\{D_S : S \in \mathcal{S}\}$ with the usual product topology. For each $x \in X$, define $f_x \in Y$ by

$$f_x(S) = \begin{cases} 0 & \text{if } x \notin S, \\ 1 & \text{if } x \in S. \end{cases}$$

Let $M = \{f_x : x \in X\}$. Put a stronger topology on the set Y by declaring all sets of the form $U \cup K$, U open in the original space Y, $K \subseteq Y - M$, to be open. The resulting space Y_M is normal and Hausdorff, but M is a discrete unseparated collection of points. (For details see Bing's original paper or e.g. the text [E]. For a proof that Y_M has character $2^{\aleph_1}$, see [T_1].)

Bing also pointed out that if there were an uncountable Q-set, the bubble space derived from it would be normal. A simple counting argument establishes that no set of power $2^{\aleph_0}$ can be a Q-set and that if $2^{\aleph_0} < 2^{\aleph_1}$, there is no uncountable Q-set. HEATH [H] proved that if there is a separable normal non-metrizable space, then there is an uncountable Q-set, while in [T_1], it is proved that if the bubble space derived from a set of B of reals is normal, then B must be a Q-set.

2. TRANSLATIONS AND CONSISTENCY RESULTS

The set-theoretic character of the *separable* case of the normal Moore space conjecture was thus quite clear by 1964. The next year - at the urging of logician John Addison - BING published [B_2] a set-theoretic translation which he had obtained some years earlier of a somewhat more general case. In effect, he translated into set theory the assertion that every normal Moore space is collectionwise Hausdorff. Bing's translation provided the impetus for the consistency results of $[T_1]$, but has not played a major role since. It was extended in $[T_1]$ and $[T_2]$ to a more complicated form sufficiently strong to encompass the normal Moore space conjecture, namely a translation was provided for the assertion that in every first countable T_1 space, every normalized collection is separated. Since it may be of interest to set-theorists we give a typical example of translation, and refer to $[T_2]$ for the proof.

DEFINITION 2. Let $G \subseteq {}^{\kappa}\omega$ (the set of functions from κ to ω). G is *doubly superior* if

$$(\forall f \in {}^{\kappa}\omega)(\exists g \in G)(\exists \alpha_0 \neq \alpha_1 \in \kappa)(g(\alpha_0) > f(\alpha_0),\ g(\alpha_1) > f(\alpha_1)).$$

G *splits* if there is a function p from κ into 2 such that

$$(\forall f \in {}^{\kappa}\omega)(\exists g \in G)(\exists \alpha_0,\alpha_1 \in \kappa)(p(\alpha_0) \neq p(\alpha_1) \text{ and}$$

$$g(\alpha_0) > f(\alpha_0),\ g(\alpha_1) > f(\alpha_1)).$$

THEOREM 3. *Every normal first countable* T_1 *space is* κ*-collectionwise Hausdorff if and only if every doubly superior* $G \subset {}^{\kappa}\omega$ *splits.*

It may occur to the reader familiar with forcing that, given a doubly superior G, one can try to adjoin a generic partition to split it. Do this then for all doubly superior collections, and prove "doubly superior implies splits". Alternatively, say for $\kappa = \omega_1$, inductively construct a partition to take care of a $\diamondsuit$-sequence of functions and argue that such a partition splits all functions. The first approach was in effect taken in $[T_1]$, the second in effect in $[F_2]$, although both phrased their work topologically. We shall return to these results later.

In 1967 while trying to prove the consistency of the existence of an uncountable Q-set, the author mentioned the problem and its relationship to Moore spaces to Jack Silver. Later that year, Silver noticed that in a model of set thoery constructed by Solovay, there existed such a set. It followed - since there was known to be a model for $2^{\aleph_0} < 2^{\aleph_1}$ - that

THEOREM 4. *The existence of a separable normal non-metrizable Moore space is consistent with and independent of the usual axioms of set theory.*

In retrospect, Silver's half of Theorem 4 was easily available since the consistency of

> P: if $\{A_\alpha\}_{\alpha<\omega_1}$ are subsets of ω with every finite intersection infinite, then there is an infinite subset A of ω such that $A - A_\alpha$ is finite for all α.

was known to set-theorists, while ROTHBERGER [RO] had proved in 1948 that P implies every set of reals of power $\aleph_1$ is a Q-set. Nowadays when all (point-set) topologists have heard of *MARTIN's Axiom* [MS], we should state specifically that

THEOREM 5. *Martin's Axiom plus* $2^{\aleph_0} > \aleph_1$ *implies* P *implies every set of reals of power* $\aleph_1$ *is a Q-set implies there is a separable normal non-metrizable Moore space.*

For a proof of the first implication, see e.g. [RU].

The natural problem that remained in 1967 - and still remains - was to prove the consistency of *every* normal Moore space being collectionwise normal and hence metrizable. It is easy to see that countable normalized collections are separated, so the simplest non-trivial question is whether it is consistent that every normal Moore space is $\aleph_1$-collectionwise Hausdorff. This was proved by the author in [T_1]; the Moore space assumption turns out to be irrelevant - restricting the character to avoid Bing's example suffices.

THEOREM 6. *There is a model of set theory plus* $2^{\aleph_0} = \aleph_1$ *plus* $2^{\aleph_1}$ = *"anything reasonable" in which every normal space of character* $<2^{\aleph_1}$ *is* $\aleph_1$*-collectionwise Hausdorff.*

The proof proceeds roughly as follows. For simplicity we will start with GCH and preserve it. The idea is to make every unseparated discrete collection of $\aleph_1$ points unnormalized, by creating a "generic" subset of ω_1 which cannot be separated from its complement in ω_1 (we identify ω_1 with the collection) by disjoint open sets. Assume that this basic step can be done. Then argue as follows. Because only $\aleph_1$ points in a space of character $\leq \aleph_1$ are at issue, without loss of generality we may assume each space in question and a basis for its topology both have cardinality $\aleph_1$. We adjoin $\aleph_2$ subsets of ω_1 to a model of set theory. Each space, base, and unseparated collection appear at some stage. By the basic step the collection becomes unnormalized at the next stage. One then shows that (with this particular variety of forcing) unnormalized collections stay that way. It follows that in the final model, unseparated collections are unnormalized, and so normal implies $\aleph_1$-collectionwise Hausdorff for spaces of character $\leq \aleph_1$. To perform the basic step, force with countable partial functions from ω_1 into 2, which are intended to approximate countable subsets of the generic set and its complement. Because countable subsets of the collection are separated while the whole collection is not, it is possible, given any assignment of basic open sets to the elements of the discrete collection, to force that two of the basic open sets intersect while one contains an element of the generic set and the other contains an element of its complement.

A somewhat longer sketch appears in [T_3]. All the gory details can be found in [T_1].

Having taken care of the simplest non-trivial case, there are two obvious directions in which next to proceed. One is to try to separate $\aleph_1$ arbitrary closed sets. An easier version of this is to take care of $\aleph_1$ closed sets, each of cardinality $\leq\aleph_1$. The other approach is to increase the size of the collection, trying to separate κ points for arbitrary κ. In [T_1] I showed that if various countability conditions were imposed on the closed sets, e.g. requiring them to be Lindelöf, indeed $\aleph_1$ of them of cardinality $\leq\aleph_1$ could be separated. (The cardinality restriction was necessary to ensure the sets "appeared at some stage".) As for the second approach, by repeating the arguments in the proof of Theorem 6, one may extend a model of set theory to obtain one in which normal spaces of character $<2^{\aleph_2}$ which are $\aleph_1$-collectionwise Hausdorff, are also $\aleph_2$-collectionwise Hausdorff. In particular, the model of Theorem 6 can be so extended, to get that normal spaces of character e.g. $\leq\aleph_1$ are $\aleph_2$-collectionwise Hausdorff. One can continue this inductive construction of models via *reverse Easton forcing,* but technical difficulties arise at singular cardinals of uncountable cofinality. I was thus only able to prove

THEOREM 7. *There is a model of set theory plus the GCH in which for every cardinal* κ *which is either regular of has countable cofinality, every normal space of character* $\aleph_1$ *which is* λ*-collectionwise Hausdorff for all* $\lambda < \kappa$*, is* κ*-collectionwise Hausdorff.*

The sticking point was thus $\aleph_{\omega_1}$. Thus in 1969 a number of difficult questions remained to be solved.

(1) What to do at singular cardinals?

(2) What about arbitrary closed sets of restricted cardinality?

(3) What about arbitrary closed sets of arbitrary cardinality?

(4) Of what use - if any - is first countability or developability?

Note that the singular cardinals problem loses its difficulty if one knows how to (inductively) separate arbitrary collections of regular cardinality, i.e. if κ is singular and for every $\lambda < \kappa$ X is λ-collectionwise normal, then X is κ-collectionwise normal.

3. TREES, $\Diamond$, AND GEORGE

In his thesis [F_1], FLEISSNER made several important contributions to the normal Moore space problem, including the solution to the first question above (for points). These were later published in [F_2] and [F_3]. His starting point was "Jones' road space" [JO_2], a well-known example of a non-metrizable Moore space, which for our purposes differs inessentially from a *special Aronszajn tree*.

DEFINITION 3. A *tree* is a partially ordered set $<T,\leq>$ such that for each $t \in T$, $\{x: x < t\}$ is well-ordered by $<$. The *height* of t is the ordinal isomorphic to $\{x: x < t\}$. The height of T is the sup of the heights of its elements. The α'th level of T is the collection of all $t \in T$ of height α. $A \subseteq T$ is an *antichain* if it is pairwise unordered. $S \subseteq T$ is a *cofinal branch* if S is totally ordered and sup{height s: $s \in S$} = height T. An ω_1*-tree* is a tree of height ω_1 with countable levels. An *Aronszajn tree* is an ω_1-tree with no cofinal branch. An ω_1-tree is *special* if it is the union of countably many antichains.

The *tree topology* on a tree $<T,\leq>$ is generated by taking the collection of all open intervals as a basis. It is not difficult (see e.g. [RU], [F_3], and below) to show that special Aronszajn trees, when endowed with the tree topology, are Moore spaces which are not $\aleph_1$-collectionwise Hausdorff. It follows from Theorem 6 that they are consistently not normal. [T_1] was sufficiently obsecure to point-set topologists that even after it had been circulated, a paper by an author who shall remain nameless was sent around purporting to show that a minor variation of such a tree was normal. Perhaps it would be more accurate to say that special Aronszajn trees are obsecure, since two excellent mathematicians later circulated proofs of the (false!) assertion that it is consistent with the continuum hypothesis that such trees be normal.

Fleissner proved

THEOREM 8. *Martin's Axiom plus* $2^{\aleph_0} > \aleph_1$ *implies every (special) Aronszajn tree is normal.* $\diamond$ *implies no special Aronszajn tree is normal.*

Recall $\diamond$ asserts the existence of a sequence of functions $\{f_\alpha : \alpha < \omega_1\}$, $f_\alpha \in {}^\alpha\alpha$, such that for each $f \in {}^{\omega_1}\omega_1$, $\{\alpha : f|\alpha = f_\alpha\}$ is *stationary,* i.e. meets every closed unbounded subset of ω_1. A stronger assertion is

$\diamond$ *for stationary systems:* Let $\{A_f : f \in {}^{\omega_1}\omega_1\}$ be stationary subsets of ω_1 such that if $f|\alpha = g|\alpha$, then $A_f \cap (\alpha+1) = A_g \cap (\alpha+1)$. Then there exist $\{f_\alpha : f_\alpha \in {}^\alpha\alpha, \alpha < \omega_1\}$ such that for each $f \in {}^{\omega_1}\omega_1$, $\{\alpha : f|\alpha = f_\alpha\}$ is a stationary subset of A_f.

Fleissner's proof that $\diamond$ implies special Aronszajn trees are not normal was a prototype for his proof of

THEOREM 9. $\diamond$ *for stationary systems implies normal spaces of character* $\leq\aleph_1$ *are* $\aleph_1$*-collectionwise Hausdorff.*

As one might expect, generalizations of $\diamond$ for stationary systems to other regular cardinals yield another proof by induction of Theorem 7. However Fleissner cleverly figured out how to make the induction continue at singular cardinals, just using the GCH. Since the GCH and the generalizations of $\diamond$ for stationary systems follow from V = L (Gödel's Axiom of Constructibility), Fleissner obtained

THEOREM 10. V = L *implies every normal space of character* $\leq\aleph_1$ *is collectionwise Hausdorff.*

A detailed proof may be found in $[F_2]$ and an intuitive sketch in $[F_5]$, but we shall indicate here the main points of the proof of Theorem 9. First consider the simpler version for the special Aronszajn tree. Antichains are closed discrete subspaces and since the tree is special, some antichain A must intersect stationary many levels. Without loss of generality assume A has at most one point on each level. It is easy to see that A is not separated - basic neighbourhoods "look back", giving rise to a regressive function on a stationary set, whence uncountably many neighbourhoods have points on the same level and hence can't be disjoint.

Let S be the subspace of the given tree T consisting of those levels intersected by A. It syffices to partition A into two disjoint sets H, K which do not have disjoint open sets about them in S. Let $A = \{a_\alpha : \alpha < \omega_1\}$, where a_α is on the α'th level of S. Let $A_\beta = \{a_\alpha : \alpha < \beta\}$ $\beta \leq \omega_1$. Thus $A = A_{\omega_1}$. Assume by induction we have partitioned $A_\gamma = H_\gamma \cup K_\gamma$. If $\beta \leq \omega_1$ is a limit, define $H_\beta = \bigcup\{H_\gamma : \gamma < \beta\}$, $K_\beta = \bigcup\{K_\gamma : \gamma < \beta\}$. The work of course takes place at successor ordinals. With each a_α associate a neighbourhood basis $\{N(\alpha,n) : n < \omega\}$ for a_α in S so that each $N(\alpha,n)$ is included in the levels up through α. For $f \in {}^{\delta}\omega$, $\alpha \leq \delta \leq \omega_1$, call α f-*greedy* if $a_\alpha \in \overline{\bigcup\{N(\beta,f(\beta)) : \beta < \alpha\}}$. For any such f, observe that the same regressive function argument as above shows that the set of α's which are not f-greedy is not stationary. Let $\{f_\alpha : \alpha < \omega_1\}$ be a $\Diamond$-sequence. We continue the inductive construction at $\beta = \gamma + 1$. Let $U_\gamma = \bigcup\{N(\alpha,f_\gamma(\alpha)) : \alpha \in H_\gamma\}$, $V_\gamma = \bigcup\{N(\alpha,f_\gamma(\alpha)) : \alpha \in K_\gamma\}$ if range $f_\gamma \subseteq \omega$, otherwise $U_\gamma = V_\gamma = \emptyset$. If range $f_\gamma \nsubseteq \omega$ or if γ is not f_γ-greedy, it doesn't matter what we do with a_γ, say $H_\beta = H_\gamma \cup \{a_\gamma\}$, $K_\beta = K_\gamma$. If γ is f_γ-greedy, then either $a_\gamma \in \bar{U}_\gamma$ or $a_\gamma \in \bar{V}_\gamma$. In the first case let $H_\beta = H_\gamma$, $K_\beta = K_\gamma \cup \{a_\gamma\}$; in the second case, $H_\beta = H_\gamma \cup \{a_\gamma\}$, $K_\beta = K_\gamma$. Suppose there were an $f \in {}^{\omega_1}\omega$ such that $\bigcup\{N(\alpha,f(\alpha)) : \alpha \in H\}$ and $\bigcup\{N(\alpha,f(\alpha)) : \alpha \in K\}$ were disjoint. By $\Diamond$ there is a γ which is f-greedy and such that $f|\gamma = f_\gamma$. But this yields a contradiction.

Essentially the same argument works in the more general case of a space of character $\leq\aleph_1$ which is not $\aleph_1$-collectionwise Hausdorff, to get it to be not normal. The increase in character causes no difficulty but complications ensue from the fact that the closure of an open set about the first α points may miss the α'th one but jump up and catch a higher one. This could not occur when we used basic neighbourhoods in the tree. Let $\{y_\alpha : \alpha < \omega_1\}$ be discrete and unseparated in some space X of character $\leq\aleph_1$. Let $\{N(\alpha,\beta) : \beta < \omega_1\}$ be a neighbourhood base at y_α such that $N(\alpha,\beta)$ contains no other $y_{\alpha'}$. Redefine α to be f-greedy, $f \in {}^{\delta}\omega_1$, if $\overline{\bigcup\{N(\beta,f(\beta)) : \beta < \alpha\}} \cap \{y_\beta : \beta \geq \alpha\} \neq \emptyset$. The crucial lemma is that stationarily many α's are f-greedy. In the tree case, closed unboundedly many were; to make use of this weaker result, $\Diamond$ must be strengthened to $\Diamond$ for stationary systems, the systems in question being the systems of greedy sets.

In another major advance, FLEISSNER $[F_4]$ showed that character $\leq\aleph_1$ was not sufficient to go from normal to collectionwise normal, and hence neither his methods nor mine were sufficiently sharp to settle the problem. His example - immodestly called George - is normal and collectionwise Hausdorff, has character $2^{\aleph_0}$, and includes an unseparated collection of $\aleph_1$ closed sets, each homeomorphic to the ordinal space ω_1. We give a description of the space due to PRZYMUSIŃSKI [P].

EXAMPLE 4. Let D be the discrete space of cardinality $\aleph_1$. Let $\mathcal{S}$ be the family of clopen subsets of the subspace $Z = \{<\alpha,\beta> \in \omega_1 \times D: \beta < \alpha\}$ of the product space $\omega_1 \times D$. For every $S \in \mathcal{S}$ let $\chi_S: Z \to \{0,1\}$ be the characteristic function of S. Giving $\{0,1\}$ the discrete topology and taking its power $\{0,1\}^{\mathcal{S}}$, let $Y = \omega_1 \times \{0,1\}^{\mathcal{S}}$. Define the "diagonal" map $F: Z \to Y$ by $F(<\alpha,\beta>) = <\alpha,<\chi_S(B): S \in \mathcal{S}>>$. From general principles (see e.g. [E, p. 110] it follows that F is a homeomorphic embedding of Z into Y. Identifying Z with F(Z), let $X = Y_Z$, i.e. the modification of Y making $Y - Z$ isolated as in Example 2 above. Then X is normal, collectionwise Hausdorff, and not collectionwise normal. Finally, define a subspace X^* of X which in addition to these properties also has character $2^{\aleph_0}$, as follows. Let $A_\alpha = \{<\gamma,\delta> \in Z: \gamma \leq \alpha\}$ and let $\sim_\alpha$ be the equivalence relation on $\mathcal{S}$ defined by $S \sim_\alpha S'$ if and only if $S \cap A_\alpha = S' \cap A_\alpha$. Let $X^* = \{<\alpha,<x_S: S \in \mathcal{S}>> \in X$: whenever $S \sim_\alpha S'$, $x_S = x_{S'}\}$.

Fleissner later pointed out that George was unlikely to provide clues for the construction of a first countable example, since by collapsing an inaccessible cardinal he proved it consistent that normal first countable spaces were collectionwise normal with respect to discrete collections of homeomorphs of ω_1 [F_6]. The inaccessible cardinal hypothesis was removed in [T_6]. Fleissner's result was noteworthy for being the first essential use of first countability (as opposed to character $\leq\aleph_1$) in dealing with normality problems. Unfortunately the proof gives little hope of extension to more general closed sets, and ω_1 cannot appear in a Moore space since such spaces have the (hereditary) property that every closed set is a G_δ, which fails for the space ω_1.

4. TEST SPACES

Despite the set-theorists, Mike Starbird thought there was a "real" normal non-metrizable Moore space. While trying to construct one, he and Mary Ellen Rudin invented some interesting "test spaces". Their theme was that if there were any normal non-metrizable Moore space, one of these test spaces was one. One can either regard this as a useful narrowing of the search for a counterexample, or as a useless amalgamation of all possible difficulties into one example. We bypass this question of taste, but wish to call attention to a byproduct of their work. In [RS] they demonstrate that in considering the question of separating λ sets, one may without loss of generality assume (a) the sets are metrizable and (b) the sets have cardinality $\leq 2^{2^\lambda}$. From the point of view of an eventual consistency proof that in first countable spaces, normalized collections are separated, this is significant progress. Stating the Rudin-Starbird result precisely we have

THEOREM 11. *For each infinite cardinal* λ *there is a Moore space* T_λ *of cardinality* 2^{2^λ} *containing a normalized collection* $\mathcal{V}$ *of* λ *metrizable subspaces, such that if there is any first countable space containing a normalized unseparated collection of power* λ, *then* $\mathcal{V}$ *is unseparated.*

We extract from [RS] a description of T_λ.

EXAMPLE 5. Let M be the metric space $(2^{2^\lambda})^\omega$, i.e. the product of countably many copies of the discrete space of power 2^{2^λ}. For each $\alpha \in \lambda$, let $M = \{f \in {}^\omega(2^{2^\lambda})$: if $A \subseteq \lambda$, then for some $n \in \omega$, $A \in f(n)\}$. Let $C_\alpha = \{<\alpha,f>: f \in M\}$, $C = \cup\{C_\alpha: \alpha \in \lambda\}$. For $n \in \omega$, let $D_n = \{<\{<\alpha,f>,<\beta,g>\},n>:$ $<\alpha,f>,<\beta,g> \in C$, and if $A \in f(i) \cap g(j)$ for some $i,j < n$, then $\alpha \in A$ if and only if $\beta \in A\}$. Let $D = \cup\{D_n: n \in \omega\}$. For each $n \in \omega$ and $<\alpha,f> \in C'$, let $U_n(<\alpha,f>) = \{<\alpha,g> \in C: g|n = f|n\} \cup \{<\{<\alpha,g>,<\beta,h>\},m> \in D: m \geq n$ and $g|n = f|n\}$. Let $T_\lambda = C \cup D$, with each point in D open and with $\{U_n(<\alpha,f>): n \in \omega\}$ as a basis for $<\alpha,f> \in C$.

A proof of Nyikos is given in [RS] which unfortunately demonstrates that if T_λ is normal, then it is metrizable. Thus there is at present no general version of Theorem 11 with a normal test space, although Rudin and Starbird get one for certain restricted cases.

5. LARGE CARDINALS AND MEASURES

"Large cardinals" are hypothetical cardinal numbers with interesting combinatorial properties. See [T_7], [KM], [J], or [DR]. Since the existence of such large cardinals implies the existence of a model of set theory, by the Gödel incompleteness theorem such cardinals cannot be proved to (consistently) exist within the framework of the usual axioms of set theory. Large cardinals were brought into the normal Moore space arena by FLEISSNER [F_6], [F_7], [F_9], but the most spectacular application has been the recent work of KUNEN [K] and NYIKOS [N], which proves that, on the consistency of the existence of a *strongly compact* cardinal, there is a model of set theory in which every normal Moore space is metrizable.

DEFINITION 4. A filter is κ-complete if it is closed under intersections of size less than κ. A cardinal κ is *strongly compact* if every κ-complete filter can be extended to a κ-complete ultrafilter.

Strongly compact cardinals are considered by set-theorists to be "large" large cardinals because they not only affect sets their own size, but also sets of all larger sizes. Large large cardinals are somewhat dubious in that it is not unreasonable to believe that they will some day be shown to engender a contradiction.

Many topologists have come across *measurable* cardinals before so let us point out that strongly compact cardinals are measurable. (A measure is κ-*additive* if "the measure of the union equals the sum of the measures" for disjoint collections of size less than κ. A cardinal κ is *measurable* if it admits a two-valued κ-additive measure defined on all its subsets. Equivalently, if there is a κ-complete non-principal ultrafilter on κ.)

The first use of measures in the attack on the normal Moore space problem was due to STARBIRD [S] in an unpublished, untitled manuscript circulated in 1977, in which he pointed to connections between the existence of a measure on a cardinal λ and the collectionwise normality of the T_λ of Example 5 above.

NYIKOS [N] in 1977 investigated what he calls the *Product Measure Extension Axiom* (PMEA), which asserts that for any cardinal λ, the usual product measure on ${}^{\lambda}2$ can be extended to a $2^{\aleph_0}$-additive measure defined on all subsets of ${}^{\lambda}2$. Nyikos proved

THEOREM 12. *PMEA implies that in any space of character less than* $2^{\aleph_0}$*, every normalized collection is separated.*

It had been known to Kunen for some time that

THEOREM 13. *If there is a model of set theory in which there is a strongly compact cardinal, then there is a model in which PMEA holds.*

See [K] for a proof that the adjunction of κ random reals does the trick. Thus

COROLLARY 14. *If there is a model of set theory in which there is a strongly compact cardinal, then there is a model in which every normal Moore space is metrizable.*

Nyikos' proof proceeds as follows. Let $\{Y_\alpha\}_{\alpha<\lambda}$ be a normalized collection. Let μ be the measure given by PMEA defined on all subsets of ${}^{\lambda}2$. For each $f \in {}^{\lambda}2$ there exist disjoint open sets $U_i(f)$ including $\bigcup\{Y_\alpha : f(\alpha) = i\}$, $i = 0,1$. For each $y \in \bigcup\{Y_\alpha : \alpha < \lambda\}$ let $\{N_\beta(y) : \beta < \kappa_y\}$ be a neighbourhood base for y, κ_y a cardinal less than $2^{\aleph_0}$. Then for each y and for each $f \in {}^{\lambda}2$ there is a $\gamma(f,y) < \kappa_y$ such that $N_{\gamma(f,y)}(y) \subseteq U_0(f)$ or $U_1(f)$ depending on the value of $f(\alpha)$, where $y \in Y_\alpha$. Let $M(\beta,y) = \{f \in {}^{\lambda}2 : \gamma(f,y) \leq \beta\}$. By $2^{\aleph_0}$-additivity and since the $N_\beta(y)$'s are a basis, there is a $\beta_y < 2^{\aleph_0}$ such that $\mu(M(\beta_y,y)) > 7/8$. For any $y,z \in \bigcup\{Y_\alpha : \alpha < \lambda\}$ then, we have $\mu(M(\beta_y,y) \cap M(\beta_z,z)) > 3/4$. Suppose $y \in Y_\alpha$ and $z \in Y_\beta$ for different α,β. Then since $\mu(\{f \in {}^{\lambda}2 : f(\alpha) = 0 \text{ and } f(\beta) = 1\}) = 1/4$, there is an $f \in M(y,\beta_y) \cap M(Z,\beta_z)$ such that $f(\alpha) = 0$ and $f(\beta) = 1$. But then $N_{\beta_y}(y) \subseteq U_0(f)$ and $N_{\beta_z}(z) \subseteq U_1(f)$, and these are disjoint. Thus the required separation is $\{\bigcup\{N_{\beta_y}(y) : y \in Y_\alpha\} : \alpha < \lambda\}$.

6. THE CURRENT SITUATION

Is then the normal Moore space problem solved? No! The assumption of the consistency of a strongly compact cardinal is too strong. It may in fact be false. What is needed is a proof that if there is a model of set theory, then there is one in which every normal Moore space is metrizable, and this is still open. A pessimistic conjecture would be that if every normal Moore space is metrizable, then there is a large cardinal. This would place the problem in a strange sort of limbo in that there would be little hope for an example and no hope for a consistency theorem.

An esthetically unsatisfactory (to some) aspect of Nyikos' result is that in Kunen's model, $2^{\aleph_0}$ is a regular limit cardinal and so the continuum hypothesis fails badly. It would seem reasonable that if it were consistent that every normal Moore space be metrizable, then that would be consistent with the GCH. That remains to be seen, even modulo large cardinals.

What then are the appropriate directions for research in an attempt to prove the consistency of the normal Moore space conjecture? A reasonable one to try is to attempt to force enough of the Product Measure Extension Axiom to hold so as to get first countable normal spaces to be collectionwise normal without obtaining the full strength of the axiom, since as Nyikos notes, PMEA implies the consistency of the existence of a measurable cardinal. One could try to push further Fleissner's use of L, but there is no clear way to proceed and indeed I incline toward Kunen's conjecture that in fact V = L implies there is a normal non-metrizable Moore space. It is still possible that in the model of [T_1] all normal Moore spaces are metrizable, but a way must be found to use first countability, and as well, methods must be devised to deal with λ closed sets, each of power 2^{2^λ}. As pointed out in [T_1], if suitable cardinal generalizations of the assertion that paralindelöf normal Moore spaces are metrizable are true, then the results obtained therein can be vastly improved. (For the latest on paralindelöf spaces, see [FR].) It therefore might be worthwhile to first try to force enough of PMEA to obtain these generalizations, and then perform the reverse Easton extension. The connection of paralindelöfness (every open cover has a locally countable open refinement) with the problem of separating discrete collections is that in [T_1] it is shown consistent that locally countable (where it matters) open covers of normalized collections of $\aleph_1$ closed sets of cardinality $\aleph_1$ exist in spaces of character $\leq\aleph_1$. If the closed sets are e.g. points, such a cover yields a separation. It is not known whether these covers yield separations in general, e.g. in Moore spaces.

Finally, there are of course other models that could be investigated.

7. RELATED WORK

This completes our survey of the principal results leading toward a solution of the normal Moore space problem. We shall now briefly touch on some related results and problems. First, one can ask whether normal Moore spaces satisfying additional conditions are metrizable. For separable ones the answer is consistent and independent as mentioned earlier. Similarly for countable chain condition ones [T_1], [Š], even ones locally satisfying that condition [AP]. Ditto for locally compact ones [FT], [F_2]. Surprisingly, without additional axioms locally compact, locally connected ones are metrizable [RZ]. The most interesting open question in this area is whether the existence of a normal non-metrizable Moore space implies the existence of a metacompact one. See [T_4] for related results.

WAGE [W] proved that if every countably paracompact Moore space is normal, then every normal Moore space is metrizable. The status of the former assertion under PMEA is not known.

DEVLIN and SHELAH [DS] improved Fleissner by showing that $2^{\aleph_0} < 2^{\aleph_1}$ implies special Aronszajn trees are not normal. SHELAH [DS] established the consistency of GCH with the existence of a normal Moore space which is not collectionwise Hausdorff, namely a modification of a special Aronszajn tree obtained by isolating some of its points. On the other hand, GCH implies that given a discrete collection of κ points in a normal space of character $\leq\aleph_1$, κ of the points can be separated [T_5]. REED [R] brought Shelah's example more in line with previous work by establishing the equivalence of a generalization of the notion of Q-set with a weakening of "Shelah's Principle" sufficient to yield the example.

Fleissner has extensively investigated whether $<\kappa$ collectionwise Hausdorff implies κ collectionwise Hausdorff. See [F_7] and the references listed there.

$E(\kappa)$ is the assertion that κ is a regular cardinal greater than $\aleph_1$ and that there is a stationary subset E of κ such that each member of E has countable cofinality and $E \cap \alpha$ is not stationary in α for any $\alpha < \kappa$. FLEISSNER [F_8] proved that Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ plus $E(\kappa)$ for some κ implies there is a normal collectionwise Hausdorff non-metrizable Moore space. These hyportheses are known to be consistent.

The combinatorial principle $\diamond^*$ follows from V = L and is strictly stronger than $\diamond$ [D].

$\diamond^*$: There exist $\{\mathcal{S}_\alpha : \alpha < \omega_1\}$, $\mathcal{S}_\alpha$ a countable collection of subsets of α, such that for every $A \subseteq \omega_1$, $\{\alpha$: for some $S \in \mathcal{S}_\alpha$, $S = A \cap \alpha\}$ includes a closed unbounded set.

Shelah has recently proven that neither $\diamond^*$ nor $\diamond$ for stationary systems implies the other, but that $\diamond^*$ also implies normal spaces of character $\leq\aleph_1$ are $\aleph_1$-collectionwise Hausdorff. The proof is similar to that for the $\diamond$ for stationary systems version, but as FLEISSNER [F_{10}] notes, generalizes differently to yield

<u>THEOREM 15</u>. *Assume* $2^{\aleph_0} = \aleph_1$ *and* $2^{\aleph_1} = \aleph_2$. *If* X *is first countable and* $\aleph_1$-*collectionwise normal, then* X *is* $\aleph_2$-*collectionwise Hausdorff.*

The point is that GREGORY [G] proved that if $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, then a generalization of $\diamond^*$ holds on the set of ordinals in ω_2 having countable cofinality.

I conjecture that $\diamond^*$ implies countably paracompact first countable Hausdorff spaces are $\aleph_1$-collectionwise Hausdorff.

I have no doubt forgotten to include several results that should be here, but directions for further study are amply indicated.

In conclusion, what is so fascinating about the normal Moore space problem is the variety of sophisticated set-theoretic techniques that have been brought to bear upon it. No other problem in topology comes close. Martin's Axiom, reverse Easton forcing, various combinatorial principles of L, iterated forcing consistent with GCH, large cardinals directly and via collapse and adding reals, generalized Martin's Axiom [ ] - the list is endless. I look forward to what is next.

ADDED IN PROOF: Next was structure theory of ideals - see [TA]!

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ON PSEUDO-BOUNDARIES AND PSEUDO-INTERIORS

M. van de Vel

1. PRELIMINARIES

A *topological convexity structure* is a pair $(X,\mathcal{C})$ consisting of a T_1-space X together with a collection $\mathcal{C}$ of nonempty closed subsets of X such that $X \in \mathcal{C}$, $\{x\} \in \mathcal{C}$ for each $x \in X$, and such that $\mathcal{C}$ is closed under nonempty intersections. This definition extends the one in [3], which was designed for compact spaces.

If $A \subset X$ is an arbitrary subset, then the *$\mathcal{C}$-convex closure of* A is the set

$$I_{\mathcal{C}}(A) = \cap\{C \mid A \subset C \in \mathcal{C}\}.$$

See [1, 2.4]. The set A is called *(weakly) $\mathcal{C}$-convex* if for each compact (finite) $K \subset A$, it is true that $I_{\mathcal{C}}(K) \subset A$, and A is called a *$\mathcal{C}$-half-space* if both A and $X \setminus A$ are $\mathcal{C}$-convex. See [7].

The existence of sufficiently many half-spaces can be derived for convexities with some separation structure. Let A_1, A_2, B_1, B_2 be subsets of a set X. Then (A_1,A_2) is said *to be separated by* (B_1,B_2) if

$$A_1 \subset B_1\backslash B_2, \quad A_2 \subset B_2\backslash B_1, \quad B_1 \cup B_2 = X.$$

Let $\mathcal{S}$ be a collection of subsets of X. A *separation structure for* $(X,\mathcal{S})$ is a binary relation SEP among subsets of X with the following properties:

(i) if $x \in X$, $S \in \mathcal{S}$ are such that $x \notin S$, then $(\{x\},S) \in$ SEP;

(ii) If $(A_1,A_2) \in$ SEP, then there exist $S_1,S_2 \in \mathcal{S}$ such that (A_1,A_2) is separated by (S_1,S_2), and such that $(A_1,S_2) \in$ SEP, $(A_2,S_1) \in$ SEP.

A topological convexity $(X,\mathcal{C})$ is called *semi-regular* if there exists a separation structure SEP for $(X,\mathcal{C})$. If SEP contains all pairs (C_1,C_2) in

$\mathcal{C}$ with $C_1 \cap C_2 = \emptyset$ and with C_1 equal to the convex closure of a finite set, then $(X,\mathcal{C})$ is called *regular*. If SEP contains all paris of a disjoint members of $\mathcal{C}$, then $(X,\mathcal{C})$ is called *normal*.

EXAMPLE. Let X be locally convex linear space, and let $\mathcal{C}$ be the collection of all nonempty closed linearly convex sets of X. Define

$$\mathrm{SEP} = \{(C_1,C_2) \mid C_1,C_2 \in \mathcal{C} \text{ and } 0 \notin \overline{C_1-C_2}\}.$$

Using a Hahn-Banach theorem (cf. e.g. [2, p. 118]), it follows that SEP is a separation structure for $(X,\mathcal{C})$ containing all pairs of disjoint sets $(C_1,C_2) \in \mathcal{C} \times \mathcal{C}$ with C_1 compact. Hence $(X,\mathcal{C})$ is a regular convexity. If $\dim X > 1$, then $(X,\mathcal{C})$ is not normal.

MODIFIED URYSOHN THEOREM. *Let* $(X,\mathcal{C})$ *be a semi-regular convexity structure. If* $x \in X$ *and* $C \in \mathcal{C}$ *are such that* $x \notin C$, *then there exists a continuous map*

$$f\colon X \to [0,1]$$

such that $f(x) = 0$, $f(C) = \{1\}$, *and* $f^{-1}[t_1,t_2] \in \mathcal{C}$ *for all* $t_1 \leq t_2$ *in* $[0,1]$.

PROOF. as in [6, Thm. 2.1].

This theorem establishes the existence of sufficiently many closed half-spaces. For instance, if $(X,\mathcal{C})$ is semi-regular, then each $C \in \mathcal{C}$ is the intersection of a family of closed half-spaces. It also motivates the introduction of so-called *convexity preserving mappings* (see [4]).

All results below will appear with detailed proofs (and with other results) in a forthcoming paper [8] of the author.

2. PSEUDO-BOUNDARIES

Let $(X,\mathcal{C})$ be a topological convexity structure. Its *pseudo-boundary* $\partial(X,\mathcal{C})$ is defined to be the set of all $x \in X$ such that there exists a closed half-space C of $(X,\mathcal{C})$ with

$$x \in C, \quad \operatorname{int} C = \emptyset.$$

EXAMPLE 1. Let α be a cardinal number, and let $X = [0,1]^{\alpha}$ be equipped with the "cubical" convexity $\mathcal{C}$, consisting of all sets of type

$$\prod_{i\in\alpha} [t_i, t_i'], \qquad 0 \leq t_i \leq t_i' \leq 1.$$

Then $(X,\mathcal{C})$ is a normal convexity, and $\partial(X,\mathcal{C})$ equals the "classical" pseudo-boundary of X.

EXAMPLE 2. Let X be a compact tree-like space. Using unicoherence and local connectedness, the collection $\mathcal{C}$ of all subcontinua of X is a normal convexity for which $\partial(X,\mathcal{C})$ is the set of all endpoints of X.

EXAMPLE 3. Let X be a closed linearly convex subset of Euclidean n-space and let $\mathcal{C}$ denote the "linear" convexity of X. Then $(X,\mathcal{C})$ is a regular convexity, and $\partial(X,\mathcal{C})$ equals the set of all endpoints of maximal line segments in C.

EXAMPLE 4. Let $(X_1,\mathcal{C}_1)$ and $(X_2,\mathcal{C}_2)$ be topological convexities. Then $(X_1 \times X_2, \mathcal{C}_1 \times \mathcal{C}_2)$, with

$$\mathcal{C}_1 \times \mathcal{C}_2 = \{C_1 \times C_2 \mid C_1 \in \mathcal{C}_1,\ C_2 \in \mathcal{C}_2\}$$

is a topological convexity, and

$$\partial(X_1 \times X_2, \mathcal{C}_1 \times \mathcal{C}_2) = \partial(X_1,\mathcal{C}_1) \times X_2 \cup X_1 \times \partial(X_2,\mathcal{C}_2).$$

THEOREM 2.1. *Let* X *be a nondegenerate continuum, and let* $(X,\mathcal{C})$ *be a semi-regular convexity with* $\mathcal{C}$ *closed in the hyperspace* H(X) *of* X. *Let* $A \subset X$ *be such that* $I_{\mathcal{C}}(A) = X$. *Then*

(i) $I_{\mathcal{C}}(A \cap \partial(X,\mathcal{C})) = X$ *if* A *is closed;*

(ii) $I_{\mathcal{C}}(A \backslash A_0) = X$ *for each compact* A_0 *disjoint from* $\partial(X,\mathcal{C})$.

In particular, $\partial(X,\mathcal{C}) \neq \emptyset$, and if $A \subset X$ is minimal with the property that $I_{\mathcal{C}}(A) = X$ then $A \subset \partial(X,\mathcal{C})$.

Let $(X,\mathcal{C})$ be a convexity structure, and let $C \in \mathcal{C}$. A point $x \in C$ is called an *extreme point*, or a *Krein-Milman point*, of C if for each nondegenerate $D \in \mathcal{C}$ with $x \in D \subset C$, x is a pseudo-boundary point of the "trace" convexity $(D,\mathcal{C}|D)$. Compare with the notion of extremality in linearly convex sets ([2, p. 130]).

The well-known Krein-Milman theorem for linear spaces can be parallelled for a class of convexities which we now describe. A convexity $(X,\mathcal{C})$ is called *continuous* if for each convex closed set C and for each convex open set 0 with $0 \cap C \neq \emptyset$ it is true that

$$\bar{0} \cap C = \overline{0 \cap C}.$$

EXAMPLE. Let X be a topological vector space equipped with its "linear" convexity $\mathcal{C}$. Then $(X,\mathcal{C})$ is continuous: this follows from the fact that if $0 \subset X$ is convex open, then the line segment joining $x \in 0$ with $y \in X\backslash 0$ meets the (topological) boundary of 0 in exactly one point. Note: a linearly convex open set is weakly $\mathcal{C}$-convex by definition, and it can be shown that such a set is even $\mathcal{C}$-convex.

THEOREM 2.2. (Generalized Krein-Milman Theorem). *Let* X *be connected, and let* $(X,\mathcal{C})$ *be a semi-regular continuous convexity with* $\mathcal{C}$ *closed in the hyperspace* H(X) *of* X. *Then each compact convex set is the* $\mathcal{C}$*-convex closure of its set of extreme points.*

REMARK 1. Let $(X,\mathcal{C})$ be a topological convexity, and let $C \in \mathcal{C}$. Define $\partial_X(C,\mathcal{C})$ to be the set of all $x \in C$ such that there exists a closed half-space D of $(X,\mathcal{C})$ with

$$C \not\subset D, \quad x \in D, \quad (\text{int } D) \cap C = \emptyset.$$

For continuous convexities $(X,\mathcal{C})$, it is true that $\partial_X(C,\mathcal{C}) \subset \partial(C,\mathcal{C}|C)$. In general, both sets are distinct. For instance, taking $X = \mathbb{R}^{\omega_0}$ (ω_0 is the cardinal number of $\mathbb{N}$), $\mathcal{C}$ = "linear" convexity of X, and C = Hilbert cube, we find that the point $x = (\frac{1}{n+1})_{n\in\omega_0}$ is in $\partial(C,\mathcal{C}|C)$ but not in $\partial_X(C,\mathcal{C}|C)$. The above sets do coincide for normal binary convexities (cf. section 3 below).

