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MATHEMATICS **12**

A THEOREM ON THE SIGN TEST WHEN TIES ARE PRESENT

BY

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1. *Introduction*

When $\mathbf{x}$ and $\mathbf{y}$ are two independently distributed random variables [1]), tests for the hypothesis H_0, that $\mathbf{x}$ and $\mathbf{y}$ have the same probability distribution, may be based on their difference $\mathbf{x}-\mathbf{y}$. Let a number (n) of pairs of independent observations $(x_1, y_1), \ldots, (x_n, y_n)$ be given, then the number of positive differences $x_i - y_i$ $(i = 1, \ldots, n)$ may be used as a test criterion. The test based on this criterion is called the sign test [2]). There seems to be some uncertainty concerning the best treatment of zero differences (often called "ties" in this context) when applying this test. W. J. DIXON and A. M. MOOD (1946) propose to count half of the ties as positive and half as negative and W. J. DIXON and F. J. MASSEY Jr., (1951) advise to omit the ties from the observations. The object of this note is to prove, that omitting the ties results in a more powerful test than dividing them equally among the positive and negative observations.

2. *Preliminary remarks*

Let $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_0 = n - \mathbf{n}_1 - \mathbf{n}_2$ denote the numbers of positive, negative and zero differences among n independent observations of $\mathbf{x}-\mathbf{y}$, and let p_1, p_2 and $p_0 = 1 - p_1 - p_2$ be the probabilities for each observation of being positive, negative and zero respectively. Then, under the hypothesis H_0 to be tested, we have

$$p_1 = p_2. \tag{1}$$

The sign test with omission of the $\mathbf{n}_0$ ties is based on the conditional probability distribution of $\mathbf{n}_1$, under the condition $\mathbf{n}_0 = n_0$ and under hypothesis H_0, which is

$$\mathrm{P}\,[\mathbf{n}_1 = n_1 \,|\, n_0; H_0] = \binom{n-n_0}{n_1} \cdot 2^{n_0-n}. \tag{2}$$

The critical region Z_1 of this test, with level of significance α_1, consists of those results of the n observations, for which one of the relations

$$n_1 \leqq d(n_0) \quad \text{or} \quad n - n_0 - d(n_0) \leqq n_1 \tag{3}$$

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[1]) Random variables are denoted by bold type symbols, values assumed by random variables by the same symbols in normal type.

[2]) The sign test is probably the oldest test in existence. It has been applied already in 1710 by JOHN ARBUTHNOT (cf. H. FREUDENTHAL (1951) and J. TODHUNTER (1865)).

holds, where $d(n_0)$ $(n_0 = 1, 2, \ldots, n)$ is the largest integer satisfying

$$(4) \qquad 2^{n_0 - n} \sum_{\nu=0}^{d(n_0)} \binom{n-n_0}{\nu} \leqq \tfrac{1}{2} \alpha_1 \text{ [1]}.$$

When the ties are divided equally among the positive and the negative observations, the statistic

$$(5) \qquad \boldsymbol{n}_1' = \boldsymbol{n}_1 + \tfrac{1}{2}\boldsymbol{n}_0$$

is used and the critical region Z_2, with level of significance α_2, is defined as the set of those results, for which one of the relations

$$(6) \qquad n_1' \leqslant d' \quad \text{or} \quad n - d' \leqq n_1'$$

holds, d' being the largest integer satisfying

$$(7) \qquad 2^{-n} \sum_{\nu=0}^{d'} \binom{n}{\nu} \leqq \tfrac{1}{2} \alpha_2.$$

In other words Z_2 is defined as though the probability distribution of $\boldsymbol{n}_1'$, under H_0, were a binomial distribution with parameters n and $p = \frac{1}{2}$. Notice that d' is independent of n_0.

The power functions of the two tests are

$$(8) \qquad \beta_i(p_1, p_2) = \mathrm{P}[(\boldsymbol{n}_1, \boldsymbol{n}_2) \in Z_i | p_1, p_2] \quad (i = 1, 2) \text{ [2]}$$

and it may easily be proved that

$$(9) \qquad \beta_1(p, p) \leqq \alpha_1$$

for every $p \leqq \frac{1}{2}$. The analogous relation for β_2 follows from the theorem which we are going to prove. Both tests thus satisfy the obvious requirement that the probability of rejecting H_0, when it is true, is at most equal to the level of significance.

