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Some remarks on the power function of  
Wilcoxon's test for the problem of two samples

I and II

BY

H. R. VAN DER VAART

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# SOME REMARKS ON THE POWER FUNCTION OF WILCOXON'S TEST FOR THE PROBLEM OF TWO SAMPLES. I

BY

H. R. VAN DER VAART

(Communicated by Prof. D. VAN DANTZIG at the meeting of February 25, 1950)

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(See further Proceedings 53, 507—520 (1950)).

## 1. INTRODUCTION

F. WILCOXON (9) gave a non-parametric solution of the problem of two samples of equal sizes. This solution was generalized and studied in detail by H. B. MANN and D. R. WHITNEY (2).

Let  $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n$ <sup>1)</sup> be  $m + n$  independent random variables, the  $\mathbf{x}_i$  all having the *continuous* (cumulative) distribution function  $F(x)$  and the  $\mathbf{y}_j$  all having the *continuous* (cumulative) distribution function  $G(x)$ .

For any set of values  $E = (x_1, \dots, x_m, y_1, \dots, y_n)$  the variables can take<sup>2)</sup> let  $U = U(x_1, \dots, x_m, y_1, \dots, y_n) = U(E)$  be defined as the number

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<sup>1)</sup> The letter which denotes a variable is printed in bold type when stress is laid upon the random character of the variable, i.e. upon the fact the variable has a distribution function.

<sup>2)</sup> Such a set  $E$  is called a "*sample point*"; the set of all sample points  $E$  which eventually might be obtained is called the "*sample space*"  $W$ .

of pairs of integers  $(i, j)$  ( $1 \leq i \leq m; 1 \leq j \leq n$ ) with  $x_i > y_j$ , provided  $x_i \neq y_j$  for every pair  $(i, j)$ . Apparently  $U$  is zero or a positive integer. Then  $\mathbf{U} = U(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n) = U(\mathbf{E})$  is a discrete random variable, defined on the sample space  $W$  everywhere with the exception of a set of probability zero consisting of all points  $E$  with  $x_i = y_j$  for any pair  $(i, j)$  ( $1 \leq i \leq m; 1 \leq j \leq n$ ).

The statistic  $U(E)$  was introduced by H. B. MANN and D. R. WHITNEY for testing the hypothesis that  $G(x) = F(x)$  holds for all  $x$  (shortly:  $G = F$ ) against the alternative that  $G(x) < F(x)$  holds for all  $x$  (shortly:  $G < F$ ).

The distribution function of  $\mathbf{U}$  under the hypothesis  $G = F$  has been computed by MANN and WHITNEY for  $m \leq 8, n \leq 8$ , and by the Computing Department of the Mathematical Centre at Amsterdam for  $m \leq 10, n \leq 10$ . For  $m \rightarrow \infty$  and  $n \rightarrow \infty$  the distribution of  $[(\mathbf{U} - \frac{1}{2}mn) / \frac{1}{12} \cdot mn(m+n+1)]^{-1}$  tends to a normal (0,1)-distribution.

The test, as given by MANN and WHITNEY, consists in rejecting the hypothesis  $G = F$  on the level of significance  $\alpha$  if and only if  $P[\mathbf{U} \leq U(E) | G = F] \leq \alpha$ , where  $E$  is the sample point corresponding with the empirical data.

As for the power function of this test, MANN and WHITNEY remarked that it presents formidable difficulties. They proved however that their test is consistent<sup>3)</sup> with respect to the class of alternatives  $G < F$ .

## 2. THE POWER FUNCTION OF A STATISTICAL TEST IN GENERAL

### 2.1. Critical region.

A statistical hypothesis  $H_0$  is tested (against an alternative hypothesis  $H$ ) by dividing the sample space  $W$  into two parts,  $w$  and  $(W - w)$ , and applying the rule that  $H_0$  is rejected if the sample point  $E$  falls in  $w$  and that  $H_0$  is not rejected if  $E$  falls in  $(W - w)$ .  $w$  is called the "critical region" of the test. (cf. J. NEYMAN and E. S. PEARSON (4) or J. NEYMAN (3)).

### 2.2. The concept of power function.

The power function of a test, a concept introduced by NEYMAN and PEARSON (5) (cf. also J. NEYMAN (3)), is the probability  $\alpha(H) = P[\mathbf{E} \in w | H]$ , that the sample point  $\mathbf{E}$  falls in the critical region  $w$ , calculated under any admissible hypothesis  $H$ .  $H_0$  denoting the hypothesis tested (to which there may be an infinity of alternatives), the critical region  $w$  is chosen so that:

$$\alpha(H_0) = P[\mathbf{E} \in w | H_0] \text{ is equal to } \alpha.$$

<sup>3)</sup> A test is called "consistent" according to A. WALD and J. WOLFOWITZ (8), if and only if the probability of rejecting the hypothesis tested (here  $G = F$ ) when it is false, tends to 1 as the sample size tends (here: sizes tend) to infinity.

Here  $\alpha$  is a given positive number, the so-called level of significance. Apparently  $\alpha$  is the probability that  $H_0$  will be rejected (cf. the definition of  $w$  in 2. 1), when  $H_0$  is true. Usually  $\alpha$  is chosen small, as will be easily understood from I below.

Clearly, if among the many possible critical regions one would exist for which:

$$\text{I. } P[\mathbf{E} \in w | H_0] = \alpha$$

$$\text{II. } P[\mathbf{E} \in w | H] = 1 \text{ for all } H \neq H_0,$$

this one would be preferred for testing  $H_0$ .

Critical regions for which I and II hold cannot be realized, however, as soon as all sample points have a positive probability under both  $H_0$  and  $H$ .

Remark.  $w$  depends, of course, on the number of dimensions  $N$  of the sample space  $W$ . Making this dependence explicit by writing  $w_N$  for  $w$ , we see that a test is consistent (cf. the note at the end of 1) if and only if:

$$\lim_{N \rightarrow \infty} P[\mathbf{E} \in w_N | H] = 1 \text{ for all } H \neq H_0.$$

Hence this asymptotic relation corresponding with II defines consistent tests.

### 2. 3. The use of the power function.

The power function is instrumental to judge the "goodness" of a test and to compare several tests.

a) A critical region  $w$  with  $P[\mathbf{E} \in w | H_0] > P[\mathbf{E} \in w | H]$  for some alternative  $H$ , leads to the hypothesis  $H_0$  being rejected more often under  $H_0$  than under this alternative  $H$ . A test based on such a critical region is called "biased" (cf. NEYMAN and PEARSON (5) or NEYMAN (3)). An unbiased test is based on a critical region  $w$  with  $P[\mathbf{E} \in w | H_0] < P[\mathbf{E} \in w | H]$  for all admissible alternatives  $H$ . Clearly an unbiased test is in general preferable to a biased one.

$\beta$ ) When two critical regions,  $w_1$  and  $w_2$  with  $P[\mathbf{E} \in w_1 | H_0] = P[\mathbf{E} \in w_2 | H_0]$ , both give unbiased tests,  $w_1$  is a closer approximation to the ideal case as sketched in 2. 2 under II, than  $w_2$  if:

$$P[\mathbf{E} \in w_1 | H] > P[\mathbf{E} \in w_2 | H] \text{ for all admissible } H \neq H_0.$$

Then the test based on  $w_1$  is called *uniformly more powerful* than the test based on  $w_2$ .