Theorem 2.2 remains valid if ∂ is replaced by ∂_X in the definition of extremality.

REMARK 2. Continuity appears to be a quite stringent condition. It has been proved in [8] that, if $(X,\mathcal{C})$ is continuous and regular with $\mathcal{C}$ closed in H(X), then each compact $C \in \mathcal{C}$ is connected provided that X is connected.

3. PSEUDO-INTERIORS

The *pseudo-interior* $\iota(X,\mathcal{C})$ of a topological convexity $(X,\mathcal{C})$ is defined to be the complement of $\partial(X,\mathcal{C})$. Note that $\iota(X,\mathcal{C})$ is a $\mathcal{C}$-convex set. The very first question to ask is: under which condition is $\iota(X,\mathcal{C})$ nonempty. This turns out to be a hard problem.

EXAMPLE. Let X be compact, and let the convexity $\mathcal{C}$ on $H(X)$ consist of all nonempty intersections of sets of type

$$\langle C,X\rangle = \{A \in H(X) \mid A \cap C \neq \emptyset\}$$

$$\langle C\rangle = \{A \in H(X) \mid A \subset C\},$$

where C ranges over $H(X)$. Then $(H(X),\mathcal{C})$ is a normal convexity with $\mathcal{C}$ closed in $H(H(X))$ (cf. [5]). It is easy to see that the sets of type $\langle C\rangle$ or $\langle C,X\rangle$ are half-spaces of $(H(X),\mathcal{C})$, and hence that $\iota(H(X),\mathcal{C}) = \emptyset$ if X has no isolated points.

The problem whether or not there exist pseudo-interior points has been solved for an important class of convexities which we describe next. A convexity $(X,\mathcal{C})$ is *binary* if for each $\mathcal{C}' \subset \mathcal{C}$, $\cap\, \mathcal{C}' \neq \emptyset$ provided that every two members of $\mathcal{C}'$ meet, and if X has the *weak topology* with respect to $\mathcal{C}$, that is: $\mathcal{C}$ is a closed subbase for the topology of X (compare with the definition of weak topology in linear spaces). In particular, X is compact by Alexander's subbase lemma.

THEOREM 3.1. *Let* $(X,\mathcal{C})$ *be a normal binary convexity. Then* $\mathcal{C}$ *is closed in* $H(X)$ *and continuous. Also:*

(i) $\iota(X,\mathcal{C})$ *is dense in* X *if it is nonempty;*

(ii) $\iota(X,\mathcal{C}) \neq \emptyset$ *if* X *is metrizable.*

There is an (unpublished) example of C.F. Mills of a normal binary convexity with no pseudo-interior points.

THEOREM 3.2. *Let* $(X,\mathcal{C})$ *be a normal binary convexity. Then the following assertions are equivalent:*

(i) *for each* $C \in \mathcal{C}$, $\iota(C,\mathcal{C}|C) \neq \emptyset$;

(ii) *if $\mathcal{O}$ is a collection of convex open sets such that for each* $O \in \mathcal{O}$ *there is a compact set* $A \subset O$ *meeting all members of $\mathcal{O}$, then* $\cap\mathcal{O} \neq \emptyset$;

(iii) [*each closed half-space of* X *is a* G_δ*-set*]: *each covering of* X *with closed half-spaces has a countable subcover.*

(iv) [*same assumption*] *if $\mathcal{O}$ is a collection of convex open sets such that each countable subfamily has a nonempty intersection, then* $\cap\mathcal{O} \neq \emptyset$.

There exist generalizations of theorem 3.2 obtained from a close inspection of its proof. Three fundamental tools are being used: first, continuity of the convexity structure. Secondly, a Hahn-Banach theorem (see [7] for the case of compact spaces, cf. also 3.3 below). Finally, a technical condition which asserts that $\partial C = \partial_X C$ (see remark 1 of section 2) for each $C \in \mathcal{C}$.

Except the third condition, the above listed tools are also available on locally convex linear spaces, suggesting the following extension of theorem 3.2. We say that X is *locally convex* with respect to a convexity $\mathcal{C}$ on X if each point of X has a nbd. base of $\mathcal{C}$-convex open sets.

THEOREM 3.2'. *Let* $(X,\mathcal{C})$ *be regular and continuous with* $\mathcal{C}$ *closed in* H(X). *If each open half-space of* X *is Lindelöf, and if each* $C \in \mathcal{C}$ *is a Baire space, then the following assertions are equivalent:*

(i) *for each* $C \in \mathcal{C}$, $\partial_X(C,\mathcal{C}) \neq \emptyset$;

(ii) *each covering of* X *with closed half-spaces has a countable subcover*

(iii) [$(X,\mathcal{C})$ *satisfies the conclusion of the Hahn-Banach theorem below*]: *if $\mathcal{O}$ is a family of convex open sets of which each countable subfamily has a nonempty intersection, then* $\cap\mathcal{O} \neq \emptyset$.

HAHN-BANACH THEOREM 3.3. *Let* $(X,\mathcal{C})$ *be a regular convexity with* $\mathcal{C} \subset H(X)$ *closed. Assume that one of the following conditions hold:*

(i) *every weakly $\mathcal{C}$-convex open set is $\mathcal{C}$-convex;*

(ii) $(X,\mathcal{C})$ *is locally convex, the $\mathcal{C}$-convex closure of a finite set is compact, and $\mathcal{C}$ is "equally spread", that is: for each closed set* $A \subset X$ *the collection*

$$\{B \mid \exists C \in \mathcal{C}: B \subset C \subset A\}$$

is closed in H(X).

If C, O *are disjoint convex sets with* C *closed and* O *open, then there exists* *a closed half-space* D *of* $(X,\mathcal{C})$ *such that*

$$C \subset D, \quad D \cap O = \emptyset.$$

For a compact space X condition (ii) follows if $(X,\mathcal{C})$ is normal with $\mathcal{C}$ closed in $H(X)$. On the other hand, locally convex linear spaces $(X,\mathcal{C})$ are regular with $\mathcal{C}$ closed in $H(X)$, and they satisfy (i) as well as (ii).

An unsatisfactory element in the above theorem is that it is unknown whether or not (ii) implies (i), and - if the answer is negative - whether there exists a suitable common generalization of (i) and (ii) which still leads to the conclusion of the Hahn-Banach theorem.

Applications of the theory of pseudo-boundaries and pseudo-interiors have been made in the direction of identification of Hilbert cubes. It has been proved in [8] that if $(X,\mathcal{C})$ is a normal binary convexity on a metric continuum X with $\partial(X,\mathcal{C})$ dense in X, then the subspace $\mathcal{C}$ of $H(X)$ is homeomorphic to the Hilbert cube. It can be proved with the same techniques that convex closed set of the superextension $\lambda(X)$ (cf. [9]) of a metric continuum X meeting both $\partial\lambda(X)$ and $\iota\lambda(X)$ is homeomorphic to the Hilbert cube.

Considerations on pseudo-interiority in superextensions also lead to an intriguing new topological property. Let X be compact. X is said to have the *shrinking property* if for each family $\mathcal{O}$ of open sets, such that for each $O \in \mathcal{O}$ there is a compact $A \subset O$ meeting all members of $\mathcal{O}$, it is possible to choose a compact $A_O \subset O$ for each $O \in \mathcal{O}$ such that the sets $\{A_O \mid O \in \mathcal{O}\}$ meet two by two.

For compact X, the shrinking property is equivalent to the statement that each convex closed set of $\lambda(X)$ has nonempty pseudo-interiority. Metrizable compacta satisfy the shrinking property. Our proof relies on the use of Whitney mappings.

The assumptions of Theorem 3.2' are valid for separable Fréchet spaces. It is therefore desirable to obtain more information on the convexity theory of such spaces. In particular, it is of interest to find neat formulas for ∂C and $\partial_X C$, C linearly convex.

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MINIMAL HAUSDORFF AND COMPACTLIKE SPACES

Johannes Vermeer

1. SOME PRELIMINARY REMARKS

All spaces under consideration are assured to be Hausdorff. A space X is called:

- Absolute closed - or H-closed - if X is a closed subset of every space Y, in which X is embedded.
- T2½, or Urysohn, if distinct points have disjoint closed neighbourhoods.
- Compactlike, if X is H-closed + T2½
- Semi-regular, if the regular open sets form a base for the topology.
- Minimal Hausdorff, if every weaker topology on the space X is no longer Hausdorff.

LEMMA 1.1.

(i) *For a space* X *the following statements are equivalent:*

a) X *is* H-*closed;*

b) *Every open filter on* X *has an accumulation point;*

c) *Every open cover* $\{U_i\}_{i\in I}$ *of* X *contains a finite subcollection* $\{U_{i_j}\}_{j=1,\dots,n}$ *such that* $X = \bigcup_{j=1}^{n} c\ell_X U_{i_j}$.

(ii) *A regular space which is* H-*closed is compact.*

(iii) *For a space* X *the following statements are equivalent:*

a) X *is minimal Hausdorff;*

b) X *is* H-*closed + semi-regular.*

Although the following proposition is known, I do not know any article with the proof of it. For the sake of completeness I will give a proof.

PROPOSITION 1.2. *For a space* X *the following statements are equivalent:*

a) X *is minimal Hausdorff +* T2½;

b) X *is compact.*

PROOF. b) $\Rightarrow$ a): clear

a) $\Rightarrow$ b): We only have to prove that X is regular.

Take a closed subset $A \subset X$ with $x \notin A$. There exists a semi-regular open neighbourhood U_x of x such that $x \in U_x \subset X \setminus A$. The set $X \setminus U_x$ is - being a regular closed subset of X - H-closed and $x \notin X \setminus U_x$. With 1.1 (i) c it is easy to construct open sets V_x and V such that $x \in V_x$, $A \subset X \setminus U_x \subset V$, $V \cap V_x = \emptyset$. □

A function $f\colon X \to Y$ is called *weakly continuous* if:

$$\forall\, x\ \forall U_{f(x)}\ \exists V_x \text{ such that: } f(c\ell\, V_x) \subset c\ell_Y V_{f(x)}.$$

- A continuous function is weakly-continuous.

A function $f\colon X \to Y$ is called a *weak-homeomorphism* if f and f^{-1} are weak-continuous. Then X and Y are called *weakly-homeomorphic*.

- Note: X and Y are weakly-homeomorphic if and only if X and Y have the same collection of regular open sets.

If $(X,\mathcal{T})$ is a topological space, then the collection $\{\mathrm{int}(c\ell\, U)\}_{U\in\mathcal{T}}$ forms a base for some topology $\mathcal{T}_s$ on X, which is called the *semi-regularization* of $(X,\mathcal{T})$.

The space $(X,\mathcal{T}_s)$ is denoted by X_s.

PROPOSITION 1.3. *Let* $(X,\mathcal{T})$ *be a topological space. Then*

(i) $\mathcal{T} = \mathcal{T}_s \Longleftrightarrow X$ *is semi-regular.*

(ii) X_s *is semi-regular.*

(iii) *The function* $\mathrm{id}_X\colon (X,\mathcal{T}) \to (X,\mathcal{T}_s)$ *is a weak-homeomorphism.*

(iv) X *is* H-*closed* $\Longleftrightarrow X_s$ *is minimal Hausdorff.*

(v) X *is compactlike* $\Longleftrightarrow X_s$ *is compact.*

PROPOSITION 1.4. *Suppose* Y *is* T2½. *If* $f\colon X \to Y$ *and* $g\colon X \to Y$ *are weakly-continuous functions which coincide on a dense set, then* $f = g$.

PROPOSITION 1.5. *Suppose* $f\colon X \to Y$ *is weakly-continuous. Then the functions*

$$\left.\begin{array}{l} f\colon X \to Y_s \\ f\colon X_s \to Y \\ f\colon X_s \to Y_s \end{array}\right\} \text{ are all weakly continuous.}$$

PROPOSITION 1.6. (see [4]). *Suppose* $(X,\mathcal{T})$ *is* H-*closed. For every* H-*closed topology* $\mathcal{L}$ *on* X *such that the topology* $\mathcal{L} \cap \mathcal{T}$ *is Hausdorff, we have* $\mathcal{T}_s = \mathcal{L}_s$.

PROPOSITION 1.7. (see [5]). *Suppose* Y *is a dense subset of* $(X,\mathcal{T})$. *Then* $(\mathcal{T}\backslash Y)_S = \mathcal{T}_S\backslash Y$.

2. EMBEDDINGS AND EXTENSIONS

For an arbitrary Hausdorff space X the following embeddings are known.

(i) X can be embedded in a semi-regular space (see [6]);

(ii) X can be embedded as a dense and open subspace of an H-closed space, the Katetov-extension K[X] (see [2]);

(iii) X can be embedded as a nowhere dense closed subspace of an H-closed space (see [7]);

(iv) X can be embedded as a dense subspace of a minimal Hausdorff space if and only if X is semi-regular (see [2]).

As far as I know the following questions are unanswered. If X is a semi-regular space, then:

1) Is it possible to embed X as an open subspace of some minimal Hausdorff space?

2) Is it possible to embed X as an open and dense subspace of some minimal Hausdorff space?

In this section we shall give the answer to [1]. I was unable to prove or disprove [2]; however, I do have some conditions on X such that the answer is affirmative.

Before I give the proof of [1], I need a proposition about H-closed extensions.

PROPOSITION 2.1. *Suppose* X *is a dense subset of* Y. *Then:* Y *is an H-closed extension of* X *iff*

- Every open filter of X has an accumulation point in Y iff
- Every regular open filter of X has an accumulation point in Y, (a regular open filter is an open filter which has a base consisting of regular open sets).

So if we want to check that some extension Y of X is H-closed, we only have to consider the (regular) open ultrafilters on X, and not the open filters on Y.

COROLLARY 2.2. *If* Y *is H-closed and* X *is a dense subset of* Y, *then the space* Y *remains H-closed if we declare* X *to be open in* Y.

The new space is called Y_X.

If X is not open in Y, then Y_X is not minimal Hausdorff. There is a similar construction possible to obtain minimal Hausdorff spaces.

Suppose X is dense in Y. Declare X to be open in Y. Take two disjoint copies of X - say X_1 and X_2 - and consider the space:

$$Y_{X_1}^{X_2} = X_1 \cup X_2 \cup Y \backslash X \qquad \text{(Notation: } U \subset X \Rightarrow U_i \subset X_i\text{)}$$

with a topology generated in the following way:

- X_1 and X_2 are open in $Y_{X_1}^{X_2}$ (i.e. if $p_i \in X_i$, then $\mathcal{U}_{p_i} = \{U_i \mid U_i \subset X_i, U_i \text{ open}, p_i \in U_i\}$ is said to be a local base for p_i).
- If U is open in Y, then U' is defined by:

$$U' = (U \cap X)_1 \cup (U \cap X_2) \cup (U \cap Y \backslash X)$$

If $p \in Y\backslash X$, then we take the collection

$$\mathcal{U}'_p = \{U'_p \mid U_p \text{ open in } Y \text{ and } p \in U_p\}$$

as a local base for p in $Y_{X_1}^{X_2}$.

CLAIM 1. If Y is semi-regular, and X is dense in Y, then the space $Y_{X_1}^{X_2}$ is also semi-regular!

- X is dense in Y, so X is semi-regular.
- If $U \subset X$ is regular open in X, then $U_1 \subset X_1$ is regular open in $Y_{X_1}^{X_2}$. So, the regular-open neighbourhoods of a point $p \in X_1$ (or X_2) are a local base.
- If $p \in Y\backslash X$, and U_p is a regular open neighbourhood of p in Y, then, U'_p is a regular open neighbourhood of p in $Y_{X_1}^{X_2}$. So $Y_{X_1}^{X_2}$ is semi-regular.

CLAIM 2. If Y is H-closed, and X is dense in Y, then the space $Y_{X_1}^{X_2}$ is also H-closed.

Take an open ultrafilter $\mathcal{F}$ of the dense set $X_1 \cup X_2$ in $Y_{X_1}^{X_2}$. Suppose $X_1 \in \mathcal{F}$. Then the collection $\mathcal{F}' = \{U \cap Y_{X_1}^{X_2} \backslash X_2 \mid U \in \mathcal{F}\}$ is an open ultrafilter on the space $X_1 \cup Y\backslash X$, which is homeomorphic to Y_{X_1}, and so $\mathcal{F}'$ has an accumulation point in Y_{X_1}:

COROLLARY.

THEOREM 2.3. *Every semi-regular space* X *is embeddable as an open subspace of some minimal Hausdorff space.*

PROOF. It is possible to embed X as a dense set of some minimal Hausdorff space Y. But then $X \simeq X_1$ is an open subspace of the semi-regular and H-closed - and so minimal Hausdorff space - $Y_{X_1}^{X_2}$:

If X is semi-regular and locally H-closed, then X is an open dense subspace of the minimal Hausdorff space $(K(X))_S$ (the semi-regularization of the Katetov-extension).

But this condition is far from necessary. The nowhere locally H-closed space Q (the rationals) is according to 2.4 embeddable as an open and dense subset of some Hausdorff space.

COROLLARY 2.4. *If* X *is a semi-regular space, such that the disjoint topological sum* $X \oplus X$ *is homeomorphic to* X*, then it is possible to embed* X *as an open and dense set of some minimal Hausdorff space.*

PROOF. $X \simeq X_1 \oplus X_2$ is open and dense in $Y_{X_1}^{X_2}$.

3. PROJECTIVE OBJECTS IN CERTAIN CATEGORIES

We define the following categories.

$HC\ell$: the category of H-closed spaces and continuous maps

$C\ell$ the category of compactlike spaces and continuous maps.

MH the category of minimal Hausdorff spaces and continuous maps.

A space X is called projective in a category $\mathcal{O}$, if for each diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "\exists h"'] & & Y \\ & Z \arrow[ur, two heads, "g"'] & \end{tikzcd}$$

(g is a bijection)

- all spaces and functions of which are in the category $\mathcal{O}$, there exists a map $h: X \to Z$ such that the diagram commutes.

In [3] C.T. LIU proved that the projective objects in $HC\ell$ are the finite spaces.

In [1] BLASZCZYK proved:

* - The projective objects in $C\ell$ are the Ǩatetov extensions of discrete spaces.

The proof of (*) was based on the following lemma, which is unfortunately false.

LEMMA. *Let* X *be a compactlike space and* $A \subset X$. *Then:*

A *is H-closed if and only if* A *is regular embedded in* X.

(A *is called regular embedded, if* $\forall\, x \notin A\ \exists$ *neighbourhood* U_x *such that* $c\ell U_x \cap A = \emptyset$.)

One implication of the lemma is correct: if A is H-closed, then A is regular embedded in X; but not the converse.

EXAMPLE. If X is discrete, then the Katětov extension $K(X)$ of X is compactlike. The discrete subspace $K(X)\backslash X$ is regular embedded in $K(X)$, but not in H-closed.

THEOREM 3.1. *The projective objects in* $C\ell$ *are precisely the finite spaces.*

PROOF. Suppose X is projective in $C\ell$.

We prove that each point of X is isolated, and so X is finite. Suppose there exists $y_0 \in X$ such that $\{y_0\}$ is not open in X. Consider $\alpha\mathbb{N} = \mathbb{N} \cup \{\infty\}$, the one point compactification of $\mathbb{N}$. The space $Y = \alpha\mathbb{N} \times X$ is compactlike. Because $\{y_0\}$ is not isolated, the subspace $A = \mathbb{N} \times X \cup \{(\infty, y_0)\}$ of Y is not open in Y. But of course A is dense. Declare A to be open, and we get the compact-like space Y_A. Define

$$g\colon X \to Y \quad \text{by} \quad g(x) = (\infty, x),$$

$$f\colon Y_A \to Y \quad \text{by} \quad f(x) = x \quad (f \text{ is a surjection}).$$

X is projective, so there exists a map $h\colon X \to Y_A'$ such that $f \circ h = g$. We have of course $h(x) = (\infty, x)$. But then $f^{-1}(A) = \{y_0\}$ is open in X, contradiction.

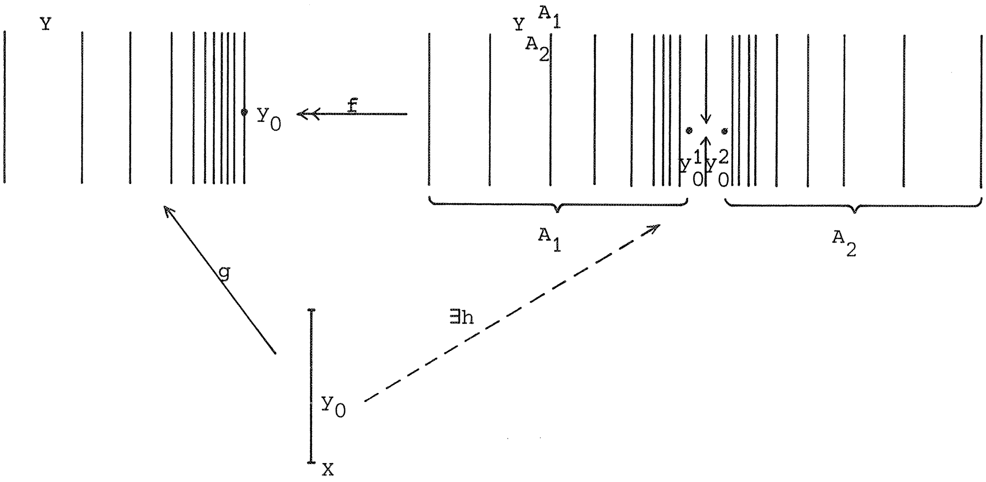
□

THEOREM 3.2. *The projective objects in* MH *are precisely the finite spaces.*

PROOF. Suppose X is projective in MH.

We prove that each point of X is isolated, and so X is finite. Suppose there exists $y_0 \in X$ such that $\{y_0\}$ is not open. $Y = \alpha\mathbb{N} \times X$ is minimal

Hausdorff, and the subspace $A = \mathbb{N} \times X \cup \{(\infty,y_0)\}$ is dense but not open in Y. Declare A open, and double A, and we get the space $Y_{A_1}^{A_2}$, which is minimal Hausdorff.



Define

$$g: X \to Y \quad \text{by} \quad g(x) = (\infty,x),$$

$$f: Y_{A_1}^{A_2} \to Y \text{ the projection (which is surjective).}$$

X is projective, so there exists a map $h: X \to Y_{A_1}^{A_2}$ such that $f \circ h = g$. Clearly,

$$\forall x \neq y_0 \Rightarrow h(x) = (\infty,x) \in Y_{A_1}^{A_2};$$

$$h(y_0) = y_0^1 \text{ or } h(y_0) = y_0^2.$$

Suppose $h(y_0) = y_0^1$. A_1 is an open subset of $Y_{A_1}^{A_2}$, so $h^{-1}(A_1)$ is open in X. But $h^{-1}(A_1) = \{y_0\}$, contradiction. □

4. COMPACTLIKE EXTENSIONS

In the paper [4] PAROVIČENKO proved the following theorem.

THEOREM 4.1. *Suppose* X *is a* T2½*-space. Then: If* X *has a compactlike extension, then there exists a maximal one.*

Furthermore, he gave an example of a regular space X which does not have a compactlike extension. The question of when a space X has a compactlike extension remained unanswered.

In this chapter we will give the answer, and we will give a construction of the maximal compactlike extension.

THEOREM 4.2. *A space* $(X,\mathcal{T})$ *has a compactlike extension if and only if the space* $(X,\mathcal{T}_S)$ *is Tychonoff.*

PROOF. Suppose $(\alpha X,\mathcal{T}^{\alpha})$ is a compactlike space which contains $(X,\ )$ as a dense subspace.

The Proposition 1.3 (V) & 1.7 imply that $(\alpha X,\mathcal{T}^{\alpha}_S)$ is a compact space, which contains $(X,\mathcal{T}_S)$ as a (dense) subspace, and so $(X,\mathcal{T}_S)$ is Tychonoff. On the other hand, suppose $X_S = (X,\mathcal{T}_S)$ is Tychonoff. The Čech-Stone compactification βX_S is of course a compactlike extension of X_S. Declare X_S to be open in βX_S.

We get the compactlike extension $\beta' X_S$. Replace X_S by X, and call the new space $\tilde{\beta}X$. Clearly X is a dense subset of $\tilde{\beta}X$, and I claim that $\tilde{\beta}X$ is H-closed (and so compactlike).

We have to check that each regular open ultrafilter in X has an accumulation point in $\tilde{\beta}X$. But X_S and X have the same collection of regular open sets, so there is nothing to check. $\tilde{\beta}X$ is a compactlike extension of X! □

Theorem 4.1 implies that, if X is a space such that X_S is Tychonoff, then there exists a maximal compactlike extension of X. We call this maximal extension $B(X)$.

LEMMA 4.3. *The extension* $B(X)$ *of* X *has the following properties*

(i) X *is an open subset of* $B(X)$.

(ii) $B(X)\backslash X$ *is a discrete subspace.*

(iii) $(B(X))_S = \beta X_S$.

(iv) *If* U *is an open and dense subset of* X*, and* $p \in B(X)\backslash X$ *then* $\{p\} \cup U$ *is a neighbourhood of* p *in* $B(X)$.

PROOF.

(i) Is clear from 2.2.

(ii) If $p \in B(X)\backslash X$ then $\mathcal{U}_p = \{U \subset B(X) \mid U \text{ open } \& \ p \in U\}$. We define a new local base $\mathcal{U}'_p$ for p by

$$\mathcal{U}'_p = \{(U \cap X) \cup \{p\} \mid U \in \mathcal{U}_p\}.$$

Because of (i) we get a topology on B(X) (and a stronger topology then the original). But clearly, this new topology on B(X) is compactlike. (If $\mathcal{F}$ is an open filter of X, which has p as accumulation in B(X), then p remains an accumulation point in the new topology). Because of the maximality of B(X), we conclude:

$$\mathcal{U}_p = \mathcal{U}'_p.$$

But this implies that $\{p\}$ is open in $B(X)\backslash X$.

(iii) As we conclude in the proof of Theorem 4.2, $(B(X))_S$ is a compactification of X_S. Say $(B(X))_S = B'(X_S)$.

Suppose αX_S is (another) compactification of X_S. Repeat the proof of 4.1, and make the compactlike extension $\widetilde{\alpha X}$ of X. (Declare X_S open in αX_S, and replace it by X.) B(X) is the maximal compactlike extension of X, so there exists a map

$$f\colon B(X) \to \widetilde{\alpha X}$$

such that $f|X = id_X$.

Proposition 1.5 implies that the map:

$$f\colon B'(X_S) \to (\widetilde{\alpha X})_S$$

is weak-continuous.

Note that on the set αX_S we have two topologies, namely: a compact topology (as a compactification of X_S), and a stronger compactlike topology (as a compactlike extension of X).

Proposition 1.6 implies: $(\widetilde{\alpha X})_S = (\alpha X_S)_S = \alpha X_S$. But αX_S is regular, and so the map:

$$f: B'(X_S) \to \alpha X_S$$

is continuous (and $f|X_S = \mathrm{id}\ X_S$). But then: $B'(X_S) = (=(B(X))_S) = \beta X_S$.

(iv) Consider the open filter $F = \{U \subset X \mid U$ is an open and dense$\}$ on X. If $p \in B(X)\backslash X$ and $U \in F$ define $U \cup \{p\}$ to be open in B(X).

From Proposition 2.1 it is easy to see that this stronger topology on B(X) is again compactlike and it contains X as a dense set. From the maximality of B(X), we can conclude that the topology does not change. So $U \cup \{p\}$ was already open in B(X). □

Lemma 4.3 gives us exactly the information we need to answer the question: how to construct B(X)?

Suppose X is a space such that X_S is Tychonoff. Consider βX_S.

(i) declare X_S open and replace it by X, we get $\tilde{\beta}(X)$

(ii) $\forall p \in \tilde{\beta} X_S \backslash X_S$ we take the collection:

$$\mathcal{U}_p = \{U \cap X \cup \{p\} \mid U \text{ is open and dense in } X \text{ or } U \text{ is open in } \tilde{\beta}(X) \text{ and } p \in U\}$$

as a local subbase for p.

It is easy to check that we get a compactlike extension of X. We call this extension $\Delta(X)$.

THEOREM 4.4. *For each space X such that X_S is Tychonoff, the compactlike extension $\Delta(X)$ of X is equivalent to the extension B(X) of X.*

PROOF.

- $\Delta(X)$ is constructed by enlarging the topology on βX_S. Proposition 1.6 implies that $(\Delta(X))_S = \beta X_S$.
- $\Delta(X)$ is a compactlike extension of X. So there exists a map

$$f: B(X) \to \Delta(X)$$

which extends the identity on X.

- $(\Delta(X))_S = \beta(X_S) = (B(X))_S$. So the idenity g on $\beta(X_S)$ is a map

$$g: (\Delta(X))_S \to (B(X))_S$$

which extends the identity on X_S.

Proposition 1.5 implies that the map

$$g\colon \Delta(X) \to B(X)$$

is weak-continuous (and extends the identity on X). Clearly:

$$f \circ g|X = g \circ f|X = id_X.$$

Proposition 1.4 implies that $g = f^{-1}$ and $f = g^{-1}$. So, the only thing we have to prove is: g is continuous.

a) X is open in $\Delta(X)$ and in $B(X)$, so g is continuous in every point of X.

b) Take a point $p \in \Delta(X)$, and a neighbourhood U_g of g(p). g is weak-continuous, so there exists an neighbourhood V_p of p such that:

$$g(c\ell_{\Delta X}V_p) \subset c\ell_{B(X)}U_{g(p)}.$$

From this we conclude that:

$$g(V_p \cap X) = V_p \cap X \subset c\ell_X U_{g(p)},$$

because g is the identity on X.

The open set $U_{g(p)} \cup X\backslash c\ell_X U_{g(p)} = W$ is open and dense in X, and so $W \cup \{p\}$ is a neighbourhood of p. But then the neighbourhood of p:

$$W_p = V_p \cap (W \cup \{p\})$$

satisfies the condition:

$$g(W_p) \subset (c\ell_X U_{g(p)} \cap W) \cup \{g(p)\} \subset U_{g(p)}$$

and so: g is continuous in p. □

THEOREM 4.5. *The extension* B(X) *of* X *is not compact, if* X *is not.*

PROOF. Suppose B(X) is compact. Then: $B(X) \simeq \beta(X)$, and so:

- X open in $\beta X \Rightarrow$ X locally compact.
- $\left.\begin{array}{l}\beta(X)\backslash X \text{ is compact}\\ \beta(X)\backslash X \text{ is discrete}\end{array}\right\} \Rightarrow \beta(X)\backslash X$ is finite $\Rightarrow$ X pseudo-compact.

- Note: because of the construction of $B[X]$, we have the following property of X, if $B[X]$ is compact.

(*) All nowhere dense and closed subsets of X are compact.

In [8] E. WATTEL and I proved the following proposition.

(**) If Y is a topological space with no isolated points such that every nowhere dense closed subsets of Y is compact, then Y is compact!

CASE 1. X has no isolated points. (*) and (**) implies: X is compact!

CASE 2. X has a subset A of isolated points.

(i) A is dense in X. Then $X\setminus A$ is nowhere dense and closed in X, so compact. There exist disjoint open neighbourhoods U and V of $\beta X\setminus X$ and $X\setminus A$. But then $U \cap A$ is a clopen, discrete and infinite subset of X, so X is not pseudo-compact. Contradiction.

(ii) A is not dense in X. Consider the subspace $\mathrm{int}_X(X\setminus A)$. This subspace has no isolated points, and so $c\ell_X(\mathrm{int}_X(X\setminus A))$ has no isolated points. (**) implies: $c\ell_X(\mathrm{int}_X(X\setminus A))$ is compact. We also have:

$$G = X\setminus(A \cup \mathrm{int}_X(X\setminus A))$$

is compact, because of (*). There exist disjoint open neighbourhoods U and V of $\beta X\setminus X$ and G, such that $U \cap c\ell_X(\mathrm{int}_X(X\setminus A)) = \emptyset$.

But then $U \cap A$ is a clopen, discrete and infinite subset of X, so X is not pseudo-compact. Contradiction. □

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REMOTE POINTS, FAR POINTS AND HOMOGENEITY OF X^*

J. Vermeer & E. Wattel

1. INTRODUCTION

Frolík has proved the following famous theorem:

THEOREM 1.1. *If* X *is not pseudo-compact then the Čech-Stone remainder* X^* *of* X *is not homogeneous.*

This theorem is proved by a cardinality argument. It does not show which points of X^* cannot be mapped onto each other by any autohomeomorphism of X^*. Our aim is to construct such types of points in X^* for a certain class of spaces. Earlier results of this type are already obtained by van DOUWEN [1], [2].

DEFINITION 1.2.

(i) A point $p \in X^*$ is called a *remote point* of X, iff p is not in the closure of any nowhere dense subset of X.

(ii) A point $p \in X^*$ is called a *far point* of X, iff p is not in the closure of any closed discrete subset of X.

Throughout this paper the cardinality of a set S will be denoted by $\#(S)$.

Note that: if $f: \beta X \to \beta X$ is an autohomeomorphism of βX such that $f[X] = X$ then $f(p)$ is a remote point (far point) iff p is.

It is not clear that for any autohomeomorphism $\phi: X^* \to X^*$ the image of a remote point (far point) should be a remote point (far point). In fact, it is not true. There exists an example of E. K. van DOUWEN [2] of a non-realcompact space X such that a remote point of X^* is mapped onto a non-remote point of X^* by an autohomeomorphism of X^*. The problem is still open for realcompact spaces; it is even still open for $X = \mathbb{R}$ cf. [2]. For a large class of spaces we solve the problem, namely for the class of nowhere locally compact spaces.

DEFINITION 1.3. A space is called *nowhere locally compact* iff no point of X has a compact neighbourhood.

Note:

(i) X is nowhere locally compact $\Longleftrightarrow X^*$ is dense in βX.

(ii) X is nowhere locally compact $\Rightarrow X^*$ is nowhere locally compact.

DEFINITION 1.4. Let X be a space and let A and B be disjoint subsets of X such that $A \cup B = X$.

(1) $p \in A$ is called *remote from* B, iff p is not in the closure of any closed nowhere dense subset of B.

(2) $p \in A$ is called *far from* B, iff p is not in the closure of any closed discrete subset of B.

If we talk about a remote point of X, then we mean a point of X^* remote from X.

2. NON-HOMOGENEITY OF X^* AND REMOTE POINTS

In this section we show that for a nowhere locally compact space X the remote points of X^* are invariant under autohomeomorphisms of X^*. From this we show that the remainders of nowhere locally compact non-pseudocompact spaces with countable π weight cannot be homogeneous because they contain remote points.

The following lemma is well known:

LEMMA 2.1. *Let* $\phi\colon X \twoheadrightarrow Y$ *be an irreducible continuous closed map. Then*

(i) *if* E *is a nowhere dense closed subset of* X, *then* f[E] *is a closed nowhere dense subset of* Y.

(ii) *if* A *is a closed nowhere dense subset of* Y, *then* $f^{-1}[A]$ *is a nowhere dense closed subset of* X.

THEOREM 2.2. *Let* X *be a nowhere locally compact space, and* $p \in X^*$. *Then* p *is a remote point from* X *if and only if* p *is remote from* $\beta X^*\backslash X^*$ *in* βX^*.

PROOF. X is nowhere locally compact so X^* is dense in βX. Therefore βX is a compactification of X^* and so there exists a mapping $f\colon \beta X^* \twoheadrightarrow \beta X$ such that $f|X^* = id_{X^*}$ and $f[\beta X^*\backslash X^*] = X$. Note that the mappings

$$f\colon \beta X^* \twoheadrightarrow \beta X \quad \text{and} \quad f\colon \beta X^*\backslash X^* \twoheadrightarrow X$$

are both irreducible and closed.

$\Leftarrow$ Suppose that p is a non-remote point.

There exists a nowhere dense closed subset E of X such that $p \in c\ell_{\beta X}(E)$. According to 2.1 the set $f^{-1}[E]$ is a closed nowhere dense subset of $\beta X^*\backslash X^*$ and we claim that $p \in c\ell_{\beta X^*}(f^{-1}[E])$. Suppose the opposite. Then there exists a neighbourhood U_p of p, such that

$$U_p \cap f^{-1}[E] = \emptyset.$$

Therefore

$$\beta X^*\backslash U_p \supset f^{-1}[E],$$

hence

$$f[\beta X^*\backslash U_p] \supset E \quad \text{and} \quad \beta X\backslash f[\beta X^*\backslash U_p] \cap E = \emptyset.$$

However: $\beta X\backslash f[\beta X^*\backslash U_p]$ is a neighbourhood of p in βX. Contradiction.

$\Rightarrow$. Suppose that p is not remote from $\beta X^*\backslash X^* = X^{**}$.

There exists a nowhere dense closed subset E of X^{**} such that $p \in c\ell_{\beta X^*}(E)$. But f[E] is a nowhere dense closed subset of X (cf. 2.1) and

$$p = f(p) \in f[c\ell_{\beta X^*}(E)] \hookrightarrow c\ell_{\beta X}(f[E])$$

and so p is a non-remote point of X^*.

COROLLARY 2.3. *Being a remote point in* X^* *- where X is a nowhere locally compact space - is a topological property of that point in* X^*.

PROOF. Suppose that $f\colon X^* \twoheadrightarrow X^*$ is an autohomeomorphism, and $p \in X^*$ is a remote point (and so a remote point of $\beta X^*\backslash X^*$). Then $f\colon X^* \to X^*$ can be extended to an autohomeomorphism $\beta f: \beta X^* \to \beta X^*$. Clearly $\beta f(p) = f(p)$ is remote from $\beta X^* \setminus X^*$, and so $f(p)$ is a remote point.

If we can prove that the growth of a non-pseudocompact nowhere locally compact space contains both remote points and non-remote points than we have shown in a constructive way, why those growths are not homogeneous.

THEOREM 2.4. *Let* X *be a space without isolated points and assume that every closed nowhere dense subspace of* X *is compact. Then* X *is itself compact.*

And so, if X is not compact, then X^* contains a non-remote point.

PROOF. Let $\mathcal{U}$ be any open cover of X. Let $\mathcal{V}$ be a subcover of minimal cardinality say $\#(\mathcal{V}) = \kappa$. Well-order $\mathcal{V}$ in such a way that $\mathcal{V} = \{V_\alpha \mid \alpha < \kappa = \#(\mathcal{V})\}$. We consider the collection of all unions over initial segments of $\mathcal{V}$. This is an increasing open cover of X. Let $\mathcal{W}$ be a subcover of this cover with minimal cardinality. The set $\mathcal{W}$ is well-ordered say $\mathcal{W} = \{W_\alpha \mid \alpha < \lambda\}$. We claim that λ has to be finite. Suppose not. We choose a point p_α from each set $W_\alpha \setminus c\ell_X(\bigcup\{W_\beta \mid \beta < \alpha\})$ whenever this set is non-empty. Let N be $c\ell_X(\{p_\alpha \mid \alpha < \lambda\})$. Clearly N is nowhere dense. Moreover, $\mathcal{W}$ covers N and each subset of $\mathcal{W}$ which does not cover X cannot cover N either. Contradiction. This shows our theorem. □

REMARK. The referee of this paper pointed out to us that this result was known [7].

Unfortunately, E.K. van DOUWEN and J. van MILL [3] constructed an example of a nowhere locally compact space X such that X^* does not contain remote points. However, we have the following result of E.K. van DOUWEN [2].

THEOREM 2.5. *If* X *is a nonpseudocompact space with countable* π*-weight, then* X^* *contains a remote point.*

The following corollary generalizes results of E.K. VAN DOUWEN [1], [2]

COROLLARY 2.6. *If* X *is a nowhere locally compact nonpseudocompact space with countable* π*-weight, then* X^* *is not homogeneous because a remote point of* X *cannot be mapped onto a non-remote point of* X *by any autohomeomorphism of* X^*.

3. NON-HOMOGENEITY OF X^* AND FAR POINTS

<u>THEOREM 3.1</u>. *Suppose that* X *is a nowhere locally compact space and that* $p \in X^*$. *Then* $p \in X^*$ *is a far point if and only if* p *is far from* $\beta X^*\backslash X^*$.

<u>PROOF</u>. ⇒ cf. [2]. Suppose that p is not far from $\beta X^*\backslash X^* = X^{**}$. Then there exists a discrete closed subset $D \subset X^{**}$ such that $p \in c\ell_{\beta X^*}(D)$. Note that the mappings

$$f\colon \beta X^* \to \beta X \quad \text{and} \quad f\colon \beta X^*\backslash X^* \to X$$

are both irreducible and closed as in 2.2. Moreover,
Moreover, f[D] is a closed subset of X and f[D] is a discrete subset of X since every subset of f[D] is closed in X. (N.B. f is closed and every subset of D is discrete and hence closed.) Of course, $p \in c\ell_{\beta X}(f[D])$, so p is not a far point.

⇐ Suppose that $p \in X^*$ is not a far point relative to X. Then there exists a closed discrete subset $E \subset X$ such that $p \in c\ell_{\beta X}(E)$. For each $e \in E$ we choose a point $x_e \in f^{-1}(e)$. Clearly $A = \{x_e \mid e \in E\}$ is a discrete closed subset of $\beta X^*\backslash X^*$.

We claim that $p \in c\ell_{\beta X^*}(A)$.
Take a neighbourhood U_p of p in βX^*. Then $f(\beta X^*\backslash U_p)$ does not contain p. Therefore $\beta X\backslash f(\beta X^*\backslash U_p)$ is a neighbourhood of p in βX, so it contains infinitely many $e \in E$. Now for infinitely many e we have that

$$e \notin f(\beta X^*\backslash U_p)$$

and

$$f^{-1}(e) \cap \beta X^*\backslash U_p = \emptyset; \quad f^{-1}(e) \subset U_p; \quad x_e \in U_p$$

This means that $p \in c\ell_{\beta X^*}(A)$.