3. *A lemma*

For the proof of our theorem we shall use the following

Lemma: *If d, k and n are positive integers satisfying*

$$(10) \qquad d' = [d + \tfrac{1}{2}k + \tfrac{1}{2}] < \tfrac{1}{2}n - \tfrac{1}{2}, \text{ [3]}$$

then

$$(11) \qquad 2^k \sum_{\nu=0}^{d} \binom{n-k}{\nu} < \sum_{\nu=0}^{d'} \binom{n}{\nu}.$$

Proof: The proof is given by induction with respect to k.

First consider even values of k, starting with $k = 2$. Then $d' = d + 1$. Using the relation

$$\binom{a}{b} - \binom{a-1}{b} = \binom{a-1}{b-1}$$

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[1]) It should be kept in mind, that $n = n_0 + n_1 + n_2$ is a given positive integer.

[2]) The symbols p_1 and p_2 behind the vertical line denote that p_1 and p_2 are the true probabilities of a positive and a negative difference respectively.

[3]) $[u]$ denoting the largest integer not surpassing u.

we find

$$4\sum_{\nu=0}^{d}\binom{n-2}{\nu}-\sum_{\nu=0}^{d+1}\binom{n}{\nu}=4\sum_{\nu=0}^{d}\binom{n-2}{\nu}-2\sum_{\nu=0}^{d}\binom{n-1}{\nu}+2\sum_{\nu=0}^{d}\binom{n-1}{\nu}-\sum_{\nu=0}^{d+1}\binom{n}{\nu}=$$

$$=2\sum_{\nu=0}^{d}\binom{n-2}{\nu}-2\sum_{\nu=0}^{d}\binom{n-2}{\nu-1}+\sum_{\nu=0}^{d}\binom{n-1}{\nu}-\sum_{\nu=0}^{d}\binom{n-1}{\nu-1}-\binom{n}{d+1}=$$

$$=2\sum_{\nu=0}^{d}\binom{n-2}{\nu}-2\sum_{\nu=0}^{d-1}\binom{n-2}{\nu}+\sum_{\nu=0}^{d}\binom{n-1}{\nu}-\sum_{\nu=0}^{d-1}\binom{n-1}{\nu}-\binom{n}{d+1}=$$

$$=2\binom{n-2}{d}+\binom{n-1}{d}-\binom{n}{d+1}=\binom{n-2}{d}\frac{2d+3-n}{d+1},$$

and this is <0 according to (10) with $d'=d+1$.

Now if (11) is true for $k=2h$ we shall prove this inequality for $k=2h+2$, given (10), which for $k=2h+2$ assumes the form

$$d'=d+h+1<\tfrac{1}{2}n-\tfrac{1}{2}. \tag{12}$$

The left hand member of the inequality to be proved is

$$2^{2h+2}\sum_{\nu=0}^{d}\binom{n-2h-2}{\nu}=2^2\left\{2^{2h}\sum_{\nu=0}^{d}\binom{n-2-2h}{\nu}\right\}.$$

The formula between curved brackets is equal to the left hand member of (11) when $n-2$ is substituted for n and $2h$ for k. Therefore (11) may be applied if (10) is satisfied. However, for $k=2h$ and with $n-2$ instead of n, (10) reduces to (12), which is a given relation. Thus

$$2^{2h+2}\sum_{\nu=0}^{d}\binom{n-2h-2}{\nu}<2^2\sum_{\nu=0}^{d+h}\binom{n-2}{\nu}.$$

The right hand member is again equal to the left hand member of (11), this time with $k=2$ and $n-2$ substituted for n. Again (10) reduces to (12) when these values are substituted and (11) may be applied once more. This gives

$$2^{2h+2}\sum_{\nu=0}^{d}\binom{n-2h-2}{\nu}<\sum_{\nu=0}^{d+h+1}\binom{n}{\nu},$$

which was to be proved.

For odd values of k ($k=2h-1$) the proof of the inequality, given (10), follows from (11) with $k=2h$ and $n+1$ instead of n:

$$2^{2h-1}\sum_{\nu=0}^{d}\binom{n-2h+1}{\nu}=2^{-1}\cdot 2^{2h}\sum_{\nu=0}^{d}\binom{n+1-2h}{\nu}<$$

$$<2^{-1}\sum_{\nu=0}^{d+h}\binom{n+1}{\nu}=2^{-1}\left\{\sum_{\nu=0}^{d+h}\binom{n}{\nu}+\sum_{\nu=0}^{d+h}\binom{n}{\nu-1}\right\}=$$

$$=\sum_{\nu=0}^{d+h}\binom{n}{\nu}-\tfrac{1}{2}\binom{n}{d+h}<\sum_{\nu=0}^{d+h}\binom{n}{\nu},$$

(10) being satisfied for $k=2h$ and with $n+1$ instead of n. This proves the lemma.