Remark.  $H$  specifies the joint distribution function of the  $N$  random coordinates of  $\mathbf{E}$ . So the power function is a function defined on a function space in the most general case, when all sorts of alternative  $H$ 's are admitted. The class of admissible hypotheses  $H$  can be restricted to various degrees. In the most simplified case the only difference between the hypotheses  $H$  consists in the value of one parameter. Denote this parameter

by  $\mu$  and the powerfunction  $P[\mathbf{E} \in w \mid \mu]$  by  $\alpha_w(\mu)$ . Let  $H_0$  consist in  $\mu = 0$  and let  $\mu$  be variable in an interval containing  $\mu = 0$ . If the first and second derivatives of  $\alpha_w(\mu)$  for  $\mu = 0$  exist and are denoted by  $\alpha'_w(0)$  and  $\alpha''_w(0)$  respectively, then a necessary condition for unbiasedness with respect to the alternatives  $\mu > 0$  is, if  $\alpha'_w(0) \neq 0$ , that  $\alpha'_w(0) > 0$  (or, if  $\alpha'_w(0) = 0$ , that  $\alpha''_w(0) > 0$ , etc.), whereas a necessary condition for unbiasedness with respect to the alternatives  $\mu \neq 0$  is, that  $\alpha'_w(0) = 0$  and, if  $\alpha''_w(0) \neq 0$ , that  $\alpha''_w(0) > 0$  (or, if  $\alpha''_w(0) = 0$ , that  $\alpha^{(3)}_w(0) = 0$  and  $\alpha^{(4)}_w(0) > 0$ , if these derivatives exist, etc.).

A necessary condition for  $w_1$  being more powerful than  $w_2$  is, with respect to the alternatives  $\mu > 0$ , that  $\alpha'_{w_1}(0) > \alpha'_{w_2}(0)$ , if  $\alpha'_{w_1}(0) \neq \alpha'_{w_2}(0)$  (or, if  $\alpha'_{w_1}(0) = \alpha'_{w_2}(0)$ , that  $\alpha''_{w_1}(0) > \alpha''_{w_2}(0)$ , etc.), and with respect to the alternatives  $\mu \neq 0$ , if  $\alpha'_{w_1}(0) \neq \alpha'_{w_2}(0)$ , that  $\alpha''_{w_1}(0) > \alpha''_{w_2}(0)$  (etc.).

### 3. THE POWER FUNCTION OF WILCOXON'S TEST

#### 3.1. General remarks.

Power functions of WILCOXON'S test will be investigated under the following general restrictions on the *distribution functions*  $F(x)$  and  $G(x)$  (cf. 1):

a)  $F(x)$  and  $G(x)$  have continuous derivatives,  $f(x)$  and  $g(x)$ , respectively, for all  $x$ -values with the exception at most of those bounding the infinite intervals (when present) for which  $f(x)$  or  $g(x)$  are zero.

b)  $G(x) = F(x - \mu)$ ;  $g(x) = f(x - \mu)$ .

These restrictions will be assumed valid throughout the rest of the paper unless the contrary is mentioned. According to the restriction b the hypotheses  $H$  specify  $\mu$ -values, but  $F$  is left unspecified.

The *critical regions* considered are defined by 1, 2 and 3 respectively:

- |   |   |  |
|---|---|--|
| 1) $\mathbf{U} \leq U_\alpha$ with the level of significance $\alpha$   | } | with $U_\alpha < \frac{1}{2} mn$ .<br>(for $U_\alpha$ of 3. 2) |
| 2) $\mathbf{U} \geq mn - U_\alpha$ with the level of significance $\alpha$                                    |   |  |
| 3) $ \mathbf{U} - \frac{1}{2} mn  \geq \frac{1}{2} mn - U_\alpha$ with the level of<br>significance $2\alpha$ |   |  |

The first region, the only one considered by MANN and WHITNEY, serves to test the hypothesis  $\mu = 0$  against  $\mu > 0$  and the second region serves to test  $\mu = 0$  against  $\mu < 0$  (one-sided sets of alternative hypotheses, shortly: "one-sided alternatives"), whereas the third region serves to test  $\mu = 0$  against  $\mu \neq 0$  ("two-sided alternatives"). Sometimes mathematical difficulties require restrictions on  $m$  and  $n$  together with restrictions on  $\alpha$  and on  $U_\alpha$ .

In order to compare the power function of WILCOXON'S test with that of STUDENT'S test for the difference of two means (under conditions which allow the use of STUDENT'S test), in addition to a) and b) the further restriction on the distribution functions involved:

$$c) \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2},$$

will be imposed in a part of the paper.

### 3. 2. The critical region of Wilcoxon's test.

I. The critical region defined by  $\mathbf{U} \leq U_\alpha$  ( $U_\alpha < \frac{1}{2} mn$ ) consists of all sample points  $\mathbf{E} = (\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n)$  satisfying the inequality  $\mathbf{U} \leq U_\alpha$ , where  $U_\alpha$  is the maximum of all integers  $U$  with  $P[\mathbf{U} \leq U | G = F] \leq \alpha$ .

This critical region,  $w$ , is the sum of  $U_\alpha + 1$  disjoint regions  $w_k$  ( $0 \leq k \leq U_\alpha$ ), where  $w_k$  is the set of all sample points for which  $U = k$ . Each region  $w_k$  consists of  $p'_{m,n}(k)$  disjoint subregions  $w_{k,q}$  ( $q = 1, \dots, p'_{m,n}(k)$ ) with constant  $x - y$ -arrangement<sup>4</sup>). For  $p'_{m,n}(k)$  MANN and WHITNEY gave the recurrence relation

$$(3. 2, 1) \quad \begin{cases} p'_{m,n}(k) = p'_{m-1,n}(k-n) + p'_{m,n-1}(k) \\ (p'_{i,j}(k) = 0 \text{ if } k < 0; p'_{i,0}(k) = p'_{0,i}(k) = \begin{cases} 0, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases} \end{cases}$$

It is seen that  $p'_{m,n}(0) = 1 = p'_{m,n}(1)$  if  $m \neq 0$ , and  $n \neq 0$ .

Clearly each subregion  $w_{k,q}$  is built up out of  $m!n!$  disjoint sub-subregions, the points of which are characterized by a constant permutation of their  $m$   $x$ -coordinates and their  $n$   $y$ -coordinates respectively in the constant  $x - y$ -arrangement corresponding to  $w_{k,q}$ .

Because of the continuity of the distribution functions of  $\mathbf{x}_i$  and  $\mathbf{y}_j$  the probability of  $\mathbf{E}$  lying in the boundary of one of the above-mentioned regions is zero, so that the boundaries of these regions may be included without changing any probability calculated.

II. The critical region defined by  $\mathbf{U} \geq mn - U_\alpha$  ( $U_\alpha < \frac{1}{2} mn$ ) is the sum of  $U_\alpha + 1$  disjoint regions  $w_{mn-k}$  ( $0 \leq k \leq U_\alpha$ ). The regions  $w_{mn-k}$  consist of subregions  $w_{(mn-k),q}$  with constant  $x - y$ -arrangement, which are built up out of sub-subregions of constant permutations, as described for the region  $\mathbf{U} \leq U_\alpha$ .

a) By the substitutions  $x_i = -x'_{i'}$ , ( $i, i' = 1, \dots, m$ );  $y_j = -y'_{j'}$ , ( $j, j' = 1, \dots, n$ ) a one-to-one correspondence is established between the points  $\mathbf{E} = (x_1, \dots, x_m, y_1, \dots, y_n)$  of the space  $W$  and the points  $\mathbf{E}' = (x'_{1'}, \dots, x'_{m'}, y'_{1'}, \dots, y'_{n'})$  of the space  $W'$ . Here a pair  $(i, j)$  with  $x_i \geq y_j$  corresponds to a pair  $(i', j')$  with  $x'_{i'} \leq y'_{j'}$ . Hence a point  $\mathbf{E}$  with  $U(\mathbf{E}) = mn - k$  corresponds to a point  $\mathbf{E}'$  with  $U(\mathbf{E}') = k$  ( $U(\mathbf{E}')$  being defined as the number of pairs  $(i', j')$  with  $x'_{i'} > y'_{j'}$ ), and a region  $w_{mn-k}$  corresponds to a region  $w'_k$ , while the region in  $W$  defined by  $\mathbf{U} \geq mn - U_\alpha$  corresponds to the region in  $W'$  defined by  $\mathbf{U} \leq U_\alpha$ .

Remark 1. It is easily seen that  $p'_{m,n}(k) = p'_{m,n}(mn - k)$ .

Remark 2. The points  $\mathbf{E}$  of  $W$  have  $m$   $x$ -coordinates and  $n$   $y$ -coordinates.