<u>COROLLARY 3.2</u>. *Being a far point in* X^* *- where* X *is a nowhere locally compact space - is a topological property of that point.*

E.K. VAN DOUWEN [2] has proved the following: If X is a normal space which is not Lindelöf, then X^* has an ω-far point.

THEOREM 3.3. *If* X *is a normal nowhere locally compact space which is not pseudo-compact and not Lindelöf, then* X^* *is not homogeneous.*

QUESTION 3.4. Let X be a non-pseudocompact nowhere locally compact space. Does X^* contain a far point?

If the answer to this question is affirmative, then 3.2 gives us the answer why X^* is non-homogeneous for a non-pseudocompact nowhere locally compact space. In that case X^* contains non-far points (because X contains a C-embedded copy of $\mathbb{N}$) and far points, and these points cannot be mapped onto each other by any autohomeomorphism.

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TOPICS IN THE THEORY OF TOPOLOGICAL TRANSFORMATION GROUPS

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1. INTRODUCTION

The general program which I want to illustrate in this paper is the following: consider general topology as the theory of topological transformation groups with a trivial acting group, and try to generalize this theory to the case where the acting group is a, not necessarily trivial, topological group G. The point is, then, that instead of all continuous functions only those continuous functions are available which commute with the actions of G (G-equivariant mappings). Stated otherwise, my program is to find in the category TOP^G analogues of properties of TOP.

A good illustration of this program has been given by Yu. M. SMIRNOV in [16], where he discussed aspects of dimension theory, extensions and retractions, and embeddings of G-spaces in linear G-spaces. In this paper I would like to discuss first the following question: can every completely regular Hausdorff G-space equivariantly be embedded in a compact Hausdorff G-space. Then I discuss a certain construction which assigns in a canonical way a G-space to every H-space, where H is a subgroup of G, and an application of the G-compactification theorem of section 2 to this construction. Finally, I present some preliminary results about extensors and coseparators in the categories TOP^G and COMP^G. Using the construction, discussed in section 3, a generalization of a result of SMIRNOV [15] is presented, concerning a necessary condition for a G-space to be a G-extensor for a certain class of pairs (see section 4 for the definitions of G-extensor, etc.).

Let me introduce now some terminology and notation. A *topological transformation group* (*ttg*) is a triple $\langle G,X,\pi\rangle$ where G is a topological group, X a topological space and $\pi\colon G \times X \to X$ a continuous surjection with the property that $\pi(t,\pi(s,x)) = \pi(ts,x)$ for all $s,t \in G$ and $x \in X$. If we put $\pi^t x := \pi(t,x) =: \pi_x t$ for $t \in G$ and $x \in X$, this means that $\pi^t \circ \pi^s = \pi^{ts}$

for all $t,s \in G$, and that $\pi^e = 1_X$, the identity mapping on X (e denotes the unit element of G). In fact, $t \mapsto \pi^t: G \to H(X)$ is a homomorphism of the group G into the full homeomorphism group of X.

If $\langle G,X,\pi\rangle$ is a ttg, I shall call X a *G-space*; the mapping π is called the *action* (*of* G, *on* X). If H is a subgroup of G, then a subset A of X is called *H-invariant* whenever $\pi^t A \subseteq A$ for every $t \in H$. In that case, $\pi|_{H\times A}$ is an action of H on A; for convenience, this action will also be denoted by π. So we have the ttg $\langle H,A,\pi\rangle$, the *restriction* of $\langle G,X,\pi\rangle$ to A. If $\langle G,X,\pi\rangle$ and $\langle G,Y,\sigma\rangle$ are two ttg's, then a mapping $f: X \to Y$ is said to be *G-equivariant* whenever $f\circ\pi^t = \sigma^t\circ f$ for every $t \in G$. A continuous, G-equivariant mapping will also be called a *morphism of G-spaces*. Notation: $f: \langle G,X,\pi\rangle \to \langle G,Y,\sigma\rangle$. Observe, that if $f: \langle G,X,\pi\rangle \to \langle G,Y,\sigma\rangle$ is a morphism of G-spaces, it is also H-equivariant for every subgroup H of G, so we have in that case the morphism of H-spaces $f: \langle H,X,\pi\rangle \to \langle H,Y,\sigma\rangle$. It is obvious that the composition of two G-equivariant mappings is again G-equivariant. Therefore, it makes sense to define the category TOP^G as the category of all G-spaces and all G-equivariant continuous mappings. If K is a subcategory of TOP, then K^G will denote the subcategory of TOP^G whose objects are the G-spaces $\langle G,X,\pi\rangle$ with X in K and whose morphisms are morphisms of K which have the additional property of being G-equivariant. In this paper I shall consider only full subcategories of TOP; then the corresponding subcategories of TOP^G are full as well. The following subcategories of TOP will be considered: HAUS (Hausdorff spaces), CR (completely regular Hausdorff spaces), MET (metrizable spaces) and COMP (compact Hausdorff spaces). An investigation of TOP^G and several of its subcategories can be found in [17].

2. COMPACTIFICATIONS

It is well-known that COMP is a reflective subcategory of TOP, i.e. the inclusion functor COMP $\to$ TOP has a left adjoint B: TOP $\to$ COMP. If the unit of adjunction is denoted by β, then this means, that for every topological space X there exists a continuous mapping $\beta_X: X \to BX$, where BX is a compact Hausdorff space, and β_X has the following universal property: for every continuous mapping $f: X \to Z$, Z compact Hausdorff, there exists a unique continuous mapping $\bar{f}: BX \to Z$ such that $f = \bar{f}\circ\beta_X$.
The proofs of the following statements can be found in every textbook on general topology:

2.1. PROPOSITION. *For every topological space* X *the following properties are equivalent*:

(i) X *is a completely regular Hausdorff space;*

(ii) β_X *is a topological embedding of* X *in* BX [1]*;*

(iii) X *can be (topologically) embedded in a compact Hausdorff space.*

If these conditions are fulfilled, then X *can be embedded in a compact Hausdorff space* X^* *such that* $w(X^*) = w(X)$. □

(Here $w(Z)$ denotes the weight of a topological space Z, i.e. the minimal cardinal number of a base for the topology of Z.)

Now the question is whether in TOP^G similar results are valid with respect to COMP^G, G being an arbitrary topological group, or under which conditions on G or the G-spaces we have similar results. It is not difficult to show that, for every topological group G, COMP^G is a reflective subcategory of TOP^G; see section 4.3 of [17]. Thus, the inclusion functor $\mathrm{COMP}^G \to \mathrm{TOP}^G$ has a left adjoint $B^G: \mathrm{TOP}^G \to \mathrm{COMP}^G$. If the unit of adjunction is denoted by β^G and if we abbreviate $\beta^G_{<G,X,\pi>}$ to β^G_π, then this has the following meaning. For every G-space $<G,X,\pi>$ there exists a morphism of G-spaces $\beta^G_\pi: <G,X,\pi> \to <G,\tilde{X},\tilde{\pi}> := B^G<G,X,\pi>$, where $\tilde{X}$ is a compact Hausdorff space, having the following universal property: for every morphism of G-spaces $f: <G,X,\pi> \to <G,Z,\zeta>$ with Z compact Hausdorff there exists a unique morphism of G-spaces $\tilde{f}: <G,\tilde{X},\tilde{\pi}> \to <G,Z,\zeta>$ such that $f = \tilde{f}\circ\beta^G_\pi$.

Before discussing the analogue of proposition 2.1, let me consider the relationship between $\beta_X: X \to BX$ and $\beta^G_\pi: X \to \tilde{X}$ for a given G-space $<G,X,\pi>$; here $\tilde{X}$ is the underlying topological space of the ttg $<G,\tilde{X},\tilde{\pi}> := B^G<G,X,\pi>$. In view of the universal property of β_X, there exists a unique continuous function $\bar{\beta}^G_\pi: BX \to \tilde{X}$ such that $\beta^G_\pi = \bar{\beta}^G_\pi\circ\beta_X$. In general, $\bar{\beta}^G_\pi$ is not a homeomorphism. Let me first explain what it exactly means that $\bar{\beta}^G_\pi$ is, or is not, a homeomorphism. To this end, observe that for every $t \in G$ there is a continuous mapping $\bar{\pi}^t := B\pi^t: BX \to BX$ such that $\bar{\pi}^t\circ\beta_X = \beta_X\circ\bar{\pi}^t$. Unicity of each $\bar{\pi}^t$, $t \in G$, implies that $\bar{\pi}^t\circ\bar{\pi}^s = \bar{\pi}^{ts}$ for every $t,s \in G$, and $\bar{\pi}^e = 1_{BX}$. Hence $\bar{\pi}: (t,z) \mapsto \bar{\pi}^t z: G \times BX \to BX$ has the properties of an action of G on BX *except possibly continuity*. However, $\bar{\pi}$ is an action of G_d (= G with its

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[1] In this case, BX is usually denoted as βX, the Stone-Čech compactification of X.

discrete topology) on BX. If $\bar{\pi}$: $G \times BX \to BX$ is continuous, then we say that the action of G on X *can continuously be extended* to an action of G on BX. Notice that in that case β_X is G-equivariant, as is $\bar{\beta}^G_\pi$.

Using the above notation, the following proposition can be formulated; its straightforward proof is left to the reader.

2.2. PROPOSITION. *Let* $\langle G,X,\pi\rangle$ *be a ttg. The following statements are equivalent:*

(i) $\bar{\pi}$: $G \times BX \to BX$ *is continuous;*

(ii) $\langle G,BX,\bar{\pi}\rangle$ *is a ttg and* β_X: $\langle G,X,\pi\rangle \to \langle G,BX,\bar{\pi}\rangle$ *is a morphism of* G-*spaces which has the universal property which characterizes the reflection of* $\langle G,X,\pi\rangle$ *in* COMP^G;

(iii) $\bar{\beta}^G_\pi$: $BX \to \tilde{X}$ *is a homeomorphism.* □

In general, $\bar{\beta}^G_\pi$: $BX \to \tilde{X}$ is a quotient mapping, which can also be described in the following way. Let X' be the space BX provided with the finest topology making $\bar{\pi}$: $G \times BX \to X'$ continuous, and let q: $X' \to \tilde{X}$ be the Hausdorff reflection of X'. By [17], 3.3.13(ii) and 4.4.3 there is a unique continuous action of G on $\tilde{X}$ making $q\circ 1_{BX}$: $BX \to \tilde{X}$ G-equivariant, and the resulting ttg is $\langle G,\tilde{X},\tilde{\pi}\rangle$ referred to above, whereas $\bar{\beta}^G_\pi = q\circ 1_{BX}$ (up to isomorphism in TOP^G).

2.3. EXAMPLES. The following examples show that in general $\bar{\pi}$: $G \times BX \to BX$ is not continuous or, equivalently, that $\bar{\beta}^G_\pi$: $BX \to \tilde{X}$ is not a homeomorphism (notation as in 2.2).

1^o. Let G be a compact Hausdorff group and consider the ttg $\langle G,G\times Y,\mu\rangle$, where Y is a completely regular Hausdorff space and $\mu^t(s,y) := (ts,y)$ for $t \in G$ and $(s,y) \in G \times Y$. In [17], 4.4.13(iv) it is shown that the reflection of $\langle G,G\times Y,\mu\rangle$ in COMP^G has the form $\beta^G_\mu = 1_G \times \beta_Y$: $\langle G,G\times Y,\mu\rangle \to \langle G,G\times BY,\tilde{\mu}\rangle$, where $\tilde{\mu}^t(s,z) := (ts,z)$ for $t \in G$ and $(s,z) \in G \times BY$. By a result of GLICKSBERG [10], the mapping $\bar{\beta}^G_\mu = \overline{1_G \times \beta_Y}$ is not a homeomorphism if X is not pseudocompact.

2^o. Consider the ttg $\langle G,G,\lambda\rangle$, where $\lambda(t,s) = ts$ for $t,s \in G$. Its reflection in COMP^G can be described as follows: the mapping β^G_λ: $G \to \tilde{G}$ is the unique compactification of G with the property that the induced mapping $h \mapsto h\circ\beta^G_\lambda$: $C(\tilde{G}) \to C(G)$ maps $C(\tilde{G})$ isometrically onto the space $\mathrm{RUC}^*(G)$ of all bounded right uniformly continuous functions on G (cf. [17], 4.4.17; also [3], [4] and [8]). It follows that $\bar{\beta}^G_\lambda$: $BG \to \tilde{G}$ is a homeomorphism if and only if $C^*(G) = RUC^*(G)$. Results of COMFORT and ROSS [6] imply that for locally compact G this is only possible if G is compact or discrete.

$\underline{3}^{\text{o}}$. (This example is adapted from [5], Theorem 4.10.) Let $\langle \mathbb{R}, X, \pi\rangle$ be an $\mathbb{R}$-space with X normal and Hausdorff. Suppose that X contains a point x whose positive limit set is empty, that is, no sequence $\{\pi(t_n,x): n \in \mathbb{N}\}$ with $t_n \to \infty$ (in $\mathbb{R}$) has a limit point in X. Then $F_1 := \{\pi(n,x): n = 2,3,\ldots\}$ and $F_2 := \{\pi(n+n^{-1},x): n = 2,3,\ldots\}$ are two disjoint closed subsets of X. Hence F_1 and F_2 have disjoint closures in βX. However, suppose that π can be extended to a continuous action $\bar{\pi}$ of $\mathbb{R}$ on βX. Putting $x_n := \pi(n,x)$, the sequence $\{x_n: n = 2,3,\ldots\}$ has a limit point y in βX, and continuity of $\bar{\pi}$ would imply that the sequence $\{\bar{\pi}(n^{-1},x_n): n = 2,3,\ldots\}$ has $\bar{\pi}(0,y) = y$ as a limit point. Since $\bar{\pi}(n^{-1},x_n) = \pi(n+n^{-1},x)$, this would imply that y is a common limit point of F_1 and F_2, which is impossible. This contradiction shows that π cannot be extended to a continuous action of $\mathbb{R}$ on βX.

We come now to the analogue of proposition 2.1. Let us first observe that for a ttg $\langle G,X,\pi\rangle$ the morphism β_π^G is an equivariant embedding of X in $\tilde{X}$ if and only if there exists a morphism of G-spaces $f: \langle G,X,\pi\rangle \to \langle G,Y,\sigma\rangle$ such that Y is compact Hausdorff and f is a topological embedding of X in Y (compare with 2.1(ii) and (iii)). Necessary for this is that X is completely regular Hausdorff; is this also sufficient? If so, how "small" can Y be chosen? The following result provides a partial answer to these questions.

2.4. THEOREM. *Let* $\langle G,X,\pi\rangle$ *be a ttg with* X *a completely regular Hausdorff space, and suppose that one of the following conditions is fulfilled*:

(i) G *is locally compact;*

(ii) *There is a uniformity* $\mathcal{U}$ *on* X, *compatible with the topology, such that* $\{\pi^t: t \in G\}$ *is pointwise* $\mathcal{U}$*-equicontinuous.*

Then there exists a morphism of G*-spaces* $f: \langle G,X,\pi\rangle \to \langle G,Y,\sigma\rangle$ *such that* f *is a dense embedding of* X *in* Y, *and* Y *is a compact Hausdorff space of weight* $w(Y) \le \max\{L(G/G_0), w(X)\}$; *here* G_0 *is the stabilizer of* $\langle G,X,\pi\rangle$ *and* $L(G/G_0)$ *is the Lindelöf degree of* G/G_0.

PROOF. For the statement involving (i), see [18] and [19], for the statement involving (ii), see [13] and [17], 7.3.12. □

For applications of this theorem I refer to section 4 of [18]. Below I shall present two additional applications: cf. 3.3(v) and 4.3.

3. TWISTED PRODUCTS

In this section, G is a topological group and H is a subgroup of G, not necessarily closed. Then restriction of actions of G to actions of H defines a functor $R: \mathrm{TOP}^G \to \mathrm{TOP}^H$. Results from general category theory and from [17] (concerning the formation of products, etc., in categories of ttg's) imply that R has always a left adjoint $L: \mathrm{TOP}^H \to \mathrm{TOP}^G$ and that, if G is locally compact, R also has a right adjoint. I shall consider here only the left adjoint L of R, or rather the unit of adjunction γ for L and R. For details, cf. [17], section 3.3.

Consider an arbitrary H-space $\langle H,X,\pi\rangle$ and define on $G \times X$ an equivalence relation $\sim$ by

$$(s,x)\sim(s',x') \iff \exists u \in H\colon su^{-1} = s' \ \&\ \pi^u x = x'.$$

Denote the corresponding quotient space $G \times X/\sim$ by $G \times_H X$ (twisted product) and the quotient map by $(s,x) \mapsto [s,x]\colon G \times_H X \to G \times_H X$. This quotient map is an open mapping (and continuous, of course) if $G \times_H X$ is given the corresponding quotient topology. It can be shown that the formula

$$\tilde{\pi}^t[s,x] := [ts,x], \qquad t \in G \text{ and } [s,x] \in G \times_H X$$

defines unambiguously a continuous action $\tilde{\pi}$ of G on $G \times_H X$. The continuous mapping

$$\gamma_X\colon x \mapsto [e,x] : X \to G \times_H X$$

turns out to be H-equivariant (note that the G-space $G \times_H X$ is, by restriction of the action, also an H-space), and it has the following universal property: if $\langle G,Y,\sigma\rangle$ is a G-space and if $f\colon \langle H,X,\pi\rangle \to \langle H,Y,\sigma\rangle = R\langle G,Y,\sigma\rangle$ is a morphism of H-spaces, then there exists a unique morphism of G-spaces $\tilde{f}\colon \langle G,G\times_H X,\tilde{\pi}\rangle \to \langle G,Y,\sigma\rangle$ such that $f = \tilde{f}\circ\gamma_X$.

$$\begin{tikzcd}
\langle H,X,\pi\rangle \arrow[rr, "\gamma_X"] \arrow[dr, "f"'] & & \langle H,G\times_H X,\tilde{\pi}\rangle \arrow[dl, dashed, "\tilde{f}"] \\
& \langle H,Y,\sigma\rangle &
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle G,G\times_H X,\tilde{\pi}\rangle \arrow[d, dashed, "\tilde{f}"] \\
\langle G,Y,\sigma\rangle
\end{tikzcd}$$

In fact, $\tilde{f}$ is given by $\tilde{f}[s,x] := \sigma^s f(x)$, $[s,x] \in G \times_H X$.

This construction not only shows that R has a left adjoint, but it also gives a realisation of the object $\mathsf{L}\langle H,X,\pi\rangle$ and the universal arrow γ_X: $\langle H,X,\pi\rangle \to \mathsf{RL}\langle H,X,\pi\rangle$. Indeed, we can take $\mathsf{L}\langle H,X,\pi\rangle = \langle G,G\times_H X,\tilde{\pi}\rangle$ for any H-space $\langle H,X,\pi\rangle$.

The following propositions are well-known. Their proofs are straightforward applications of quotient topologies.

3.1. PROPOSITION. *Let* f: $\langle H,X,\pi\rangle \to \langle H,Y,\sigma\rangle$ *be a morphism of* H-*spaces, and consider the corresponding morphism* $\mathsf{L}(f)$: $\langle G,G\times_H X,\tilde{\pi}\rangle \to \langle G,G\times_H Y,\tilde{\sigma}\rangle$ *of* G-*spaces. If* f *is a topological embedding, then so is* $\mathsf{L}(f)$. *If* H *is closed in* G *and* f *is a closed embedding, then so is* $\mathsf{L}(f)$. □

3.2. PROPOSITION. *For every* H-*space* $\langle H,X,\pi\rangle$, $\gamma_X : X \to G \times_H X$ *is a topological embedding. The subset* $S := \gamma_X[X]$ *of* $G \times_H X$ *has the following properties*:

(i) S *is* H-*invariant*;

(ii) S *generates* $G \times_H X$ *in the sense that* $\tilde{\pi}[G\times S] = G \times_H X$;

(iii) *For every* $t \in G$: $\tilde{\pi}^t S \cap S = \emptyset \iff t \notin H$.

Moreover, S *is a closed subset of* $G \times_H X$ *if* H *is closed in* G. □

Recall that a subset A of G is called *relatively dense* in G whenever there exists a compact subset K of G such that G = KA. If a subgroup H of G is relatively dense then G/H is compact. Conversely, if G is locally compact and G/H is compact, then H is relatively dense in G (cf. [11], 5.24(b)).

3.3. PROPOSITION. *Let* G *be a Hausdorff group and let* H *be a closed subgroup of* G. *Then for every* H-*space* $\langle H,X,\pi\rangle$ *the following is true*:

(i) *If* X *is a Hausdorff space, then so is* $G \times_H X$.

(ii) *If* X *is metrizable,* G *is metrizable and* H *is a compact subgroup of* G, *then* $G \times_H X$ *is metrizable.*

(iii) *If* G *and* X *are locally compact, then so is* $G \times_H X$.

(iv) *If* X *is compact and* H *is relatively dense in* G, *then* $G \times_H X$ *is compact.*

(v) *If* G *is locally compact and* X *is a completely regular Hausdorff space, then* $G \times_H X$ *is a completely regular Hausdorff space.*

(vi) *If* X *is completely regular Hausdorff,* $\{\pi^t : t \in H\}$ *is equicontinuous with respect to a uniformity for* X *and* H *is relatively dense in* G, *then* $G \times_H X$ *is completely regular and Hausdorff.*

PROOF. The proofs of (i) through (iv) are applications of standard results on quotient topology. For (i), see [7], Chap. VII, 1.6 (the equivalence relation $\sim$ on $G \times X$ is a closed subset of $(G\times X)\times(G\times X)$). For (ii), see [7], Chap. XI, 5.1 (in this situation, the quotient mapping $(s,x) \mapsto [s,x] : G \times X \to G \times_H X$ is perfect), and (iii) is an obvious consequence of the fact that this quotient mapping is open. The proof of (iv) is an obvious application of 3.2 above. Finally, we prove (v) and (vi) as follows.

Let $f: \langle H,X,\pi\rangle \to \langle H,X^*,\pi^*\rangle$ be an H-equivariant embedding, where X^* is compact Hausdorff (cf. 2.4). Now we have the following commutative diagram of continuous functions

$$\begin{tikzcd}
X \arrow[r, "\gamma_X"] \arrow[d, "f"'] & G\times_H X \arrow[d, "L(f)"'] \\
X^* \arrow[r, "\gamma_{X^*}"] & G\times_H X^*
\end{tikzcd}$$

By (iii) or (iv) and (i), $G \times_H X^*$ is locally compact and Hausdorff, and by 3.1, $L(f)$ is a topological embedding. So $G \times_H X$ is homeomorphic with a subset of a locally compact Hausdorff space, hence completely regular and Hausdorff. □

The results of 3.3 are well-known. For (v), see [14], where Haar measure on G has been used in the proof. Our proof avoids any reference to non-topological means.

4. COSEPARATORS AND EXTENSORS

It is possible to formulate the notions which are important for this section in a strictly categorical way and to prove our results as consequences of certain preservation properties of adjoint functors. However, I have chosen not to present the results as corollaries of abstract theorems of category theory: for category theorists the underlying categorical ideas will yet be apparant, while a strictly categorical setting would perhaps frustrate many others.

A topological space X is called an *extensor for a pair of spaces* (A,Y) whenever A is a closed subspace of Y and every continuous mapping $f: A \to X$ has a continuous extension $\bar{f}: Y \to X$. A topological space S is called a *coseparator for a class* K *of topological spaces* (or rather, for a full subcategory of TOP) whenever for every $X \in \mathsf{K}$ and every pair of distinct points $x,y \in X$ there exists a continuous mapping $f\colon X \to S$ such that $f(x) \neq f(y)$.

It is well-known that the closed unit interval $I := [0,1]$ is an extensor for the class of all pairs (A,Y) with Y a normal space. In addition, I is a coseparator for the class of all functionally Hausdorff spaces (which includes CR). Observe, that I is an object in COMP which is both a coseparator for COMP and an extensor for all pairs in COMP, i.e. all pairs (A,Y) with Y in COMP. It is the property of being a coseparator which makes I a very non-trivial extensor (this in contradistinction to a one-point space).

The modifications of the above definitions for G-spaces is quite straightforward. If G is an arbitrary topological group, then a G-space $\langle G,X,\pi\rangle$ is called a *G-extensor for a pair* $(A,\langle G,Y,\sigma\rangle)$ whenever A is a closed G-invariant subset of Y and every G-equivariant continuous mapping $f\colon A \to X$ has a G-equivariant continuous extension $f : Y \to X$. A G-space $\langle G,X,\pi\rangle$ is called a *G-coseparator for a class* M *of G-spaces* whenever for every $\langle G,Y,\sigma\rangle \in \mathsf{M}$ and every pair of distinct points $x,y \in Y$ there exists a G-equivariant mapping $f : Y \to X$ such that $f(x) \neq f(y)$.

For the next proposition I need one additional definition. Let $C_c(G,I)$ be the set of all continuous functions from G into I endowed with the compact-open topology. Define $\tilde{\rho}\colon G \times C_c(G,I) \to C_c(G,I)$ by $\tilde{\rho}^t f(s) := f(st)$ for $f \in C_c(G,I)$ and $s,t \in G$. Then $\langle G_d, C_c(G,I), \tilde{\rho}\rangle$ is a ttg (G_d is the group G with its discrete topology). If G is locally compact, then $\tilde{\rho}$ is continuous and $\langle G, C_c(G,I), \tilde{\rho}\rangle$ is a ttg. This is why in the next proposition local compactness of G is needed. In fact, if $\tilde{\rho}\colon G \times C_c(G,I) \to C_c(G,I)$ is continuous, then the evaluation mapping $(f,t) \mapsto f(t) := \tilde{\rho}^t f(e) : C_c(G,I) \times G \to I$ is continuous, and by a result of ARENS [2], G is locally compact.

4.1. PROPOSITION. *Let* G *be a locally compact topological group. Then*:

(i) $\langle G, C_c(G,I), \tilde{\rho}\rangle$ *is a G-extensor for all pairs* $(A,\langle G,Y,\sigma\rangle)$ *such that* I *is an extensor for the pair* (A,Y).

(ii) $\langle G, C_c(G,I), \tilde{\rho}\rangle$ *is a G-coseparator for every class* K^G *of G-spaces such that* I *is a coseparator for* K.

PROOF. I shall only prove (i), the proof of (ii) being similar to the proof of (i) (see also [17], 6.4.8, where a categorical proof of (ii) has been given). Consider a closed, G-invariant subset A of the G-space $\langle G,Y,\sigma\rangle$

and let $f\colon A \to C_c(G,I)$ be a continuous G-equivariant mapping. The continuous mapping $x \mapsto f(x)(e)\colon A \to I$ has, by assumption, a continuous extension $f\colon Y \to I$. Now the mapping $y \mapsto f\circ\sigma_y\colon Y \to C_c(G,I)$ is easily seen to be a continuous, G-equivariant extension of f. □

The following proposition implies a sort of converse of 4.1, because the set of invariant points in $\langle G, C_c(G,I), \tilde{\rho}\rangle$ is homeomorphic to I.

4.2. PROPOSITION. *Let* G *be an arbitrary topological group, and let* $\mathcal{K}$ *be a class of topological spaces. Consider a* G*-space* $\langle G,X,\pi\rangle$ *and set* $X_G := \{x \in X\colon \pi^t x = x$ *for all* $t \in H\}$. *Then:*

(i) *If* $\langle G,X,\pi\rangle$ *is a* G*-extensor for all pairs* $(A,\langle G,Y,\sigma\rangle)$ *with* $Y \in \mathcal{K}$, *then* X_G *is an extensor for all pairs* (A,Y) *with* $Y \in \mathcal{K}$.

(ii) *If* $\langle G,X,\pi\rangle$ *is a* G*-coseparator for* $\mathcal{K}^G$, *then* X_G *is a* G*-coseparator for* $\mathcal{K}$.

PROOF. Again, we prove only (i), leaving (ii) for the reader. Let $Y \in \mathcal{K}$, A a closed subset of Y and $f\colon A \to X_G$ a continuous function. Consider Y as a G-space under the trivial action $\tau\colon (t,x) \mapsto x$ of G on Y. Then A is an invariant subset, and f is G-equivariant, because the action of G on $f[A]$ is trivial, $f[A]$ being a subset of X_G. By assumption, f has a continuous, G-equivariant extension $\tilde{f}\colon \langle G,Y,\tau\rangle \to \langle G,X,\pi\rangle$. Obviously, one has $\tilde{f}[Y] \subseteq X_G$, so the corestriction of $\tilde{f}$ to X_G is the desired continuous extension of f. □

It follows from proposition 4.1 that for a locally compact group G the G-space $\langle G,C_c(G,I),\tilde{\rho}\rangle$ is a G-extensor for all pairs $(A,\langle G,Y,\sigma\rangle)$ with Y normal, and that it is a G-coseparator for the class of all functionally Hausdorff G-spaces (which includes $\mathbf{CR}^G$). In particular, it is both a G-coseparator for $\mathbf{COMP}^G$ and a G-extensor for all pairs in $\mathbf{COMP}^G$. The problem is, whether of there exists an object in $\mathbf{COMP}^G$ having these two properties. It is obvious that $\mathbf{COMP}^G$ contains a G-extensor for all pairs in $\mathbf{COMP}^G$ (namely, the trivial one-point G-space), and we shall see that $\mathbf{COMP}^G$ contains a G-coseparator, but the problem is to find an object in $\mathbf{COMP}^G$ which can play a role similar to the role of I in $\mathbf{COMP}$. This problem is still open.

4.3. PROPOSITION. *If* G *is locally compact then* $\mathbf{COMP}^G$ *contains an object which is a* G*-coseparator for* $\mathbf{COMP}^G$.

PROOF. Apply 2.4(i) to $\langle G,C_c(G,I),\tilde{\rho}\rangle$, and use 4.1(i). □

Sufficient conditions (none of which applies to the object, referred to in proposition 4.3) for G-spaces to be G-extensors for certain pairs of G-spaces can be found in [9], [12] and [1]. A necessary condition occurs in [15]. The following result is based on a generalization of the method, used in [15].

4.4. THEOREM. *Let* G *be an arbitrary topological group, let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be classes of topological spaces, and suppose that the* G*-space* $\langle G,X,\pi\rangle$ *is a* G*-extensor for all pairs in* $\mathcal{K}_1^G$ *(i.e. for all pairs* $(A,\langle G,Y,\sigma\rangle)$ *with* $Y \in \mathcal{K}_1$*). If* H *is a closed subgroup of* G *such that for every* H*-space* $\langle H,Z,\zeta\rangle$,

$$\langle H,Z,\zeta\rangle \in \mathcal{K}_2^G \Rightarrow G \times_H Z \in \mathcal{K}_1,$$

then the H*-space* $\langle H,X,\pi\rangle$ *is an* H*-extensor for all pairs in* $\mathcal{K}_2^H$.

PROOF. Consider an object $\langle H,Z,\zeta\rangle$ in $\mathcal{K}_2^H$, a closed, H-invariant subset A of Z and a continuous, H-equivariant mapping $f\colon A \to X$. If $i\colon A \to Z$ denotes the inclusion mapping, then by 3.1, $\mathcal{L}(i)\colon \langle G,G\times_H A,\tilde{\zeta}\rangle \to \langle G,G\times_H Z,\tilde{\zeta}\rangle$ is a closed G-equivariant embedding. Now consider the following diagram, in which the solid arrows are H-equivariant mappings, the dotted arrows are G-equivariant mappings, and where $p\colon \langle G,G\times_H X,\tilde{\pi}\rangle \to \langle G,X,\pi\rangle$ is defined by $p[s,x] := \pi^s x$ for $[s,x] \in G \times_H X$ (see section 3 for notation)

$$\begin{tikzcd}
\langle H,A,\zeta\rangle \arrow[rr, "\gamma_A"] \arrow[dr, "f"] \arrow[dd, "i"'] & & \langle G,G\times_H A,\tilde{\zeta}\rangle \arrow[dr, dashed, "\mathcal{L}(f)"] \arrow[dd, dashed, "\mathcal{L}(i)"'] & & \\
& \langle H,X,\pi\rangle \arrow[rr, "\gamma_X"] & & \langle G,G\times_H X,\tilde{\pi}\rangle \arrow[dr, dashed, "p"] & \\
\langle H,Z,\zeta\rangle \arrow[rr, "\gamma_Z"] & & \langle G,G\times_H Z,\tilde{\zeta}\rangle & & \langle G,X,\pi\rangle
\end{tikzcd}$$

By our assumptions, $p\circ\mathcal{L}(f)\colon \langle G,G\times_H A,\tilde{\zeta}\rangle \to \langle G,X,\pi\rangle$ has a continuous G-equivariant extension $\tilde{p}\colon \langle G,G\times_H Z,\tilde{\zeta}\rangle \to \langle G,X,\pi\rangle$. Now $\tilde{p}\circ\gamma_Z$ turns out to be an H-equivariant extension of f. □

4.5. COROLLARY 1. *If* G *is a metrizable topological group,* H *a compact subgroup of* G *and* $\langle G,X,\pi\rangle$ *a* G*-extensor for all pairs in* MET^G*, then* $\langle H,X,\pi\rangle$ *is an* H*-extensor for all pairs in* MET^H*. If* G *is a Hausdorff group,* H *a locally compact and relatively dense subgroup of* G [1] *and* $\langle G,X,\pi\rangle$ *is a*

---

1) The existence of such a subgroup implies that G is locally compact. Indeed, G is the image of the locally compact space $K \times H$ under the continuous and open mapping $(s,t) \mapsto st$; here K denotes a compact set such that $G = KH$

G-extensor for all pairs in COMP^G*, then* $\langle H,X,\pi\rangle$ *is an* H*-extensor for all pairs in* COMP^H. □

4.6. COROLLARY 2. *Let* G, H, K_1 *and* K_2 *satisfy the conditions of Theorem* 4.4. *If* $\langle G,X,\pi\rangle$ *is a* G*-extensor for all pairs in* K_1^G*, then* X_H*, the set of all* H*-fixed points in* X*, is an extensor for all pairs in* K_2.

PROOF. Combine 4.4. and 4.2.(i). □

The hypothesis in 4.6 can slightly be weakened. Indeed, if the proof of 4.4 is re-written for the special case, considered in 4.6, namely trivial actions of H (cf. also 4.2(i)), then we may replace $G \times_H Z$ by $(G/H) \times Z$. So the conclusion of 4.6 already holds under the following condition on G, H, K_1 and K_2: $Z \in K_2 \Rightarrow (G/H) \times Z \in K_1$. In this form, the result is due to SMIRNOV [15]. Consequently, taking $K_1 = K_2 = \mathrm{MET}$, G a metrizable group and H a closed subgroup of G one shws: if $\langle G,X,\pi\rangle$ is a G-extensor for MET^G, then X_H is an extensor for all pairs in MET (by [11], 8.14, G/H is metrizable in this case). Similarly, if G is a Hausdorff group, H a closed, relatively dense subgroup of G and $\langle G,X,\pi\rangle$ an G-extensor for all pairs in COMP^G, then X_H is an extensor for all pairs in COMP.

ADDED IN PROOF. An interesting corollary of Theorem 4.4 is obtained by taking $K_1 := \mathrm{COMP}$ and $K_2 := T_4$, the class of all normal Hausdorff spaces: *If* G *is a locally compact Hausdorff group,* H *is a compactly generated closed subgroup of* G *and* $\langle G,X,\pi\rangle$ *is a* G*-extensor for all pairs in* COMP^G*, then* $\langle H,X,\pi\rangle$ *is an* H*-extensor for all pairs in* T_4^H*; consequently,* X_H *is an extensor for all pairs in* T_4

To prove this, it is sufficient to show that for every compact Hausdorff H-space Z, the space $G \times_H Z$ (which is Hausdorff by Theorem 3.3) is paracompact. The argument is a modification of [20; Cor.14]. We know that $H = \bigcup_{n=1}^{\infty} K^n$ for some compact subset K of G. Let H be a compact neighbourhood of e in G, set $K_1 := KU \cup U^{-1}K^{-1}$, and $H_1 := \bigcup_{n=1}^{\infty} K_1^n$. Then H_1 is an open, σ-compact subgroup of G such that $H \subseteq H_1$, and $G \times Z$ is the disjoint union of open, σ-compact subspaces, each of which saturated w.r.t. the equivalence relation ∿ (cf. section 3), viz. sets of the form $sH_1 \times Z$ for some $s \in G$. Since the quotient mapping $(s,z) \mapsto [s,z]: G \times Z \to G \times_H Z$ is open, it follows that the locally compact Hausdorff space $G \times_H Z$ (cf. Theorem 3.3 (i) and (iii)) is the disjoint union of open, σ-compact subsets. By

[7; XI,7.3], $G \times_H Z$ is paracompact, hence normal.

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SUPEREXTENSIONS EMBEDDED IN CUBES

E. Wattel

0. INTRODUCTION

In this note we investigate the superextensions defined by DE GROOT [4]. The basic papers in this area are VERBEEK's monograph [17] and van MILL's thesis [8]. The construction of a superextension is the natural way to embed a space with a given subbase in a supercompact space. (cf. [4]).

Many compact Hausdorff spaces are supercompact, (e.g. all compact metric spaces, cf. [16], elementary proofs of this fact were recently discovered by MILLS and van DOUWEN [2], [14], and MILLS [15] also proved that all compact topological groups are supercompact), but in 1976 BELL constructed compact Hausdorff spaces which are not supercompact. (cf. [1]). The supercompact spaces with a normal subbase seem to be of particular interest because they carry a natural convexity structure, (cf. [11], [13], [8], [12]), which can be used to construct a nice category, (cf. [19]).

All the known papers construct superextensions by means of linked systems, cf. definition 2.1, but in this note we give an equivalent construction by means of a type of closure of the evaluation of the space in a product of the ranges of a set of separating functions called Urysohn maps. In [13] we have already embedded a supercompact space as a triple convex subset of a product of intervals. In this note we make an extension of a space by considering the "triple convex closure" of the evaluation of the space in the considered product of image intervals.

In the first section we investigate the basic definitions and constructions of [13] and prepare them for our purposes. Although we list a host of definitions, none of them is essentially new. In the second section we obtain our main results, the third section is devoted to conclusive remarks and possible extensions of the theory.

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1. TRIPLE CONVEX SETS

DEFINITION 1.1. All topological spaces under consideration are assumed to be T_1. Let X be a topological space. A closed subbase $\mathcal{S}$ for X is called *binary*, iff every subsystem of $\mathcal{S}$ in which every two members meet has a non-empty intersection. $\mathcal{S}$ is called *normal*, iff every two disjoint members S_1 and S_2 of $\mathcal{S}$ can be separated with two complements $X\backslash S_3$ and $X\backslash S_4$ of members of $\mathcal{S}$. A subbase is called T_1 iff for every $x \in X$ and $S_1 \in \mathcal{S}$ with $x \notin S_1$ there exist an $S_2 \in \mathcal{S}$ such that $x \in S_2$ and $S_1 \cap S_2 = \emptyset$.

A space with a binary subbase is called *supercompact*. In the sequel we will only consider T_1 normal subases, unless it is stated explicitely otherwise.

Let X be a topological space with T_1 normal subbase $\mathcal{S}$. A set C is called *closed convex* [11] iff it is an intersection of subbase members of $\mathcal{S}$. The collection of non-empty closed convex sets is usually denoted by $\mathcal{H}(X,\mathcal{S})$. A mapping f between two spaces X and Y with closed subbases $\mathcal{S}$ and $\mathcal{T}$ is called *convexity preserving* or *cp*, iff inverse images of subbase members of $\mathcal{T}$ belong to $\mathcal{H}(X,\mathcal{S}) \cup \{\emptyset\}$ cf. [12].

A mapping is called *van Mill* iff inverse images of disjoint members T_1 and T_2 of $\mathcal{T}$ can be separated by a pair of disjoint subsets S_1 and S_2 of $\mathcal{S}$ i.e.

$$f^{-1}[T_1] \subset S_1; \quad f^{-1}[T_2] \subset S_2; \quad S_1 \cap S_2 = \emptyset.$$

A mapping is called *Jensen* iff inverse images of subbase members belong to the subbase, (cf. [8], p. 77 and [5]).

For an arbitrary set A of a space X with subbase $\mathcal{S}$ we define the *convex closure* of A to be the intersection of all subbase members of $\mathcal{S}$ which contain A. This set is denoted by $\mathcal{I}_S(A)$, or if no confusion is possible, by $\mathcal{I}(A)$, cf [3]. In the same way we define the *interval* between two points x and y to be $\mathcal{I}(\{x,y\})$ or shortly $\mathcal{I}[x,y]$, cf. [10]. For an ordered space we will always use the subbase consisting of all halfspaces of the types:

$$L_a = \{x \mid x \leq a\} \quad \text{and} \quad R_a = \{x \mid x \geq a\}.$$

For products we always use the subbase of the inverse images under projections of subbase members. In the ordered case the interval structure is the ordinary interval notion although the $I[z,x]$ is defined even if $z > x$ and equal to $I[x,z]$.

Let X be a space with binary normal T_1 subbase $\mathcal{S}$. It can be shown that for every three points x, y, z the intersection

$$I[x,y] \cap I[x,z] \cap I[y,z]$$

consists of a single point cf. [9]. We define the *triple point* or *mean* w of x, y, and z by

$$\{w\} = \{m(x,y,z)\} = I[x,y] \cap I[x,z] \cap I[y,z].$$

cf. [9].

The function m can be considered as a function of three variables on X onto X. It is not strictly necessary that $\mathcal{S}$ is binary and normal to make the construction of m work. We will in the sequel also consider subbases which admit a mean function without binarity. In the case of an ordered space with the usual halfline subbase the triple point of x, y and z is the middle of the three.

In the same way we find the triple point in a product of ordered spaces. In that case for each coordinate α the point w_α is the middle of the three x_α, y_α and z_α.

A space X with a closed subbase $\mathcal{S}$ which admits a mean function with respect to $\mathcal{S}$ for every triple is called *triple convex*. A subset T of a triple convex space X is called *triple convex* provided that with every three points in T also the triple point is contained in T, (cf. [13]).