4. *A theorem about the power of the two tests*

Theorem: *If* $\alpha_2 \leqq \alpha_1 < 1$, *then* $\beta_2(p_1, p_2) \leqq \beta_1(p_1, p_2)$ *for every* p_1 *and* p_2 *with* $p_1 + p_2 \leqq 1$.

Proof: Consider the subset of possible results where $n_0 = k$ and denote the intersection of this set and Z_i by $Z_{i,k}$ $(i = 1, 2;\ k = 1, \ldots, n)$. According to the definition of Z_1 and Z_2 the regions $Z_{1,0}$ and $Z_{2,0}$ coincide if $\alpha_1 = \alpha_2$, and $Z_{1,0} \supset Z_{2,0}$ when $\alpha_1 \geqq \alpha_2$. We shall prove the latter relation for $k > 0$. It then follows that $Z_1 \supset Z_2$ when $\alpha_1 \geqq \alpha_2$ and this proves the theorem according to the definition of β_1 and β_2 (cf. (8)).

According to (3) and (4) $Z_{1,k}$ consists of two parts, situated symmetrically with respect to the point $n_1 = \frac{1}{2}(n-k)$, which we denote, for simplicity, by

$$n_1 \leqq d \quad \text{and} \quad n-k-d \leqq n_1, \tag{13}$$

where d is the largest integer satisfying

$$2^{k-n} \sum_{\nu=0}^{d} \binom{n-k}{\nu} \leqq \tfrac{1}{2}\alpha_1. \tag{14}$$

From (5) and (6) we find that $Z_{2,k}$ consists of the two parts

$$n_1 \leqq d' - \tfrac{1}{2}k \quad \text{and} \quad n - \tfrac{1}{2}k - d' \leqq n_1, \tag{15}$$

which are also situated symmetrically with respect to $n_1 = \frac{1}{2}(n-k)$, with d' defined by (7).

The proof of the relation $Z_{1,k} \supset Z_{2,k}$ will be given indirectly by proving that α_1 would be smaller than α_2 if this relation was untrue. Supposing $Z_{2,k}$ to contain more values of n_1 than $Z_{1,k}$, we consider the region $Z^*_{1,k}$, defined by the relation $Z^*_{1,k} = Z_{2,k}$, which contains $Z_{1,k}$. Denoting the value of d pertaining to $Z_{1,k}$ by d^*, we remark that $d^* > d$ and we find from (13) and (15)

$$d' = [d^* + \tfrac{1}{2}k + \tfrac{1}{2}]. \tag{16}$$

Now $d' < \frac{1}{2}n - \frac{1}{2}$, for $d' \geqq \frac{1}{2}n - \frac{1}{2}$ would imply $\alpha_2 = 1$ according to (7), and this has been excluded in the theorem. Consequently the lemma proved above may be applied and multiplying both sides of (11) by 2^{-n} we get

$$2^{k-n} \sum_{\nu=0}^{d^*} \binom{n-k}{\nu} < 2^{-n} \sum_{\nu=0}^{d'} \binom{n}{\nu}. \tag{17}$$

From $d^* > d$, from the fact that d is the largest integer satisfying (14) and from (7) and (17) it follows that $\alpha_2 > \alpha_1$, which contradicts the premises of the theorem.

5. *Remarks*

Numerical computation of special cases shows that in general, when $\alpha_1 = \alpha_2$, Z_1 contains more points than Z_2, so that $\beta_1 > \beta_2$ for every p_1 and p_2 $(p_1 + p_2 \leqq 1)$. It may be proved, using the normal appoximation of the binomial distribution, that the difference $\beta_1 - \beta_2$, given p_1/p_2 and the level of significance $\alpha_1 = \alpha_2$, increases with increasing p_0.

We may remark further, that the application of the sign test is not limited to the situation described in the introduction but may be used whenever the equality of the probabilities of two of the possible results of an experiment has to be tested. The test is consistent if and only if these probabilities are different.

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