The points  $\mathbf{E}'$  of  $W'$  have  $m$   $x'$ -coordinates and  $n$   $y'$ -coordinates.

To make this fact explicit one can write  $W_{m,n}$  instead of  $W$  and  $W'_{m,n}$  instead of  $W'$ .

<sup>4</sup>) By this term is meant, that the same ordered arrangement of  $x$  and  $y$  corresponds to every point  $\mathbf{E}$  of  $w_{k,q}$ , when the coordinates  $x_i$  ( $i = 1, \dots, m$ ) and  $y_j$  ( $j = 1, \dots, n$ ) of  $\mathbf{E}$  are arranged in order according to increasing magnitude and the suffices of  $x$  and  $y$  are omitted.

$\beta$ ) A one-to-one correspondence between the points  $E = (x_1, \dots, x_m, y_1, \dots, y_n)$  of  $W_{m,n}$  and the points  $E' = (x'_1, \dots, x'_n, y'_1, \dots, y'_m)$  of  $W'_{n,m}$  is established by the substitutions  $x_i = y'_{j'} + \mu$  ( $i, j' = 1, \dots, m$ ) and  $y_j = x'_{i'} + \mu$  ( $j, i' = 1, \dots, n$ ). Now a pair  $(i, j)$  with  $x_i \geq y_j$  corresponds to a pair  $(i', j')$  with  $x'_{i'} \leq y'_{j'}$ . By the same argument as under  $a$ ) it is seen that the region of  $W_{m,n}$  defined by  $U \geq m n - U_\alpha$  corresponds to the region of  $W'_{m,n}$  defined by  $U \leq U_\alpha$ .

Remark 3. Clearly  $p'_{n,m}(k) = p_{m,n}(m n - k)$ . Because of remark 1. it is seen that  $p_{n,m}(k) = p_{m,n}(k)$ .

III. The critical region defined by  $|U - \frac{1}{2} m n| \geq \frac{1}{2} m n - U_\alpha$  ( $U_\alpha < \frac{1}{2} m n$ ) is the sum of the two disjoint regions defined by  $U \leq U_\alpha$  (cf. 3. 2, I) and by  $U \geq m n - U_\alpha$  (cf. 3. 2, II), respectively.

### 3. 3. General expressions for the power function of Wilcoxon's test.

Under the restrictions  $a$ ) and  $b$ ) of 3.1 imposed on  $F(x)$  and  $G(x)$ , the power function of WILCOXON'S test is given by

$$(3. 3, 1) \quad \alpha_+(\mu) = \int \dots \int_{U \leq U_\alpha} \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j - \mu) dy_j\}$$

for testing  $\mu = 0$  against  $\mu > 0$ ,

$$(3. 3, 2a) \quad \alpha_-(\mu) = \int \dots \int_{U \geq m n - U_\alpha} \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j - \mu) dy_j\}$$

$$(3. 3, 2b) \quad = \int \dots \int_{U \leq U_\alpha} \prod_{i=1}^m \{f(-x_i) dx_i\} \prod_{j=1}^n \{f(-y_j - \mu) dy_j\}$$

$$(3. 3, 2c) \quad = \int \dots \int_{U \leq U_\alpha} \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j + \mu) dy_j\}$$

for testing  $\mu = 0$  against  $\mu < 0$ ,

((3. 3, 2b) follows from (3. 3, 2a) by the substitutions of 3. 2, II,  $a$ ); (3. 3, 2c) from (3. 3, 2a) by 3. 2, II,  $\beta$ )) and

$$(3. 3, 3) \quad \alpha_\pm(\mu) = \alpha_+(\mu) + \alpha_-(\mu)$$

for testing  $\mu = 0$  against  $\mu \neq 0$ .

When  $U_\alpha = 0$ , one finds, denoting in this case  $\alpha_+(\mu)$  by  $\alpha_+^{(0)}(\mu)$ , that

$$(3. 3, 4) \quad \left\{ \begin{array}{l} \alpha_+^{(0)}(\mu) = m! n! \int \dots \int_R \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j - \mu) dy_j\} \\ \text{where } R \text{ is defined by the inequalities:} \\ -\infty < x_1 < x_2 < \dots < x_m < y_1 < \dots < y_n < +\infty. \end{array} \right.$$

From (3.3, 2c) it follows that

$$(3.3, 5a) \left\{ \begin{array}{l} \alpha_{-}^{(0)}(\mu) = m! n! \int_R \dots \int \prod_{i=1}^n \{f(x_i) dx_i\} \prod_{j=1}^m \{f(y_j + \mu) dy_j\}, \\ \text{with } R \text{ defined by: } -\infty < x_1 < \dots < x_n < y_1 < \dots < y_m < +\infty \end{array} \right.$$

or, by replacing  $x_i$  by  $x_i - \mu$  and  $y_j$  by  $y_j - \mu$ , that

$$(3.3, 5b) \left\{ \begin{array}{l} \alpha_{-}^{(0)}(\mu) = m! n! \int_R \dots \int \prod_{i=1}^n \{f(x_i - \mu) dx_i\} \prod_{j=1}^m \{f(y_j) dy_j\} \\ \text{with } R \text{ defined as in (3.3, 5a).} \end{array} \right.$$

Remark. Even without the restriction b) of 3.1, writing again  $g(y)$  for  $f(y - \mu)$ , the  $(m+n)$ -fold integral (3.3, 4) can be reduced to a single one. In fact, writing  $x$  for  $x_m$ , it is seen that  $\alpha_{+}^{(0)}(\mu) =$

$$= m! n! \int_{-\infty}^{+\infty} f(x) dx \int_{-\infty < x_1 < \dots < x_{m-1} < x} \prod_{i=1}^{m-1} \{f(x_i) dx_i\} \int_{x < y_1 < \dots < y_n < +\infty} \prod_{j=1}^n \{g(y_j) dy_j\}$$

Because of

$$\int_{a < x_1 < \dots < x_k < b} \prod_{i=1}^k \{\varphi(x_i) dx_i\} = \frac{1}{k!} \left[ \int_a^b \varphi(x) dx \right]^k$$

(cf. (7)) one finds:

$$(3.3, 6) \left\{ \begin{array}{l} \alpha_{+}^{(0)}(\mu) = m \int_{-\infty}^{+\infty} f(x) \cdot F^{m-1}(x) \cdot [1 - G(x)]^n dx = \\ = \int_{-\infty}^{+\infty} [1 - G(x)]^n \cdot d[F^m(x)] = - \int_{-\infty}^{+\infty} F^m(x) \cdot d\{[1 - G(x)]^n\}. \end{array} \right.$$

In the same way it is found from (3.3, 5b) that

$$(3.3, 7) \left\{ \begin{array}{l} \alpha_{-}^{(0)}(\mu) = n \int_{+\infty}^{+\infty} g(x) \cdot G^{n-1}(x) \cdot [1 - F(x)]^m dx = \\ = \int_{-\infty}^{+\infty} [1 - F(x)]^m \cdot d[G^n(x)] = - \int_{-\infty}^{+\infty} G^n(x) \cdot d\{[1 - F(x)]^m\} \end{array} \right.$$

For  $\alpha_{+}^{(0)}(\mu)$  one can write instead of (3.3, 4):

$$(3.3, 8) \quad \alpha_{+}^{(0)}(\mu) = m! n! \int_{-\infty}^{+\infty} f(x_1) dx_1 \int_{x_1}^{+\infty} f(x_2) dx_2 \int_{x_2}^{+\infty} \dots \int_{x_m}^{+\infty} f(y_1 - \mu) dy_1 \int_{y_1}^{+\infty} \dots \int_{y_{n-1}}^{+\infty} f(y_n - \mu) dy_n.$$

This expression will be written more shortly by the use of some operators and notations which will be defined in (3.3, 9, 9a, 9b and 10):

$$(3.3, 9) \left\{ \begin{array}{l} \int_y^{+\infty} \varphi_1(x_1) dx_1 \int_{x_1}^{+\infty} \varphi_2(x_2) dx_2 \int_{x_2}^{+\infty} \dots \int_{x_{p-1}}^{+\infty} \varphi_p(x_p) dx_p = \\ = [(I\varphi_1)(I\varphi_2) \dots (I\varphi_p)](y) = \left[ \prod_{\nu=1}^p (I\varphi_{\nu}) \right](y). \end{array} \right.$$