PROPOSITION 1.2. *Let* X *be a space with a normal subbase* $\mathcal{S}$. *Then for each pair of disjoint subbase members* P *and* Q *there exist a van Mill-mapping* f *from* X *to the unit interval* [0,1] *with the usual subbase of closed half-spaces, such that* $f[P] = \{0\}$ *and* $f[Q] = \{1\}$.

For the proof of this result we refer to [13].

Van Mill maps from a space onto the unit interval or the two point space {0,1} will be called *Urysohn maps* in the rest of this note.

PROPOSITION 1.3. *Let* X *be a space with binary normal subbase* $\mathcal{S}$. *Let* $\mathcal{F}$ *be the system of all cp-mappings* f *from* X *onto the closed unit interval* $[0,1]_f$ *or to the two point space* $\{0,1\}_f$. *Then the evaluation mapping* e *embeds* X *as* *a closed triple convex subset of*

$$\prod_{f \in F} [0,1]_f .$$

Moreover, the convexity structure on X *defined by* S *coincides with the restriction onto* e[X] *of the usual convexity structure on the product of intervals.*

For a proof of this result we refer to [13]. In the same place it is shown that the closed triple convex subspaces of cubes have a binary normal subbase, namely the restriction of the product subbase which is binary and normal.

PROPOSITION 1.4. *Let* Q *be a triple convex space with respect to a* T_1 *normal subbase (e.g. a product of ordered spaces). Let* m *be the mapping which assigns to three points* x, y *and* z *their triple point* w = m(x,y,z). *Then* $m: Q * Q * Q \to Q$ *is a continuous function.*

PROOF. According to 1.2 we can embed Q cp isomorphically in a product of ordered spaces, (cf. [13]). Let f be a Urysohn map and let $[0,1]_f$ be its coordinate space in such an embedding.

Assume that w is the mean of x, y and z with $f(w) < a_f$. Then we get a basic open neighbourhood 0_f of w if we consider the set of all $q \in Q$ with $f(q) < a_f$. Now at least two of the three points f(x), f(y) and f(z) are smaller than a_f. The set $0_f * 0_f * Q$, is a neighbourhood of (x,y,z) in the space $Q * Q * Q$ which is mapped entirely into 0_f by the function m. All the other cases can be treated similarly and we conclude that the function is continuous because inverse images of open subbase members are open. □

Observe that for every subset X of a cube Q there exist a minimal triple convex subset T containing X and a minimal closed triple convex subset T^- containing X. T can be defined inductively by taking iterative images of X under the function m of the previous proposition.

We define

$$T_1 = m[X^3].$$

In the same way we define inductively

$$T_{i+1} = m[T_i^3], \qquad \text{for } i \in \mathbb{N}.$$

The set T is the union of the T_i. Since

$$T_{i+1} \supset T_i \supset X$$

it follows that T is the minimal triple convex subset containing X. We define T^- to be the closure of T. From the continuity of m it follows that T^- is triple convex and it is clearly the intersection of all triple convex closed subsets of Q containing X.

PROPOSITION 1.5. *Let* $(X,\mathcal{S})$ *and* $(Y,\mathcal{T})$ *be spaces which are triple convex w.r.t.* $\mathcal{S}$ *and* $\mathcal{T}$ *respectively and let* $f\colon X \to Y$ *be cp then* $f[X]$ *is a triple convex subset of* $(Y,\mathcal{T})$.

PROOF. Let x, y, z be three points of f[X] in Y; let w be their triple point. Assume that x', y', z' are three points of X such that f(x') = x, f(y') = y, f(z') = z. Let w' be the triple point of x', y' and z'. We prove now that f(w') = w.

Suppose not. Then we can find a subbase member T in $\mathcal{T}$ such that $w \in T$ and $f(w') \notin T$. Since $\mathcal{T}$ is T_1 we find a set $T' \in \mathcal{T}$ such that $T \cap T' = \emptyset$ and $f(w') \in T'$. We now separate the sets T' and T by two subbasic members W' and W such that $T \subset W$; $T' \subset W'$; $T' \cap W = \emptyset$; $T \cap W' = \emptyset$; $W \cup W' = X$. Next we separate T and W' by two subbase members V and V' such that $T \subset V$; $W' \subset V'$; $T \cap V' = \emptyset$; $W' \cap V = \emptyset$; and $V' \cup V = Y$. Since w is in T at least two of the three points x, y and z must be in V, since otherwise the m(x,y,z) would be in V'. The two sets $f^{-1}(W)$ and $f^{-1}(W')$ are convex subsets in X. Since $x' \notin f^{-1}(W)$ and $f^{-1}(W') \cup f^{-1}(W) = X$ it follows that two of the three points x', y' and z' are in $f^{-1}(W')$ and so two of the three points f(x'), f(y'), f(z') must be in W'. However, two of the three points x, y, z are in V and $V \cap W' = \emptyset$. Contradiction. This proves our proposition. □

The following corollary generalizes the main result of [13].

COROLLARY 1.6. *The space* $(X,\mathcal{S})$ *is triple convex with respect to* $\mathcal{S}$ *if and only if* X *is cp isomorphic to some triple convex subset of a cube* Q. *Moreover, in this case the image* f[X] *of* X *under any cp map into a cube* Q^* *is triple convex.*

2. SUPEREXTENSIONS

DEFINITION 2.1. A collection of sets L is called *linked,* iff every two members have a non-empty intersection. Usually we will consider linked subsystems of the current subbase. Throughout this paper all linked systems of the space will be subbasial unless explicitely stated otherwise. A linked system is called a *maximal linked system* or shortly an *mls,* iff it is not properly contained in a larger subbasial linked system. From Zorn's lemma it follows that every linked system L is contained in some maximal linked system. If L is contained in precisely one mls, then L is called a *pre-mls.*

Let X be a space with closed subbase S. The collection of all maximal linked systems in X with respect to S is called $\lambda(X,S)$. It can be made into a topological space with a closed subbase S^+ if we put:

$$S^+ = \{s^+ \mid s \in S\}$$

in which

$$s^+ = \{L \mid L \in \lambda(X,S) \text{ and } s \in L\}.$$

Every point of X defines a unique mls, namely the collection of all subbase members containing it. We can embed X in $\lambda(X,S)$ by assigning to every point its obvious linked systems. It is easy to see that the restriction of S^+ to the image of X yields the original subbase S. Therefore $\lambda(X,S)$ contains a homeomorphic image of X which is usually identified with X. For more details see e.g. [17].

THEOREM 2.2. *Let S be a normal subbase for the space* X. *Let F be the corresponding collection of all Urysohn mappings. Then*
(a) *the evaluation map* $e\colon X \to I^F$ *is an embedding;*
(b) e *can be extended to a cp-isomorphism* $e^+\colon \lambda(X,S) \to T$, *where* T *is the smallest closed triple convex set containing* e[X].

PROOF.
(a) Trivial. For simplicity, we indentify X and e[X].
(b) We take an arbitrary member $f \in F$. Then f can be extended to a mapping $\lambda(f)\colon \lambda(X,S) \to I_f$. Indeed, we define $\lambda(f)$ by

$$(*) \qquad \{\lambda(f)(\mathcal{M})\} = \bigcap_{M\in\mathcal{M}} \mathcal{I}_f^{\#}(f[M]),$$

in which $\mathcal{I}_f^{\#}$ denotes the convex closure in the interval $[0,1]_f$ resp. in $\{0,1\}_f$.

First of all we show that $\lambda(f)$ is well defined. Let us first prove the weaker statement that

$$\bigcap_{M\in\mathcal{M}} \mathcal{I}_f^{\#}(F(M)) \neq \emptyset \qquad \text{for all } \mathcal{M} \in \lambda(X,\mathcal{S}).$$

Since $[0,1]_f$ resp. $\{0,1\}_f$ is supercompact w.r.t. the collection of all closed intervals and for each pair M_1, M_2 in $\mathcal{M}$ we have

$$M_1 \cap M_2 \neq \emptyset, \qquad f(M_1) \cap F(M_2) \neq \emptyset,$$

it follows that

$$\mathcal{I}_f^{\#}(f(M_1)) \cap \mathcal{I}_f^{\#}(f(M_2)) \neq \emptyset.$$

We now prove that this is intersection is a singleton. If s and r are both in $\lambda(f)(\mathcal{M})$ and say $s < r$, then there exists an ε such that $s+\varepsilon < r+\varepsilon$. Each M in $\mathcal{M}$ contains a point m_s with $f(m_s) \in [0,s+\varepsilon]$ and a point m_r with $f(m_r) \in [r-\varepsilon,1]_f$. Moreover, $[0,s+\varepsilon]_f$ and $[r-\varepsilon,1]_f$ are subbase members of $[0,1]_f$. Since f is van Mill there exist an S and an R in $\mathcal{S}$, such that $S \cap R = \emptyset$, and $f^{-1}([0,s+\varepsilon]_f) \subset S$; $f^{-1}([r-\varepsilon,1]_f) \subset R$. This would mean that both S and R are linked with $\mathcal{M}$ which is impossible since $\mathcal{M}$ is maximal linked.

$$(**) \qquad \begin{array}{l} e^+ : \lambda(X,\mathcal{S}) \to I^{\mathcal{F}} \quad \text{by} \\ e^+(\mathcal{M})_f := \lambda(f)(\mathcal{M}). \end{array}$$

Then e^+ is clearly well defined. We claim that e^+ is a cp-isomorphism between $\lambda(X,\mathcal{S})$ and T. In order to show this we observe that:

(i) e^+ is one to one.

(ii) e^+ is cp and hence continuous.

(iii) $e^+[\lambda(X,\mathcal{S})] = T$.

Then it follows from 1.5 and from Corollary 1.6 in [12] that T and $\lambda(X,\mathcal{S})$ are cp-isomorphic.

(i) e^+ is one to one.

Indeed take $\mathcal{M}$, $\mathcal{N} \in \lambda(X,\mathcal{S})$. Then there are an $M \in \mathcal{M}$ and $N \in \mathcal{N}$ such that $M \cap N = \emptyset$. Let $f \in \mathcal{F}$ be such that $f[M] = 1$ and $f[N] = 0$. By the definition of $\lambda(f)$ it follows that $\lambda(f)(\mathcal{M}) = 1$ and $\lambda(f)(\mathcal{N}) = 0$. Therefore

$$\{e^+(\mathcal{M})\}_f \neq \{e^+(\mathcal{N})\}_f.$$

We conclude that e^+ is one to one.

(ii) e^+ is cp and hence continuous.

Take $t \in [0,1]$ and $f \in \mathcal{F}$. We show that

$$(e^+)^{-1}(f^{-1}[0,t]_f) \quad \text{and} \quad (e^+)^{-1}(f^{-1}[t,1]_f)$$

are closed convex, which obviously implies that e^+ is cp.

We claim that

$$(***) \qquad (e^+)^{-1}(f^{-1}[0,t]_f) = \bigcap_{\varepsilon>0} \{S^+ \mid f^{-1}[0,t+\varepsilon]_f \subset S\}.$$

Let $\mathcal{M} \in (e^+)^{-1}[f^{-1}[0,t]_f]$ and suppose that there is an $\varepsilon > 0$ and an $S \in \mathcal{S}$ such that $f^{-1}[0,t+\varepsilon]_f \subset S$ and $S \notin \mathcal{M}$. Take an $M \in \mathcal{M}$ which misses S. Then

$$\lambda(f)[S] \cap [0,t+\varepsilon]_f = \emptyset.$$

and consequently,

$$I_f^{\#}[\lambda(f)[S]] \subset [t+\varepsilon,1].$$

Since $[t+\varepsilon,1]_f \cap [0,t]_f = \emptyset$ this contradicts $\lambda(f)(\mathcal{M}) \in [0,t]_f$.

Now suppose that $\mathcal{M} \in \bigcap_{\varepsilon>0} \{S^+ \mid f^{-1}[0,t+\varepsilon]_f \subset S\}$ while moreover $\lambda(f)(\mathcal{M}) > t$. For simplicity, write $s = \lambda(f)(\mathcal{M})$. Choose $\varepsilon > 0$ such that $[0,t+\varepsilon]_f \cap [s-\varepsilon,1]_f = \emptyset$. Since f is van Mill, there is some $S_0 \in \mathcal{S}$ such that

$$f^{-1}[0,t+\varepsilon]_f \subset S_0 \subset X\backslash f^{-1}[s-\varepsilon,1]_f.$$

Then $S_0 \in \mathcal{M}$ by assumption, and consequently

$$s \in I_f^{\#}[f[S_0]] \subset [0,s-\varepsilon]_f,$$

which is a contradiction.

In precisely the same way it can be verified that

$$(e^+)^{-1}(f^{-1}[t,1]_f) = \bigcap_{\varepsilon>0} \{s^+ \mid f^{-1}[t-\varepsilon,1]_f \subset S\}.$$

We conclude that e^+ is convexity preserving and therefore continuous.

(iii) $e^+[\lambda(X,\mathcal{S})] = T$.

Since $\mathcal{S}$ is normal, so is the subbase $\{S^+ \mid S \in \mathcal{S}\}$ for $\lambda(X,\mathcal{S})$. According to Proposition 1.6 we obtain that $e^+[\lambda(X,\mathcal{S})]$ is a triple convex subset of $I^{\mathcal{F}}$ and hence it trivially contains T.

Now assume that there is a point $w \in e^+[\lambda(x,\mathcal{S})]\backslash T$. Take $\mathcal{L} \in \lambda(X,\mathcal{S})$ such that $e^+(\mathcal{L}) = w$. Since T is closed in $I^{\mathcal{F}}$ there must be some basic neighbourhood W of w in $I^{\mathcal{F}}$ disjoint from T.

Suppose that W is defined by

$$W = \bigcap\{W_{f_i} \mid i = 1,2,\ldots,n\},$$

in which f_i is a suitable projection for $i = 1,2,\ldots.n$ and

$$W_{f_i} = f_i^{-1}[(a_i,b_i)] \quad \text{with} \quad a_i < f_i(w) < b_i.$$

Since the mapping f_i is an Urysohn map there exist disjoint sets S_{a1} and S_{a2} in $\mathcal{S}$ such that

$$f_i^{-1}[[0,a_i]_f] \cap X \subset S_{a1}$$

and

$$f_i^{-1}[[f_i(w),1]_f] \cap X \subset S_{a2}.$$

In the same way there exist disjoint sets S_{b1} and S_{b2} in $\mathcal{S}$ such that

$$f_i^{-1}[[0,f_i(w)]_f] \cap X \subset S_{b1}$$

and

$$f_i^{-1}[[b_i,1]_f] \cap X \subset S_{b2}.$$

Both S_{a2} and S_{b1} belong to L and so their intersection is non-empty. Let c_i be the infimum of $f_i[S_{a2}]$ and d_i be the supremum of $f_i[S_{b1}]$. Then we obtain two sets

$$L^+ = f_i^{-1}[d_i,1]_f \quad \text{and} \quad L^- = f_i^{-1}[0,c_i]_f$$

in Q which both contain images of members of L. This can be done for all $i < r$. We obtain a finite linked subcollection of halfspaces of Q. We call it $L^\#$. Observe that $\cap L^\#$ is contained in W but every two members L^+ and L^- contain a point of e[X].

We show by induction that the intersection of finitely many members of $L^\#$ contains a point of T. This is already established for every pair in $L^\#$. Suppose that

$$t_1 \in T \quad \text{and} \quad t_1 \in L_1 \cap \ldots \cap L_{j-1}$$

$$t_2 \in T \quad \text{and} \quad t_2 \in L_1 \cap \ldots \cap L_{j-2} \cap L_j$$

$$t_3 \in T \quad \text{and} \quad t_3 \in L_{j-1} \cap L_j.$$

Then

$$m(t_1,t_2,t_3) \in L_1 \cap \ldots \cap L_j.$$

Therefore the intersection of L contains a point of T, and hence W contains a point of T. Contradiction. We conclude that every linked system in X has its image in T.

We now have shown (i), (ii) and (iii), and we conclude that e^+ is a cp homeomorphism between $\lambda(X,S)$ and T. From a theorem of VAN MILL and VAN DE VEL ([12] Corollary 1.6) it follows that $(e^+)^{-1}$ is also a cp map. Hence T and $\lambda(X,S)$ are cp isomorphic. □

3. CONCLUSIVE REMARKS

REMARK 3.1. In the Theorem 2.1 the normality of the subbase cannot be deleted. It is not just necessary for the existence of sufficiently many Urysohn mappings, but also for the fact that the embedding e^+ is one to one. We show by means of the following two examples that there exist rather nice spaces with natural subbases whose superextensions cannot be embedded in a cube with trace subbase.

For spaces X with binary normal subbase $\mathcal{S}$ the embedding of $\lambda(X,\mathcal{S})$ in a product of intervals can without loss of essential information be generalized to embeddings into products of general spaces with binary normal subbase in which the function m is the usual mean with resepct to the current product subbase.

EXAMPLE 3.2. Let X be the subbase of [0,1] * [0,1] defined by

$$X = \{(x,y) \mid 0 < x < y < 1\}.$$

Let $\mathcal{S}$ be the natural subbase of halfspaces of [0,1] * [0,1] restricted to X. Then the set

$$S_x = \{(x,y) \mid y \le \frac{1}{2}\}$$

cannot be separated from the set

$$S_y = \{(x,y) \mid x \ge \frac{1}{2}\}.$$

However, the mls in [0,1] * [0,1] of all halfspaces containing $(\frac{1}{2},\frac{1}{2})$ is not maximal linked in X. It extends to two maximal linked systems with respect to $\mathcal{S}$ one containing S_x and one containing S_y. In the $\lambda(X,\mathcal{S})$ those two mls's cannot be separated by disjoint open sets. We conclude that the embedding in a cube has to fail because $\lambda(X,\mathcal{S})$ is not Hausdorff.

EXAMPLE 3.3. Let X be the circumference of the circle and let $\mathcal{S}$ be the collection of all closed connected quarter circumferences. This space is supercompact and Hausdorff but there are not even enough Urysohn maps to separate points and closed subbase members. Since in any subspace of a space with binary normal subbase the points and the subbase members can be separated with Urysohn maps, the circle circumference cannot be embedded in a cube with a suitable trace subbase.

DEFINITION 3.4. Let X be a space and let $\mathcal{S}$ be a subbase for X. Assume that there are sufficiently many Urysohn maps in X to separate points and members of the subbase. A subcollection of $\mathcal{S}$ is called *near* iff no two members of the collection can be separated by an Urysohn function.

PROPOSITION 3.5. *Let* X *be a space with closed subbase* $\mathcal{S}$. *Let* $\mathcal{F}$ *be a collection of Urysohn mappings which separates points and subbase members. Let* e *be the evaluation map of* X *into* $I^{\mathcal{F}}$ *and let* $\lambda^n(X,\mathcal{S})$ *be the collection of all maximal near systems of* X *w.r.t.* $\mathcal{S}$, *supplied with a subbase* $\mathcal{S}^+$ *of all* S^+ *in which*

$$S^+ = \{\mathcal{L} \mid \mathcal{L} \text{ is maximal near in } \mathcal{S} \text{ and } S \in \mathcal{L}\}.$$

Then $\lambda^n(X,\mathcal{F})$ *is cp-isomorphic to the smallest triple convex set of* $I^{\mathcal{F}}$ *containing* e[X].

The proof of this proposition is an obvious modification of the proof of Theorem 2.2. The consequences of this approach are not yet clear but it seems to fit in the general theory of Nearness cf. [7]; [18]; and [6].

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A HEREDITARILY SEPARABLE COMPACT ORDERED SPACE X FOR WICH λX IS NOT FIRST COUNTABLE

E.Wattel

For all undefined terms we refer to VERBEEK [2] and van MILL [1].

Let Y be the Alexandroff double arrow space, i.e. $[0,1]*\{0,1\}$ with the topology generated by the lexicographical ordering. It is well-known that $X = Y\backslash\{<0,0>,<1,1>\}$ is compact, perfect and hereditarily separable. We claim that X is as required.

The point $<0,1>$ will be denoted by 0 and the point $<1,0>$ by 1.

Let $G \subset X$ be closed; we say that $\mu(G) = r$, whenever the Lebesgue measure of the natural projection of G into $[0,1]$ is equal to r. For each closed set M_a we define

$$m_a = \inf\{x \mid x \notin M_a\}.$$

Now let $\mathcal{L}$ be the collection of all closed sets M_a in X which satisfy:

(α) $\mu(M_a) = \frac{1}{2}$;

(β) if $1 \in M_a$ then $m_a \in M_a$;

(γ) $M_a\backslash\{m_a\}$ is clopen.

Let $\mathcal{C}_0 = \{C \subset X \mid X \text{ is clopen}, \mu(C) = \frac{1}{2}, 1 \notin C\}$. Observe that $\mathcal{C}_0 \subset \mathcal{L}$. Define $\mathcal{C} = \mathcal{C}_0 \cup \{\emptyset\}$. We claim that the collection $\mathcal{L}$ satisfies the following properties:

(a) For each $C \in \mathcal{C}$ we have that $\mathcal{L}(C) = (\mathcal{L}\backslash\{C\}) \cup (X\backslash C)$ is linked; (notice that $\mathcal{L}(\emptyset) = \mathcal{L} \cup \{X\}$);

(b) $\mathcal{L}(C)$ is a pre-mls, i.e. $\mathcal{L}(C)$ extends to precisely one point of λX;

(c) the mls generated by is not a G in X, hence X is not first countable.

To prove (a) we first show that L is linked. Suppose, to the contrary, that this is not true, i.e. some $M_1, M_2 \in L$ have an empty intersection. Since $\mu(M_1) = \frac{1}{2} = \mu(M_2)$ it easily follows that $M_1 \cup M_2 = X$. Without loss of generality $1 \in M_1$. Hence, by the definition of L, $m_1 \in M_1$. Since $M_1 \cap M_2 = \emptyset$, both sets are clopen. By (γ), m_1 is isolated in M_1, hence m_1 is isolated in X, since M_1 is clopen; Contradiction. So L is linked.

Now take $C \in C_0$. To prove the linkedness of $L(C)$ we only need to show that each $L_0 \in L\backslash\{C\}$ intersects $X\backslash C$. Since $\mu(L_0) = \frac{1}{2}$ and $\mu(X\backslash C) = \frac{1}{2}$ it follows that L_0 is not properly contained in C, i.e. L_o intersects $X\backslash C$.

To prove (b), to the contrary, we suppose that there are closed sets G and H such that $L(C) \cup \{G\}$ is linked and $L(C) \cup \{H\}$ is linked while $G \cap H = \emptyset$. If $\mu(G) < \frac{1}{2}$ then $\mu(G \cup \{1\}) < \frac{1}{2}$ and consequently there is a clopen $C_0 \subset X$ such that $C_0 \cap (G \cup \{1\}) = \emptyset$ and $\mu(C_0) = \frac{1}{2}$. If $C_0 \in L(C)$ then this contradicts the linkedness of $L(C) \cup \{G\}$, which is a contradiction. So $C_0 \notin L(C)$, i.e. $C_0 = C$.

<u>CASE 1</u>. $G \cup \{1\} = X\backslash C$.

Then $1 \in G$ since $G \cup \{1\}$ is clopen and G is closed. Hence $G \in L(C)$ which contradicts the linkedness of $L(C) \cup \{H\}$.

<u>CASE 2</u>. $G \cup \{1\} \neq X\backslash C$.

Take a clopen $E \subset X\backslash(G \cup \{1\} \cup (X\backslash C))$ such that $0 < \mu(E) < \frac{1}{2}$. Also, take a clopen $F \subset C_0$ such that $\mu(F) = \frac{1}{2} - \mu(E)$. Then

$$\mu(E \cup F) = \mu(E) + \mu(F) = \frac{1}{2},$$

while in addition $1 \notin E \cup F$ and $E \cup F \neq C$. Therefore $E \cup F \in L(C)$ and since $G \cap (E \cup F) = \emptyset$, this is a contradiction.

We conclude that $\mu(G) \geq \frac{1}{2}$ and in the same way $\mu(H) \geq \frac{1}{2}$; Therefore $G \cup H = X$, so, in particular, both G and H are clopen. Without loss of generality $1 \notin G$. If $G \neq C$ then $G \in L(C)$ which contradicts the linkedness of $L(C) \cup \{H\}$. In case $G = C$, $H = X\backslash C \in L(C)$ which contradicts the linkedness of $L(C) \cup \{G\}$. Anyway we have derived a contradiction.

To prove (c), let x be the (unique) mls generated by L. We have to show that λX is not first countable at x but we prove the stronger statement that x is not countably generated, i.e. no countable subcollection of X is a pre-mls for x. It is enough to prove this, for assume that $\{U_n \mid n < \omega\}$ is a countable neighbourhood base for x. For each $n < \omega$ there is a finite $\mathcal{E}(n) \subset x$ such that

$$\cup\{E^+ \mid E \in \mathcal{E}(n)\} \subset U_n.$$

It is now easily seen that the collection $\{E \mid \exists n < \omega\colon E \in \mathcal{E}(n)\}$ is a countable pre-mls for x. So we only need to prove the following

FACT. x is not countably generated.

To the contrary, assume that $\{G_i \mid i < \omega\}$ is a countable pre-mis for x. Without loss of generality we may assume that each G_i is clopen. For each $C \in \mathcal{C}_0$ let $x(C)$ be the mls generated by $\mathcal{L}(C)$. Since $x(C) \neq x$ there is some $E \in \mathcal{L}(C)$ and an $i < \omega$ such that

$$E \cap G_i = \emptyset.$$

Clearly $E = X\backslash C$ since each element of $\mathcal{L}(C)\backslash\{X\backslash C\}$ is in $\mathcal{L}$, so intersects each G_i. Since $X\backslash G_i \notin x$, some $C_0 \in \mathcal{L}$ does not intersect $X\backslash G_i$. Each $F \in \mathcal{L}\backslash\{C\}$ intersects $X\backslash C$ (by (a)); therefore $C_0 = C$. We conclude that $C = G_i$. So for each $C \in \mathcal{C}_0$ there is some $i < \omega$ such that $C = G_i$. Since $|\mathcal{C}_0| = 2^{\omega}$ and $|\{G_i \mid i < \omega\}| = \omega$. This is a contradiction.

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A SURVEY OF ABSOLUTES OF TOPOLOGICAL SPACES

R. Grant Woods

§1. PRELIMINARIES

I wish to thank Professor J. van Mill for inviting me to write this survey paper on the development of the theory of absolutes of topological spaces. I have attempted herein to trace the development of several aspects of this theory; however, this article is far from comprehensive, and of necessity reflects my own interests and preoccupations, perhaps to the exclusion of other topics and ideas that might well have been discussed.

I wish to thank Professors Jack Porter and Eric van Douwen for their many halpful comments and suggestions.

In this opening section we give a rapid summary of some of the concepts and theorems we shall use in subsequent sections. It is assumed that the reader has prior familiarity with these ideas; however, we provide references to standard texts and/or papers where further details can be found. No proofs of results quoted in §1 are included. In subsequent sections proofs of theorems are provided if the results are fundamental, the proofs are not too lengthy, and the proofs that appear in the literature are brief or obscure.

Throughout this paper all hypothesized topological spaces are assumed to be Hausdorff, unless the contrary is explicitly stated. Thus the word "space" means "Hausdorff topological space".

1.1. Extremally disconnected spaces

A space is *extremally disconnected* if its open subsets have open closures; equivalently, if disjoint open subsets of the space have disjoint closures. If X is extremally disconnected and U and V are open sets of X then $c\ell_X(U\cap V) = c\ell_X U \cap c\ell_X V$. An extremally disconnected regular space is zero-dimensional; i.e. its open-and-closed (clopen) subsets form a base for its topology. Dense subspaces of extremally disconnected spaces are extremally disconnected. See [GJ] and [WAL] for more details.

1.2. βX and νX.

Each completely regular (i.e. Tychonoff) space X can be embedded densely in a compact space βX called its *Stone-Čech compactification*. βX is characterized among compactifications of X by the fact that X is C^*-embedded in it, i.e. each bounded continuous real-valued function on X can be continuously extended to βX. A dense subspace of an extremally disconnected space is C^*-embedded in it, so each compact extremally disconnected space is the Stone-Čech compactification of each of its dense subspaces. (See 1H and 6M of [GJ]).

A Tychonoff space is *real-compact* if it is homeomorphic to a closed subspace of a power of the space of real numbers. Each Tychonoff space X can be densely embedded in a real-compact space νX called its *Hewitt real-compactification*. νX is characterized among real-compactifications of X by the fact that X is C-embedded in νX, i.e. each real-valued continuous function on X can be extended continuously to X. For such X one has $X \subseteq \nu X \subset \beta X$, and so $\beta(\nu X) = \beta X$. See [GJ] and [WAL] for more details.

1.3. Stone spaces of Boolean algebras

Let L be a lattice with 0 and 1. A subset F of L is called a *filter* if $0 \notin F$, $a \leq b$ and $a \in F$ implies $b \in F$, and if $a,b \in F$ implies $a \wedge b \in F$. An *ultrafilter* is a filter not properly contained on any other filter. A *filter base* is a subset S of L such that $0 \notin S$ and if $s_1, s_2 \in S$ then $s_3 \leq s_1 \wedge s_2$ for some $s_3 \in S$. By Zorn's lemma every filter base is contained in some ultrafilter.

If B is a Boolean algebra, let S(B) denote the set of ultrafilters on B. If $b \in B$, let $\lambda(b) = \{\alpha \in S(B) : b \in \alpha\}$. Then $\{\lambda(b) : b \in B\}$ is a base for a compact Hausdorff zero-dimensional topology τ on S(B). The space $(S(B),\tau)$ is called the *Stone space* of B. The map $b \to \lambda(b)$ is a Boolean algebra isomorphism from B onto the Boolean algebra of clopen subsets of S(B). This construction is due to STONE [ST_1]; a detailed discussion appears in [WAL].

A subset A of a space X is *regular closed* if $A = c\ell_X(\mathrm{int}_X A)$. The set $R(X)$ of all regular closed subsets of X, when partially ordered by inclusion, is a complete Boolean algebra. If $\{A_i : i \in I\} \subseteq R(X)$, then $\vee_{i \in I} A_i =$

$c\ell_X[\cup_{i\in I} A_i]$, $\wedge_{i\in I} A_i = c\ell_X \mathrm{int}_X[\cap_{i\in I} A_i]$, and $A_i' = c\ell_X(X\backslash A_i)$. Since $\mathcal{R}(X)$ is complete, its Stone space is extremally disconnected (see Theorem 2.5 of [WAL]). The complement of a regular closed set is called *regular open*; the collection $\mathcal{RO}(X)$ of regular open sets of X, partially ordered by inclusion, is a Boolean algebra and the map $A \to \mathrm{int}_X A$ is a Boolean algebra isomorphism from $\mathcal{R}(X)$ onto $\mathcal{RO}(X)$.

1.4. Semi-regularization of a topology

A space is *semi-regular* if its regular open sets form a base for its topology. If (X,τ) is a space, the *semi-regularization* of this space is the space (X,τ^*), where τ^* is the topology for which $\{\mathrm{int}_\tau c\ell_\tau V: V \in \tau\}$ is a base. τ^* is a Hausdorff topology (because τ is), and (X,τ^*) is a semi-regular space. It is routine to verify that τ is extremally disconnected iff τ^* is.

1.5. Perfect maps and θ-continuous maps.

Let X and Y be spaces. A map $f: X \to Y$ is *perfect* if it is closed and compact (i.e. point inverses are compact subsets of X). We do not require perfect maps to be continuous or onto. A map $f: X \to Y$ is *irreducible* if $f[X] = Y$ and if, for each proper closed subset A of X, $f[A] \neq Y$. An easy application of Zorn's lemma shows that if f is a compact map from X onto Y, there is a closed subset A of X such that $f|A$ is irreducible (see Lemma 5 of [STR]). If f is perfect, $f|A$ is also perfect. If $f: X \to Y$ is irreducible and continuous and if S is a dense subset of Y, then $f^{\leftarrow}[S]$ is a dense subset of X.

A map $f: X \to Y$ is *θ-continuous at a point* x of X if, whenever U is open in Y and $f(x) \in U$, there is an open set V of X such that $x \in X$ and $f[c\ell_X V] \subseteq c\ell_Y U$. A function is *θ-continuous* if it is θ-continuous at each point of its domain. The concept of θ-continuity was introduced by FOMIN [FO]; detailed discussions of it appear in [I1] and [PV]. A θ-continuous function with regular range is continuous.

The following result will be fundamental in our later work. A special case of it appears in 2.3 of [GI]; the more general formulation given below appears implicitly in Theorem 11 of [IF].

THEOREM. *An irreducible closed θ-continuous function from a space onto an extremally disconnected space is one-to-one. If the function is continuous, it is a homeomorphism.*

1.6. H-closed spaces and the Katětov extension

A space X is called *H-closed* if, whenever Y is a space and X is a subspace of Y, X is a closed subset of Y. (We re-emphasize that all hypothesized spaces are Hausdorff.) A space X is H-closed iff, whenever $\mathcal{C}$ is an open cover of X, there is a finite subcollection $\mathcal{F}$ of $\mathcal{C}$ such that $\cup\{F: F \in \mathcal{F}\}$ is dense in X.

An *open ultrafilter* on a space X is an ultrafilter on the lattice of open subsets of X. An open ultrafilter μ of X *converges* to a point x of X if $\cap\{c\ell_X U: U \in \mu\} \supseteq \{x\}$. An open ultrafilter either converges to exactly one point, or to no point at all; in the latter case we say it is *non-convergent*. A space X is H-closed iff each open ultrafilter on X converges.

If (X,τ) is a space let $\kappa X = X \cup \{\mu: \mu$ is a non-convergent open ultrafilter on $X\}$. Let $\tau' = \tau \cup \{V \subset \kappa X:$ if $\mu \in (\kappa X\backslash X) \cap V$, then $V \cap X \in \mu\}$. Then τ' is an H-closed Hausdorff topology on κX, and X is a dense open subspace of κX. If T is any other H-closed space that contains X as a dense subspace, the embedding $i: X \to T$ can be extended continuously to κX.

κX is called the *Katetov H-closed extension* of X. It was first constructed by KATETOV [KA]; a detailed discussion of its properties, and of the properties of H-closed spaces mentioned above, may be found in [PV]. We note that κX is extremally disconnected iff X is.

A *p-cover* of a space Y is an open cover of Y that has a finite subcollection whose union is dense in Y. A function $f: X \to Y$ is a *p-map* if it is continuous and if $\{f^{\leftarrow}[C]: C \in \mathcal{C}\}$ is a p-cover of X whenever $\mathcal{C}$ is a p-cover of Y. HARRIS [HAR] introduced p-maps and proved that a continuous function $f: X \to Y$ can be extended continuously to a function $f^{\kappa}: \kappa X \to \kappa Y$ iff f is a p-map.

2. THE CONSTRUCTION OF THE ABSOLUTE

The theory of the absolutes of Hausdorff spaces had its beginnings in 1937 with the publication of Stone's seminal papers on Boolean algebras and compact zero-dimensional spaces ([ST_1], [ST_2]). However, the modern theory begins in 1958 with the publication of [GL]. In this paper Gleason showed that associated with each compact space X there is an essentially unique extremally disconnected compact space EX which can be mapped onto X by an irreducible continuous function k_X. Explicitly, EX is the Stone space of the Boolean algebra $\mathcal{R}(X)$, and k_X maps each point of EX - i.e. each ultrafilter on $\mathcal{R}(X)$ - to the unique point of X to which it converges. Thus if $\alpha \in EX$, $k_X(\alpha) = \cap\{A \in \mathcal{R}(X): A \in \alpha\}$. If $f: K \to X$ is another irreducible continuous surjection from a compact extremally disconnected space K, Gleason shows there is a homeomorphism $h: EX \to K$ such that $k_X = f \circ h$. Thus the pair (EX, k_X) is "unique up to homeomorphism".

During the next six years Gleason's construction of EX was extended, for a variety of spaces and by a variety of methods, by a number of authors, notably ILIADIS ([IL]), PONOMAREV ([PO_1], [PO_2]), FLACHSMEYER ([FL]). In each case the author associates with each space X drawn from a certain class an extremally disconnected regular (and therefore Tychonoff) space EX that can be mapped onto X by a perfect irreducible θ-continuous mapping k_X; EX is unique (up to homeomorphism) with respect to these properties. In [PO_1] PONOMAREV called EX the *absolute* of X, and this term has since come into widespread use.

In [PO_1] and [PO_2] PONOMAREV constructs EX as the inverse limit of an inverse system $\mathcal{S}$ of spaces. The individual spaces of $\mathcal{S}$ are finite covers $\mathcal{C}$ of X with the property that if C_1 and C_2 are distinct members of $\mathcal{C}$, then $\text{int}_X(c\ell_X C_1 \cap c\ell_X C_2) = \emptyset$; $\mathcal{C}$ is given the discrete topology, and $\mathcal{S}$ is partially ordered by refinement, with the bonding maps induced in the natural way by the refinement relationship. In [PO_1] PONOMAREV constructs the absolute of a paracompact space in this way, and shows that it is paracompact; in [PO_2] he constructs the absolute of an arbitrary (Hausdorff) space. In each case he constructs the map k_X discussed above, and verifies that it is perfect, irreducible, and θ-continuous.

In [IL], ILIADIS topologies the set S of all convergent open ultrafilters (which he calls "maximal centred systems of open sets") on a space X as follows. If U is open in X, let $O(U) = \{\alpha \in S: U \in \alpha\}$; then $\{O(U):$ U open in X$\}$ is a base for a topology τ on S, and (S,τ) is a regular extremally disconnected (Hausdorff) space. The map $k_X: S \to X$ defined by $k_X(\alpha) = \cap\{c\ell_X U: U \in \alpha\}$ is an irreducible perfect θ-continuous surjection. Thus (S,τ) is the absolute EX of X. Iliadis verifies the uniqueness of EX and notes that k_X is continuous iff X is regular.

Let X be an arbitrary space. In [FL] FLACHSMEYER constructs EX by a method that closely parallels Gleason's original construction. The underlying set for EX in this construction is the set of convergent ultrafilters on the Boolean algebra $\mathcal{R}O(X)$; thus $EX = \{\alpha \in S(\mathcal{R}O(X)): \cap\{c\ell U: U \in \alpha\} \neq \emptyset\}$, equipped with the subspace topology induced on it by $S(\mathcal{R}O(X))$. Then EX is regular extremally disconnected space. The map $\alpha \to \cap\{c\ell_X U: U \in \alpha\}$ turns out to be a well-defined perfect irreducible θ-continuous mapping.

As $\mathcal{R}(X)$ is isomorphic to $\mathcal{R}O(X)$, one can obviously use $\mathcal{R}(X)$ in Flachsmeyer's construction. As the map $\alpha \to \{\mathrm{int}_X c\ell_X U: U \in \alpha\}$ is a bijection from the convergent open ultrafilters on X to the convergent ultrafilters on $\mathcal{R}O(X)$, it is easy to set up a correspondence between Iliadis and Flachsmeyer's constructions.

Each of the above-mentioned authors notes that if X is completely regular, then $E(\beta X) = \beta(EX)$. To see this consider the subspace $k_{\beta X}^{\leftarrow}[X]$ of $E(\beta X)$. As $k_{\beta X}$ is irreducible, $k_{\beta X}^{\leftarrow}[\beta X]$ is a dense subspace of $E(\beta X)$ (see 1.5) and thus extremally disconnected (see 1.2) and C^*-embedded in $E(\beta X)$. Thus $\beta(k_{\beta X}^{\leftarrow}[X]) = E(\beta X)$. But it is easily verified that $k_{\beta X} \mid k_{\beta X}^{\leftarrow}[X]$ is a perfect irreducible continuous surjection from $k_{\beta X}^{\leftarrow}[X]$ onto X; thus (up to homeomorphism) $k_{\beta X}^{\leftarrow}[X]$ is EX, $k_{\beta X} \mid k_{\beta X}^{\leftarrow}[X]$ is k_X, and $\beta(EX) = E(\beta X)$.

In addition to the papers cited above, we note three other papers concerning the construction of absolutes that appeared in the late 1950's and 1960's. In [RA] RAINWATER duplicated the work done by GLeason [GL], but used a quite different (and ingenious) method of construction. The survey paper by ILIADIS and FOMIN [IF] gives an overview of the work done on absolutes up to 1966, as well as a comprehensive survey of the theory of H-closed spaces as it existed at that time. In [STR] STRAUSS essentially rediscovered work done by Iliadis, Ponomarev and FLACHSMEYER in the case in which the original space is regular.

In order to have a fixed description of EX with which to work, we now give a detailed construction of EX to which we will adhere for the remainder of this article. Our approach is essentially that of FLACHSMEYER [FL] and STRAUSS [STR].

2.1.

THEOREM. *Let* X *be a space. Then:*

(1) *There exists a regular extremally disconnected space* EX *and a perfect irreducible* θ*-continuous map* k_X *from* EX *onto* X.

(2) k_X *is continuous iff* X *is regular.*

PROOF. Let S denote the Stone space of $\mathcal{R}(X)$ and let $EX = \{\alpha \in S: \cap\{A: A \in \alpha\} \neq \emptyset\}$, with the subspace topology induced by S. Then EX is a dense subspace of S; to prove this we must show that if $\emptyset \neq A \in \mathcal{R}(X)$, then $\lambda(A) \cap EX \neq \emptyset$ (see 1.3). Let $x \in \mathrm{int}_X A$ and put $\eta(x) = \{B \in \mathcal{R}(X): x \in \mathrm{int}_X B\}$. Then $\eta(x)$ is a filter on $\mathcal{R}(X)$, so there is an ultrafilter α on $\mathcal{R}(X)$ that contains $\eta(x)$, and hence A. Thus $\alpha \in \lambda(A)$. If $B \in \mathcal{R}(X)$ and $x \notin B$, then $c\ell_X(X\backslash B) = B' \in \eta(x)$, so $B \notin \alpha$. It follows that $\cap\{B: B \in \alpha\} \subseteq \{x\}$, and as $\eta(x) \subset \alpha$ and X is Hausdorff, this implies that $\cap\{B: B \in \alpha\} = \{x\}$. Thus $\alpha \in \lambda(A) \cap EX$ and EX is dense in S. Now S is extremally disconnected (see 1.3), so its dense subspace EX is also extremally disconnected (see 1.2).