If  $\varphi_\nu(x) = f(x)$  ( $\nu = 1, \dots, k$ ), the following abbreviation is used:

$$(3.3, 9a) \quad \left[ \prod_{\nu=1}^p (I\varphi_\nu) \right] (y) = [(If)^p] (y).$$

If  $\varphi_\nu(x) = f(x)$  for all integers  $\nu \neq i$  with  $1 \leq \nu \leq p$  and  $\varphi_i(x) = \{f(x)\}^\omega$ , then the abbreviation used is:

$$(3.3, 9b) \quad \left[ \prod_{\nu=1}^p (I\varphi_\nu) \right] (y) = [(If)^{i-1} (If^\omega) (If)^{p-i}] (y)$$

Furthermore, the following notations are used:

$$(3.3, 10) \quad \begin{cases} f(x-\mu) = f_{-\mu} & ; & f(x+\mu) = f_{+\mu} & ; & f'(x) = f' & ; \\ f'(x-\mu) = f'_{-\mu} & ; & f'(x+\mu) = f'_{+\mu} & \text{and } f''(x) = f'' & . \end{cases}$$

With (3.3, 9, 9a and 10) the expression (3.3, 8) becomes:

$$(3.3, 11) \quad \alpha_+^{(0)}(\mu) = m! n! [(If)^m (If_{-\mu})^n] (-\infty)$$

$$(3.3, 11a) \quad = m! n! [(If_{+\mu})^m (If)^n] (-\infty)$$

((3.3, 11a) follows from (3.3, 11) by the substitutions

$$x_i = x'_i + \mu \quad (1 \leq i \leq m) \text{ and } y_j = y'_j + \mu \quad (1 \leq j \leq n).)$$

In a similar way one finds for  $\alpha_+(\mu)$ , for any (integer) value of  $U$  which is  $\leq \frac{1}{2} m n$ :

$$(3.3, 12) \quad \alpha_+(\mu) = m! n! \sum_{U=0}^{U_\alpha} \left[ \prod_{h=1}^r \{(If)^{m_h} (If_{-\mu})^{n_h}\} \right] (-\infty)$$

$$(3.3, 12a) \quad = m! n! \sum_{U=0}^{U_\alpha} \left[ \prod_{h=1}^r \{(If_{+\mu})^{m_h} (If)^{n_h}\} \right] (-\infty)$$

$$(3.3, 12b) \quad \left\{ \begin{array}{l} \text{Here } m_1 \geq 0 \text{ and } m_h > 0 \text{ for } h > 0, \\ n_h > 0 \text{ for } h < r \text{ and } n_r \geq 0; \\ \sum_{h=1}^r m_h = m, \sum_{h=1}^r n_h = n \text{ and } U = \sum_{i=1}^r n_i \sum_{j=i+1}^r m_j \end{array} \right.$$

$$(3.3, 12c) \quad \left\{ \begin{array}{l} \text{The summations (over } U) \text{ in (3.3, 12) and (3.3, 12a)} \\ \text{are to be extended over all combinations of } r, \text{ and of} \\ m_h \text{ and } n_h \text{ (} 1 \leq h \leq r) \text{ which give a value of } U \leq U_\alpha. \end{array} \right.$$

A similar expression follows for  $\alpha_-(\mu)$  by the use of (3.3, 2c);  $\alpha_\pm(\mu)$  then follows from (3.3, 3).

#### 4. SOME PROPERTIES OF THE POWER FUNCTION OF WILCOXON'S TEST WHEN ALTERNATIVES ARE ONE-SIDED

##### 4.1. General theorems.

##### 4.1.1. A theorem on the unbiasedness of the test.

Theorem 1. Under the restrictions a) and b) of 3.1 the power function

$\alpha_+(\mu)$  given by (3.3, 1), when  $a$ ,  $m$  and  $n$  are constant, is a monotonous non-decreasing function of  $\mu$ .

Proof.<sup>5)</sup> By the substitutions  $x_i = x'_{i'}$ ;  $y_j = y'_{j'} + \mu$  ( $i, i' = 1, \dots, m$ ;  $j, j' = 1, \dots, n$ ) a one-to-one correspondence is established between the points  $E = (x_1, \dots, x_m, y_1, \dots, y_n)$  of the space  $W$  and the points  $E' = (x'_{1'}, \dots, x'_{m'}, y'_{1'}, \dots, y'_{n'})$  of the space  $W'$ . When  $U = U(E)$  is defined as in 1. and  $U_\mu(E')$ , for each point  $E'$  with  $x'_{i'} \neq y'_{j'} + \mu$  for each pair  $(i', j')$ , is defined as the number of pairs  $(i', j')$  ( $1 \leq i' \leq m$ ;  $1 \leq j' \leq n$ ) with  $x'_{i'} > y'_{j'} + \mu$ , then  $U(E) = U_\mu(E')$  if  $E$  and  $E'$  are corresponding points.

Hence

$$(4.1, 1) \quad \alpha_+(\mu) = \int \dots \int_{U(E) \leq U_a} \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j - \mu) dy_j\}$$

$$(4.1, 2) \quad = \int \dots \int_{U_\mu(E') \leq U_a} \prod_{i'=1}^m \{f(x'_{i'}) dx'_{i'}\} \prod_{j'=1}^n \{f(y'_{j'}) dy'_{j'}\}.$$

Now the integrand of (4.1, 2) is independent of  $\mu$  and non-negative for every  $E'$ . From the definition of  $U_\mu(E')$  follows that, in a fixed point  $E'$ ,  $U_\mu(E')$  is a monotonous non-increasing function of  $\mu$ . Hence the set of all points with  $U_\mu(E') \leq U_a$  cannot decrease when  $\mu$  increases. So  $\alpha_+(\mu)$  is a monotonous non-decreasing function of  $\mu$ .

Corollary. For all distribution functions  $F$  and  $G$  satisfying a) and b) of 3.1, the critical region  $U \leq U_a$  provides an unbiased test of the hypothesis  $\mu = 0$  against all alternatives  $\mu > 0$  (cf 2.3, a).

Remark. The same holds for the critical region  $U \geq mn - U_a$  when the alternatives are given by  $\mu < 0$ .

#### 4.12. On the interchangeability of $m$ and $n$ in the power function.

Theorem 2. Under the restrictions a) and b) of 3.1 the power functions  $\alpha_+(\mu)$  as well as  $\alpha_-(\mu)$  given by (3.3, 1) and by (3.3, 2) respectively, when  $a$  is constant, are identical with those obtained when  $m$  and  $n$  are interchanged, provided  $f(x - c)$  is an even function of  $x$  for some  $c$ .

Proof. Without loss of generality one may suppose  $c = 0$ . Now the expression (3.3, 2b) is equal to the expression (3.3, 2c), when  $f(x)$  is an even function. The proof of this equality given in 3.2, II, a) and 3.2, II, b), holds good when  $+\mu$  is replaced by  $-\mu$ . The theorem can also be proved directly from (3.3, 1) by the substitutions  $x_i = -y'_{i'} + \mu$  ( $i, i' = 1, \dots, m$ ) and  $y_j = -x'_{j'} + \mu$  ( $j, j' = 1, \dots, n$ ).

#### 4.2. Calculation of $\alpha'_+(0)$ for Wilcoxon's test with specialization to the normal distribution.

<sup>5)</sup> This proof is due to Mr J. HEMELRIJK.

From the expression (3. 3, 12a) for  $\alpha_+(\mu)$ ,  $\alpha'_+(\mu) = \frac{d\alpha_+(\mu)}{d\mu}$  is easily calculated.