If $\alpha \in EX$, then we claim that $\cap\{A: A \in \alpha\}$ contains exactly one element. By definition of EX this intersection contains at least one element. If x_0 and x_1 are distinct points of X, find disjoint open sets U and V of X such that $x_0 \in U$ and $x_1 \in V$. Then $c\ell_X U \wedge c\ell_X V = \emptyset$, so not both $c\ell_X U$ and $c\ell_X V$ are in α. If $c\ell_X U \notin \alpha$, the maximality of α implies that $(c\ell_X U)' \in \alpha$; i.e. $c\ell_X(X - c\ell_X U) \in \alpha$. But $U \cap c\ell_X(X - c\ell_X U) = \emptyset$ so $x_0 \notin \cap\{A: A \in \alpha\}$ and our claim is verified.

If $\alpha \in EX$, define $k_X(\alpha)$ to be the unique point in $\cap\{A: A \in \alpha\}$. By the preceding paragraph k_X is well-defined. If $x \in X$ the argument of the paragraph before last shows that there is an element α of EX that contains $\eta(x)$, and so $k_X(\alpha) = x$. Thus k_X maps EX onto X.

Next note that if $A \in \mathcal{R}(X)$, then $k_X[EX \cap \lambda(A)] = A$. To prove this, first suppose that $x \in A$. Then $\{A\} \cup \eta(x)$ generates a filter base on $\mathcal{R}(X)$, and hence is contained in an ultrafilter α on $\mathcal{R}(X)$. As above, one shows that $\alpha \in EX$ and $k_X(\alpha) = x$. Since $A \in \alpha$, $\alpha \in \lambda(A)$. Thus $A \subseteq k_X[EX \cap \lambda(A)]$. Conversely, if $\alpha \in EX \cap \lambda(A)$ then $A \in \alpha$ so $k_X(\alpha) \in A$ by definition of k_X.

Let $x \in X$ and let $\{A_i: i \in I\}$ be a subfamily of $\mathcal{R}(X)$ such that $k_X^{\leftarrow}(x)\backslash\cup\{\lambda(A_i): i \in F\} \neq \emptyset$ for each finite subset F of I. Let $\mathcal{S} = \{(\vee\{A_i: i \in F\})': F \text{ is a finite subset of } I\}$. Then $\mathcal{S}$ is a subfamily of $\mathcal{R}(X)$ closed under finite meets, and by hypothesis $x \in S$ for each $S \in \mathcal{S}$.

Thus $\mathcal{S} \cup \eta(x)$ generates a filter base on $\mathcal{R}(X)$, so there is an ultrafilter on X containing both $\mathcal{S}$ and $\eta(x)$. Thus $\alpha \in k_X^{\leftarrow}(x)$ and $A_i \notin \alpha$ for each $i \in I$. Thus $\alpha \in EX\backslash\cup\{\lambda(A_i): i \in I\}$. It follows that $k_X^{\leftarrow}(x)$ is a compact subset of EX, and so k_X is a compact map.

Now let F be a closed subset of EX, and suppose $x \in X\backslash k_X[F]$. Then the compact set $k_X^{\leftarrow}(x)$ is disjoint from the closed set F, so there exists $B \in \mathcal{R}(X)$ such that $k_X^{\leftarrow}(x) \subseteq \lambda(B) \subseteq EX\backslash F$. Then $x \in \mathrm{int}_X B$; for if not then $x \in c\ell_X(X\backslash B) = B'$, and $\{B'\} \cup \eta(x)$ would then generate a filter base and thus be contained in some ultrafilter γ. Evidently $\gamma \in k_X^{\leftarrow}(x) \cap \lambda(B') = k_X^{\leftarrow}(x)\backslash\lambda(B)$, which would be a contradiction. Furthermore, if $y \in \mathrm{int}_X B$ then $\lambda(B') \cap k_X^{\leftarrow}(y) = \emptyset$, so $k_X^{\leftarrow}[\mathrm{int}_X B] \subset EX\backslash F$. Thus $\mathrm{int}_X B$ is a neighbourhood of x disjoint from $k_X[F]$, and so $k_X[F]$ is closed. Thus k_X is a closed map, and thus perfect.

If $\alpha \in EX$ and $k_X(\alpha) \in V$, where V is open in X, then $V \cap A \neq \emptyset$ for each $A \in \alpha$. By the maximality of α it follows that $c\ell_X V \in \alpha$, and so $\alpha \in \lambda(c\ell_X V)$. As seen earlier, $k_X[\lambda(c\ell_X V)] = c\ell_X V$, and it follows that k_X is θ-continuous.

If X is regular, then k_X has a regular range and thus is continuous (see 1.5). Conversely, if k_X is continuous, then X is the image of a regular space under a perfect map and thus is regular (see, for example, Lemma 3 of [STR]). Thus k_X is continuous.

Finally, if $\emptyset \neq A \in R(X)$, $k_X[EX\backslash\lambda(A)] = k_X[\lambda(A')] = A' \neq X$, so k_X is irreducible. □

We now prove that the absolute of a Hausdorff space is unique up to homeomorphism. Our proof is a modification of the proof presented in Theorem 11 of [IF].

<u>LEMMA 2.2</u>. *Let* $f\colon X \to Y$ *be an irreducible closed surjection. If* U *is a non-empty open set of* X *then* $\mathrm{int}_Y f[U] \neq \emptyset$.

<u>PROOF</u>. If $\mathrm{int}_Y f[U] = \emptyset$ then $Y = Y\backslash \mathrm{int}_Y f[U] = c\ell_Y[Y - f[U]]$. But $Y - f[U]$ is contained in $f[X\backslash U]$, which is a closed subset of Y. This contradicts the irreducibility of f. □

<u>LEMMA 2.3</u>. *Let* $f\colon X \to Y$ *be an irreducible closed* θ*-continuous surjection from* X *onto* Y. *Then* $B \to f[B]$ *is a Boolean algebra isomorphism from* $R(X)$ *onto* $R(Y)$.

<u>PROOF</u>. Let $B \in R(X)$ and suppose that $f[B] \backslash c\ell_Y \mathrm{int}_Y f[B] \neq \emptyset$. There exists $b \in B$ such that $f(b) \in Y\backslash c\ell_Y \mathrm{int}_Y f[B]$. As f is θ-continuous there is an open subset V of X such that $b \in V$ and $f[c\ell_X V] \subseteq c\ell_Y[Y - c\ell_Y \mathrm{int}_Y f[B]]$. Then as $(\mathrm{int}_Y B) \cap V \neq \emptyset$, by 2.2 $\mathrm{int}_Y f[B \cap V]$ is a non-empty open subset of Y contained in $f[V] \cap \mathrm{int}_Y f[B]$. This is a contradiction; thus $f[B] \subseteq c\ell_Y \mathrm{int}_Y f[B]$. As f is a closed map, this implies that $f[B] \in R(Y)$.

If $A \in R(Y)$, let $p \in f^{\leftarrow}[\mathrm{int}_Y A]$. As f is θ-continuous there is an open set V(p) of X such that $p \in V(p)$ and $f[c\ell_X V(p)] \subset A$. If $V(p)\backslash c\ell_X f^{\leftarrow}[\mathrm{int}_Y A] \neq \emptyset$, then by 2.2 $\mathrm{int}_Y f[V(p) - c\ell_X f^{\leftarrow}[\mathrm{int}_Y A]]$ is a non-empty open subset of Y contained in A and disjoint from $\text{int}_Y A$, which is a contradiction; thus $V(p) \subseteq c\ell_X f^{\leftarrow}[\text{int}_Y A]$. It follows that $c\ell_X f^{\leftarrow}[\text{int}_Y A] = c\ell_X[\cup\{V(p): p \in f^{\leftarrow}[\text{int}_Y A]\}] \in \mathcal{R}(X)$. Furthermore, $f[c\ell_X f^{\leftarrow}[\text{int}_Y A]] = A$; for as f is closed, $f[c\ell_X f^{\leftarrow}[\text{int}_Y A]] \supseteq A$. Conversely if $p \in X$ and $f(p) \notin A$, by θ-continuity there is an open set V of X such that $p \in V$ and $f[c\ell_X V] \subseteq c\ell_Y(Y - A) = Y - \text{int}_Y A$. Thus $V \cap f^{\leftarrow}[\text{int}_Y A] = \emptyset$, so $p \notin c\ell_X f^{\leftarrow}[\text{int}_Y A]$.

From the above remarks we infer that the correspondence $B \to f[B]$ maps $\mathcal{R}(X)$ onto $\mathcal{R}(Y)$. To show it is one-to-one, suppose B_1 and B_2 are distinct members of $\mathcal{R}(X)$. Without loss of generality assume that $(\text{int}_X B_1) - B_2 \neq \emptyset$. Then as f is irreducible, there exists $p \in Y - f[(X - \text{int}_X B_1) \cup B_2]$. Thus there exists $b \in \text{int}_X B_1$ such that $f(b) = p \notin f[B_2]$. It follows that $f[B_1] \neq f[B_2]$.

Obviously $f[B_1 \cup B_2] = f[B_1] \cup f[B_2]$, and from this it follows that $B \to f[B]$ is an order isomorphism from $\mathcal{R}(X)$ onto $\mathcal{R}(Y)$. Since a Boolean algebra is determined by its order structure, it follows that this correspondence is a Boolean algebra isomorphism. □

<u>THEOREM 2.4</u>. *Let* X *be a space. Let* Z *be a regular extremally disconnected space and let* $g: Z \to X$ *be a perfect irreducible* θ*-continuous surjection. Then there is a homeomorphism* $h: EX \to Z$ *such that* $g \circ h = k_X$.

<u>PROOF</u>.As Z is extremally disconnected, $\mathcal{R}(Z)$ is just the collection $\mathcal{B}(Z)$ of clopen sets of Z. By 2.3 $B \to g[B]$ is a Boolean algebra isomorphism from $\mathcal{B}(Z)$ onto $\mathcal{R}(X)$.

Let $\alpha \in EX$ and let $\alpha' = \{B \in \mathcal{B}(Z): g[B] \in \alpha\}$. By 2.3 α' is an ultrafilter of clopen sets of Z, so $\cap\{B: B \in \alpha'\}$ contains at most one point. Obviously $\{B \cap g^{\leftarrow}(k_X(\alpha)): B \in \alpha'\}$ has the finite intersection property, so as g is a compact map, $\cap\{B \cap g^{\leftarrow}(k_X(\alpha)): B \in \alpha'\} \neq \emptyset$. Let $h(\alpha)$ denote the unique point in this intersection; then h is a well-defined map from EX to Z and $g \circ h = k_X$.

If $z \in Z$ let $\alpha(z) = \{g[B]: B \in \mathcal{B}(Z) \text{ and } z \in B\}$. Now $\{B \in \mathcal{B}(Z): z \in B\}$ is an ultrafilter on $\mathcal{B}(Z)$, so by 2.3, $\alpha(z)$ is an ultrafilter on $\mathcal{B}(X)$ converging to $g(z)$. Evidently $h(\alpha(z)) = z$, so h maps EX onto Z.

If $\{A_i: i \in I\} \subset \mathcal{R}(X)$, one can verify directly that $h[\cap_{i\in I}\lambda(A_i) \cap EX] = \cap_{i\in I} c\ell_Z g^{\leftarrow}[\text{int}_X A_i]$, and so h is a closed map. As k_X is irreducible and g is a surjection, it follows that h is irreducible. If $\alpha \in EX$ and $h(\alpha) \in B \in \mathcal{B}(Z)$, then $\alpha \in \lambda(g[B])$ and $h[\lambda(g[B])] \subset B$; thus h is continuous. Since distinct ultrafilters on $\mathcal{R}(X)$ correspond to distinct ultrafilters on $\mathcal{B}(Z)$, it follows that h is one-to-one. Hence h is a homeomorphism. □

In the mid-1970s ULJAN'OV [U] and ŠAPIRO [S] constructed what they called the "absolute" of an arbitrary (not necessarily Hausdorff) topological space. This "absolute" is not a regular Hausdorff space in general, and hence does not conform to our use of the term. Since it is a direct generalization of the "projective cover" of a Hausdorff space constructed by BANASCHEWSKI [BA_2], we postpone our discussion of it until after our discussion of Banaschewski's work in §3.

3. ABSOLUTES AND PROJECTIVE COVERS; CATEGORIAL CONSIDERATIONS

Let $\mathcal{C}$ be a category. An object X of $\mathcal{C}$ is said to be *projective* (in $\mathcal{C}$) if, whenever Y and Z are ojects of $\mathcal{C}$, $g: X \to Z$ is a morphism in $\mathcal{C}$, and $f: Y \to Z$ is an epimorphism in $\mathcal{C}$, there is a morphism $h: X \to Y$ of $\mathcal{C}$ such that $g = f \circ h$ (see, for example, [HS], of Chapter 10 of [WAL]). Let $\mathcal{P}$ be a topological property, and let $\mathcal{C}$ be a category whose objects are all Hausdorff spaces with $\mathcal{P}$, and whose morphisms are some subclass of the class of θ-continuous maps. Let C be an object of $\mathcal{C}$. A pair (C',k) is called a *projective cover* of C (in $\mathcal{C}$) if C' is a projective object of $\mathcal{C}$, k is a morphism of $\mathcal{C}$, and k is an irreducible map from C' onto C.

Projective objects and covers in topological categories have been the subject of much study, beginning with Gleason. It happens that for a wide class of topological categories the projective objects are precisely the extremally disconnected objects, and the projective cover of an object X is either the pair (EX,k_X) discussed in §2 or the pair (PX,k_X), where PX is the space whose underlying set is the underlying set of EX, and whose topology is generated by the topology EX together with the family $\{k_X^{\leftarrow}[V]: V$ open in $X\}$. In this section we develop the theory of projective objects and covers of such categories, and relate it to the theory of absolutes developed in §2. Most of the results in this section are due to GLEASON [GL], FLASCHMEYER [FL], STRAUSS [STR], BANASCHEWSKI [BA_2], and MIODUSZEWSKI and RUDOLF [MR].

The first results concerning projective objects and covers were obtained by GLEASON [GL], who proved the following three theorems.

THEOREM 3.1. *Let $\mathcal{C}$ be a category satisfying the following conditions:*

(a) *All objects are Hausdorff spaces and all morphisms are continuous maps.*

(b) *All homeomorphisms are morphisms, and if X is an object of $\mathcal{C}$ and Y is homeomorphic to X, then Y is an object of $\mathcal{C}$.*

(c) *The disjoint union (coproduct) of finitely many copies of an object* A *of* $\mathcal{C}$ *is an object of* $\mathcal{C}$, *and the projection from this disjoint union onto* A *is a morphism of* $\mathcal{C}$.

(d) *If* A *is an object of* $\mathcal{C}$ *and* $B \in \mathcal{R}(A)$, *then* B *is an object of* $\mathcal{C}$ *and the embedding map* $i: B \to A$ *is a morphism of* $\mathcal{C}$.

Then, every projective object in $\mathcal{C}$ *is extremally disconnected.*

(In fact Gleason assumed that each closed subspace of an object of $\mathcal{C}$ is in $\mathcal{C}$, but his proof works under the weaker assumption (d) above.)

THEOREM 3.2. *In the category* $\mathcal{K}$ *of compact spaces and continuous maps, the projective objects are precisely the extremally disconnected spaces.*

THEOREM 3.3. *Let* X *be a compact space. Then* (EX,k_X) *is a projective cover of* X, *and is "unique up to homeomorphism" in the following sense: if* (Y,f) *is another projective cover of* X *in* $\mathcal{K}$, *there is a homoemorphism* $h: Y \to EX$ *such that* $k_X \circ h = f$.

The topological categories in which each object has a projective cover turn out to be categories in which the morphisms are perfect maps of some description. This is in fact true of the results of Gleason stated above, since a continuous map between compact spaces is of necessity perfect. Gleason's results were generalized to the category $\mathcal{R}$ of regular spaces and perfect continuous maps by FLACHSMEYER [FL], and later STRAUSS [STR], who essentially proved the following result.

THEOREM 3.4. *The projective objects of* $\mathcal{R}$ *are precisely the regular extremally disconnected spaces. If* X *is a regular space then* (EX,k_X) *is a projective cover of* X, *and this projective cover is unique up to homeomorphism (in the sense of* 3.3*). (Hence we shall speak of* <u>the</u> *projective cover in* $\mathcal{R}$ *of an object of* $\mathcal{R}$*).*

In proving 3.4 one implicitly uses the fact that in $\mathcal{R}$, the epimorphisms are precisely the onto maps; a slight adaptation of the proof of 10.18 of [WAL] shows this to be true.

The question of the existence of projective covers in well-behaved full subcategories of $\mathcal{R}$ is settled by the following theorem and its corollary.

THEOREM 3.5. *Let* $\mathcal{P}$ *be a topological property and let* $\mathcal{R}_{\mathcal{P}}$ *be the category of regular spaces with* $\mathcal{P}$ *and perfect continuous maps. The following are equivalent:*

(1) *Each object in* R_P *has a projective cover, and each projective object in* R_P *is extremally disconnected.*

(2) *If* X *is a regular space with* P, *then* EX *has* P.

PROOF. (1) ⇒ (2): Let (Y,f) be a projective cover of X in R_P. By hypothesis Y is extremally disconnected, so by 3.4 (Y,f) is a projective cover of X in R. Again by 3.4, Y is homemorphic to EX. Thus EX has P.

(2) ⇒ (1): Since EX is projective in R, and since each object of R_P is an object of R, it follows that EX is projective in R_P. Thus (EX,k_X) is a projective cover of X in R_P. If Y is a projective object in R_P, since EX is an object of R_P there is a perfect map h: Y → EX such that $k_X \circ h = 1_Y$ (the identity on Y). Obviously h is irreducible and onto, so by 1.5 h is a homeomorphism. Hence Y is extremally disconnected. □

COROLLARY 3.6. *Let* P *be a topological property of regular spaces with the following properties: the free union of two regular spaces with* P *has* P, *and regular closed subsets of spaces with* P *have* P. *Let* R_P *be as above. The following are equivalent.*

(1) *Each object in* R_P *has a projective cover in* R_P.

(2) *If* X *is regular and has* P, *then* EX *has* P.

PROOF. This follows from 3.1 and 3.5.

Let P be a topological property with the following properties: closed subspaces of spaces with P have P, and if X has P and K is compact then X × K has P. If X is Tychonoff and has P, and if Y is a pre-image of X under a perfect map, then Y has P (see Prop. 2 of [HvdS]). Thus if J_P is the category of Tychonoff spaces with P and perfect maps, by 3.6 each object in J_P has a projective cover in J_P.

EXAMPLE 3.7.

(a) The category PS of pseudocompact Tychonoff spaces and perfect maps is not closed-hereditary, but as EX is pseudocompact (and Tychonoff) whenever X is (see 2.5 of [WO_3]), each object in PS has a projective cover in PS, and the projective objects in PS are precisely the extremally disconnected ones.

(b) The category η of normal spaces and perfect maps satisfies the hypotheses on P given in 3.6; however there are normal spaces whose absolutes are not normal (see §6). Thus all projective objects in η are extremally disconnected, but some objects of η do not have projective covers in η. □

We now consider projective objects and projective covers for categories of Hausdorff spaces that include non-regular spaces among their objects. Since the perfect continuous image of a regular space is regular (see, for example, [STR]) if we consider categories $\mathcal{C}$ for which the morphisms are perfect continuous maps, then (EX,k_X) cannot be the projective cover of a non-regular object X of $\mathcal{C}$. Thus two lines of investigation appear; one can investigate categories $\mathcal{C}$ for which (EX,k_X) will be the projective cover of X in $\mathcal{C}$, or one can attempt to characterize the projective objects and covers in categories in which the morphisms are perfect continuous maps. The first approach leads to the consideration of categories in which the morphisms are perfect θ-continuous maps, and is briefly considered towards the end of this section. The second approach seems more fruitful; a theory of projective objects and covers for the category $\mathcal{H}$ of Hausdorff spaces and perfect continuous maps has been developed by FLACHSMEYER [FL], BANASCHEWSKI [BA_2], and MIODUSZEWSKI and RUDOLF [MR]. We present this theory now. We include the proof of the next result as we wish to modify it later.

THEOREM 3.8. [FL]. *The projective objects in the category $\mathcal{H}$ of Hausdorff spaces and perfect continuous maps are precisely the extremally disconnected spaces.*

PROOF. By 3.1 each projective object in $\mathcal{H}$ is extremally disconnected. Conversely, let E be an extremally disconnected space, let $g\colon E \to X$ be perfect and continuous, and let $f\colon Y \to X$ be a perfect continuous surjection. Let $P = \{(e,y) \in E \times Y\colon g(e) = f(y)\}$ and let $\phi = \pi_E|P$ (π_E is the projection map from $E \times Y$ onto E). Obviously ϕ is continuous, and is a surjection since f is. If $e \in E$, then $\phi^{\leftarrow}(e) = \{e\} \times f^{\leftarrow}(g(e))$; thus ϕ is a compact map since f is. If F is a closed subset of P and $e \in E - \phi[F]$, then F and $\{e\} \times f^{\leftarrow}(g(e))$ are disjoint closed sets of P, one of which is compact. Hence there are open sets $D_1,\ldots,D_n$ of E and $W_1,\ldots,W_n$ of Y such that $e \in \cap_{i=1}^n D_i = D$ and $\{e\} \times f^{\leftarrow}(g(e)) \subset \cup_{i=1}^n D_i \times W_i \subseteq (E \times Y)\backslash F$. Let $W = \cup_{i=1}^n W_i$. Then $f^{\leftarrow}(g(e)) \subseteq W$, so $g(e) \in X - f[Y-W]$, which is open in X as f is a closed map. Let $U = D \cap g^{\leftarrow}[X - f[Y-W]]$. Then $e \in U$ and $U \cap \phi[F] = \emptyset$. Thus ϕ is a closed map, and thus is a perfect surjection onto E. Similarly one proves that $\psi = \pi_Y|P$ is a perfect map from P onto Y.

Let A be a closed subset of P such that $\phi|A$ is an irreducible map onto E (see 1.5). Then $\phi|A$ is a homeomorphism (see 1.5). Let $h = \psi \circ (\phi|A)^{\leftarrow}$. Then h is a perfect continuous map from E to Y and $g = f \circ h$. Hence E is projective in $\mathcal{H}$. □

BANASCHEWSKI ($[BA_1]$, $[BA_2]$) defined the concept of a projective cover in a purely category-theoretic setting and obtained a number of results containing the existence and uniqueness of projective covers in certain categories. He then applied these results to categories whose morphisms are perfect continuous maps and whose objects are certain subclasses of the class of Hausdorff spaces. Similar results were obtained by more purely topolgical means by MIODUSZEWSKI and RUDOLF [MR]. The next few theorems appear in $[BA_2]$ and/or [MR].

THEOREM 3.9. *Each object of the cateogry $\mathcal{H}$ has a projective cover in $\mathcal{H}$, and this projective cover is unique up to homeomorphism (in the sense of* 3.3*).*

SKETCH OF THE PROOF. Let X be an object of $\mathcal{H}$ and (EX,k_X) its absolute. Let $\mathcal{B} = \{k_X^{\leftarrow}[V] \cap \lambda(A)\colon A \in \mathcal{R}(X)$ and V open in X$\}$. Then $\mathcal{B}$ is a base for a topology τ on the underlying set of EX. Let PX denote this set, topologized by τ; thus PX is the underlying set of EX, topologized with the smallest topology that both includes the topology of EX and makes k_X continuous. We claim that (PX,k_X) is a projective cover of X in $\mathcal{H}$.

Obviously PX is Hausdorff and $k_X\colon PX \to X$ is a continuous surjection. It is easy to verify that $k_X\colon PX \to X$ is compact and irreducible. To show it is closed, note that a closed subset F of PX has the form $\cap\{[PX \cap \lambda(A_i)] \cup k_X^{\leftarrow}[X\backslash V_i]\colon i \in I\}$, where $A_i \in \mathcal{R}(X)$ and V_i is open in X. It is easy to verify that $k_X[F] \subseteq \cap\{A_i \cup (X\backslash V_i)\colon i \in I\}$; conversely, if $x_0 \in \cap\{A_i \cup (X\backslash V_i)\colon i \in I\}$, let $I_0 = \{i \in I\colon x_0 \in A_i\}$ and $I_1 = \{i \in I\colon x_0 \in X\backslash V_i\}$. Arguing as in 2.1, one shows that there exists $\alpha \in k_X^{\leftarrow}(x_0) \cap \cap\{\lambda(A_i)\colon i \in I_0\}$, so $\alpha \in \cap\{k_X^{\leftarrow}[X\backslash V_i]\colon i \in I_1\}$. Thus $k_X[F] = \cap\{A_i \cup (X\backslash V_i)\colon i \in I\}$ and so $k_X\colon PX \to X$ is closed, and hence perfect.

It is a consequence of 2.3 that the semi-regularization of (PX,τ) is just the absolute EX of X. Hence by 1.4 and 3.8 (PX,k_X) is a projective cover of X in $\mathcal{H}$. The uniqueness of the projective cover can be deduced from 3.8 as follows: if (Y,f) is a projective cover of X in $\mathcal{H}$, by 3.8 there is a perfect continuous map h from PX onto Y such that $k_X = f \circ h$. As $k_X\colon PX \to X$ is irreducible, so is h. Hence by 1.5 h is a homeomorphism. □

Having characterized projective objects and projective covers in $\mathcal{H}$, we consider full subcategories of $\mathcal{H}$. The following theorem is due to BANASCHEWSKI (see Proposition 3 of $[BA_2]$); in our proof, as in his, the projective cover is constructed as an inverse limit.

THEOREM 3.10. *Let $\mathcal{P}$ be a topological property of (Hausdorff) spaces with the following properties:*

(a) *Closed subspaces of spaces with $\mathcal{P}$ have $\mathcal{P}$.*

(b) *Products of spaces with $\mathcal{P}$ have $\mathcal{P}$.*

(c) *There exist Hausdorff spaces with $\mathcal{P}$ having more than one point.*

Let $\mathcal{H}_{\mathcal{P}}$ denote the category of Hausdorff spaces with $\mathcal{P}$, and perfect continuous maps. Then:

(1) *The projective objects of $\mathcal{H}_{\mathcal{P}}$ are precisely the extremally disconnected objects of $\mathcal{P}$.*

(2) *Each object in $\mathcal{H}_{\mathcal{P}}$ has a unique projective cover in $\mathcal{H}_{\mathcal{P}}$, which is identical to its projective cover in $\mathcal{H}$.*

PROOF. (1) By 3.1 and the hypotheses on $\mathcal{P}$, each projective object in $\mathcal{H}_{\mathcal{P}}$ is extremally disconnected. By 3.8 every extremally disconnected object in $\mathcal{H}_{\mathcal{P}}$ is projective.

(2) Uniqueness is proved as in 3.9. To prove existence, we prove that is X is a space with $\mathcal{P}$, then PX also has $\mathcal{P}$. Fix such an X and let $f_\alpha: X_\alpha \to X$ be a prefect irreducible continuous surjection. By 2.3 $\mathcal{R}(X_\alpha)$ and $\mathcal{R}(X)$ are isomorphic, so as X is Hausdorff, $|X_\alpha| \leq 2^{|\mathcal{R}(X)|}$. Let S be a set of cardinality $2^{|\mathcal{R}(X)|}$ containing X and let $\mathcal{S} = \{(Y,f): Y$ is a space with $\mathcal{P}$ whose underlying set is a subset of S and f is a perfect irreducible continuous surjection from Y onto X$\}$. Then $\mathcal{S}$ is a set (not a proper class), and if Y has $\mathcal{P}$ and $g: Y \to X$ is a perfect irreducible continuous surjection, there is a pair $(Z,k) \in \mathcal{S}$ and a homeomorphism $h: Z \to Y$ such that $k = g \circ h$.

A subfamily $\mathcal{F}$ of $\mathcal{S}$ is a *special set* if there is a well-ordering $(X_\alpha,f_\alpha)_{0\leq\alpha<\lambda}$ (where λ is some ordinal) such that: (1) $X_0 = X$. (2) If $\alpha < \beta < \lambda$ there is a perfect irreducible surjection $f_{\beta\alpha}: X_\beta \to X_\alpha$ such that $f_\alpha \circ f_{\alpha\beta} = f_\beta$, and $f_{\beta\alpha}$ is not a homeomorphism. Let σ denote the collection of special sets, and partially order σ as follows: $(X_\alpha,f_\alpha)_{0\leq\alpha<\lambda} \leq (X'_\alpha,f'_\alpha)_{0\leq\alpha<\lambda'}$, if $\lambda \leq \lambda'$, and $\alpha < \lambda$ implies $X_\alpha = X'_\alpha$ and $f_\alpha = f'_\alpha$. Then σ is inductive and there is a maximal special set $\mathcal{F}_0 = (X_\alpha,f_\alpha)_{0\leq\alpha<\lambda_0}$. Let $P = \pi\{X_\alpha: 0 \leq \alpha < \lambda_0\}$ and let $T = \{<x_\alpha> \in P: \alpha, \beta < \lambda$ implies $f_\alpha(x_\alpha) = f_\beta(x_\beta)\}$. The fact that X is Hausdorff and that each f_α is continuous implies that T is closed in P. Hence by hypothesis T has $\mathcal{P}$, and thus is an object of $\mathcal{H}_{\mathcal{P}}$.

Define $f: T \to X$ by: $f(<x_\alpha>) = f_\alpha(x_\alpha)$. By definition of T this is independent of α and hence f is well-defined. For each $\alpha \in I$ let $g_\alpha = \pi_\alpha|T$ (π_α is the α-th projection map on P). Thus $f = f_\alpha \circ g_\alpha$ for each α and so f is continuous. If $x_\alpha \in X_\alpha$, then $\overset{\leftarrow}{g_\alpha}(x_\alpha)$ is the non-empty compact space

$\pi\{f_\beta^{\leftarrow}(f_\alpha(x_\alpha)): \beta \in I\}$, so g_α is a compact surjection. Now suppose F is a closed subset of T and let $x_\alpha \in X_\alpha \backslash g_\alpha[F]$. Then $\{x_\alpha\} \times \pi\{f_\beta^{\leftarrow}(f_\alpha(x_\alpha)): \beta \in I\backslash\{\alpha\}\}$ is a compact subset of P disjoint from the closed subset F of P. Thus there is a finite family $(W_j)_{j=1}^n$ of basic open sets of P such that $\{x_\alpha\} \times \pi\{f_\beta^{\leftarrow}(f_\alpha(x_\alpha)): \beta \in I\backslash\{\alpha\}\} \subseteq \cup_{j=1}^n W_j \subseteq P\backslash F$. Let $W_j = \pi\{V_{j,\beta}: \beta \in I\}$ where, for each j, $V_{j,\beta} = X_\beta$ for all but finitely many β. Let $V = \cap_{j=1}^n V_{j,\alpha}$, let $T_j = X\backslash\cup\{f_\beta[X_\beta\backslash V_{j,\beta}]: \beta \in I\backslash\{\alpha\}\}$, and let $M = \cup_{j=1}^n f_\alpha^{\leftarrow}[T_j]$. Then $V \cap M$ is an open set of X_α containing x_α and disjoint from $g_\alpha[F]$. Thus g_α is a closed (and therefore perfect) surjection onto X_α.

Now let A be a closed subset of T such that $f|A$ is an irreducible surjection onto X. Then A has $\mathcal{P}$ by hypothesis. Suppose that U is open in A and that $c\ell_A U \cap c\ell_A(A - c\ell_A U) \neq \emptyset$. Let Y be the free union of the subspaces $c\ell_A U$ and $c\ell_A(A - c\ell_A U)$, and let $\phi: Y \to A$ be the natural surjection. Then ϕ is a perfect irreducible non-homeomorphic continuous map, so $f \circ \phi: Y \to X$ is also. Hence there exists $(Z,k) \in \mathcal{S}$ and a homeomorphism $h: Z \to Y$ such that $f \circ \phi \circ h = k$. Let $X_{\lambda_0} = Z$, $f_{\lambda_0} = k$, and $f_{\lambda_0\alpha} = g_\alpha \circ \phi \circ h$. Then $f_\alpha \circ f_{\lambda_0\alpha} = f_{\lambda_0}$, and $f_{\lambda_0\alpha}$ is not a homeomorphism (as ϕ isn't). Thus $\mathcal{F}_0 \cup \{(X_{\lambda_0}, f_{\alpha_0})\} \in \sigma$ and properly contains $\mathcal{F}_0$, contradicting the maximality of $\mathcal{F}_0$. Thus $c\ell_A U \cap c\ell_A(A - c\ell_A U) = \emptyset$.

Thus A is an extremally disconnected space, and thus projective in $\mathcal{H}$, and thus projective in $\mathcal{H}_{\mathcal{P}}$. As $f|A: A \to X$ is perfect and irreducible, it follows that $(A, f|A)$ is the projective cover of X in $\mathcal{H}_{\mathcal{P}}$, and also its projective cover in $\mathcal{H}$. □

As noted in prior to 2.2, if X is a completely regular space then $E(\beta X) = \beta(EX)$. It turns out that the Katětov H-closed extension is to PX as the Stone-Čech compactification is to EX. The next result, and its proof, appears in §4 of [MR].

THEOREM 3.11. *Let* X *be a space. Then* $\kappa(PX) = P(\kappa X)$.

PROOF. Let k_{PX} be the perfect continuous irreducible surjection from $P(\kappa X)$ onto κX, let $Y = k_{PX}^{\leftarrow}[X]$, and let $f = k_{PX}|Y$. As k_{PX} is irreducible and perfect, Y is a dense subspace of $P(\kappa X)$ (see 1.5) and thus extremally disconnected, as $P(\kappa X)$ is (see 1H of [GJ] and 1.6). It is straightforward to check that f is a perfect continuous irreducible map from Y onto X. It follows by 3.9 that $Y = PX$. As k_{PX} is perfect, $P(\kappa X)$ is H-closed. If $\alpha \in P(\kappa X)\backslash Y$, then $\{\{\alpha\} \cup k_{PX}^{\leftarrow}[V]: V \in k_{PX}^{\leftarrow}(\alpha)\}$ is a neighborhood base at α in $P(\kappa X)$. Hence $P(\kappa X) = \kappa Y$. □

We now consider projective objects in subcategories of the category θH of Hausdorff spaces and perfect θ-continuous maps. Such objects have not been fully characterized; however, we record some partial results. Lemma 3.12 below essentially appears in [BA_2] and [MR].

LEMMA 3.12. *Let P be a topological property such that if* X *is a space with* P, *then* EX *has* P. *Let* $\theta(H_P)$ *be the category of (Hausdorff) spaces with* P *and perfect* θ*-continuous maps. If* X *is a projective object in* $\theta(H_P)$, *then* X *is regular and extremally disconnected.*

PROOF. If X is a projective object in $\theta(H_P)$, then there exists a perfect θ-continuous map h: X → EX such that $k_X \circ h = 1_X$ (the identity map on X). As EX is regular, h is continuous. As k_X is irreducible, so is h; hence by 1.5 h is a homeomorphism. □

A space X is Urysohn if distinct points of X are contained in disjoint closed neighbourhoods. Let $\theta\mu$ denote the category of H-closed Urysohn spaces and perfect θ-continuous maps. The following lemma is easily proved; see III, §3 of [MR].

LEMMA 3.13. *If* f: X → Y *is a perfect* θ*-continuous irreducible surjection and* Y *is* H*-closed, then* X *is* H*-closed.*

THEOREM 3.14. *The projective objects in* $\theta\mu$ *are precisely the compact extremally disconnected spaces.*

PROOF. First note that each extremally disconnected space is Urysohn. Thus by 3.12 a projective object in $\theta\mu$ is H-closed, regular (and hence compact), and extremally disconnected.

Conversely, let K be a compact extremally disconnected space, let X and Z be H-closed Urysohn spaces, and let g: K → X and f: Z → X be perfect θ-continuous maps, with f being a surjection. Let A be a closed subspace of Z such that $f|A$ is irreducible (and perfect and θ-continuous). Then $f \circ k_A$: EA → Z is irreducible perfect, and θ-continuous, and EA is H-closed (by 3.14), regular, and thus compact.

Now mimic the proof of 3.8. Let $P = \{(e,y) \in EZ \times K\colon (f \circ k_A)(e) = g(y)\}$. Then P is closed in EA × K, for if $(a,b) \in (EA \times Y)\setminus P$, $(f \circ k_A)(a) \neq g(b)$ so as X is Urysohn there are open sets U and V of X such that $(f \circ k_A)(a) \in U$, $g(b) \in V$, and $c\ell_X U \cap c\ell_X V = \emptyset$. By θ-continuity there exist clopen sets B and C of EA and K respectively such that $a \in A$, $b \in B$, $(f \circ k_A)[A] \subset c\ell_X U$, and $g[B] \subset c\ell_X V$. Thus $(A \times B) \cap P = \emptyset$ and so P is closed.

As EA × K is compact, so is P. Hence the projection restrictions $\pi_K|P$ and $\pi_{EA}|P$ are perfect continuous maps; the former is a surjection as $f \circ k_A$ is. As in 3.8 find a closed subset F of P such that $\pi_K|F$ is a homeomorphism. Then $f \circ (k_A \circ \pi_{EA} \circ (\pi_K|F)^{\leftarrow}) = g$, and K is projective. □

There remain several open questions in this area; the following are typical.

PROBLEM 3.15. Chracterize the projective objects in (i) the category of Hausdorff spaces and perfect θ-continuous maps (ii) the category of H-closed spaces and perfect θ-continuous maps.

If X and Y are compact spaces and if $f: X \to Y$ is continuous, it follows from the projectivity of EX that there is a continuous map $\bar{f}: EX \to EY$ such that $k_Y \circ \bar{f} = f \circ k_X$. In [HJ], HENRIKSEN and JERISON prove that there is a *unique* mapping $\bar{f}: EX \to EY$ satisfying $k_Y \circ \bar{f} = f \circ k_X$ iff $\text{int}_X f^{\leftarrow}[c\ell_Y V] = \text{int}_X c\ell_X f^{\leftarrow}[V]$ for each $V \in RO(Y)$. Maps (not necessarily continuous) satisfying this condition are called HJ-*maps* in [MR]; if the condition holds for all open subsets V of Y (not just regular open sets), f is called a *skeletal map*. In [MR] categories in which the morphisms are skeltal maps are studied, and are related to the study of projective objects and covers.

We now consider "absolutes" of arbitrary (non-Hausdorff in general) topological spaces and continuous maps. These spaces are constructed and studied in [BL], [U], and [S]. Our description below follows that given in [PS] and [S], to which we refer the reader for further details.

DEFINITION 3.16. ([PS], [S]). A continuous function $f: X \to Y$ is *separated* if, whenever $y \in Y$ and x_1 and x_2 are distinct points of $f^{\leftarrow}(y)$, there are disjoint neighbourhoods of x_1 and x_2 in X.

Let X be a space (not necessarily Hausdorff), let EX denote the convergent ultrafilters on $\mathcal{R}(X)$, and let $\tilde{X} = \{(\alpha,x) \in EX \times X: x \in \cap\{A: A \in \alpha\}$, endowed with the subspace topology inherited from the product space EX × X. Let $k_X = \pi_X|\tilde{X}$, where π_X is the projection from EX × X onto X. Then $\tilde{X}$ turns out to be an extremally disconnected space and k_X is a separated perfect irreducible continuous surjection from $\tilde{X}$ onto X. Furthermore, the pair $(\tilde{X},k_X)$ is unique with respect to these properties. Šapiro calls $(\tilde{X},k_X)$ the *absolute* of X. It is easily seen that if $\tilde{X}$ is Hausdorff, then X is just the space PX constructed in 3.9.

If X and Y are arbitrary spaces and $f\colon X \to Y$ is continuous, Šapiro shows that there is a continuous function $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ such that $k_Y \circ \tilde{f} = f \circ k_X$. Šapiro calls such an $\tilde{f}$ an *absolute of* f. He proves that $\tilde{f}$ is compact, perfect onto, or separated iff f has the corresponding property, and that if f is a separated perfect irreducible surjection, then $\tilde{f}$ is a homeomorphism.

In general a function will have many absolutes. Extending the concept and theorem of Henriksen and Jerison quoted above, Šapiro says a continuous function $f\colon X \to Y$ *satisfies the condition* HJ if $X - \overset{\leftarrow}{f}[\mathrm{bd}_Y A]$ is dense in X for ecah $A \in \mathcal{R}(Y)$ ($\mathrm{bd}_Y A$ is the topological boundary of A). Such functions are also called C-maps.

He then proves that a function has a unique absolute iff it satisfies the condition HJ. Explicitly, in this case the function $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ is defined as follows: $\tilde{f}((\alpha,x)) = (\alpha',f(x))$, where $\alpha' = \{f[A]\colon A \in \alpha\}$.

4. ALGEBRAIC AND LATTICE-THEORETIC PROPERTIES OF EX

In this section we present some of the results linking rings and lattices of functions on a space X to rings and lattices of functions on EX. These questions lead naturally to a study of the relationships between the spaces $E(\nu X)$ and $\nu(EX)$ (see 1.2). Although these results are not particularly new they have not, to my knowledge, been collected in one place and decsribed systematically before now. Proofs are not included for reasons of space, and because the original sources are easily accessible and the proofs therein are clear and complete. Ther reader is referred to [GJ] for background material relevant to this section, and also to Chapter 4 of [WE] for a detailed study of the preservation of real-compactness by various types of mappings. Throughout this section, unless the contrary is explicitly stated, the word "space" means "completely regular Hausdorff space" (i.e. "Tychonoff space").

Let X be a space. Since $X \subset \nu X \subset \beta X$ and $E(\beta X) = \beta(EX)$ (see §2), it follows that $EX \subset E(\nu X) \subset \beta(EX)$. Since the inverse image of a real-compact space under a perfect map is real-compact (see [DY_1]), $E(\nu X)$ is real-compact. It follows that $\nu(EX) \subset E(\nu X)$; we shall be concerned with when this inclusion is in fact an equality.