Defining the operator  $I^{-1}$  by

$$(4. 2, 1) \quad I^{-1}\varphi(x) = I^{-1}\varphi = -\varphi'(x) = -\varphi'$$

it follows from the definition of the operator  $I$  in (3. 3, 9) that

$$(4. 2, 2) \quad I^{-1}(I\varphi) = \varphi.$$

Furthermore defining

$$(4. 2, 3) \quad [\varphi_1(I\varphi_2)](y) = \varphi_1(y) \int^{\infty} \varphi_2(x) dx$$

and

$$(4. 2, 4) \quad [(I\varphi_2)\varphi_1](y) = \int_y^{\infty} \varphi_2(x) \varphi_1(x) dx$$

it is seen that

$$(4. 2, 5) \quad \begin{cases} [(If'_{+\mu})(I\varphi)](y) = -[f_{+\mu}(I\varphi)](y) + [If_{+\mu}\varphi](y) = \\ = -[I^{-1}(If_{+\mu})(I\varphi)](y) + [(If_{+\mu})I^{-1}(I\varphi)](y). \end{cases}$$

Hence

$$(4. 2, 6) \quad \left\{ \begin{aligned} \left[ \frac{\partial}{\partial \mu} \{(If_{+\mu})^m\} \chi \right] (y) &= \sum_{l=0}^{m-1} [(If_{+\mu})^l (If'_{+\mu}) (If_{+\mu})^{m-l-1} \chi] (y) = \\ &= -\sum_{l=0}^{m-1} [(If_{+\mu})^l I^{-1} (If_{+\mu})^{m-l} \chi] (y) + \\ &\quad + \sum_{l=0}^{m-1} [(If_{+\mu})^{l+1} I^{-1} (If_{+\mu})^{m-l-1} \chi] (y) = \\ &= -[I^{-1} (If_{+\mu})^m \chi] (y) + [(If_{+\mu})^m I^{-1} \chi] (y). \end{aligned} \right.$$

Now

$$\alpha'_+(\mu) = m! n! \sum_{U=0}^{U_\alpha} \sum_{k=1}^r \left[ \prod_{h=1}^{k-1} \{(If_{+\mu})^{m_h} (If)^{n_h}\} \cdot \frac{\partial}{\partial \mu} \{(If_{+\mu})^{m_k}\} (If)^{n_k} \prod_{h=k+1}^r \{(If_{+\mu})^{m_h} (If)^{n_h}\} \right] (-\infty).$$

Hence, by means of (4. 2, 6):

$$(4. 2, 7) \quad \alpha'_+(\mu) = \begin{cases} -m! n! \sum_{U=0}^{U_\alpha} \sum_{k=1}^r \left[ \prod_{h=1}^{k-1} \{(If_{+\mu})^{m_h} (If)^{n_h}\} I^{-1} (If_{+\mu})^{m_k} (If)^{n_k} \prod_{h=k+1}^r \{(If_{+\mu})^{m_h} (If)^{n_h}\} \right] (-\infty) \\ + m! n! \sum_{U=0}^{U_\alpha} \sum_{k=1}^r \left[ \prod_{h=1}^{k-1} \{(If_{+\mu})^{m_h} (If)^{n_h}\} (If_{+\mu})^{m_k} I^{-1} (If)^{n_k} \prod_{h=k+1}^r \{(If_{+\mu})^{m_h} (If)^{n_h}\} \right] (-\infty). \end{cases}$$

If the expression between [ ] begins or ends with  $I^{-1}$  for some  $k$ , then the corresponding term is to be considered as zero.

From (4. 2, 7) it follows that:

$$(4. 2, 8) \quad \alpha'_+(0) = \begin{cases} -m! n! \sum_{U=0}^{U_\alpha} \sum_{k=1}^r [(If)^{L_k-1} (If^2) (If)^{m_k+n_k-1+M_k}] (-\infty) \\ + m! n! \sum_{U=0}^{U_\alpha} \sum_{k=1}^r [(If)^{L_k+m_k-1} (If^2) (If)^{n_k-1+M_k}] (-\infty). \end{cases}$$

Here  $L_k = \sum_{h=1}^{k-1} (m_h + n_h)$  and  $M_k = \sum_{h=k+1}^r (m_h + n_h)$ ; for  $m_h, n_h, r$  and  $\sum_U$  cf. (3.3, 12b) and (3.3, 12c). Those terms in the sums of (4.2, 8) in which the first or the last symbolic power has a negative exponent are to be considered as zero.

The expression (4.2, 8) for  $\alpha'_+(0)$  will be calculated for  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-ix^2}$  (cf. c) in 3.1) from the Appendix; cf. (A.1, 1) and (A.1, 9). It is seen from A.1, Remark 2, that the calculations will be restricted to  $\tau \leq 4$ , i.e. to  $m+n \leq 5$ . Furthermore, only relatively low values of  $\alpha$  are considered (cf. 2.2). If  $U_\alpha = 0$  determines an  $\alpha > 0,15$ , then only  $U_\alpha = 0$  is considered, otherwise  $U_\alpha = 1$ , too, will be considered. Cases with  $m=1$  or  $n=1$  are omitted as being trivial. By means of (5.2, 8), (A.1, 1) and (A.1, 9) the following results are obtained:

$$(4.2, 9) \left\{ \begin{array}{l} \text{When } U_\alpha = 0, \alpha'_+(0) = m! n! [(If)^{m-1} (If^2) (If)^{n-1}] (-\infty) = \\ \\ = \frac{m! n! (m+n-1) \cdot \Gamma\left(\frac{m+n-1}{2}\right)}{2^2 \cdot \pi^{\frac{1}{2}(m+n)}} \cdot V_{1^{m-1}, 2, 1^{n-1}} \end{array} \right.$$

$$(4.2, 10) \left\{ \begin{array}{l} \text{When } U_\alpha = 1, \alpha'_+(0) = \\ = +m! n! [(If)^{m-2} (If^2) (If)^n] (-\infty) + m! n! [(If)^m (If^2) (If)^{n-2}] (-\infty) = \\ = \frac{m! n! (m+n-1) \cdot \Gamma\left(\frac{m+n-1}{2}\right)}{2^2 \cdot \pi^{\frac{1}{2}(m+n)}} \cdot \{V_{1^{m-2}, 2, 1^n} + V_{1^m, 2, 1^{n-2}}\} \end{array} \right.$$

From (4.2, 9) and (4.2, 10) the following results are obtained by means of (A.3, 2) and (A.4, 4):

$m$	$n$	$U_\alpha$	$\alpha$	$\alpha'_+(0)$
2	2	0	$\frac{1}{6}$	$\frac{3}{2\pi\sqrt{\pi}} V_{1,2,1} = \frac{1}{\pi\sqrt{\pi}} \arccos \frac{1}{3} = 0,22106$
2	3	0	$\frac{1}{10}$	$\frac{12}{\pi^2\sqrt{\pi}} V_{1,2,1,1} = \frac{3}{2\pi\sqrt{\pi}} \arcsin \frac{\sqrt{3}}{3} = 0,16580$
2	3	1	$\frac{1}{6}$	$\frac{12}{\pi^2\sqrt{\pi}} (V_{2,1,1,1} + V_{1,1,2,1}) =$ $= \frac{3}{2\pi\sqrt{\pi}} \left( \arcsin \frac{\sqrt{18}-\sqrt{3}}{6} + \arcsin \frac{\sqrt{3}}{3} \right) = \frac{1}{2\sqrt{\pi}} = 0,28209.$

Remark 1. For the values of  $\alpha$  cf. MANN and WHITNEY or 3.2.

Remark 2. Because of theorem 2 (4.12) it is not necessary to calculate separately the case  $m=3, n=2$ .

4.3. A comparison with  $\alpha'_+(0)$  for Student's test for the difference of two means.

The alternatives to the hypothesis tested ( $\mu = 0$ ) are  $\mu > 0$  (therefore the notation  $\alpha_+$  is used, cf. (3.3, 1)). Let  $x_1, \dots, x, y_1, \dots, y_n$  be

$m + n = N$  independent random variables, the  $x_i$  all having the distribution function  $F(x)$  and the  $y_j$  all having the distribution function

$F(x - \mu)$  with  $F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2} dt$ . The critical region for testing,

according to STUDENT's test, the hypothesis  $\mu = 0$  against  $\mu > 0$  at the level of significance  $\alpha$  is given by  $t \geq t_\alpha$ .

Here  $t$  is defined by

$$t = \sqrt{\frac{mn(N-2)}{N}} \cdot \frac{\bar{y} - \bar{x}}{\sqrt{ms_x^2 + ns_y^2}},$$

where

$$\bar{y} = \frac{1}{n} \sum_{j=1}^n y_j; \quad \bar{x} = \frac{1}{m} \sum_{i=1}^m x_i; \quad ms_x^2 = \sum_{i=1}^m (x_i - \bar{x})^2$$

and  $ns_y^2 = \sum_{j=1}^n (y_j - \bar{y})^2$ , whereas  $t_\alpha$  is defined by:

$$\alpha = \frac{1}{\sqrt{(N-2)\pi}} \cdot \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)} \int_{t_\alpha}^{\infty} \left(1 + \frac{x^2}{N-2}\right)^{-\frac{N-1}{2}} dx.$$

$t_\alpha^1$  can be found from the tables of STUDENT's distribution. For the values of  $\alpha = 1/6, 1/10$  and  $1/5$   $t_\alpha$  has been tabulated in 3 decimal places only, or not at all. Therefore the quantity  $\eta_\alpha = \frac{N-2}{N-2+t_\alpha^2}$  was calculated directly from

$$(4.3,1) \quad \alpha = \frac{1}{2} I_{\eta_\alpha} \left( \frac{N-2}{2}, \frac{1}{2} \right),$$

where  $I_x(p, q)$  is the incomplete  $B$ -function tabulated by K. PEARSON (6).

The power function of the critical region  $t \geq t_\alpha$  is, analogously to P. L. HSU (1):

$$(4.3,2) \quad \alpha_+(\mu) = \begin{cases} \frac{e^{-\frac{1}{2} \frac{mn}{N} \mu^2}}{2\sqrt{\pi} \cdot \sqrt{N-2} \cdot \Gamma\left(\frac{N-2}{2}\right)} \cdot \sum_{k=0}^{\infty} \left[ \frac{2mn}{N(N-2)} \right]^{\frac{k}{2}} \\ \cdot \frac{\mu^k \cdot \Gamma\left(\frac{k+N-1}{2}\right)}{k!} \int_{t_\alpha^2}^{\infty} x^{\frac{k-1}{2}} \left(1 + \frac{x}{N-2}\right)^{-\frac{k+N-1}{2}} dx \end{cases}$$

Hence

$$(4.3,3) \quad \alpha'_+(0) = \sqrt{\frac{mn}{2\pi N}} \cdot \left( \frac{N-2}{N-2+t_\alpha^2} \right)^{\frac{N-2}{2}} = \sqrt{\frac{mn}{2\pi N}} \cdot \eta_\alpha^{\frac{N-2}{2}}$$

From (4.3, 3) and (4.3, 1) the following results are obtained:

$m$	$n$	$\alpha$	$\eta_\alpha$	$\alpha'_+(0)$	$\alpha'_+(0)_{St} - \alpha'_+(0)_{Wi}$
2	2	1/6	5/9	0,22163	0,00057
2	3	1/10	0,527963	0,16765	0,00185
2	3	1/5	0,758072	0,28845	0,00636

In order to facilitate the comparison with  $\alpha'_+(0)$  for WILCOXON's test, a column is added containing the difference of  $\alpha'_+(0)$  for the test of STUDENT and for the test of WILCOXON. It is seen that STUDENT's test and WILCOXON's test satisfy a necessary condition that STUDENT's test is more powerful than WILCOXON's test ( $\alpha'_+(0)_{St} - \alpha'_+(0)_{Wi} > 0$ , cf. the end of the remark in 2.3), but clearly the difference is very small.

SOME REMARKS ON THE POWER FUNCTION OF WILCOXON'S  
TEST FOR THE PROBLEM OF TWO SAMPLES. II

BY

H. R. VAN DER VAART

(Communicated by Prof. D. VAN DANTZIG at the meeting of February 25, 1950)

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5. SOME PROPERTIES OF THE POWER FUNCTION OF WILCOXON'S TEST WHEN ALTERNATIVES ARE TWO-SIDED

5.1. *General theorems when alternatives are two-sided.*

5.11. *Properties of symmetry of the power function.*

**Theorem 3.** *Under the restrictions a) and b) of 3.1 the power function  $\alpha_{\pm}(\mu)$  given by (3.3, 3), when  $\alpha$ ,  $m$  and  $n$  are constant, is an even function of  $\mu$ , either when 1)  $f(x - c)$  is an even function of  $x$  for some  $c$ , or when 2)  $m = n$  (or when both conditions are satisfied, of course).*

**Proof.**

1. From (3.3, 3, 1 and 2b) one obtains:

$$\begin{aligned} \alpha_{\pm}(\mu) = & \int_{v \leq v_{\alpha}} \dots \int \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j - \mu) dy_j\} + \\ & + \int_{v \leq v_{\alpha}} \dots \int \prod_{i=1}^m \{f(-x_i) dx_i\} \prod_{j=1}^n \{f(-y_j - \mu) dy_j\}. \end{aligned}$$

Without loss of generality  $c$  may be supposed to be equal to zero.

Then  $f(x)$  is an even function. Hence:

$$(5.1, 1) \quad \left\{ \begin{aligned} \alpha_{\pm}(\mu) &= \int_{U \leq U_a} \dots \int \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j - \mu) dy_j\} + \\ &+ \int_{U \leq U_a} \dots \int \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j + \mu) dy_j\}. \end{aligned} \right.$$

By changing  $+\mu$  into  $-\mu$  the first term of the second member of (5.1, 1) passes into the second one and vice versa. Hence  $\alpha_{\pm}(-\mu) = \alpha_{\pm}(+\mu)$ , q.e.d.

2. From (3.3, 3, 1 and 2c) one obtains:

$$\alpha_{\pm}(\mu) = \int_{U \leq U_a} \dots \int \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j - \mu) dy_j\} + \int_{U \leq U_a} \dots \int \prod_{i=1}^n \{f(x_i) dx_i\} \prod_{j=1}^m \{f(y_j + \mu) dy_j\}.$$

When  $m = n$ , it follows that:

$$(5.1, 2) \quad \left\{ \begin{aligned} \alpha_{\pm}(\mu) &= \int_{U \leq U_a} \dots \int \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^m \{f(y_j - \mu) dy_j\} + \\ &+ \int_{U \leq U_a} \dots \int \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^m \{f(y_j + \mu) dy_j\}. \end{aligned} \right.$$

By changing  $+\mu$  into  $-\mu$  the first term of the second member of (5.1, 2) passes into the second one and vice versa. Hence  $\alpha_{\pm}(-\mu) = \alpha_{\pm}(+\mu)$ , q.e.d.

*Corollary.* If the conditions of theorem 3 are satisfied and  $a'_{\pm}(0)$  exists, then  $a'_{\pm}(0) = 0$ .