We recall briefly the definition and properties of the Dedekind-MacNeille completion of a partially ordered set (henceforth abbreviated to "poset"). Recall that a lattice is *complete* if each of its subsets has a supremum and an infimum. If $(P,\leq)$ is a poset there is a complete lattice L(P) and an embedding $j: P \to L(P)$ with the following properties: (1) If $x,y \in P$ then $x \leq y$ iff $j(x) \leq j(y)$. (2) If $a \in L(P)$ then $a = \sup\{j(x): x \in P$ and $j(x) \leq a\} = \inf\{j(x): x \in P$ and $j(x) \geq a\}$. The pair $(L(P),j)$ is unique up to order isomorphism over S and is called the *Dedekind-MacNeille completion* of $(P,\leq)$. Now assume P has no greatest element or smallest element. If one removes 1 and 0 (the greatest and smallest elements of $L(P)$) from $L(P)$, one obtains the *conditional completion* of $L(P)$, denoted $L^*(P)$; if A is a subset of P, then $j[A]$ has a supremum (resp. infimum) in $L^*(P)$ iff A is bounded above (resp. below) in P. See [L] for details.

If X is a space, both $C(X)$ and $C^*(X)$ are lattice-ordered algebras, where all algebraic and order-theoretic operations are defined pointwise (see Chapter 1 of [GJ]), and neither has a greatest or smallest element. It is natural to ask whether $L^*(C(X))$ and $L^*(C^*(X))$ have "natural" representations as lattices of functions on some space, and whether there is an algebraic structure on these lattices compatible with the order-theoretic structure. Since $C^*(X) = C(\beta X)$, it suffices to consider only $C(X)$ in the ensuing discussion.

A *Φ-algebra* is a real archimedian lattice-ordered algebra with an identity element that is a weak-order unit. Evidently $C(X)$ is a Φ-algebra. In [J_1] JOHNSON shows that if A is a Φ-algebra, then so is $L^*(A)$; hence $L^*(C(X))$ does have an algebraic structure compatible with the order structure.

We now investigate under what circumstances $L^*(C(X))$ can be identified with $C(EX)$. To do this, we consider the lattice of normal upper semicontinuous functions on X. Let f be a real-valued function on a space X. Define

$$f^*(x) = \inf\{\sup\{f(y): y \in U\}: U \in \eta(x)\}$$

and

$$f_*(x) = \sup\{\inf\{f(y): y \in U\}: U \in \eta(x)\}$$

(here $\eta(x)$ denotes the set of neighbourhoods of x in X).

Properties of f^* and f_* are given in [DI] and [K]. In general f^* and f_* take values in $\mathbb{R} \cup \{+\infty,-\infty\}$, the two-point compactification of $\mathbb{R}$; they are both real-valued iff f is *locally bounded,* i.e. iff each point of X has a neighbouhood on which f is bounded. A real-valued function f is called *normal upper semicontinuous* (resp. *normal lower semicontinuous*) if

$f = (f_*)^*$ (resp. $f = (f^*)_*$). Let $\mathrm{NUSC}(X) = \{f \in \mathbb{R}^X : f = (f_*)^*\}$ and $\mathrm{NLSC}(X) = \{f \in \mathbb{R}^X : f = (f^*)_*)\}$. These are both lattices under pointwise operations; i.e. $(f \vee g)(x) = \max\{f(x), g(x)\}$, etc. The following is 2.1 of [MJ].

THEOREM 4.1. *The mapping* $f \to (f \circ k_X)_*$ *is a lattice isomorphism* σ *of* $\mathrm{NUSC}(X)$ *onto* $C(EX)$.

As $C(X)$ is a sublattice of $\mathrm{NUSC}(X)$, if $f \in C(X)$ then $\sigma(f) = f \circ k_X$. Obviously $C(EX) = L^*(C(X))$ up to isomorphism, where σ is the canonical embedding, iff $\mathrm{NUSC}(X) = L^*(C(X))$. Mack and Johnson prove that this occurs precisely when X is a "weak cb space".

DEFINITION 4.2. A space X is *weak* cb if each locally bounded lower semicontinuous function is bounded above by a continuous function.

In 3.1 of [MJ] a list of characterizations of weak cb spaces is given. The most interesting of these is the following.

THEOREM 4.3. *A space* X *is weak* cb *iff given a decreasing sequence* $\{A_n : n \in N\}$ *of* $\mathcal{R}(X)$ *with* $\cap\{A_n : n \in N\} = \emptyset$ *, there is a decreasing sequence* $\{Z_n : n \in N\}$ *of zero-sets of* X *with* $A_n \subseteq Z_n$ *and* $\cap\{Z_n : n \in N\} = \emptyset$.

The culmination of these ideas is the following, which is 4.2 and 4.3 of [MJ].

THEOREM 4.4. *If* X *is a weak* cb *space then* $L^*(C(X))$ *is isomorphic, as a lattice-ordered ring, to* $C(EX)$. *Conversely, if* $L^*(C(X))$ *is isomorphic, as a lattice-ordered ring, to* $C(Y)$ *for some space* Y, *then* X *is weak* cb.

Since $C(X)$ and $C(\nu X)$ are isomorphic. if X is weak cb then by 4.4, $L^*(C(\nu X)) \simeq C(EX)$. Hence by the other half of 4.4, νX is weak cb. If X is weak cb and $f \in C(EX)$, by 4.1 there exists a unique $f' \in \mathrm{NUSC}(X)$ such that $f = f' \circ k_X$. It can be shown that since X is weak cb, f' can be extended to $(f')^\nu \in \mathrm{NUSC}(\nu X)$. Then by 4.1 the map $f \to (f')^\nu \circ k_{\nu X}$ is a ring isomorphism from $C(EX)$ into $C(E(\nu X))$. Thus EX is dense and C-embedded in the real-compact space $E(\nu X)$, so $\nu(EX) = E(\nu X)$. Hence if X is weak cb, then $\nu(EX) = E(\nu X)$. The converse is untrue, as there are real-compact spaces that are not weak cb (see the final example in [MJ]).

Since compact space are weak cb, we have as a special case of 4.4 that $L^*(C^*(X)) \simeq L^*(C(\beta X)) = C(E(\beta X)$; this result was originally proved by DILWORTH [DI].

Weak cb spaces are important in other contexts; they play an important role in the investigation of weak normality properties in product spaces. See [M] for details.

We now consider the question of characterizing those spaces X for which $E(\nu X) = \nu(EX)$. This leads us to consider two classes of spaces introduced respectively by FROLÍK [FR] and DYKES [DY_2], namely almost real-compact spaces and c-real-compact spaces.

DEFINITION 4.5. [FR]. A Hausdorff space X is *almost real-compact* if each ultrafilter $\mathcal{A}$ on $\mathcal{R}(X)$ with the countable intersection property (C.I.P.) converges to a point of X.

Let X and Y be regular spaces and let $f: X \to Y$ be a perfect continuous surjection. FROLÍK [FR] proves that X is almost real-compact iff Y is. DYKES ([DY_1], Theorem 1.2) proves that an almost real-compact weak cb Tychonoff space is real-compact. Since extremally disconnected spaces are obviously weak cb, the following theorem, characterizing those regular spaces with real-compact absolutes, is an immediate consequence.

THEOREM 4.6. ([DY_1], 1.7). *Let* X *be a regular space. Then* X *is almost real-compact iff* EX *is real-compact.*

Hence if one wants a space X such that $E(\nu X) \neq \nu(EX)$, it suffices to find an X that is almost real-compact but not real-compact. Until recently virtually the only example of such a space was a complicated one due to MROWKA (see [MR]). However, KATO [KAT] has now produced a wealth of examples of almost real-compact, non-real-compact Tychonoff spaces. We briefly describe one of these.

EXAMPLE 4.7. [KAT]. Consider the ordinal space $\omega_1 + 1$ with the interval topology (ω_1 is the first uncountable ordinal). Let D be the set of isolated points (non-limit ordinals) of $\omega_1 + 1$ and let $D' = D \cup \{\omega_1\}$, with the subspace topology inherited from $\omega_1 + 1$. Let $\underline{N} \cup \{\omega\} = \alpha\underline{N}$ be the one-point compactification of the countable discrete space $\underline{N}$. Put $T_D = (D' \times \alpha\underline{N}) - \{(\omega_1, \omega)\}$; T_D is the so-called "Dieudonné plank". Kato shows that T_D is almost real-compact but not real-compact; in fact $\nu T_D = D' \times \alpha\underline{N}$. □

We now consider c-real-compact spaces.

DEFINITION 4.8. [DY_2]. A space X is c-*real-compact* if for each $p \in \beta X \backslash X$, there exists $f \in NLSC(\beta X)$ such that $f(p) = 0$ and f is positive on X.

DYKES proves in [DY_2] that each almost real-compact space is c-real-compact, and that weak cb, c-real-compact spaces are real-compact.

Since products of almost real-compact Hausdorff spaces are almost real-compact and closed subspaces of regular almost real-compact spaces are almost real-compact (see [FR]), the category of almost real-compact spaces and continuous maps forms an epireflective subcategory of the category of Tychonoff spaces and continuous maps (see [HvdS]). Hence each space X has an almost real-compact epireflection aX such that $X \subseteq aX \subseteq \beta X$; aX is the smallest subspace of βX that is almost real-compact. Although the c-real-compact spaces do not form such an epireflective subcategory, there is a smallest c-real-compact space uX such that $X \subseteq uX \subseteq \beta X$. Function-theoretic characterizations of uX appear in [DY_2] and [HW]. The following more "topological" characterizations appear in [WO_4] and [HW] respectively.

THEOREM 4.9. [WO_4]. *If* X *is a space let* $a_1 X = \{x \in \beta X$: *there exists an ultrafilter* μ *on* $\mathcal{R}(X)$ *with* C.I.P. *such that* $\{x\} = \cap\{c\ell_{\beta X} A: A \in \mu\}\}$. *Let* $a_n X = a_1(a_{n-1}X)$ *for each* $n \geq 2$. *Then* $aX = \cup\{a_n X: n \in N\}$.

THEOREM 4.10. [HW]. $uX = \{x \in \beta X$: *if* μ *is an ultrafilter on* $\mathcal{R}(X)$ *and* $\{x\} = \cap\{c\ell_{\beta X} A: A \in \mu\}\}$ *then* μ *has* C.I.P.

The following theorem relates the question of when $E(\nu X) = \nu(EX)$ to the extensions uX and aX. It is an amalgam of 3.2 and 3.3 of [WO_4], and 2.4 and 2.7 of [HW]. An ultrafilter $\mathcal{A}$ on $\mathcal{R}(X)$ is *stable* if each $f \in C(X)$ is bounded on some member of $\mathcal{A}$ (see §5 of [MAN]).

THEOREM 4.11.

(a) uX *is the largest subspace* T *of* βX *such that* $ET \subset \nu(EX)$; aX *is the smallest subspace* T *of* βX *such that* $\nu(EX) \subset ET$. *Thus* $\nu(EX) = ET$ *for some* $T \subset \beta X$ *iff* $uX = aX$; *in this case* $\nu(EX) = E(aX) = E(a_1 X) = E(uX)$.

(b) *the following are equivalent for a space* X:

 (i) $E(\nu X) = \nu(EX)$

 (ii) $uX = \nu X$

 (iii) *If* $\{A_n; n \in N\}$ *is a decreasing sequence of members of* $\mathcal{R}(X)$, *then* $c\ell_{\nu X}[\cap\{A_n : n \in N\}] = \cap\{c\ell_{\nu X} A_n ; n \in N\}$.

 (iv) *Each stable ultrafilter on* $\mathcal{R}(X)$ *has* C.I.P.

ISIWATA [IS] calls spaces satisfying the hypotheses of 4.11 "weak cb^* spaces".

EXAMPLE 4.12. We briefly describe an example of a space X such that $\nu(EX) \neq ET$ for any T such that $X \subset T \subset \beta X$. This example is discussed in ([MJ], pg. 240). Let $Y = \{(\sigma,\tau) \in \omega_1 \times (\omega_1 + 1): \sigma \leq \tau\}$. Let X be the quotient space of $Y \times N$ obtained by identifying $(\sigma,\sigma,2n)$ with $(\sigma,\sigma,2n-1)$, and $(\sigma,\omega_1,2n)$, with $(\sigma,\omega_1,2n-1)$, for each $\sigma < \omega_1$ and positive integer n. Then X turns out to be a locally compact non-weak cb space, while νX is weak cb and not locally compact (in fact $|\nu X \backslash X| = 1$). In 1.11 of [HW] it is proved that X is weak cb iff uX is weak cb. It follows that $uX = X \neq \nu X$.

Although $E(\nu X) = \nu(EX)$ only in special circumstances, it is interesting to note that $\beta(EX)\backslash E(\nu X)$ is *always* dense in $\beta(EX) - \nu(EX)$; this is proved in 2.8 of [WO_3].

Further information concerning various rings and lattices of functions on a space X that are related to $L^*(C(X))$ may be found in [HA_1] and [HA_2]. Other papers discussing algebraic modfications of the ring C(X) - particularly the complete ring of quotients of C(X) - that involve a consideration of EX include [FGL], [H], and [J_2].

Let $\mathcal{P}$ be a topological property that is preserved by taking closed subspaces and products, and that is possessed by all compact spaces. In [HvdS] HERRLICH and VAN DER SLOT prove that the category $\mathcal{PJ}$ of Tychonoff spaces with $\mathcal{P}$ and continuous maps in an epireflective subcategory of the category $\mathcal{J}$ of all Tychonoff spaces and continuous maps. If $\mathcal{P}X$ is the epireflection in $\mathcal{PJ}$ of X, then $X \subset \mathcal{P}X \subset \beta X$ (see pg. 528 of [HvdS]). It is also known (see Prop. 2 of [HvdS]) that if $f: X \to Y$ is a perfect continuous surjection and Y has $\mathcal{P}$, then X has $\mathcal{P}$. Thus for any Tychonoff space X, $E(\mathcal{P}X) = k_{\beta X}^{\leftarrow}[\mathcal{P}X]$ and so $E(\mathcal{P}X)$ has $\mathcal{P}$. Hence $EX \subseteq \mathcal{P}(EX) \subseteq E(\mathcal{P}X) \subseteq \beta(EX)$, and one can ask under what conditions, both on $\mathcal{P}$ and on X, $\mathcal{P}(EX) = E(\mathcal{P}X)$. The preceding discussion has essentially been an analysis of this problem when $\mathcal{P}$ is real-compactness; the general problem is untouched.

Finally, in [WH] WHEELER has applied the foregoing theory of weak cb spaces, and the relationship between $C^*(X)$ and $C^*(EX)$, and between $E(\nu X)$ $\nu(EX)$, to the study of topological measure theory. Roughly speaking, Wheeler relates various spaces of measures on X to corresponding spaces of measures on EX, and then uses the fact that Baire sets in extremally disconnected spaces have a relatively simple structure to analyze these spaces of measures on EX. The results obtained are "mapped back" to X to obtain results concerning the structure of spaces of measures on X. The bibliography of [WH] lists earlier papers containing results in this field.

5. CO-ABSOLUTES OF OUTGROWTHS OF STONE-CECH COMPACTIFICATIONS; REMOTE POINTS

Two regular spaces are said to be *co-absolute* if their absolutes are homeomorphic. During the last fifteen years much work has been done on characterizing the family of co-absolutes of "well-known" spaces or types of spaces, and investigating which topological properties are preserved in passing from a given space to one co-absolute with it. Thus, for example, co-absolutes of metric spaces have been studied by PONOMAREV [PO_3], EFIMOV [E], and ŠAPIROVSKIǏ [SA_2], among others; CHERTANOV [CH] and more recently WILLIAMS [WI] have investigated co-absolutes of ordered spaces. The survey paper by PONOMAREV and ŠAPIRO [PS] gives a comprehensive overview of results obtained in the above areas, together with an extensive bibliography and a list of open problems. In this section we concentrate on a more specialized problem not considered in any detail in [PS], namely the problem of calculating the co-absolutes of $\beta X\backslash X$ for a Tychonoff space X. Much of this work was done in the early 1970's, but in the last two years there has been a considerable advance in this area, largely due to new results on the properties of "remote points" of certain Tychonoff spaces.

Throughout this section the word "space" means "completely regular Hausdorff space". If X is a space, we denote the space $\beta X\backslash X$ by X^*. We use the following simple consequence of the uniqueness of the absolute of a space: if f is a perfect irreducible continuous surjection from X to Y, then X and Y are co-absolute.

If X is a space, then $E(\beta X)$ is mapped onto βX by the perfect irreducible mapping $k_{\beta X}$. Since we can identify EX with $k_{\beta X}^{\leftarrow}[X]$ and $E(\beta X)$ with $\beta(EX)$, the restriction $k_{\beta X}|(EX)^*$ (henceforth denoted k^*) is a perfect map from $(EX)^*$ onto X^*. If it were also irreducible, by the result noted above X^* and $(EX)^*$ would be co-absolute. In [WO_2] and [WO_3] the author studied successively more general conditions on a space X that imply that k^* is irreducible. Since the structure of EX depends heavily on the structure of $\mathcal{R}(X)$, the following result (which combines 2.3 and 3.1 of [WO_1]) is perhaps not surprising. If $A \in \mathcal{R}(X)$, denote $c\ell_{\beta X}A\backslash X$ by A^*.

<u>THEOREM 5.1</u>. *The following conditions on a space* X *are equivalent:*

(a) $c\ell_{X^*}(X^*\backslash A^*) = [c\ell_X(X\backslash A)]^*$ *for each* $A \in \mathcal{R}(X)$.

(b) *The map* $A \to A^*$ *is a Boolean algebra homomorphism from* $\mathcal{R}(X)$ *into* $\mathcal{R}(X^*)$.

(c) *The map* k^* *is irreducible.*

The conditions of 5.1 are satisfied by many spaces; the next results are immediate consequences of 2.7 and 5.1 of [WO_1].

THEOREM 5.2. *If* X *is real-compact, or metric, or nowhere locally compact, then* k^*: $(EX)^* \to X^*$ *is irreducible, and* X^* *and* $(EX)^*$ *are co-absolute.*

THEOREM 5.3. *Let* X *and* Y *be real-compact, or metric, or nowhere locally compact. If* X *and* Y *are co-absolute, then so are* X^* *and* Y^*.

REMARKS 5.4.

(a) If X is nowhere locally compact then EX is nowhere locally compact and so $(EX)^*$ is dense in $\beta(EX)$. Hence by 1.1 $(EX)^*$ is extremally disconnected, so by 5.3 $E(X^*) = (EX)^*$.

(b) If X and Y are locally compact metric spaces, without isolated points, then they are co-absolute iff they have the same density character (see, for example, [PO_3]). In this case X^* and Y^* are co-absolute; in fact, by 2.6 of [WO_2] X^* and Y^* have dense homeomorphic subspaces.

Since X is dense in νX, the map $A \to c\ell_{\nu X}A$ is a Boolean algebra isomorphism from $\mathcal{R}(X)$ onto $\mathcal{R}(\nu X)$. We combine the observation with 5.1 and 5.2 and obtain the following result, which is 2.2 of [WO_3].

THEOREM 5.5. *For any space* X, *the mapping* $A \to c\ell_{\beta X}A\backslash\nu X$ *is a Boolean algebra homomorphism from* $\mathcal{R}(X)$ *into* $\mathcal{R}(\beta X\backslash\nu X)$, *and* $(E(\nu X))^*$ *is co-absolute with* $(\nu X)^*$.

We now consider the kernels of the homomorphisms described in 5.1(b) and 5.5. The kernel of the former homomorphism is obviously the set of compact numbers of $\mathcal{R}(X)$. In 4.1 of [CO_1], COMFORT proves that $A \to c\ell_{\nu X}A$ is a bijection from the set of pseudocompact members of $\mathcal{R}(X)$ onto the set of compact members of $\mathcal{R}(\nu X)$. Thus the kernel of the homomorphism described in 5.5 is the ideal I of pseudocompact members of $\mathcal{R}(X)$. In 2.9 of [WO_3] it is proved that the Stone space of the factor algebra $\mathcal{R}(X)/I$ is homeomorphic to $c\ell_{\beta(EX)}[\beta(EX)\backslash E(\nu X)]$, which by 2.8 of [$WO_3$] is just $c\ell_{\beta(EX)}[\beta(EX)\backslash\nu(EX)]$.

Let $\mathcal{C}$ denote the class of spaces X for which $(\nu X)^*$ is dense in X^*. Suppose $X \in \mathcal{C}$. By 5.2, $k_{\beta X}|(E(\nu X))^*$: $(E(\nu X))^* \to (\nu X)^*$ is irreducible. It follows that k^*: $(EX)^* \to X^*$ is also irreducible, and so X^* and $(EX)^*$ are co-absolute. The class $\mathcal{C}$ contains many non-real-compact spaces; in particular $\mathcal{C}$ contains $\mathcal{C}'$, the class of spaces Y for which the pseudocompact closed subsets of Y are compact. $\mathcal{C}'$ contains all almost real-compact spaces and all P-spaces (see 4K of [GJ]); recall that a P-space is a Tychonoff space whose G_δ-sets are all open. Also, closed subspaces of spaces in $\mathcal{C}'$, and products of spaces in $\mathcal{C}'$, are again in $\mathcal{C}'$.

Let $f: X \to Y$ be a perfect continuous surjection. If X has a dense subspace S such that $f|S$ is a homeomorphism from S onto $f[S]$, then f is irreducible; the converse is untrue in general. Let X be realcompact. By 5.2 $k^*: (EX)^* \to X^*$ is irreducible, and one might ask whether $(EX)^*$ has a dense subspace S such that $k^*|S$ is a homeomorphism from S onto $k^*[S]$. In general the answer is "no", but the work of GATES ($[GA_1]$, $[GA_2]$) and DOUWEN $[vD_1]$ shows that it *is* true whenever X has countable π-weight (defined below). In this case the set TX of "remote points" of X is homeomorphic to $k_{\beta X}^{\leftarrow}[TX]$, and $k_{\beta X}|k_{\beta X}^{\leftarrow}[TX]: k_{\beta X}^{\leftarrow}[TX] \to TX$ is a homeomorphism.

DEFINITION 5.6. A point p of βX is called a *remote point* of X if $p \notin \cup\{c\ell_{\beta X}F$: F is a closed nowhere dense subset of X$\}$. The set of remote points of X is denoted TX.

Remote points were first introduced by FINE and GILLMAN [FG], who proved using the continuum hypothesis (henceforth abbreviated CH) that $T\mathbb{R}$ is a dense subspace of $\mathbb{R}^*$. They also proved that $T\mathbb{Q} \neq \emptyset$ ($\mathbb{Q}$ denotes the space of rationals). PLANK [PL] proved, using CH, that if X is a separable locally compact space without isolated points, then TX is dense in X^*; ROBINSON [RO] removed the assumption of separability. The author $[WO_2]$ proved that if X and Y are locally compact metric spaces without isolated points and of the same density character, then TX and TY are homeomorphic. This, combined with Robinson's result, implied that under CH X^* and Y^* had homeomorphic dense subspaces. In [PW] it was shown under CH that if X is a nowhere locally compact separable metric space, then TX is the largest extremally disconnected subspace of βX.

The above results are all subsumed by the following recent results of GATES ($[GA_1]$, $[GA_2]$) and VAN DOUWEN $[VD_1]$. Recall that a π-*base* of a space X is a family $\mathcal{R}$ of non-empty open subsets of X such that each non-empty open subset of X contains a member of $\mathcal{R}$. The π-*weight* of X, denoted $\pi w(X)$, is $\min\{\kappa$: X has a π-base of cardinality $\kappa\}$. Note the absence of any set theoretic assumptions in the following. We denote 2^ω by c.

THEOREM 5.7. (4.2 and 4.4 of [vD_1]). *Let* X *be a space of countable* π*-weight. Then:*

(a) *If* G *is a non-empty* G_δ*-set of* βX *and* $G \subseteq X^*$, *then* G *contains* 2^c *remote points of* X.

(b) *If* X *is real-compact then each non-empty open subset of* X^* *contains* 2^c *remote points of* X.

It follows from (a) that if $\pi w(X) = \omega$ and X is not pseudocompact, then $TX \neq \emptyset$. The following combines 2.2, 2.3 and 2.4 of [GA_2].

THEOREM 5.8. *Let* f *be an irreducible closed continuous surjection from* X *onto* Y. *Then* $f^\beta | (f^\beta)^\leftarrow[TY]$: $(f^\beta)^\leftarrow[TY] \to TY$ *is a homeomorphism and* $(f^\beta)^\leftarrow[TY] \subseteq TX$. *If in addition* Y *is normal,* $(f^\beta)^\leftarrow[TY] = TX$ *and so* TX *and* TY *are homeomorphic.*

If we combine 5.7 and 5.8 and note that $\pi w(X) = \pi w(EX)$ we obtain the following:

THEOREM 5.9. *Let* X *be a real-compact space of countable* π*-weight. Then* X^* *and* $(EX)^*$ *have homeomorphic dense subspaces, namely* TX *and* $(f^\beta)^\leftarrow[TX]$, *and thus are co-absolute. If in addition* X *is normal, then* TX *and* T(EX) *are homeomorphic, and dense in* X^* *and* $(EX)^*$ *respectively.*

In some sense 5.9. tells us why X^* and $(EX)^*$ are co-absolute when X is real-compact and of countable π-weight; namely, X^* and $(EX)^*$ contain homeomorphic dense subspaces. However, there are real-compact spaces of large π-weight with no remote points. In [$vDvM_2$] van DOUWEN and van MILL construct a locally compact, σ-compact, non-compact space without remote points, which we describe now. If κ is a cardinal (with the discrete topology), let $U(\kappa)$ denote the space of uniform ultrafilters on κ; i.e. $U(\kappa) = \{\alpha \in \beta\kappa\colon A \in \alpha \text{ implies } |A| = \kappa\}$. Let $Y = U(\omega_2) \times \omega$; van Douwen and van Mill prove that $TY = \emptyset$. Nonetheless, by 5.2 Y^* and $(EY)^*$ are co-absolute. This suggests the following problem.

PROBLEM 5.10. Characterize those (real-compact) spaces X for which X^* and $(EX)^*$ have dense homeomorphic subspaces.

Recall (see [GJ] or [WAL]) that a space X is an *F-space* if its cozero-sets are C^*-embedded. Let (*) denote the following statement: if K is a compact zero-dimensional F-space without isolated points, if the weight of K is c, and if each zero-set of K belongs to $\mathcal{R}(K)$, then K is homeomorphic to $\beta N \backslash N$. In [P_2] PAROVIČENKO proved that CH implies (*), and recently van DOUWEN and van MILL [$vDvM_1$] have proved that (*) is equivalent to CH. This theorem can be used to prove that for many spaces X, CH implies that X^* is homeomorphic to $\beta N \backslash N$. A typical result is the following theorem, which is a special case of 3.1 of [WO_3].

THEOREM 5.11. [CH]. *Let X be a locally compact real-compact non-compact space such that* $|\mathcal{R}(X)| = c$. *Then* X^* *is co-absolute with* $\beta N \backslash N$.

Recently van Mill has proved that CH is equivalent to the following statement: if X is locally compact and real-compact and $|C^*(X)| = c$ then X^* os co-absolute with $\beta N \backslash N$.

Another fascinating result concerning co-absolutes of outgrowths of Stone-Čech compactifications is the following theorem of BALCAR and VOPĚNKA [BV], which is proved in detail in chapter 12 of [CN_2].

THEOREM 5.12. *Let* κ *be a regular uncountable cardinal with the discrete topology and suppose that* $2^\kappa = \kappa^+$. *Then* $U(\kappa)$ *is co-absolute with* $\beta((\kappa^+)^\omega)$, *where* κ^+ *has the discrete topology.*

We conclude this section be summarizing some of the results recently obtained by WILLIAMS ([WI]). The first set of results concerns co-absolutes of linearly ordered spaces.

THEOREM 5.13. (2.3, 2.6 and 2.10 of [WI]). *Let X be a space. Then:*

(a) βX *is a co-absolute with a linearly ordered space iff the poset* $(\mathcal{R}(X) - \{X\}, \subseteq)$ *contains a cofinal tree (a tree is a poset P such that if* $x \in P$ *then* $\{y \in P\colon y \leq x\}$ *is well-ordered).*

(b) *If X is a Moore space, then* βX *is co-absolute with a linearly ordered space iff it has a dense metrizable subspace iff it has a dense metrizable linearly ordered space.*

(c) *If X is dyadic then it is co-absolute with a linearly ordered space iff it is separable and metrizable*

The second set of results concern co-absolutes of X^*. Theorem 5.14(b) should be contrasted with 5.11 and the result of van Mill that follows 5.11.

THEOREM 5.14. *Let X be a locally compact non-compact metric space. Then:*

(a) X^* *is co-absolute with a compact linearly ordered space having a dense set of P-points.*

(b) *It is consistent with the negation of the continuum hypothesis that* X^* *and* $\beta N\backslash N$ *are co-absolute whenever* X *has density character no greater than* c.

6. MISCELLANEOUS RECENT RESULTS

In this section we consider a varied collection of recent results on absolutes. These fall into two (non-disjoint) sets. First, we consider results that pertain to the following very difficult problem: for which X is EX a normal space? Second, we describe solutions discovered in the last few years to several of the problems posed in the 1976 survey article of PONOMAREV and ŠAPIRO [PS]. Most of the results in this section have a much more set-theoretic flavour than those of previous sections; many of them were discovered by members or adherents of the "Madison school" of set-theoretic topology.

We begin with a very general open problem.

PROBLEM 6.1. Characterize those topological properties $\mathcal{P}$ such that a Tychonoff space X has $\mathcal{P}$ iff EX has $\mathcal{P}$.

Let us call a topological property $\mathcal{P}$ such that a Tychonoff space X has $\mathcal{P}$ iff EX has $\mathcal{P}$ a *co-absolute invariant* property. Co-absolute-invariant properties seem to fall roughly into three classes: covering properties, properties that are equivalent to Boolean-algebraic properties of $\mathcal{R}(X)$, and properties that can be described in terms of the convergence behaviour of certain open filters or ultrafilters. Most covering properties are preserved directly and inversely by perfect maps and are hence co-absolute-invariant. Examples are compactness, local compactness, σ-compactness, the Lindelof property and its various higher-cardinality analogues, paracompactness, metacompactness and so on. Many of the topological properties that are described in terms of cardinal functions involving the open sets of a space X are equivalent to properties of $\mathcal{R}(X)$ that are invariant under Boolean-algebra isomorphism; such properties are obviously co-absolute-invariant. Examples are celluarlity (or Souslin number) and π-weight, (see [RU] for a discussion of these). Such properties are obviously co-absolute-invariant and have the property that if X is dense in T, then X has the property iff

T does. Co-absolute-invariant properties that are described in terms of the convergence of open (ultra)filters include almost real compactness, pseudocompactness, and the property of being H-closed. It will be a challenge to find a common generalization of these three types of properties.

We now concentrate on one particular property, the property of being normal. In [STR] STRAUSS says "I have not been able to determine whether the normality of X implies that of EX". Shortly thereafter WARREN [WAR] proved that $E\omega_1$ is not normal even though ω_1 is (here, and in the following discussion, ordinals are assumed to have the order topology). In [MA_1] MALYKHIN generalized Warren's result by proving that if κ is a singular cardinal of uncountable cofinality, or a regular but inaccessible cardinal, then $E\kappa$ is not normal. Recently KUNEN and PARSONS [KP] characterized those ordinals whose absolutes are normal. We state their main results below. A cardinal κ is *weakly compact* iff it is strongly inaccessible and there is no κ-Aronszajan tree (see [RU] or [KU_2] for a discussion of Aronszajan trees, and [KU_2] for various characterizations of weakly compact cardinals). It is consistent with the usual axioms of set theory that weakly compact cardinals do not exist; if they exist they must be very large (see [KU_2]). The *cofinality* of an ordinal α is denoted $cf(\alpha)$.

THEOREM 6.1. (see 1.11 of [KP]). *Let* α *be an ordinal. Then* $E\alpha$ *is normal iff one of the following conditions holds.*

(a) α *is a successor (and thus compact)*

(b) $cf(\alpha) = \omega$ *(thus* α*, and therefore* $E\alpha$*, is* σ*-comapct)* .

(c) $cf(\alpha) = \kappa > \omega$*, where* κ *is weakly compact and there is a* β *such that* $\alpha = \beta + \kappa$.

If (a) *or* (b) *holds, then* $E\alpha$ *is paracompact; if* (c) *holds,* $E\alpha$ *is not paracompact.*

A subset S of an ordinal α is called a *stationary set* if $S \cap A \neq \emptyset$ for each closed unbounded subset A of α. See [RU] for a discussion of stationary sets. The following is 2.3 of [KP].

THEOREM 6.2. *If* S *is a stationary set of an ordinal* α *that contains the isolated points (i.e. non-limit ordinals) of* α*, and* $E\alpha$ *is not normal, then* ES *is not normal.*

The above theorems give a large collection of locally compact normal spaces whose absolutes are not normal.

The most general easily described topological property P such that each space with P has a normal absolute is paracompactness; if X is paracompact then so is EX, and thus EX is normal. This raises the following problem, which was posed by Ponomarev and quoted as Problem 9 in [PS]. Is there an example of an extremally disconnected normal non-paracompact space? In [MA_2] MALYKHIN constructs an example of such a space assuming Martin's axiom plus the negation of the continuum hypothesis (henceforth abbreviated $MA + \neg CH$). In fact, if it is assumed that $2^{\omega} = 2^{\omega_1}$ (which is a consequence of $MA + \neg CH$; see [RU]), then $\beta N \backslash N$ has a C^*-embedded discrete subspace D of cardinality ω_1 (see [E]). Then $N \cup D$, regarded as a subspace of βN, is a normal extremally disconnected separable space that is not collectionwise Hausdorff and hence not paracompact. More recently, KUNEN [KU_1] constructed, without any special set-theoretic axioms, a normal non-paracompact extremally disconnected space, which we describe below. This example answers Problem 9 of [PS].

EXAMPLE 6.3. [KU_1]. Let D be the discrete space of cardinality ω_1, and let $\{A_\alpha : \alpha < 2^{\omega_1}\}$ be a family of 2^{ω_1} independent subsets of D (i.e., if Γ and Σ are disjoint finite subsets of 2^{ω_1}, then $\cap\{A_\alpha : \alpha \in \Gamma\} \cap \cap\{D \backslash A_\alpha : \alpha \in \Sigma\} \neq \emptyset$). Let μ be a finitely additive probability measure on $P(D)$, the power set of D, such that $\mu(A_\alpha) = \frac{1}{2}$ for each $\alpha < 2^{\omega_1}$. Enumerate $P(D)$ as $\{H_\alpha : \alpha < 2^{\omega_1}\}$ and for each $\zeta < \omega_1$, let $\sigma_\zeta \in \beta D \backslash D$ be chosen so that:
(1) $\mu(B) > 0$ for each $B \in \sigma_\zeta$, (2) for each $\alpha < 2^{\omega_1}$, $A_\alpha \in \sigma_\zeta$ iff $\zeta \in H_\alpha$. Let $Y = \{\mu_\zeta : \zeta < \omega_1\}$. Then $D \cup Y$, regarded as a subspace of βD, is normal (as the A_α's allow disjoint subsets of Y to be separated) but not collectionwise Hausdorff (as μ prevents the points of Y from being separated) and thus not paracompact.

We pose a more restricted version of Problem 9 of [PS]; this was first posed in [WO_7], and to our knowlwdge is still open.

PROBLEM 6.3. Is it consistent with the usual axioms of set theory that each normal locally compact extermally disconnected space is paracompact?

According to 6.1, normal extremally disconnected locally compact non-paracompact spaces exist if weakly compact cardinals exist. The following result which is 1.1(b) of [WO_7], shows the examples of such spaces that do not depend on special set-theoretic axioms must be "large" (since each extremally disconnected space is an F-space.)

THEOREM 6.4. *Assume* CH. *If* X *is a normal locally compact* F-*space and* $|C^*(X)| = c$, *then* X *is* σ-*compact.*

Recently van DOUWEN [vD$_2$] has shown that the conclusion of 6.5 is true iff CH holds. We now consider the example by which he proves this assertion.

EXAMPLE 6.5. Let $\omega_2 + 1$ have the order topology, let $X = (\omega_2+1)\setminus\{p \in \omega_2 + 1:$ p is a limit ordinal of countable cofinality$\}$, and let $\Phi = \beta X - \{\omega_2\}$. VAN DOUWEN [VD$_2$] proves that Φ is a normal, locally compact, countably compact, non-compact basically disconnected space and that $|C^*(\Phi)| = \omega_2 \cdot c$. (A space is basically disconnected if each cozero-set has an open closure; see 1H of [GJ]. Each basically disconnected space is an F-space.) It immediately follows that the conclusion of 6.5 holds iff CH holds. However, Φ is not extremally disconnected so the following problem remains open.

PROBLEM 6.6. Are the following assertions equivalent?

(1) CH

(2) Each normal, locally compact extremally disconnected space X such that $|C^*(X)| = c$ is σ-compact.

It follows from 6.4 that if CH is assumed and $p \in \beta N\setminus N$, then $\beta N\setminus\{p\}$ is not normal (this result was known prior to the proof of 6.4; it is implied by the combined results of COMFORT and NEGREPONTIS [CN$_1$] and WARREN [WAR]). However, it is not known whether this is true in the absence of CH.

PROBLEM 6.7. Is it true, without assuming any special set-theoretic axioms, that if $p \in \beta N - N$ then $\beta N - \{p\}$ is not normal?

It has been proved, without any special set-theoretic axioms, that there exist many points p of $\beta N - N$ such that $\beta N - \{p\}$ is not normal (see [SA$_1$]). Recently BALCAR and SIMON [BS] have, without special set-theoretic hypotheses, shown the existence of a discrete subspace D of βN of cardinality ω_1, and a point p of $\beta N - D$, such that each neighbourhood of p meets all but countably many mambers of D. This implies that D is not C^*-embedded in βN, answering a question posed in [WO$_6$]. Since countable subsets of extremally disconnected spaces are C^*-embedded, this implies that D is C^*-embedded in $c\ell_{\beta N}D\setminus\{p\}$; in fact $c\ell_{\beta N}D$ is homeomorphic to the one-point compactification of the space of non-uniform ultrafilters on the discrete space of cardinality ω_1. Obviously $\beta N\setminus\{p\}$ is not normal.

We conclude this section by mentioning recent solutions to two other problems posed in [PS]. In Problem 4, page 145 of [PS], it is asked whether it is true, without any special set-theoretic axioms, that an infinite compact extremally disconnected space is non-homogeneous. (A space X is homogeneous if, given two points p and q of X, there is a homeomorphism $h: X \to X$ such that $h(p) = q$.) This was answered in the affirmative by Kunen and Frolik; an excellent discussion of the proof of their theorem (6.8 below) appears in §8 of [CO$_2$].

THEOREM 6.8. *No infinite compact space in which each countable discrete subset is* C^**-embedded is homogeneous.*

KUNEN [KU$_3$] has recently proved a variant of this result as follows.

THEOREM 6.9. *Let* $\{X_i: i \in I\}$ *be a set of compact spaces. Assume that each* X_i *is either first countable or an infinite compact* F*-space, and that at least one* X_i *is an infinite compact* F*-space. Then* $\pi\{X_i: i \in I\}$ *is not homogeneous.*

Finally, in Problem 8, page 147 of [PS], the following problem is attributed to Malykhin; is there an uncountable hereditarily separable regular extremally disconnected space? In [WA], WAGE uses the set-theoretic axiom $\clubsuit$ to construct an extremally disconnected hereditarily separable (and hence hereditarily normal) non-Lindelof space in which each closed set is a G_δ. ($\clubsuit$ is a consequence of the axiom of constructability; see [RU] for a discussion of it.) The method of construction is similar to that used by Ostaszewski to construct his well-known example of hereditarily separable, normal, countably compact, non-compact space; see [RU].

In closing we recommend the bibliography of [HE] as a source of references to work on absolute prior to 1971, and the bibliography of [PS] as a source of work done on absolutes in the Soviet Union between 1965 and 1975.

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DISJOINTNESS AND QUASIFACTORS IN TOPOLOGICAL DYNAMICS

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0. INTRODUCTION

Several authors ([4], [5], [7], [9], [10] and [16]) have used quasifactors in the structure theory of minimal topological transformation groups (from now on the term topological transformation group(s) will be abbreviated by ttg('s)). In this paper we shall make extensive use of the properties of quasifactors in the determination of disjoint classes.

The first section will be devoted to some basic statements about quasifactors and extensions in relation to disjointness. It contains a result important for the rest of this paper (1.5), which states, that given a distal extension $\phi: X \to Y$, every quasifactor of X which is disjoint from Y is distal. In Section 2 we characterize in terms of quasifactors the collection $\mathcal{K}^{\perp}$ of all minimal ttg's that are disjoint from every member of a certain family $\mathcal{K}$ of minimal ttg's. For nice families $\mathcal{K}$ this gives us relatively smooth descriptions for $\mathcal{K}^{\perp}$ (2.9, 2.10, 2.12, 2.13, 5.4). Section 3 provides a generalization of [11], 4.5, where KEYNES states that for abelian phase groups two minimal ttg's are disjoint whenever the product of their Ellis groups equals $\mathcal{G} = uM$ (3.4). In Section 4 we characterize (without countability assumptions) the incontractible weakly mixing minimal ttg's as the incontractible minimal ttg's without distal factors; this result is extended in the fifth section to the minimal ttg's disjoint from every PI-ttg. The last section deals with common factor problems, and results in the observation that for arbitrary phase groups and minimal ttg's X and Y such that one of them is regular and one of them is in $\mathcal{D}^{\perp\perp}$ ($\mathcal{D}$ is the family of minimal distal ttg's), the disjointness of X and Y is equivalent to having no nontrivial common factor.

We will now introduce some basic notions and notations. For a more comprehensive treatment we refer to [4], [16] and (with more notational resemblance) [9]. Under a *ttg* we shall understand here an action of T on X, with T an arbitrary topological group and X a compact Hausdorff space. The action of T on X is a continuous mapping $(t,x) \to tx : T \times X \to X$ such that $(ts)x = t(sx)$ and $ex = x$ for all $s,t \in T$ and $x \in X$ (e is the unity of T). Mostly we shall consider T to be understood and denote the ttg by its phase space only. A *homomorphism* of ttg's $\phi: X \to Y$ is a continous map which commutes with the action of T on X and Y; i.e. $\phi(tx) = t\phi(x)$ for all $t \in T$ and $x \in X$. If ϕ is a homomorphism onto we shall call ϕ an *extension* and also we shall call X an extension of Y if there is an extension $\phi: X \to Y$.

Let (T,Q,q) be the universal ambit for T (e.g.[17], 4.4.16) and (T,M) the universal minimal ttg for T ([3]). Q is isomorphic to its enveloping semigroup $E(Q)$ and M is isomorphic to any minimal left ideal in Q. M is a semigroup itself and accordingly it acts on every minimal ttg (T,X). Denote the collection of idempotents in M by J. Then $M = \cup\{vM \mid v \in J\}$ and every vM is a subgroup in M, while $\{vM \mid v \in J\}$ is a partition of M. For $u \in J$ we can define a nice compact T_1-topology on uM, the so-called τ-topology. In uM we consider the stabilizer for (T,X) with respect to the point x_0, chosen such that $ux_0 = x_0$: $\mathcal{G}(X,x_0) = \{\alpha \in uM \mid \alpha x_0 = x_0\}$. It is called the *Ellis group* of (T,X,x_0) and it plays an important role in the structure theory of minimal ttg's ([7], [9]). The Ellis groups are τ-closed subgroups of uM and moreover every τ-closed subgroup of uM can be obtained as an Ellis group.

Every ttg (T,X) induces a *hypertransformation group* $(T,2^X)$ where $2^X = \{A \subseteq X \mid A = \bar{A} \neq \emptyset\}$ (for an explicit treatment of possible topologies on 2^X see [13], we will use the Vietoris topology on 2^X). Since X is compact Hausdorff, 2^X is compact Hausdorff and the action of T on 2^X defined by $(t,A) \to tA = \{ta \mid a \in A\}$ is continuous ([12]). Every homomorphism $\phi: X \to Y$ induces a homomorphism $\phi^*: 2^X \to 2^Y$ defined by $\phi^*(A) = \phi[A]$. If ϕ is open there are also homomorphisms $\phi^{\leftarrow *}: Y \to 2^X$ defined by $\phi^{\leftarrow *}(y) = \phi^{\leftarrow}(y)$ and $\phi^{\leftarrow **}: 2^Y \to 2^X$ by $\phi^{\leftarrow **}(B) = \phi^{\leftarrow}[B]$. Clearly $\phi^{\leftarrow *}$ and $\phi^{\leftarrow **}$ are embeddings. There exist two "actions" of Q on 2^X, namely one defined by $pA := \{pa \mid a \in A\}$ and one defined by $p \circ A := \lim t_i A$ ($\{t_i\}_i$ a net in Q converging to p) for $p \in Q$ and $A \in 2^X$; see [9]. Always $pA \subseteq p \circ A$ but in general the inclusion is strict. If X is a minimal ttg, then a *quasifactor* of X is a minimal sub-ttg of 2^X. It can be shown that every quasifactor has the form $QF(A,X) = \{p \circ A \mid p \in M\}$ for some $A \in 2^X$, where $QF(A,X)$ is non-trivial iff $u \circ A \neq X$ iff $X \notin QF(A,X)$; obviously X is a non-trivial quasifactor of itself iff X is non-trivial ($X \cong QF(\{x\},X)$).

Finally recall that two minimal ttg's X and Y are called *disjoint* $(X \perp Y)$

if $X \times Y$ is minimal. For a family $\mathcal{K}$ of minimal ttg's we denote by $\mathcal{K}^{\perp}$ the collection of minimal ttg's which are disjoint from every member of $\mathcal{K}$, and $\mathcal{K}^{\perp\perp}$ means $(\mathcal{K}^{\perp})^{\perp}$. With $X \not\perp Y$ we mean that X and Y are not disjoint.

1. QUASIFACTORS AND EXTENSIONS

For a homomorphism $\phi: X \to Y$ of minimal ttg's (hence ϕ is an extension) we define $2^{\perp\phi}$ by $2^{\perp\phi} = \{A \in 2^X \mid \phi[A] = Y\}$. Then $2^{\perp\phi}$ is a closed and invariant subset of 2^X. The easy proof of the following lemma will be omitted.

LEMMA 1.1. *Let* $\phi: X \to Y$ *be a homomorphism of minimal ttg's. Then*
(a) *if* ϕ *is an open map then every quasifactor of* Y *is a quasifactor of* X;
(b) $\phi^*[QF(A,X)] = QF(\phi[A],Y)$;
(c) *for every quasifactor* $\mathcal{X}$ *of* X, $\phi^*[\mathcal{X}]$ *is trivial iff* $\mathcal{X} \subseteq 2^{\perp\phi}$ *iff* $\mathcal{X} \cap 2^{\perp\phi} \neq \emptyset$.

REMARK 1.2. If a ttg X contains a nonempty proper subset Z which is invariant under M (i.e. for every point in Z, Z contains a minimal subset of its orbit closure), then X is not minimal. As a consequence we have

THEOREM 1.3. *Let* $\phi: X \to Y$ *be a homomorphism of minimal ttg's. Then*
(a) $\mathcal{Y} \not\perp X$ *for every nontrivial quasifactor* $\mathcal{Y}$ *of* Y;
(b) $\mathcal{X} \not\perp Y$ *for every nontrivial quasifactor* $\mathcal{X}$ *of* X *with* $\mathcal{X} \cap 2^{\perp\phi} = \emptyset$.

PROOF.
(a) Since $\mathcal{Y}$ is a nontrivial quasifactor of Y, there exist $B \in \mathcal{Y}$ and $x_0 \in X$ with $\phi(x_0) \notin B$. So the non-empty subset $\mathcal{A} = \{(x,A) \in X \times \mathcal{Y} \mid \phi(x) \in A\}$ of $X \times \mathcal{Y}$ is a proper subset. Let $(x,A) \in \mathcal{A}$; then $\phi(px) = p\phi(x) \in pA \subseteq p \circ A$ for all $p \in M$, so $p(x,A) = (px, p \circ A)$ is in $\mathcal{A}$ and $\mathcal{A}$ is invariant under M. By 1.2, $X \times \mathcal{Y}$ is not minimal, thus $X \not\perp \mathcal{Y}$.
(b) Define a subset $\mathcal{A}$ of $\mathcal{X} \times Y$ by $\mathcal{A} = \{(A,y) \in \mathcal{X} \times Y \mid y \in \phi[A]\}$. Then $\mathcal{A} \neq \emptyset$ and because there exist $B \in \mathcal{X}$ and $y_0 \in Y$ with $y_0 \in \phi[B]$, $\mathcal{A}$ is a proper subset of $\mathcal{X} \times Y$. Also $\mathcal{A}$ is invariant under M; indeed, if $x \in \phi[A]$ and $p \in M$ then $px \in p\phi[A] \subseteq p \circ \phi[A] = \phi[p \circ A]$ (ϕ^* is a homomorphism!). □

The following are easy consequences of 1.3: a ttg is never disjoint from its non-trivial quasifactors and if ϕ is a highly proximal extension, then $\mathcal{X} \not\perp Y$ for every non-trivial quasifactor $\mathcal{X}$ of X. (See the beginning of Section 2 for the definition of highly proximal extensions and 2.1.)

For our next result we need a definition: Let $\phi\colon X \to Y$ be a homomorphism of minimal ttg's. Call a quasifactor $\mathcal{X}$ of X *ϕ-sectional* whenever $\phi[A] = Y$ and $\phi[A^c] = Y$ for every $A \in \mathcal{X}$. In particular $\mathcal{X} \subseteq 2^{\perp\phi}$.

THEOREM 1.4. *Let $\phi\colon X \to Y$ be an open homomorphism of minimal ttg's. Then $\mathcal{X} \not\perp Y$ for every non-trivial quasifactor $\mathcal{X}$ of X that is not ϕ-sectional.*

PROOF. Let $\mathcal{X}$ be a quasifactor of X. For the case that $\mathcal{X} \cap 2^{\perp\phi} = \emptyset$ see 1.3.b. Let $\mathcal{X} \subseteq 2^{\perp\phi}$ be non-trivial and not ϕ-sectional. Define the subset $\mathcal{A}$ of $\mathcal{X} \times Y$ by $\mathcal{A} = \{(A,y) \in \mathcal{X} \times Y \mid \overset{\leftarrow}{\phi}(y) \subseteq A\}$. Then $\mathcal{A}$ is a nonempty proper subset of $\mathcal{X} \times Y$. Indeed, there are $B \in \mathcal{X}$ and $y_0 \in Y$ with $\phi[B] = Y$ and $y_0 \notin \phi[B^c]$; hence $\overset{\leftarrow}{\phi}(y_0) \subseteq B$. Moreover for $x \in A \in \mathcal{X}$ we have $\overset{\leftarrow}{\phi}(\phi(x)) \not\subseteq A$. Finally $\mathcal{A}$ is invariant under M, because if $(A,y) \in \mathcal{A}$ then $\overset{\leftarrow}{\phi}(py) = p \circ \overset{\leftarrow}{\phi}(y)$ ($\overset{\leftrightarrow *}{\phi}$ is a homomorphism, since ϕ is open) so $\overset{\leftarrow}{\phi}(py) \subseteq p \circ A$ and $p(A,y) = (p \circ A, py) \in \mathcal{A}$ for all $p \in M$. Application of 1.2 on $\mathcal{X} \times Y$ and $\mathcal{A}$ concludes the proof. □

For the main theorem of this section we have to recall the following fact. Let $\phi\colon X \to Y$ be a homomorphism of minimal ttg's and fix $u \in J$. Choose $x_0 \in X$ with $ux_0 = x_0$ and put $y_0 = \phi(x_0)$. Let $\mathcal{F}$ be the Ellis group of Y relative y_0; then ϕ is distal iff $\overset{\leftarrow}{\phi}(py_0) = p\mathcal{F}x_0$ for every $p \in M$. In particular $p\mathcal{F}x_0 = p \circ \mathcal{F}x_0$ (e.g. [9], I. 4.1).

THEOREM 1.5. *Let $\phi\colon X \to Y$ be a distal homomorphism of minimal ttg's. Then every non-trivial quasifactor of X that is disjoint from Y is distal.*

*)PROOF. Every distal homomorphism is open, so by 1.4 we may restrict our attention to ϕ-sectional quasifactors $\mathcal{X}$ of X. So let $\mathcal{X}$ be a quasifactor of X and $\mathcal{X} \perp Y$, and fix $x_0 \in X$, $y_0 \in Y$ and $\mathcal{F} \subseteq uM$ as above. Since $\mathcal{X}$ is ϕ-sectional we have for every $B \in \mathcal{X}$ and $p \in M$.

$$(1.6) \qquad B \cap p\mathcal{F}x_0 \neq \emptyset \quad \text{and} \quad B^c \cap p\mathcal{F}x_0 \neq \emptyset.$$

Now $\mathcal{X} = \mathrm{QF}(A,X)$ for an $A \in \mathcal{X}$ with $x_0 \in A = u \circ A$; for $\mathcal{X} = \mathrm{QF}(B,X)$ for some $B \in 2^X$ so by 1.6 there is an $f \in \mathcal{F}$ with $fx_0 \in u \circ B$, and $A = f^{-1} \circ B \in \mathcal{X}$ satisfies the requirement. Since $\mathcal{X} \times Y$ is minimal we may write $\mathcal{X} \times Y = \{(p \circ A, py_0) \mid p \in M\}$, and (A,y_0) is a u-invariant element of $\mathcal{X} \times Y$.

Define $\psi\colon \mathcal{X} \times Y \to 2^X$ by $(p \circ A, py_0) \mapsto (p \circ A) \cap \overset{\leftarrow}{\phi}(py_0)$. Clearly ψ is well defined and equivariant (commutes with the actions on $\mathcal{X} \times Y$ and 2^X). We claim that ψ is continuous too, and so ψ is a homomorphism of minimal

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*) See page 19 Added in proof.

ttg's. First observe that $(p \circ A) \cap \overset{\leftarrow}{\phi}(py_0) = (p \circ A) \cap p\mathcal{F}x_0 = p(A \cap \mathcal{F}x_0) = p \circ (A \cap \mathcal{F}x_0)$. For let $x \in (p \circ A) \cap \overset{\leftarrow}{\phi}(py_0) = (p \circ A) \cap p\mathcal{F}x_0$; say $x = pfx_0 \in p \circ A$ for some $f \in \mathcal{F}$. Since $fx_0 \in up^{-1}(p \circ A) \subseteq up^{-1} \circ (p \circ A) = u \circ A = A$ it follows that $x \in p(A \cap \mathcal{F}x_0)$. As it is clear that $p(A \cap \mathcal{F}x_0) \subseteq p \circ (A \cap \mathcal{F}x_0) \subseteq (p \circ A) \cap (p \circ \mathcal{F}x_0) = (p \circ A) \cap p\mathcal{F}x_0$ this proves our observation. It follows that $\psi[\mathcal{X} \times Y] = \{p \circ (A \cap \mathcal{F}x_0) \mid p \in M\} = \mathcal{QF}(A \cap \mathcal{F}x_0, X)$. Let $\alpha: M \to \mathcal{X} \times Y$ be defined by $\alpha(p) = (p \circ A, py_0)$ and $\gamma: M \to \mathcal{QF}(A \cap \mathcal{F}x_0, X)$ by $\gamma(p) = p \circ (A \cap \mathcal{F}x_0)$. Then α and γ are quotient maps and $\psi \circ \alpha = \gamma$, so ψ is continuous, which proves our claim.

Now define $\tilde{\phi}: \mathcal{QF}(A \cap \mathcal{F}x_0, X) \to Y$ by $\tilde{\phi}((p \circ A) \cap p\mathcal{F}x_0) = py_0$. Since $(p \circ A) \cap p\mathcal{F}x_0 = (q \circ A) \cap q\mathcal{F}x_0$ implies $p\mathcal{F}x_0 \cap q\mathcal{F}x_0 \neq \emptyset$ and so $py_0 = qy_0$, it follows that $\tilde{\phi}$ is well defined. In addition $\tilde{\phi}$ is equivariant and by a similar argument as for the continuity of ψ, $\tilde{\phi}$ is continuous and so $\tilde{\phi}$ is a homomorphism of minimal ttg's. Next we show that $\tilde{\phi}$ is not injective. To this end, use 1.6 in order to choose $p \in M$ and $f \in \mathcal{F}$ with $pfx_0 \notin p \circ A$. There exists a $q \in M$ with $q \circ A \neq p \circ A$ and $pfx_0 \in q \circ A$ so $(p \circ A) \cap p\mathcal{F}x_0 \neq (q \circ A) \cap p\mathcal{F}x_0$. As $(q \circ A, py_0) \in \mathcal{X} \times Y$ we can find an $r \in M$ with $(q \circ A, py_0) = r(A, y_0) = (r \circ A, ry_0)$ and consequently $(q \circ A) \cap p\mathcal{F}x_0 = (r \circ A) \cap r\mathcal{F}x_0$. It follows that $p\mathcal{F}x_0 \cap r\mathcal{F}x_0 \neq \emptyset$ and so $py_0 = ry_0$, whereas $(p \circ A) \cap p\mathcal{F}x_0 \neq (r \circ A) \cap r\mathcal{F}x_0 = (q \circ A) \cap p\mathcal{F}x_0$.

Finally we show that $\tilde{\phi}$ is distal, or equivalently, $\overset{\leftarrow}{\tilde{\phi}}(py_0) = p\mathcal{F}(A \cap \mathcal{F}x_0)$ for every $p \in M$. If $f \in \mathcal{F}$ then $\tilde{\phi}(pf(A \cap \mathcal{F}x_0)) = \tilde{\phi}((pf \circ A) \cap p\mathcal{F}x_0) = py_0$, so $p\mathcal{F}(A \cap \mathcal{F}x_0) \subseteq \overset{\leftarrow}{\tilde{\phi}}(py_0)$. Conversely consider $q \in M$ with $qy_0 = \tilde{\phi}(q(A \cap \mathcal{F}x_0)) = py_0$. Then $p\mathcal{F}x_0 = q\mathcal{F}x_0$ and $up^{-1}q \in \mathcal{F}$. Now choose $v \in J$ with $vp = p$ then; $vq = pf$ for $f = up^{-1}q \in \mathcal{F}$, and $vq(A \cap \mathcal{F}x_0) = pf(A \cap \mathcal{F}x_0)$. Since $q(A \cap \mathcal{F}x_0) \subseteq q\mathcal{F}x_0 = p\mathcal{F}x_0 = vp\mathcal{F}x_0 \subseteq vMx_0$ we know that $vq(A \cap \mathcal{F}x_0) = q(A \cap \mathcal{F}x_0)$, so $q(A \cap \mathcal{F}x_0) = pf(A \cap \mathcal{F}x_0) \subseteq p\mathcal{F}(A \cap \mathcal{F}x_0)$; therefore $\overset{\leftarrow}{\tilde{\phi}}(py_0) \subseteq p\mathcal{F}(A \cap \mathcal{F}x_0)$ and ϕ is distal.

If we consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{X} \times Y \arrow[rr, "\psi"] \arrow[dr, "\pi_2"'] & & \mathcal{QF}(A \cap \mathcal{F}x_0, X) \arrow[dl, "\tilde{\phi}"] \\
& Y &
\end{tikzcd}$$

with $\pi_2(p \circ A, py_0) = py_0$ for $p \in M$, we may conclude from Lemma II.3 of [1] that $\mathcal{X}$ has a non-trivial distal factor, say W. In fact W is obtained as a

quasifactor of $\mathrm{QF}(A\cap Fx_0,X)$ as follows:
Define $\psi^{\#}\colon \mathcal{X} \to 2^{\mathrm{QF}(A\cap Fx_0,X)}$ by $\psi^{\#}(p\circ A) = \{(p\circ A)\cap rFx_0 \mid r\in M \text{ and } r\circ A = p\circ A\}$. Then $\psi^{\#}$ is a homomorphism and W is defined as $\psi^{\#}(\mathcal{X})$. But $\psi^{\#}$ is injective; for let $p\circ A \neq q\circ A$, $x\in (p\circ A)\setminus(q\circ A)$ and $r\in M$ with $x\in rFx_0$. Then $(p\circ A)\cap rFx_0 \neq (q\circ A)\cap rFx_0$. Since $\{mFx_0 \mid m\in M\}$ is a partition of X we may conclude that $(p\circ A)\cap rFx_0 \notin \psi^{\#}(q\circ A)$ and so $\psi^{\#}(p\circ A)\neq\psi^{\#}(q\circ A)$. The compactness of $\mathcal{X}$ and W now gives $\mathcal{X}\cong W$, and consequently $\mathcal{X}$ is distal. □

We do not have to hope for an analogue of 1.3(a) where we compare quasifactors $\mathcal{X}$ of X with Y. The following example shows that if $\phi\colon X\to Y$ is distal, then a ϕ-sectional quasifactor $\mathcal{X}$ of X can be disjoint from Y. Let S be the unit circle and define the transformation group $(\mathbb{R}, S_p)$ by $(t,e^{i\psi}) \mapsto e^{i(\psi+pt)}$. Choose α irrational; then $(\mathbb{R},\mathbb{T}) = (\mathbb{R}, S_1\times S_\alpha)$ is a minimal torus action, and it is equicontinuous. Let $\phi\colon \mathbb{T}\to S_1$ be the projection in the first coordinate; ϕ is a homomorphism of minimal ttg's and clearly ϕ is distal. Define $A = \bar{A} = \{(e^{i\psi},1) \mid 0\le\psi<2\pi\}$. It is easy to see that $\mathrm{QF}(A,\mathbb{T})$ is a ϕ-sectional quasifactor of $\mathbb{T}$ and that it is isomorphic with $(\mathbb{R}, S_\alpha)$. Since $(\mathbb{R},\mathbb{T})$ is minimal it follows that $(\mathbb{R}, S_1)\perp(\mathbb{R}, S\ )$ and so $(\mathbb{R},S_1)\perp$ $\perp(\mathbb{R}, \mathrm{QF}(A,\mathbb{T}))$. Observe that the obvious fact that $\mathrm{QF}(A,\mathbb{T})$ is distal is in accordance with 1.5.

2. QUASIFACTORS AND DISJOINTNESS CLASSES

In [2] the authors gave a fruitful generalization of almost one-to-one extensions, the so-called highly proximal extensions (h.p. extension for short). We shall first summarize a few aspects of it that are useful for our purpose: the characterization of disjointness classes in terms of quasifactors.

Let $\phi\colon X\to Y$ be a homomorphism of minimal ttg's; ϕ will be called *highly proximal* (h.p.) if for some $y\in Y$ there is a net $\{t_n\}$ in T, such that the net $\{t_n\overset{\leftarrow}{\phi}(y)\}$ tends to a singleton in the hyperspace topology. For the proofs of the following lemmas we refer to [2].

LEMMA 2.1. *For a homomorphism* $\phi\colon X\to Y$ *of minimal ttg's the following are equivalent:*

(a) ϕ *is an* h.p. *extension.*

(b) *Every non-empty open subset of* X *contains a fiber* $\overset{\leftarrow}{\phi}(y)$ *for some* $y\in Y$.

(c) $2^{\perp\phi} = \{X\}$.

(d) *If* $y\in Y$, $x\in\overset{\leftarrow}{\phi}(y)$ *and* $p\in M$, *then* $p\circ\overset{\leftarrow}{\phi}(y) = \{px\}$.

The collection of all minimal ttg's can be partitioned in h.p. equivalence classes; two minimal ttg's are called h.p. equivalent if they have a common extension via h.p. extensions. Every equivalence class contains a unique maximal element: the maximal h.p. extension of each of the members of the equivalence class. Such a minimal ttg will be called *maximally highly proximal*. If $\gamma: M \to X$ is an extension, then for every $x_0 \in X$, $X^* = \mathrm{QF}(\gamma^{\leftarrow}(x_0), M)$ is the maximally highly proximal extension of all members of the equivalence class of X (X^* is independent of the choice of $x_0 \in X$). Similar to [5], Prop. 8.3, we have:

LEMMA 2.2. *The following are equivalent for a minimal ttg* X.
(a) X *is maximally highly proximal (i.e.* $X = X^*$*).*
(b) X *is an open image of* M.
(c) *Every homomorphism* $\phi: Y \to X$ *of minimal ttg's is open.*

The relation between h.p. extensions and disjointness is given by

LEMMA 2.3. *Let* X_1, X_2 *and* Y_1, Y_2 *be two h.p. equivalent pairs of minimal ttg's; then* $X_1 \perp Y_1$ *iff* $X_2 \perp Y_2$. *In particular* $X \perp Y$ *iff* $X^* \perp Y$.

THEOREM 2.4. *Let* X *and* Y *be minimal ttg's.*
(a) *if* Y *is a factor of* X, *then* Y^* *is a factor of* X^*.
(b) $X \not\perp Y$ *iff* Y *has a non-trivial quasifactor, which is a factor of* X^*.

PROOF. (a) is Theorem I.1(iii) of [2], and the "only if" in (b) is Lemma II.4 of [2]. The "if-part" of (b) is a simple Corollary of 1.3 and 2.3. □

Finally we recall Corollary II.1 of [2]: Let X, X_1 and Y be minimal ttg's.

2.5. If X_1 is a proximal extension of X (an extension via a proximal homomorphism) and X_1 has a distal factor Y, then Y is a factor of X.

Let $\mathcal{K}$ be a family of minimal ttg's; we denote by $[\mathcal{K}]$ the smallest collection $\mathcal{L}$ of minimal ttg's with:
(i) $\mathcal{K} \subseteq \mathcal{L}$;
(ii) If $X \in \mathcal{L}$ and $\phi: Y \to X$ is an h.p. extension, then $Y \in \mathcal{L}$;
(iii) If $X \in \mathcal{L}$ and $\phi: X \to Y$ is a homomorphism, then $Y \in \mathcal{L}$.
The following lemma characterizes $[\mathcal{K}]$.

LEMMA 2.6.

(a) $[\mathcal{K}] = \{Z \mid Z$ *is a factor of* Y^* *for some* $Y \in \mathcal{K}\}$;

(b) $[\mathcal{K}^{\perp}] = \mathcal{K}^{\perp}$ *and* $[\mathcal{K}] \subseteq \mathcal{K}^{\perp\perp}$

PROOF.

(a) Clearly $\{Z \mid Z$ is a factor of Y^* for some $Y \in \mathcal{K}\}$ is closed under factors and contains $\mathcal{K}$. Let $\phi\colon Y^* \to Z$ be a homomorphism for some $Y \in \mathcal{K}$ and let $\psi\colon Z' \to Z$ be an h.p. extension for a minimal Z'. From 2.4(a) we know that Z^* is a factor of Y^*. Since Z' and Z are h.p. equivalent, Z' is a factor of Z^*, hence of Y^*. Now $\{Z \mid Z$ is a factor of Y^* for some $Y \in \mathcal{K}\}$ satisfies (i), (ii) and (iii), and clearly it is minimal under these conditions.

(b) Follows from 2.3 and the obvious fact that if $X \perp Y$ then every factor of X is disjoint from Y.

EXAMPLES.

(i) Let $\mathcal{D}$ be the collection of minimal distal ttg's and D the universal minimal distal ttg (the phase group is fixed and understood). Then $[\mathcal{D}] = [\{D\}] = \{Z \mid Z$ is a factor of $D^*\}$.

(ii) Let $\mathcal{P}$ be the collection of minimal proximal ttg's and P the universal minimal proximal ttg. Then $\mathcal{P} = [\mathcal{P}] = [\{P\}]$.

(iii) Let F be a τ-closed subgroup of $\mathsf{G} = uM$. Then $\mathsf{QF}(u \circ \mathsf{F}, M)$ is the universal minimal proximal extension of minimal ttg's with Ellis group F ([9], IX. 3.3(2)). Now $[\{\mathsf{QF}(u \circ \mathsf{F}, M)\}] = \mathcal{M}(\mathsf{F}) = \{Y \mid Y$ is minimal and $\mathfrak{G}(Y,y_0) \supseteq \mathsf{F}$ for some $y_0 = uy_0 \in Y\}$.

To prove this we need the following definition and fact ([9], X.1.1). Let $\phi\colon X \to Y$ be a homomorphism of minimal ttg's that respects the base points $x_0 = ux_0 \in X$ and $y_0 = uy_0 \in Y$ and let $\mathsf{F} = \mathfrak{G}(Y,y_0)$. ϕ is called a RIC-*extension* if for every $p \in M$, $\phi^{\leftarrow}(py_0) = p \circ \mathsf{F}x_0$. Every RIC-extension is an open map. Since $\mathsf{QF}(u \circ \mathsf{F}, M)$ is an image of M under a RIC-extension it follows from 2.2 that $\mathsf{QF}(u \circ \mathsf{F}, M)$ is maximally highly proximal. Since for every τ-closed subgroup $\mathsf{F}' \supseteq \mathsf{F}$, $\mathsf{QF}(u \circ \mathsf{F}', M)$ is an image of $\mathsf{QF}(u \circ \mathsf{F}, M)$ (under a RIC-extension defined by $p \circ \mathsf{F} \to p \circ \mathsf{F}'$), and every minimal Y with Ellis group F' is an image of $\mathsf{QF}(u \circ \mathsf{F}', M)$ (under a proximal extension), it follows that $[\{\mathsf{QF}(u \circ \mathsf{F}, M)\}] = \mathcal{M}(\mathsf{F})$. Note that $\mathsf{QF}(u \circ \mathsf{G}, M) = P$ and $\mathcal{M}(\mathsf{G}) = \mathcal{P}$.

Also observe that if $\mathsf{QF}(u \circ \mathsf{G}, M) \neq \{*\}$ then for every τ-closed subgroup K of G we have that $\mathsf{QF}(u \circ \mathsf{K}, M) \notin \mathcal{M}(\mathsf{F})^{\perp}$, for $\mathsf{QF}(u \circ \mathsf{K}, M)$ and $\mathsf{QF}(u \circ \mathsf{F}, M)$ have $\mathsf{QF}(u \circ \mathsf{G}, M)$ as a non-trivial common factor.

THEOREM 2.7. *Let* $\mathcal{K}$ *be a family of minimal ttg's. For a minimal ttg* X *the following are equivalent:*

(a) $X \in \mathcal{K}^{\perp}$;

(b) $X \in [\mathcal{K}]^{\perp}$;

(c) $\mathcal{X} \notin [\mathcal{K}]$ *for every non-trivial quasifactor* $\mathcal{X}$ *of* X.

PROOF. (c) ⇒ (b) Assume the existence of a $Z \in [\mathcal{K}]$ with $X \not\perp Z$. Then (2.4(b)) X has a non-trivial quasifactor $\mathcal{X}$ which is a factor of Z^* and consequently $\mathcal{X} \in [\mathcal{K}]$. (b) ⇒ (a) Since $\mathcal{K} \subseteq [\mathcal{K}]$ we know $[\mathcal{K}]^{\perp} \subseteq \mathcal{K}^{\perp}$. (a) ⇒ (c) Suppose that $\mathcal{X} \in [\mathcal{K}]$ for some non-trivial quasifactor $\mathcal{X}$ of X. Then there is a $Y \in \mathcal{K}$ such that $\mathcal{X}$ is a factor of Y^*, so $X \not\perp Y$ (2.4(b)) and $X \notin \mathcal{K}^{\perp}$. □

Denote the family of almost periodic minimal ttg's with $\mathcal{AP}$.

LEMMA 2.8.

(a) $X \in \mathcal{D}$ *iff every non-trivial quasifactor of* X *is distal.*

(b) $X \in \mathcal{AP}$ *iff every non-trivial quasifactor of* X *is almost periodic.*

PROOF.

(a) Theorem 1.5 with Y trivial.

(b) By [12], $X \in \mathcal{AP}$ iff 2^X is almost periodic (and X minimal). In addition, if 2^X is almost periodic then every non-trivial quasifactor of X is almost periodic, and this in turn implies $X \in \mathcal{AP}$. □

THEOREM 2.9. *Let* $\mathcal{K}$ *be* $\mathcal{D}$ *or* $\mathcal{AP}$ *and let* K *be the universal minimal* $\mathcal{K}$*-ttg. Then for minimal* X *the following are equivalent:*

(a) $X \in \mathcal{K}^{\perp}$;

(b) $X \perp K$;

(c) X *admits no non-trivial factors in* $\mathcal{K}$;

(d) X *admits no non-trivial quasifactors in* $[\mathcal{K}]$

PROOF. The equivalence of (a), (b) and (d) is just 2.7, and (a) ⇒ (c) is trivial. (c) ⇒ (a) Suppose $X \notin \mathcal{K}^{\perp}$ then $X \not\perp Y$ for some $Y \in \mathcal{K}$. According to 2.4(b) there exists a non-trivial quasifactor $\mathcal{Y}$ of Y, which is a factor of X^*. Since $\mathcal{Y} \in \mathcal{K} \subseteq \mathcal{D}$ (2.8) and using (2.5) $\mathcal{Y}$ turns out to be a non-trivial $\mathcal{K}$-factor of X. □

COROLLARY 2.10. *Let* $\mathcal{K}$ *be* $\mathcal{D}$ *or* $\mathcal{AP}$, *and let* X *be a minimal ttg. Then* $X \in \mathcal{K}^{\perp\perp}$ *iff every non-trivial quasifactor of* X *has a non-trivial* $\mathcal{K}$*-factor.*

COROLLARY 2.11. *$\mathcal{D}^{\perp\perp} = \mathcal{AP}^{\perp\perp}$ and consequently $\mathcal{D}^{\perp} = \mathcal{AP}^{\perp}$.*

PROOF. Since $\mathcal{AP} \subseteq \mathcal{D}$ it is obvious that $\mathcal{AP}^{\perp\perp} \subseteq \mathcal{D}^{\perp\perp}$. Let $X \in \mathcal{D}^{\perp\perp}$; then every non-trivial quasifactor $\mathcal{X}$ of X has a non-trivial distal factor. In [6] ELLIS proved (without countability assumptions) that every non-trivial minimal point-distal ttg (and, consequently, every non-trivial element of $\mathcal{D}$) admits a non-trivial $\mathcal{AP}$-factor. From this and 2.10 it is clear that $\mathcal{D}^{\perp\perp} \subseteq \mathcal{AP}^{\perp\perp}$ so $\mathcal{D}^{\perp\perp} = \mathcal{AP}^{\perp\perp}$ and $\mathcal{D}^{\perp} = \mathcal{D}^{\perp\perp\perp} = \mathcal{AP}^{\perp\perp\perp} = \mathcal{AP}^{\perp}$.

Note that every non-trivial minimal distal ttg also has a non-trivial $\mathcal{AP}$-quasifactor. For let $\{*\} \neq X \in \mathcal{D}$ and let Y be a non-trivial $\mathcal{AP}$-factor of X. Then Y is an open image of X and so Y is a quasifactor of X.

THEOREM 2.12. *Let F be a τ-closed subgroup of $G = uM$, and let X be minimal. The following statements are equivalent.*

(a) $X \in \mathcal{M}(F)^{\perp}$;

(b) $X \perp QF(u \circ F, M)$;

(c) X *has no non-trivial quasifactor with Ellis group* $F' \supseteq F$;

(d) $u \circ Fx = X$ *for all* $x \in X$.

If F is a τ-closed normal subgroup of G we may replace (d) *by:*

(d') *There is an* $x \in X$ *with* $u \circ Fx = X$.

PROOF. The equivalence of (a), (b) and (c) is trivial from 2.7 and the foregoing example (iii). (c) ⇒ (d) For arbitrary $x \in X$, $QF(u \circ Fx, X)$ is a quasifactor of X whose Ellis group contains F, for $QF(u \circ Fx, X)$ is a factor of $QF(u \circ F, M)$ by way of the homomorphism defined by $p \circ F \to p \circ Fx$. Our assumption forces $QF(u \circ Fx, X)$ to be trivial. It then follows that $u \circ Fx = X$.
(d) ⇒ (c) Suppose that $\mathcal{X}$ is a quasifactor of X with Ellis group $F' \subseteq F$ and so a factor of $QF(u \circ F, M)$. Then we may assume that $\mathcal{X} = QF(D, X)$ for some $D \in 2^X$ with $u \circ D = D$ and that the homomorphism from $QF(u \circ F, M)$ onto $QF(D, X)$ is given by $p \circ F \to p \circ D$. Since $q \in p \circ F$ iff $q \circ F = p \circ F$, we know that $(p \circ F) \circ D = \bigcup\{q \circ D \mid q \in p \circ F\} = p \circ D$, so for every $p \in M$: $p \circ FD \subseteq (p \circ F) \circ D = {}$ $= p \circ D$. Choose $x \in D$; then $p \circ Fx \subseteq p \circ D$ and $p \circ Fx = p \circ (u \circ Fx) = p \circ X = X$. Therefore $X \subseteq D$ and $QF(D, X)$ is trivial. Now let F be a τ-closed normal subgroup of G. Since (d) trivially implies (d') we only have to check (d')⇒(c) as follows: Let $x_0 \in X$ be such that $u \circ Fx_0 = X$. F is normal, so for all $\alpha \in G$, $F = \alpha^{-1}F\alpha \subseteq \alpha^{-1} \circ F\alpha$, hence $\alpha \circ F \subseteq \alpha\alpha^{-1} \circ F\alpha = u \circ F\alpha$. With notation as in the proof of (d) ⇒ (c), we choose $x \in D$, say $x = px_0$. Then $p = v\alpha$ for some $v \in J$ and $\alpha \in G$ ([9], I.2.4). Since $\alpha \circ Fx_0 \subseteq u \circ F\alpha x_0 = u \circ Fv\alpha x_0 =$

$= u \circ \mathfrak{F}x \subseteq u \circ \mathfrak{F}D$ and $\alpha \circ \mathfrak{F}x_0 = \alpha \circ (u \circ \mathfrak{F}x_0) = \alpha \circ X = X$ it follows that $\mathrm{QF}(D,X)$ is trivial. □

Recall the definition of a RIC-extension in Example (iii). We define a minimal ttg to be *incontractible* if the trivial homomorphism $\phi: X \to \{*\}$ is a RIC-extension or equivalently X is incontractible iff $u \circ \mathfrak{G}x = X$ for some $x \in X$.

COROLLARY 2.13.

(a) *For a minimal ttg* X, *the following are equivalent:*

(i) $X \in \mathcal{P}^{\perp}$;

(ii) X *is incontractible;*

(iii) X *has no non-trivial proximal quasifactor.*

(b) *Let* X *be minimal; then* $X \in \mathcal{P}^{\perp\perp}$ *iff every non-trivial quasifactor of* X *has a non-trivial proximal quasifactor.*

PROOF. Put $\mathfrak{F} = \mathfrak{G}$ in 2.12 and remember that $\mathfrak{G}$ is a τ-closed normal subgroup of $\mathfrak{G}$.

3. EXTENSIONS, $\mathcal{D}^{\perp}$ AND $\mathcal{P}^{\perp}$

THEOREM 3.1.

(a) *If* $X \in \mathcal{D}^{\perp}$ *and* $\phi: X \to Y$ *is distal then* $X^{\perp} = Y^{\perp}$.

(b) *If* $\phi: X \to Y$ *is distal then* $\mathcal{D}^{\perp} \cap X^{\perp} = \mathcal{D}^{\perp} \cap Y^{\perp}$.

PROOF.

(a) Clearly $X^{\perp} \subseteq Y^{\perp}$. Conversely let Z be minimal and $Z \perp Y$, and suppose that $Z \not\perp X$. Then by 2.4(b) there exists a non-trivial quasifactor $\mathcal{X}$ of X, which is a factor of Z^*. Since $X \in \mathcal{D}^{\perp}$ it follows that $\mathcal{X}$ is not distal (2.9), hence 1.5 implies that $\mathcal{X} \not\perp Y$. So Y has a non-trivial quasifactor $\mathcal{Y}$ which is a factor of $\mathcal{X}^*$. Then by 2.4(a), $\mathcal{Y}$ is a factor of Z^*, so by 2.4(b) $Y \not\perp Z$, which contradicts the assumption.

(b) Let $Z \in \mathcal{D}^{\perp} \cap Y^{\perp}$ and suppose $Z \not\perp X$. Then X has a non-trivial quasifactor $\mathcal{X}$ which is a factor of Z^*. From $Z \perp Y$ we conclude that $Z^* \perp Y$ and so $\mathcal{X} \perp Y$. By 1.5 $\mathcal{X}$ must be distal, but as $\mathcal{X}$ is a factor of an element of $\mathcal{D}^{\perp}$ this is impossible.

COROLLARY 3.2. (Theorem II.1 of [2]) $\mathcal{D}^{\perp\perp}$ *is closed under distal extensions.*

PROOF. Let $\phi\colon X \to Y$ be distal, with $Y \in \mathcal{D}^{\perp\perp}$ then $\mathcal{D}^{\perp} \cap Y^{\perp} = \mathcal{D}^{\perp}$. By 3.1(b) $\mathcal{D}^{\perp} = \mathcal{D}^{\perp} \cap X^{\perp} \subseteq X^{\perp}$ and $X \in \mathcal{D}^{\perp\perp}$. □

We will now obtain the same kind of result with distal replaced by proximal (3.5), thus generalizing a result of SHAPIRO ([15], 2.4) by way of a generalization of [11], 4.5. First remember that for a (not necessarily minimal) ttg X, $x \in X$ is called an *almost periodic point* if its orbit closure is minimal.

LEMMA 3.3. *Let* X *and* Y *be minimal with Ellis groups* H *and* F *respectively in* $G = uM$. *Then* $X \perp Y$ *iff* $HF = G$ *and* $X \times Y$ *contains a dense subset of almost periodic points.*

PROOF. Recall that $x_0 \in X$ and $y_0 \in Y$ with $ux_0 = x_0$ and $uy_0 = y_0$, and that H and F are the Ellis groups of X and Y relative to x_0 and y_0 respectively. Assume $X \perp Y$; then $X \times Y$ is minimal and every element of $X \times Y$ is almost periodic. In particular $X \times Y$ is the orbit closure of $(x_0,y_0) = u(x_0,y_0)$. Choose $\gamma \in G$; then there exists a $p \in M$ with $p(x_0,y_0) = (x_0,\gamma y_0)$, so $px_0 = x_0$ and $py_0 = \gamma y_0$. Since $up \in H$ and $up^{-1}\gamma \in F$ it follows that $\gamma = u\gamma = up \cdot up^{-1}\gamma \in HF$, and $G \subseteq HF$.

Now suppose $HF = G$ and $X \times Y$ has a dense subset of almost periodic points. We prove that all periodic points are in the orbit closure of (x_0,y_0), which is a minimal ttg since $u(x_0,y_0) = (x_0,y_0)$ ([4], 3.7). The minimality of $X \times Y$ is then obvious. Let $(x,y) = (px_0,qy_0)$ be almost periodic. Then there is a $v \in J$ with $vpx_0 = px_0$ and $vqy_0 = qy_0$. Choose $\alpha,\beta \in G$ with $vp = v\alpha$ and $vq = v\beta$. Then $(x,y) = v\beta(\beta^{-1}\alpha x_0,y_0)$, and as $\beta^{-1}\alpha \in G = HF$ we may choose $h \in H$ and $f \in F$ with $\beta^{-1}\alpha = fh$. Now $(x,y) = v\beta(fhx_0,y_0) =$ $= v\beta(fx_0,y_0) = v\beta f(x_0,f^{-1}y_0) = v\beta f(x_0,y_0)$ and (x,y) is an element of the orbit closure of (x_0,y_0). □

There are several situations where $X \times Y$ has a dense set of almost periodic points; for instance if $X \times Y$ is a distal extension of a minimal ttg. Another situation is the basis of the following theorem, with notations as above.