When  $f(x)$  is a non-symmetric function and  $m \neq n$ , then  $\alpha_{\pm}(\mu)$ , defined by (3.3, 3), need not be an even function of  $\mu$ , as follows from theorem 4

**Theorem 4.** *Under the restrictions a) and b) of 3.1 a non-symmetric function  $f(x)$  and a value of  $U_a$  can be given, such that  $a'_{\pm}(0)$  exists and is  $\neq 0$ , if  $m \neq n$ .*

**Proof.**

$$\text{Let } F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 - e^{-x} & \text{for } x \geq 0 \end{cases}, G(x) = F(x - \mu) \text{ and } U_a = 0.$$

Then by somewhat laborious calculations it can be proved, using (3.3, 6 and 7), that:

$$(5.1, 3) \quad \alpha_{\pm}(\mu) = \begin{cases} \frac{m!n!}{(m+n)!} e^{n\mu} + (1 - e^{\mu})^n + n e^{-m\mu} \int_0^{\mu} (1-x)^{n-1} x^m dx, & \text{if } \mu < 0. \\ \frac{m!n!}{(m+n)!} e^{-m\mu} + (1 - e^{-\mu})^m + m e^{n\mu} \int_0^{-\mu} (1-x)^{m-1} x^n dx, & \text{if } \mu > 0. \end{cases}$$



Hence

$$(5.1, 4) \quad \alpha'_{\pm}(0) = \frac{m!n!}{(m+n)!} (n-m) \neq 0, \text{ q. e. d.}$$

5.12 *A theorem on possible biasedness of the test when alternatives are two-sided.*

As the critical region given by  $|U - \frac{1}{2}mn| \geq \frac{1}{2}mn - U_{\alpha}$  cannot possibly provide an unbiased test for the hypothesis  $\mu = 0$  against all alternatives  $\mu \neq 0$ , when  $\alpha'_{\pm}(0)$  exists and is  $\neq 0$ , one should, if possible, take  $m = n$  in the applications of the test. In this way one secures the symmetry of the power function. One might hope that unbiasedness then would be secured, too. One might think that as general a theorem would hold good for two-sided alternatives, as theorem 1 and its corollary proved in 4.1, for onesided alternatives. That this is impossible in such a generality, is shown by theorem 5:

**Theorem 5.** *Under the restrictions a) and b) of 3.1 for every  $m$  and  $n = m$  an even function  $f(x)$  and a value of  $U_{\alpha}$  can be given, such that  $\alpha''_{\pm}(0)$  exists and is  $< 0$ .*

**Proof.** Let  $U_{\alpha} = 0$ . By (3.3, 3, 1 and 2b) the power function  $\alpha_{\pm}(\mu)$ , when  $f(x)$  is an even function and  $m = n$ , is equal to (5.1, 2), where now " $U \leq U_{\alpha}$ " is to be replaced by " $U = 0$ ". By a reduction as described in the remark of 3.3 (under (3.3, 5b)) it is found that:

$$(5.1, 5) \quad \left\{ \begin{aligned} \alpha_{\pm}(\mu) &= \int_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1 - F(x - \mu))^m] dx + \\ &+ \int_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1 - F(x + \mu))^m] dx. \end{aligned} \right.$$

Hence, writing  $\alpha''_{\pm}(\mu)$  for  $\frac{d^2 \alpha_{\pm}(\mu)}{d\mu^2}$ ,

$$(5.1, 6) \quad \alpha''_{\pm}(0) = 2 \int_{-\infty}^{+\infty} [F^m(x)]' \cdot [(1 - F(x))^m]'' dx,$$

the primes denoting differentiation with respect to  $x$ . Because of  $f(x)$  being even,  $F(-y) = 1 - F(y)$ . Hence:

$$(5.1, 7) \quad \int_{-\infty}^0 [F^m(x)]' \cdot [(1 - F(x))^m]'' dx = - \int_0^{+\infty} [(1 - F(y))^m]' \cdot [F^m(y)]'' dy,$$

as is seen by the substitution  $x = -y$ . (The primes in the second member of (5.1, 7) denote differentiation with respect to  $y$ ). By partial integration the second member of (5.1, 7) is seen to be equal to

$$+ \int_0^{+\infty} [F^m(y)]' \cdot [(1 - F(y))^m]'' dy,$$

so that (cf. (5.1, 6) and (5.1, 7))

$$(5.1, 8) \quad \alpha''_{\pm}(0) = 4 \int_0^{+\infty} [F^m(x)]' \cdot [(1 - F(x))^m]'' dx.$$

Now  $[F^m(x)]' \geq 0$  for every  $x$  with  $0 \leq x \leq \infty$ . Hence a sufficient condition for  $\alpha_{\pm}''(0)$  being  $< 0$  is that  $[(1 - F(x))_{\pm}^m]'' \leq 0$  for  $0 \leq x \leq \infty$ , where the equality-sign does not hold in the whole interval  $(0, \infty)$ .

Such a function  $F(x)$ , the derivative  $f(x)$  of which is an even function, is defined by

$$(5.1, 9) \quad F(x) = \left\{ \begin{array}{ll} 0 & \text{for } x \leq -1 \\ \frac{1}{2}(1-x^2)^{1/m} & \text{for } -1 \leq x \leq 0 \\ 1 - \frac{1}{2}(1-x^2)^{1/m} & \text{for } 0 \leq x \leq +1 \\ 1 & \text{for } x \geq +1 \end{array} \right\}$$

For  $[(1 - F(x))^m]'' = 2^{-m}(1-x^2)'' = -2^{1-m}$ , if  $0 \leq x \leq +1$ . Hence  $\alpha_{\pm}''(0) = -2^{3-m} \left(1 - \frac{1}{2^m}\right) < 0$ .

*Conclusion.* The necessary condition for unbiasedness (consisting in  $\alpha_{\pm}''(0)$  not being  $< 0$ , cf. 2.3, remark) is not satisfied in general. Hence the critical region defined by  $|\mathbf{U} - \frac{1}{2} m n| \geq \frac{1}{2} m n - U_{\alpha}$  does not provide an unbiased test for the hypothesis  $\mu = 0$  against  $\mu \neq 0$  without further restrictions being imposed upon  $F$  and  $G$ .

5.13. *On the interchangeability of  $m$  and  $n$  in the power function when alternatives are two-sided.*

**Theorem 6.** *Under the restrictions a) and b) of 3.1 the power function  $\alpha_{\pm}(\mu)$ , given by (3.3, 3), when  $a$  is constant, is identical with the power function  $\alpha_{\pm}(\mu)$  obtained when  $m$  and  $n$  are interchanged, provided  $f(x-c)$  is an even function of  $x$  for some  $c$ .*

*Proof.* (3.3, 3) and theorem 2.

5.2. *Calculation of  $\alpha_{\pm}''(0)$  for Wilcoxon's test with specialization to the normal distribution.*

Throughout 5.2  $f(x)$  is assumed to be an even function. After (5.2, 8b)  $f(x)$  is taken to be equal to  $\frac{1}{\sqrt{2\pi}} e^{-1/2 x^2}$ . From (3.3, 3, 1 and 2b) it follows that ( $f(x)$  being an even function):  $\alpha_{\pm}(\mu) = \alpha_{+}(\mu) + \alpha_{-}(\mu) =$

$$\begin{aligned} &= \int \dots \int_{U \leq U_{\alpha}} \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j - \mu) dy_j\} + \\ &+ \int \dots \int_{U \leq U_{\alpha}} \prod_{i=1}^m \{f(x_i) dx_i\} \prod_{j=1}^n \{f(y_j + \mu) dy_j\}. \end{aligned}$$

Hence  $\alpha_{\pm}(\mu) = \alpha_{+}(\mu) + \alpha_{+}(-\mu)$ , so that  $\alpha_{\pm}(0) = 2\alpha_{+}(0) = 2\alpha$   
 $\alpha'_{\pm}(0) = \alpha'_{+}(0) - \alpha'_{+}(0) = 0$  and  $\alpha''_{\pm}(0) = 2\alpha''_{+}(0)$ .