THEOREM 3.4. *If* $X \in \mathcal{P}^{\perp}$ *and* Y *is minimal then* $X \perp Y$ *iff* $HF = G$.

PROOF. We only have to prove that the incontractibility of X implies that $X \times Y$ has a dense set of almost periodic points. This follows immediately from the remark on page 814 of [16]. □

We define a topological group T to be *strongly amenable* if there does not exist any non-trivial minimal proximal ttg with phase group T, or equivalently, if every non-trivial minimal ttg on T is incontractible ([9], II.3). For instance if T is abelian or nilpotent then T is strongly amenable From 3.4 it is obvious that for strongly amenable groups, the disjointness of X and Y is equivalent with $\mathcal{H}\mathcal{F} = \mathcal{G}$. (For T abelian see [11], 4.5).

THEOREM 3.5.

(a) *If* $X \in \mathcal{P}^{\perp}$ *and* $\phi: X \to Y$ *is proximal then* $X^{\perp} = Y^{\perp}$.

(b) *If* $\phi: X \to Y$ *is proximal then* $\mathcal{P}^{\perp} \cap X^{\perp} = \mathcal{P}^{\perp} \cap Y^{\perp}$.

PROOF.

(a) Assume $Z \perp Y$ and Z minimal. Then $\phi \times 1_Z: X \times Z \to Y \times Z$ is proximal. Since $Y \times Z$ is minimal it follows from [9], II.1 that $X \times Z$ contains a unique minimal sub-ttg. By the proof of 3.4 $X \times Z$ has a dense set of almost periodic points and so $X \times Z$ is minimal and $X \perp Z$.

(b) Suppose $Z \in \mathcal{P}^{\perp}$ and $Z \perp Y$. Let the Ellis groups of X, Y and Z in $\mathcal{G}$ be respectively $\mathcal{H}$, $\mathcal{F}$ and $\mathcal{K}$. By 3.4 $\mathcal{K}\mathcal{F} = \mathcal{G}$ and [9], I.4.1(2) implies that $\mathcal{H} = \mathcal{F}$ for a suitable choice of $x_0 = ux_0 \in X$ and $y_0 = uy_0 \in Y$. But then it follows, that $\mathcal{K}\mathcal{H} = \mathcal{G}$ and $Z \perp X$.

COROLLARY 3.6. $\mathcal{P}^{\perp\perp}$ *is closed under proximal extensions.*

PROOF. Similar to 3.2. □

4. DISJOINTNESS AND WEAK MIXING

In [14] PETERSEN characterizes the weakly mixing minimal ttg's with abelian phase group as the minimal ttg's which admit no non-trivial almost periodic factor. We shall generalize this result slightly: see 4.3 below. Recall that a ttg X is *ergodic* if X is the only closed invariant subset of X with non-empty interior, and that X is *weakly mixing* if $X \times X$ is ergodic. We denote the collection of weakly mixing minimal ttg's with $\mathcal{WM}$. We need the following two results:

THEOREM 4.1. (ELLIS [6], 1.9). *Every distal and ergodic ttg is minimal.*

THEOREM 4.2. *Let* $X \in \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp}$ *and let Y be ergodic having a dense set of almost periodic points; then* $X \times Y$ *is ergodic. In particular this applies to the case that Y is minimal.*

PROOF. This is a reformulation of [16], 2.1.6, using the fact that $\mathcal{D}^{\perp} = \mathcal{AP}^{\perp}$, 2.9 and 2.13(a). □

THEOREM 4.3. *$\mathcal{WM} \subseteq \mathcal{D}^{\perp}$ and $\mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{WM}$.*

PROOF. Let $X \in \mathcal{WM}$ and let Z be a distal factor of X. Then $Z \times Z$ is distal and ergodic (for $X \times X$ is ergodic and ergodicity is preserved under factors). So $Z \times Z$ is minimal by 4.1, hence Z is trivial and $X \in \mathcal{D}^{\perp}$ by 2.9. Now let $X \in \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp}$, then by 4.2 $X \times X$ is ergodic, so $X \in \mathcal{WM}$, and since $\mathcal{P}^{\perp} \cap \mathcal{WM} \subseteq$ $\subseteq \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} \subseteq \mathcal{WM}$ it follows that $\mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{WM}$. □

COROLLARY 4.4. *Let T be strongly amenable; then $\mathcal{WM} = \mathcal{D}^{\perp} = \mathcal{AP}^{\perp}$.*

We conclude this section with an observation about distal extensions of weakly mixing minimal ttg's but first:

LEMMA 4.5.

(a) *$\mathcal{D}^{\perp}$ is closed under proximal extensions.*

(b) *$\mathcal{P}^{\perp}$ is closed under distal extensions.*

PROOF.

(a) Let $\phi: X \to Y$ be a proximal homomorphism of minimal ttg's and $Y \in \mathcal{D}^{\perp}$. Suppose that $X \notin \mathcal{D}^{\perp}$ then by 2.9 X has a non-trivial distal factor Z. By 2.5, Z is a factor of Y, which contradicts $Y \in \mathcal{D}^{\perp}$.

(b) Let $\phi: X \to Y$ be a distal homomorphism of minimal ttg's and $Y \in \mathcal{P}^{\perp}$. Choose $Z \in \mathcal{P}$ and suppose $X \not\perp Z$. Then there is a non-trivial quasifactor $\mathcal{X}$ of X which is a factor of Z^{*}. If $\mathcal{X} \not\perp Y$, then there is a non-trivial quasifactor $\mathcal{Y}$ of Y which is a factor of $\mathcal{X}^{*}$, hence of Z^{*}. This contradicts $Z \perp Y$, so $\mathcal{X} \perp Y$. Then $\mathcal{X}$ is distal by 1.5 so Z^{*} has a distal factor, but this is impossible since $Z^{*} \in \mathcal{P} \subseteq \mathcal{D}^{\perp}$. □

COROLLARY 4.6. *Let $Y \in \mathcal{P}^{\perp} \cap \mathcal{WM}$. Then every minimal distal extension of Y without distal factor is weakly mixing.*

PROOF. Let $\phi: X \to Y$ be distal and $X \in \mathcal{D}^{\perp}$. Since $Y \in \mathcal{P}^{\perp}$ it follows from 4.5(b) that $X \in \mathcal{P}^{\perp}$, so $X \in \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{WM}$. □

5. DISJOINTNESS AND (*H*)*PI*

Let $\phi: X \to Y$ be a homomorphism of minimal ttg's. ϕ is called *strictly*-PI or X is called a *strictly*-PI *extension* of Y if there exist an ordinal ν and for every ordinal $\alpha \leq \nu$ a minimal ttg (W_α, w_α) with u-invariant base point w_α such that

(1) $W_0 = Y$ and $W_\nu = X$;

(2) for every $\alpha < \nu$, (W_α, w_α) is a factor of $(W_{\alpha+1}, w_{\alpha+1})$ under a homomorphism ϕ_α which is either proximal or almost periodic;

(3) if α is a limit ordinal then $(W_\alpha, w_\alpha) = V\{(W_\beta, w_\beta) \mid \beta < \alpha\}$;

(4) ϕ is the inverse limit of $\{\phi_\alpha\}_{\alpha<\nu}$.

Here $V\{(W_\beta, w_\beta) \mid \beta < \alpha\}$ denotes the (minimal!) orbit closure of $(w_\beta)_{\beta<\alpha}$ in $\Pi\{W_\beta \mid \beta < \alpha\}$. We shall refer to such a system $\{(W_\alpha, w_\alpha), \phi_\alpha\}$ as a *tower*.

The homomorphism $\phi: X \to Y$ of minimal ttg's is called PI or X is a PI *extension* of Y, if there exist a minimal ttg Z, a strictly-PI homomorphism $\psi: Z \to Y$ and a proximal homomorphism $\theta: Z \to X$, such that the next diagram commutes.

$$\begin{tikzcd}
Z \arrow[r, "\psi"] \arrow[d, "\theta"'] & Y \\
X \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Observe that in [16] a PI extension is what we call a strictly-PI extension. The reason for our denomination is the following:

A minimal ttg X is called (*strictly*-)PI if X is a (strictly-)PI extension of the trivial ttg $\{*\}$, and this is equivalent to the definition of a (strictly)-PI ttg in [7], [9]. In the same way we define (*strictly*-)HPI homomorphisms and -ttg's, by replacing proximal by highly proximal in (2) of the description of the tower. For more details see [7], [9], [16] and [2].

We intend to determine $PI^\perp$ and $HPI^\perp$, where PI and HPI denote the collections of all PI- and all HPI ttg's respectively.

First, define for a τ-closed subgroup F of G, H(F) to be the smallest τ-closed normal subgroup of F such that F/H(F) with the quotient topology is a compact Hausdorff topological group. Let F_∞ be the τ-closed normal subgroup of F that is the inverse limit of the sequence H(F),H(H(F)),... .

THEOREM 5.1. *Let* $\phi\colon X \to Y$ *be a homomorphism of minimal ttg's. Then* ϕ *is* PI *iff* $H \supseteq F_\infty$, *where* H *and* F *are the Ellis groups of* X *and* Y *respectively.*

PROOF. This is a relativized version of X.4.2 of [9]. □

From 5.1 it follows immediately that if $\phi = \theta \circ \psi$ is PI then also θ is PI.

LEMMA 5.2.

(a) $\mathcal{PI} = [\mathcal{PI}] = \mathcal{M}(G_\infty)$.

(b) $\mathcal{HPI} = [\mathcal{HPI}]$.

(c) $X \in \mathcal{PI}^\perp$ *iff* $u \circ G_\infty x_0 = X$.

(d) $\mathcal{PI}^\perp \subseteq \mathcal{P}^\perp \cap \mathcal{D}^\perp$ *and* $\mathcal{HPI}^\perp \subseteq \mathcal{D}^\perp$.

PROOF.

(a) [9], X.4.2.

(b) [2], Corollary III.1.

(c) Follows from (a) and 2.12(d').

(d) Since $G \supseteq G_\infty$ it follows from (c) and 2.13(a) that $\mathcal{PI}^\perp \subseteq \mathcal{P}^\perp$. In [6] ELLIS proved that every minimal distal ttg is an inverse limit of almost periodic extensions and so $\mathcal{D} \subseteq \mathcal{HPI} \subseteq \mathcal{PI}$, thus $\mathcal{PI}^\perp \subseteq \mathcal{HPI}^\perp \subseteq \mathcal{D}^\perp$ and $\mathcal{PI}^\perp \subseteq \mathcal{P}^\perp \cap \mathcal{D}^\perp$. □

We shall now prove a PI-analogue of 3.1 and 3.5. But first observe that if $\mathcal{K}$ is a collection of minimal ttg's, then $\mathcal{K}^\perp$ is closed under inverse limits, as $X \times \vee Y_\alpha \cong \vee(X \times Y_\alpha)$ for $X \in \mathcal{K}$ and $Y_\alpha \in \mathcal{K}^\perp$.

THEOREM 5.3.

(a) *If* $X \in \mathcal{PI}^\perp$ *and* $\phi\colon X \to Y$ *is strictly*-PI *then* $X^\perp = Y^\perp$.

(b) *If* $\phi\colon X \to Y$ *is strictly*-PI, *then* $\mathcal{PI}^\perp \cap X^\perp = \mathcal{PI}^\perp \cap Y^\perp$.

PROOF.

(a) Clearly every (W_α, w_α) in the tower of ϕ is in $\mathcal{PI}^\perp$. Since $\mathcal{PI}^\perp \subseteq \mathcal{P}^\perp \cap \mathcal{D}^\perp$ and every ϕ_α is either proximal or almost periodic (hence distal) it follows from 3.1(a) and 3.5(a) that for every α, $W_\alpha^\perp = Y^\perp$ and so $X^\perp = Y^\perp$.

(b) Follows in a similar way from 3.1(b) and 3.5(b). □

COROLLARY 5.4. $\mathcal{PI}^{\perp\perp}$ *is closed under* PI *extensions.*

PROOF. Similarly to 3.2 it can be shown that $\mathcal{PI}^{\perp\perp}$ is closed under strictly-PI extensions. Let $\phi\colon X \to Y$ be a PI extension, then there exist a minimal ttg Z and a strictly-PI extension $\psi\colon Z \to Y$, such that X is a factor of Z. Now we may conclude that if $Y \in \mathcal{PI}^{\perp\perp}$ then also $Z \in \mathcal{PI}^{\perp\perp}$ and so $X \in \mathcal{PI}^{\perp\perp}$. □

THEOREM 5.5.
(a) $\mathcal{D}^{\perp} = \mathcal{HPI}^{\perp}$.
(b) $\mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{PI}^{\perp}$.

PROOF. We only have to prove the converse of 5.2(d). Let $W \in \mathcal{D}^{\perp}$ and let X be a strictly-HPI ttg. If we apply 3.1(b) to the almost periodic steps in the tower of X and 2.3 to the highly proximal ones, it follows that $X \perp W$. By 5.2(b) it is clear that this implies $\mathcal{HPI}^{\perp} \supseteq \mathcal{D}^{\perp}$.

Let $W \in \mathcal{D}^{\perp} \cap \mathcal{P}^{\perp}$ and let X be a strictly-PI ttg. Using 3.1(b) for the almost periodic steps and 3.5(b) for the proximal steps, we see that $X \perp W$, therefore $\mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} \subseteq \mathcal{PI}^{\perp}$. □

COROLLARY 5.6.
(a) $\mathcal{PI}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{WM} = \mathcal{P}^{\perp} \cap \mathcal{D}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{AP}^{\perp} = \mathcal{P}^{\perp} \cap \mathcal{HPI}^{\perp}$.
(b) *If* T *is strongly amenable then* $\mathcal{PI}^{\perp} = \mathcal{WM} = \mathcal{D}^{\perp} = \mathcal{AP}^{\perp} = \mathcal{HPI}^{\perp}$
(c) *If* $X \in \mathcal{P}^{\perp}$, *then* $X \in \mathcal{WM}$ *iff* $\mathfrak{H}\mathfrak{G}_{\infty} = \mathfrak{G}$ ($\mathfrak{H}$ *is the Ellis group of* X).

PROOF. (a), (b) are clear; (c) follows from 5.2(a) and 3.4. □

6. DISJOINTNESS AND RELATIVE-PRIMENESS

We will now turn to some variations on the theme of whether relative primeness implies disjointness. It is well-known that in general this is not true, so the problem is to search for conditions which are sufficient for this implication to hold. As before $\mathfrak{G} = uM$, and Ellis groups are subgroups of $\mathfrak{G}$. Let $\mathcal{K}^{\perp c}$ denote the complement of $\mathcal{K}^{\perp}$ in the collection of all minimal ttg's and remember that two minimal ttg's are called *relatively prime* if they have no non-trivial common factor.

LEMMA 6.1. *Let* X *and* Y *be minimal with Ellis groups* $\mathfrak{H}$ *and* $\mathfrak{F}$ *and let* $\mathfrak{K}$ *be the smallest* τ*-closed subgroup of* $\mathfrak{G}$ *containing* $\mathfrak{H} \cup \mathfrak{F}$. *If* $\mathrm{QF}(u \circ \mathfrak{K}, M) \in \mathcal{D}^{\perp c}$ *then* X *and* Y *are not relatively prime by a non-trivial common distal factor.*

PROOF. Since $\mathrm{QF}(u \circ \mathfrak{K}, M) \in \mathcal{D}^{\perp c}$ it has a non-trivial distal factor $\tilde{\phi}\colon \mathrm{QF}(u \circ \mathfrak{K}, M) \to Z$. Define $\phi\colon X \to Z$ by $px_0 \mapsto \tilde{\phi}(p \circ \mathfrak{K})$ and $\psi\colon Y \to Z$ by

$py_0 \mapsto \tilde{\phi}(p \circ K)$. It suffices to prove that ϕ and ψ are well defined, for then they are obviously continuous, equivariant surjections (preserving base-points). Let p and q in M be such that $px_0 = qx_0$. Then $up^{-1}q \in H \subseteq K$, so $p \circ K$ and $q \circ K$ are proximal in $QF(u \circ K, M)$. Since Z is distal it follows that $\tilde{\phi}(p \circ K) = \tilde{\phi}(q \circ K)$ and $\phi(px_0) = \phi(qx_0)$. Similarly ψ is well defined. □

LEMMA 6.2. *Each of the following conditions implies that* $QF(u \circ K, M) \in \mathcal{D}^{\perp c}$.

(a) $\mathcal{D}^{\perp\perp} \cap M(K)^{\perp c} \neq \emptyset$.

(b) $M(G_\infty K) \cap \mathcal{P}^{\perp}$ *contains a non-trivial ttg.*

PROOF. Let $Z \in \mathcal{D}^{\perp\perp} \cap M(K)^{\perp c}$. By 2.12 there exists a $\bar{z} \in Z$ with $u \circ K\bar{z} \neq Z$. So $QF(u \circ K\bar{z}, Z)$ is non-trivial and by 2.10 it has a non-trivial distal factor. Obviously this implies that $QF(u \circ K, M)$ has a non-trivial distal factor and $QF(u \circ K, M) \in \mathcal{D}^{\perp c}$. Let $Z \in M(G_\infty K) \cap \mathcal{P}^{\perp}$ be non-trivial. Its Ellis group contains $G_\infty K$ and so it contains G_∞. Therefore Z is an incontractible PI ttg (5.2(a)) and has a non-trivial almost periodic factor ([9], X.4.4). This is also a factor of $QF(u \circ K, M)$, hence $QF(u \circ K, M) \in \mathcal{D}^{\perp c}$. □

For the following theorems we need the introduction of regular minimal ttg's. We call a minimal ttg X *regular* if its Ellis group H is a normal subgroup of G. In that case for all $x \in X$ with $ux = x$ we have $\mathfrak{G}(X,x) = H$. For an explicit treatment of regular minimal ttg's we refer to [1].

THEOREM 6.3. *Let* X *and* Y *be minimal ttg's with* X *or* Y *regular and* $X \in \mathcal{D}^{\perp\perp}$. *Then* $X \perp Y$ *iff* X *and* Y *are relatively prime.*

PROOF. Suppose X of Y is regular, $X \in \mathcal{D}^{\perp\perp}$ and $X \not\perp Y$. With notation as in 6.1 it is clear that $K = HF = FH$ is a τ-closed subgroup of G ([9], IX.1.10). As $X \not\perp Y$ we have $X \not\perp QF(u \circ K, M)$, since Y is a factor of $QF(u \circ K, M)$. In the case that F is a normal subgroup $X \neq u \circ Fx_0 = u \circ FHx_0 = u \circ Kx_0$ (2.12 (d')). If H is normal then $\alpha H \alpha^{-1} = H$ for all $\alpha \in G$. In this case, there exists by 2.12(d) an $\bar{x} \in X$ with $u \circ F\bar{x} \neq X$. Let $w \in J$ and $\alpha \in G$ be such that $\bar{x} = w\alpha x_0$. Then $\mathfrak{G}(X, u\bar{x}) = \alpha H \alpha^{-1} = H$ and $X \neq u \circ F\bar{x} = u \circ Fu\bar{x} = u \circ FH\bar{x} = u \circ K\bar{x}$. So in both cases we can find an $\bar{x} \in X$ such that $\mathcal{X} = QF(u \circ K\bar{x}, X)$ is a non-trivial quasifactor of X. Since $X \not\perp \mathcal{X}$ (1.3) and $\mathfrak{G}(\mathcal{X}, u \circ K\bar{x}) \supseteq K$ it follows that $\mathcal{X} \in M(K)$ and $X \in M(K)^{\perp c}$. The proof is finished by applying 6.1 and 6.2(a). The other way around is trivial. □

The following consequence of 6.2(b) can also be found in [8] in a somewhat weaker version.

THEOREM 6.4. *Let* X *and* Y *be minimal with Ellis groups* H *and* F, *with* X *or* Y *regular, such that* $\mathsf{G}_\infty \subseteq \mathsf{HF}$ *and every quasifactor of* X *is incontractible. Then* $X \perp Y$ *iff* X *and* Y *are relatively prime.*

PROOF. Let the notation be as before. Similarly to the proof of 6.3 we get a non-trivial quasifactor $\mathcal{X} = \mathsf{QF}(u \circ \mathsf{K}\bar{x}, X)$ of X. Our assumptions guarantee its incontractibility. Since $\mathcal{G}(\mathcal{X}, u \circ \mathsf{K}\bar{x}) \supseteq \mathsf{G}_\infty$ it follows that $\mathcal{X} \in \mathcal{M}(\mathsf{G}_\infty \mathsf{K}) \cap \mathcal{P}^\perp = \mathcal{M}(\mathsf{K}) \cap \mathcal{P}^\perp$. Now apply 6.1 and 6.2(b).

Observe that the condition of each quasifactor of X being incontractible is trivially fulfilled if T is strongly amenable.

Added in proof:

Recently J. AUSLANDER gave the following easy proof of Thm. 1.5 (personal communication).

THEOREM. *Let* $\phi: X \to Y$ *be a homomorphism of minimal ttg's and let* $\mathcal{X} \perp Y$ *for a non-trivial quasifactor* $\mathcal{X}$ *of* X. *Let* (A,B) *be a proximal pair in* $\mathcal{X}$ *with* $A \neq B$. *Then there are* $x \in A$ *and* $x' \in B$ *with* $x \neq x'$, $\phi(x) = \phi(x')$ *and* (x,x') *is a proximal pair in* X.

PROOF. Let N be a minimal left ideal in Q such that $p \circ A = p \circ B$ for every $p \in N$. Assume $B \setminus A \neq \emptyset$. Choose $x' \in B \setminus A$ and put $y = \phi(x')$. Since $\mathcal{X} \times Y$ is minimal, (A,y) is an almost periodic point. So there is an idempotent $u \in N$ with $u(A,y) = (u \circ A, uy) = (A,y)$. Put $x = ux'$; then $\phi(x) = \phi(ux') = u\phi(x') = = uy = y = \phi(x')$ and $ux' \in u \circ B = u \circ A = A$. Obviously x' and x are proximal and $x' \neq x$. □

COROLLARY. *If* ϕ *is distal,* $\mathcal{X}$ *is distal.*

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GO-SPACES AND (GENERALIZED) METRIZABILITY

J.M. van Wouwe

This article gives a survey of some results about GO-spaces that have been obtained by M.J. Faber and the author at the Vrije Universiteit, Amsterdam.

The first section is introductory. The results of the second section, about metrizability of GO-spaces, are due to M.J. Faber. He gives, among other things two nice characterizations of metrizability of a GO-space. In the next sections due to the author, these are used for the characterization of several generalizations of metrizability in GO-spaces, by means of the metrizability of certain quotient spaces.

1. INTRODUCTION

A *generalized ordered space* (abbreviated: *GO-space*) is a triple $(X,\leq,\tau)$, where X is a set, $\leq$ a linear order in it, and τ is a toplogy on X such that

(i) the *order topology* $\lambda(\leq)$ is contained in τ

(ii) τ has a base consisting of convex sets

If $\lambda(\leq) = \tau$, then the triple $(X,\leq,\lambda(\leq))$ is called a *linearly ordered topological space* (LOTS). Clearly every subspace of a LOTS is a GO-space; the converse also is true: each GO-space can be embedded in a LOTS.

GO-spaces were studied by D.J. LUTZER in [8]. He proved several interesting theorems about metrizability and paracompactness in GO-spaces. STEEN ([12]) proved that each LOTS, and hence each GO-space is hereditarily collectionwise normal.

Let $X = (X,\leq,\tau)$ be a GO-space. An ordered pair (A,B) of subsets of X such that

(i) $X = A \cup B$

(ii) $a < b$ for all $a \in A$, $b \in B$

(iii) $A,B \in \tau$

is called a *gap* if A has no right and B has no left endpoint.
left pseudogap if A ($\neq \emptyset$) has no right and B has a left endpoint.
right pseudogap if A has a right and B ($\neq \emptyset$) has no endpoint.
jump if A has a right and B has a left endpoint.

Furthermore

$$E(X) := \{x \in X \mid [x,\rightarrow[\ \in \tau \text{ or}]\leftarrow,x] \in \tau\}$$

$$H(X) := \{x \in X \mid [x,\rightarrow[\ \in \tau\backslash\lambda(\leq) \text{ or}]\leftarrow,x] \in \tau\backslash\lambda(\leq)\}.$$

Clearly there exist no pseudogaps in a LOTS X (or equivalently: $H(X) = \emptyset$). The following propositions are well-known.

THEOREM 1.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*
(i) X *is compact* $\Longleftrightarrow$ X *has no gaps or pseudogaps.*
(ii) X *is connected* $\Longleftrightarrow$ X *has no gaps (except endgaps), no pseudogaps, and no jumps.*

If $\xi = (A,B)$ is a gap or pseudogap, we may regard ξ as a virtual element of X, satisfying $a < \xi < b$ for all $a \in A$, $b \in B$. If we add all these (pseudo-) gaps to X and give the resulting set the extended order and corresponding topology, we obtain a compact LOTS X^+. Since X is dense in X^+ and the subspace X of X^+ is the space we started with, X^+ is called the *Dedekind compactification* of the GO-space $(X,\leq,\tau)$ (see [6] and [8]).

A subpace $A \subset X$ is called *relatively discrete* if the relative topology on A is discrete and A is called *discrete (in* X*)* if A is closed in X and relatively discrete. A set that is the union of countably many discrete subsets is called *σ-discrete*.

2. CHARACTERIZATIONS OF METRIZABILITY AND SOME OTHER PROPERTIES IN GO-SPACES

The following theorems, due to Faber, make use of the notions defined above.

THEOREM 2.1. ([5]). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

X *is paracompact* $\Longleftrightarrow$ *For each (pseudo-)gap* (A,B) *in* X *there is a discrete subset* L *cofinal in* A *and a discrete subset* R *coinitial in* B.

(Compare this to the notion of a Q-gap in a LOTS (see GILLMAN & HENRIKSEN [6]).

THEOREM 2.2. ([5]). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then* X *is a Lindelöf space if and only if the following two propositions hold:*

(i) *Each discrete subset of* X *is countable.*

(ii) *For each (pseudo-)gap* (A,B) *in* X *there exists a countable subset* L *cofinal in* A *and a countable subset* R *coinitial in* B.

THEOREM 2.3. ([5]). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the following propositions are equivalent:*

(i) X *is perfectly normal.*

(ii) *Each collection of disjoint convex open sets is the union of countably many discrete collections.*

(iii) *Each relatively discrete subset of* X *is* σ*-discrete (in* X*).*

From Theorem 2.3 it follows easily (with the help of 2.1) that a perfectly normal GO-space is paracompact (LUTZER [8]) and furthermore 2.3 implies that a GO-space that has a σ-discrete dense subset, is perfectly normal. The converse of this last proposition does not hold for a Suslin line. It is not known if it is false without any set-theoretical assumptions.

LUTZER ([7]) proved that a LOTS with a G_δ-diagonal is metrizable. Here "LOTS" cannot be replaced by "GO-space" since the non-metrizable Sorgenfrey line has a G_δ-diagonal. The following theorem (which is essentially due to Faber though it is somewhat reformulated) shows which condition must be added to the existence of a G_δ-diagonal to make a GO-space metrizable.

THEOREM 2.4. ([5]). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the following properties are equivalent*

(i) *is metrizable.*

(ii) *has a* σ*-discrete dense subset and* $E(X)$ *is* σ*-discrete.*

(iii) *has a* G_δ*-diagonal and* $H(X)$ *is* σ*-discrete.*

To prove Theorem 2.4, the following lemma is very convenient. It says in a way, that if we have a sequence of covers then the stars may be "bad" in points of some σ-discrete set without doing any harm, since we then can modify the sequence a bit such that the stars are "nice" in all points.

LEMMA 2.5. ([13]). *Let* $(\mathcal{U}(n))_{n=1}^{\infty}$ *be a sequence of open covers of a topological space* X, D *a* σ*-discrete subset of* X*, and let, for each* $d \in D$,

$(V(d,n))_{n=1}^{\infty}$ *be a sequence of neighbourhoods of* d. *Then there is a sequence* $(\mathcal{U}'(n))_{n=1}^{\infty}$ *of open covers of* X *with the following properties:*

(i) $St(x,\mathcal{U}'(n)) \subset St(x,\mathcal{U}(n)) \quad \forall x \in X$.

(ii) *For each* $d \in D$ *there exists a natural number* n(d) *such that* $St(d,\mathcal{U}'(n)) \subset V(d,n)$ *if* $n \geq n(d)$.

With the help of lemma 2.5 the implications (ii) → (iii) and (iii) → (i) are relatively easy to prove (note that for the metrizability of X we only have to show that X is a Moore space, since every GO-space is collectionwise normal.

By BINGS theorem ([3]), a T_1-space is metrizable if and only if it has a σ-discrete base. A GO-space has a base consisting of convex subsets and it is not very difficult to prove that a metrizable GO-space has a σ-discrete base of convex sets. Now every LOTS has a base consisting of open intervals, so one would expect that a metrizable LOTS has a σ-discrete base of open intervals. However, this is not the case; Faber proved that the lexicographic product of ω_0 copies of $\omega_1^* + \omega_0$ is metrizable, but cannot be covered by a σ-disjoint collection of open intervals.

3. GENERALIZED ORDERED p- AND M-SPACES

There are quite a few generalizations of metrizability that for a LOTS or a GO-space imply metrizability. We already encountered the notion of a G_δ-diagonal, which for a LOTS (though not for a GO-space) is equivalent to metrizability, and LUTZER proved in [8] that a semi-stratifiable GO-space is metrizable. We now consider some generalizations of metrizability that do not imply metrizability in a GO-space or a LOTS, namely p- and M-spaces, Σ-spaces, and some properties derived from these.

If X and Y are topological spaces such that $X \subset Y$, then a *pluming for* X *in* Y is a sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of coverings of X by sets open in Y, such that

$$\bigcap_{n=1}^{\infty} St(x,\mathcal{U}(n)) \subset X \quad \text{for every } x \in X.$$

A completely regular space X is a p-*space* [1] if it has a pluming in its Čech-Stone compactification, or equivalently, in any of its Hausdorff compactifications. Hence a GO-space is a p-space if and only if it has a pluming in its Dedekind compactification.

DEFINITION. Let $X = (X,\leq,\tau)$ be a GO-space. Then $\mathcal{G}_X$ is the equivalence relation on X defined by

$$x\ \mathcal{G}_X\ y \iff \text{the closed interval between } x \text{ and } y \text{ is compact} \quad (x,y \in X).$$

The decomposition $X/_{\mathcal{G}X}$ is denoted by gX and $g_X\colon X \to gX$ is the quotient map. With regard to the obvious order on gX and the quotient topology, gX is a GO-space. The mapping g_X is closed and order preserving.

In general we will drop the subscript X on $\mathcal{G}$ and g when confusion is not possible. We then have the following result.

THEOREM 3.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is a p-space} \iff gX \textit{ is metrizable}$$

In two cases the decomposition space gX has an especially simple form: when X is locally compact the space gX is discrete, and when there is a (pseudo-)gap between every two distinct points of X then gX is homemorphic to X; hence a space with this property is a p-space if and only if it is metrizable. This implies that for instance the Sorgenfrey line is not a p-space.

From Theorem 3.1 also follows the next proposition:

THEOREM 3.2. *Let* $X = (X,\leq,\tau)$ *be a GO-space such that* $H(X)$ *is* σ*-discrete in* X. *Then*

$$X \textit{ has a } \sigma\textit{-discrete dense subset} \Rightarrow X \textit{ is a (paracompact) p-space.}$$

M-spaces were defined by K. MORITA in [9]. A space X is a wΔ-space [4] *(an M-space)* if it admits a (normal) sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of open covers of X such that a sequence $(x(n))_{n=1}^{\infty}$ clusters whenever it has the following property: there is a $p \in X$ such that $x(n) \in St(p,\mathcal{U}(n))$ for each $n \in \mathbb{N}$.

THEOREM 3.3. (MORITA [9]). *A space* X *is an* M*-space if and only if it can be mapped onto a metrizable space by a quasi-perfect map.*

In general p-spaces and M-spaces are quite different, though they are equivalent for paracompact spaces.

Clearly M-spaces have something to do with clustering sequences and quasi-perfect mappings, so with countably-compactness. This motivates the following definition:

DEFINITION. Let $X = (X,\leq,\tau)$ be a GO-space and $\mathcal{C}_X$ the equivalence relation on X defined by

$$x\ \mathcal{C}_X\ y \Longleftrightarrow \text{the closed interval between } x \text{ and } y \text{ is countably compact } (x,y \in X).$$

Like with $\mathcal{G}_X$, the equivalence classes of $\mathcal{C}_X$ are convex, closed sets; hence the quotient space $cX := X/\mathcal{C}_X$ is a GO-space with regard to the obvious order and quotient topology. The quotient mapping $c_X\colon X \to cX$ is closed and order preserving. (Again we shall mostly drop the subscript X.)

THEROEM 3.4. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the following properties are equivalent*

(i) X *is a* wΔ-*space*

(ii) X *is an* M-*space*

(iii) cX *is metrizable.*

From this easily follows:

THEOREM 3.5. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is a p-space} \Rightarrow X \textit{ is an M-space.}$$

(The mapping $c\circ g^{-1}\colon gX \to cX$ is closed by the foregoing: hence cX is metrizable if gX is metrizable, since closed mappings preserve semi-stratifiability, which is equivalent to metrizability for GO-spaces.)

The converse of Theorem 3.5 is not true: Let ω_1 be the set of all countable ordinals, with the usual order, and let ω_1^* be the same set with reversed order. Replace in ω_1 every non-limit ordinal by a copy of $\omega_1^* + \omega_1$ and order the resulting set X lexicographically by $\leq$. Then it is easy to see that the LOTS $X = (X,\leq,\lambda(\leq))$ is an M-space but not a p-space, since $cX \simeq \{0\}$ and $gX \simeq \omega_1$.

Another way in which M-spaces can be characterized is the following: Make a compactification of a GO-space X by placing two points in each non-end gap and one point in each pseudogap and in each endgap. The resulting set is called X^{++}. Then define X^c by:

$$X^c := X \cup \{\xi \in X^{++}\backslash X \mid \xi \text{ is the limit of a (countable) sequence in } X\}.$$

Then the subspace X^c of X^{++} is a *countably-compactification* ([10]) of X, i.e. X^c is countably compact, contains X as a dense subspace, and each countably compact closed set in X is also closed in X^c.

THEOREM 3.6. *Let* $X = (X,\le,\tau)$ *be a GO-space. Then*

$$X \textit{ is an M-space} \Longleftrightarrow X \textit{ has a pluming in } X^c.$$

4. GENERALIZED ORDERED Σ-SPACES

Let X be a topological space, and H a cover of X. For $x \in X$ we define $C(x,H) := \cap\{H \mid x \in H \in H\}$.

A *Σ-network* for a space X is a σ-locally finite closed cover $F = \bigcup_{n=1}^{\infty} F(n)$ of X (where each $F(n)$ is locally finite), such that for each $x \in X$

(i) the set $C(x,F)$ is countably compact

(ii) F *contains an outer network for* $C(x,F)$, i.e. whenever U is an open set containing $C(x,F)$ then $C(x,F) \subset F \subset U$ for some $F \in F$.

A space that admits a Σ-network is called a Σ-space (NAGAMI [11]).

The Σ-property in GO-spaces was first studied by LUTZER [8]. He proved that every locally compact GO-space is a Σ-space. Actually each generalized ordered p-space is a Σ-space, since each generalized ordered p-space is an M-space by 3.5, and each M-space is a Σ-space

When we want to characterize the Σ-property in GO-spaces, the quotient space cX of the previous section is not very useful, since one easily constructs subsets X(n) (n = 1,2,...) of a GO-space X such that

(i) Each X(n) is closed in X.

(ii) $c|X(n)$ is quasi-perfect.

(iii) $c[X(n)] = cX$

(iv) $X = \bigcup_{n=1}^{\infty} X(n)$.

Because the property of having a Σ-net is preserved by quasi-perfect mappings both ways and a space that is the union of countably many closed Σ-spaces is a Σ-space itself, this gives the following:

THEOREM 4.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \text{ is a } \Sigma\text{-space} \Longleftrightarrow cX \text{ is a } \Sigma\text{-space}.$$

We can though, with the help of this theorem, construct GO-spaces that have a Σ-net, but that are not M-spaces.

EXAMPLE. Define $X := \{(x,y) \in (\omega_1+1) \times]-1,1[\mid y = 0 \text{ if } x \text{ is a limit ordinal}\}$. Give X the lexicographic order and corresponding topology. Then $cX \simeq \omega_1$ hence X is a (paracompact) Σ-space that is not an M-space.

However, Σ-spaces and M-spaces coincide in the presence of perfect normality:

THEOREM 4.2. *Let* $X = (X,\leq,\tau)$ *be a perfectly normal GO-space. Then*

$$X \text{ is a } \Sigma\text{-space} \Longleftrightarrow X \text{ is an M-space}$$

One of the difficulties in the handling of Σ-spaces is that a GO-space can be a Σ-space without having a convex Σ-net, i.e. a Σ-net consisting of convex sets. This is made clear by the next theorem.

THEOREM 4.3. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \text{ has a convex } \Sigma\text{-net} \Longleftrightarrow X \text{ is an M-space}.$$

Evidently, every regular σ-space is an M-space. A *pre-σ-space* ([11]) is a space that can be mapped onto a σ-space by a quasi-perfect map. Hence the following implications exist.

$$\text{M-space} \Rightarrow \text{pre-}\sigma\text{-space} \Rightarrow \Sigma\text{-space}.$$

In general none of these can be reversed (see [11]).

THEOREM 4.4. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \text{ is a pre-}\sigma\text{-space} \Longleftrightarrow X \text{ is an M-space}.$$

We now want to make a decomposition of a given GO-space X such that the quotient space is metrizable if X is a Σ-space. To that end we make the elements of the decomposition cX bigger. In what follows, spaces with a *countable Σ-network* (i.e. a Σ-network $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$, where each $\mathcal{F}(n)$ is countable) play an important role (compare the Σ-($\aleph_0$)spaces of NAGAMI [11]).

DEFINITION. Let $X = (X,\leq,\tau)$ be a GO-space. We define an equivalence relation $\mathcal{M} = \mathcal{M}_X$ on X by

$$x \; \mathcal{M} \; y \Longleftrightarrow \text{the closed interval between } x \text{ and } y \text{ has a countable } \Sigma\text{-net } (x,y \in X).$$

The decomposition space $X/_{\mathcal{M}}$ is denoted by mX and $m(=m_X): X \to mX$ is the quotient map. The elements of mX are convex, but not necessarily closed. However, they are closed if X is a Σ-space. In that case, mX is a GO-space again.

We can prove the following theorem:

THEOREM 4.5. *Let $X = (X,\leq,\tau)$ be a GO-space. Then*

$$X \text{ is a } \Sigma\text{-space} \Longleftrightarrow mX \text{ is a metrizable GO-space and each element of } mX \text{ is a } \Sigma\text{-space.}$$

The decomposition $X/_{\mathcal{M}}$ however is not an easy one to construct, since it is far from clear when a GO-space has a countable Σ-network. The situation improves when we look at paracompact spaces, since we have the following theorem, due to Nagami.

THEOREM 4.6. ([11], theorem 4.6). *Let X be a (not necessarily GO) paracompact Σ-space. Then*

$$X \text{ has a countable } \Sigma\text{-network} \Longleftrightarrow X \text{ is a Lindelöf space}$$

Fortunately, we can in a sense restrict ourselves to paracompact spaces because of the following observations: For a given GO-space X define a paracompact extension pX of X by:

$$pX := X \cup \{\xi \in X^{++} \mid \xi \text{ is not a limit point of any discrete subset of } X\}.$$

Then pX as a subspace of X^{++} is a paracompact GO-space containing X as a dense subspace, and each paracompact closed subset of X is closed in pX.

A straightforward though lengthy argument shows that if $\mathcal{F}$ is a Σ-network for X, then $\{Cl_{pX}(F) \mid F \in \mathcal{F}\}$ is a Σ-network for pX, and if $\mathcal{G}$ is a Σ-network for pX then $\{G \cap X \mid G \in \mathcal{G}\}$ is a Σ-network for X. Hence we have

THEOREM 4.7. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \text{ is a } \Sigma\text{-space} \Longleftrightarrow pX \text{ is a } \Sigma\text{-space}$$

Moreover, X has a convex Σ-network if and only pX has a Σ-network by the argument above, hence

COROLLARY. X *is an* M-*space* $\Longleftrightarrow$ pX *is an* M-*space*.

We now define another equivalence relation $L = L_X$ on a GO-space X by

$$x \, L \, y \Longleftrightarrow \text{the closed interval between } x \text{ and } y \text{ is a Lindelöf space} \quad (x,y \in X).$$

The equivalence classes are convex closed sets; hence the quotient space $\ell X := X/_L$ is a GO-space; the quotient mapping ℓ $(=\ell_X)$: $X \to \ell X$ is closed and order preserving. It follows easily from 4.6 that if each element of ℓX has a Σ-network, and X is paracompact then L is equal to M. Hence we have

THEOREM 4.8. *Let* $X = (X,\leq,\tau)$ *be a paracompact GO-space. Then*

$$X \text{ is a } \Sigma\text{-space} \Longleftrightarrow \ell X \text{ is metrizable and each } L \in \ell X \text{ is a } \Sigma\text{-space}.$$

Moreover, if X is paracompact then an element L of ℓX is a Σ-space if and only if each $[x,y]$ $(x,y \in L,\ x < y)$ is a Σ-space. Since such a subspace is a Lindelöf space by the definition of L, thus reduces the problem of characterizing generalized ordered Σ-space to the question: when does a Lindelöf GO-space admit a (countable) Σ-network?

BENNET and LUTZER ([2]) proved that a GO-space that is hereditarily an M-space (p-space) is metrizable. We do not know if the same holds for a GO-space that is hereditarily a Σ-space. However, theorem 4.8 can be used to show that the following conjectures are equivalent.

Conjecture (i) Each GO-space that is hereditarily a Σ-space is metrizable

Conjecture (ii) Each Lindelöf GO-space that is hereditarily a Σ-space, is hereditarily Lindelöf.

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