Now  $\alpha''_{+}(0)$  is calculated, starting from (4.2, 7):

The operator  $I^{-2}$  is defined by

$$(5.2, 1) \quad I^{-2} \varphi(x) = I^{-1} I^{-1} \varphi = -I^{-1} \varphi' = +\varphi''(x) = +\varphi''.$$

From the definitions in 4. 2 it then follows that

$$(5. 2, 2) \quad \left\{ \begin{aligned} \left[ \frac{\partial}{\partial \mu} \{ I^{-1} (If_{+\mu})^m \} \chi \right] (y) &= \left[ I^{-1} \frac{\partial}{\partial \mu} \{ (If_{+\mu})^m \} \chi \right] (y) = (by (4. 2, 6)) = \\ &= - [ I^{-2} (If_{+\mu})^m \chi ] (y) + [ I^{-1} (If_{+\mu})^m I^{-1} \chi ] (y). \end{aligned} \right.$$

Furthermore from (5. 2, 1) and (4. 2, 6):

$$(5. 2, 3) \quad \left[ \frac{\partial}{\partial \mu} (If_{+\mu})^m I^{-1} \chi \right] (y) = - [ I^{-1} (If_{+\mu})^m I^{-1} \chi ] (y) + [ (If_{+\mu})^m I^{-2} \chi ] (y).$$

From (4. 2, 7), by means of (4. 2, 6), (5. 2, 2) and (5. 2, 3), one obtains:  
 $\alpha''_{\pm}(0) = 2 \alpha''_{+}(0) =$

$$(5. 2, 4) = \left\{ \begin{aligned} &4 m! n! \sum_{U=0}^{U_a} \sum_{z=2}^{2r-1} \sum_{w=1}^{z-1} [ (-1)^{w+z} (If)^{P_w} I^{-1} (If)^{Q_{w,z}} I^{-1} (If)^{R_z} ] (-\infty) \\ &+ 2 m! n! \sum_{U=0}^{U_a} \sum_{w=1}^{2r-1} [ (If)^{P_w} I^{-2} (If)^{S_w} ] (-\infty). \end{aligned} \right.$$

Here:

$$P_w = \left\{ \begin{aligned} &\sum_{h=1}^{\frac{1}{2}w} (m_h + n_h), \quad \text{if } w \text{ is even.} \\ &\sum_{h=1}^{\frac{1}{2}(w-1)} (m_h + n_h) + m_{\frac{1}{2}(w+1)}, \quad \text{if } w \text{ is odd.} \end{aligned} \right.$$

$$R_z = \sum_{h=1}^z (m_h + n_h) - P_z = \left\{ \begin{aligned} &\sum_{h=\frac{1}{2}z+1}^z (m_h + n_h), \quad \text{if } z \text{ is even.} \\ &n_{\frac{1}{2}(z+1)} + \sum_{h=\frac{1}{2}(z+3)}^z (m_h + n_h), \quad \text{if } z \text{ is odd.} \end{aligned} \right.$$

$$Q_{w,z} = \sum_{h=1}^z (m_h + n_h) - P_w - R_z.$$

$$S_w = \sum_{h=1}^r (m_h + n_h) - P_w.$$

If in (5. 2, 4) the expression between [ ] begins or ends with  $I^{-1}$  or  $I^{-2}$  for some  $w$  or  $z$ , then the corresponding term is to be considered as zero.

Remark 1. The summation variables  $w$  and  $z$  should not be confounded, of course, with the  $w$  of 2. 1 (critical region) and the  $z$  (coordinates) of the Appendix.

As to the meaning of  $m_h$ ,  $n_h$  and  $r$ , cf. (3. 3, 12b).

The summations over  $U$  in (5. 2, 4) are to be extended over *all* combinations of  $r$ , and of  $m_h$  and  $n_h$  ( $1 \leq h \leq r$ ) which give a value of  $U \leq U_a$ . If  $U = 0$ , then  $r = 1$  and the sum over  $z$  in (5. 2, 4) is empty.

Hence in the first term of (5. 2, 4)  $\sum_{U=0}^{U_a}$  can be replaced by  $\sum_{U=1}^{U_a}$ .

Remark 2. Introducing the notations:

$$d_{2h-1} = m_n, \quad d_{2h} = n_n, \quad \text{and}$$

$$f_{\mu,j} = \begin{cases} f (= f(x)), & \text{if } j \text{ is even,} \\ f_{+\mu} (= f(x + \mu)), & \text{if } j \text{ is odd,} \end{cases}$$

$\alpha''_{\pm}(\mu)$  can be written as:

$$(5.2, 5) \quad \left\{ \begin{aligned} & 4m! n! \sum_{U=1}^{U_a} \sum_{z=2}^{2r-1} \sum_{w=1}^{z-1} \left[ \prod_{j=1}^w \{(If_{\mu,j})^{d_j}\} I^{-1} \prod_{j=w+1}^z \{(If_{\mu,j})^{d_j}\} I^{-1} \prod_{j=z+1}^{2r} \{(If_{\mu,j})^{d_j}\} \right] (-\infty) \\ & + 2m! n! \sum_{U=0}^{U_r} \sum_{w=1}^{2r-1} \left[ \prod_{j=1}^w \{(If_{\mu,j})^{d_j}\} I^{-2} \prod_{j=w+1}^{2r} \{(If_{\mu,j})^{d_j}\} \right] (-\infty) \text{ (end of Remark 2.)} \end{aligned} \right.$$

In the applications of (5.2, 4 or 5) the following reductions are to be used:

$$(5.2, 6) \quad \left\{ \begin{aligned} & [(If)^K I^{-1} (If)^L I^{-1} (If)^M] (-\infty) = \\ & = \begin{cases} [(If)^{K-1} (If)^2 (If)^{L-2} (If)^2 (If)^{M-1}] (-\infty), & \text{if } L \geq 2, \\ [(If)^{K-1} (If)^3 (If)^{M-1}] (-\infty), & \text{if } L = 1 \end{cases} \end{aligned} \right.$$

and:  $[(If)^K I^{-2} (If)^L] (-\infty) =$

$$(5.2, 7) \quad \left\{ \begin{aligned} & = [(If)^K I^{-1} f(If)^{L-1}] (-\infty) = \\ & = -[(If)^K f'(If)^{L-1}] (-\infty) + [(If)^K f f(If)^{L-2}] (-\infty) = \\ & = -[(If)^{K-2} (If) (If)' (If)^{L-1}] (-\infty) + [(If)^{K-1} (If)^3 (If)^{L-2}] (-\infty) = \\ & = +\frac{1}{2} [(If)^{K-2} (If)^3 (If)^{L-1}] (-\infty) + \frac{1}{2} [(If)^{K-1} (If)^3 (If)^{L-2}] (-\infty). \end{aligned} \right.$$

By means of (5.2, 6) and (5.2, 7) it follows from (5.2, 4) that  $\alpha''_{\pm}(0) =$

$$(5.2, 8) \quad \left\{ \begin{aligned} & 4m! n! \sum_{U=1}^{U_a} \sum_{z=2}^{2r-1} \sum_{w=1}^{z-1} [(-1)^{w+z} (If)^{Pw-1} (If)^2 (If)^{Qw,z-} (If)^2 (If)^{Rz-1}] (-\infty) \\ & + m! n! \sum_{U=0}^{U_a} \sum_{w=1}^{2r-1} [(If)^{Pw-2} (If)^3 (If)^{Sw-1}] (-\infty) \\ & + m! n! \sum_{U=0}^{U_a} \sum_{w=1}^{2r-1} [(If)^{Pw-1} (If)^3 (If)^{Sw-2}] (-\infty). \end{aligned} \right.$$

For  $P_w$ ,  $Q_{w,z}$ ,  $R_z$  and  $S_w$  and for the summations over  $U$  see the indications under (5.2, 4).

If  $Q_{w,z} = 1$  for some  $w$  or  $z$ , then the corresponding term in the first sum of the second member of (5.2, 8) is to be replaced by:

$$(5.2, 8a) \quad 4m! n! [(-1)^{w+z} (If)^{Pw-1} (If)^3 (If)^{Rz-1}] (-\infty).$$

In (5.2, 6), (5.2, 7) and (5.2, 8) those terms in which the first or the last symbolic power has a negative exponent, are to be considered as zero.

The expression (5. 2, 8) for  $\alpha''_{\pm}(0)$  will be calculated for  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2 x^2}$  (cf. c) in 3. 1) by means of the Appendix; cf. (A. 1, 1) and (A. 1, 9). It is seen from A. 1, Remark 2, that the calculations will be restricted to  $\tau \leq 4$ , i.e. to  $m+n \leq 6$ . Furthermore, only relatively low values of  $2\alpha$  are considered (cf. 2. 2;  $2\alpha = \alpha_{\pm}(0)$ , see 5. 2, some lines before (5. 2, 1)). If  $U_{\alpha} = 0$  determines a value of  $2\alpha > 0,15$ , then only  $U_{\alpha} = 0$  is considered, otherwise  $U_{\alpha} = 1$ , too, will be considered. Cases with  $m=1$  or  $n=1$  are omitted as being trivial. By means of (5. 2, 8), (A. 1, 1) and A. 1, 9) the following results are obtained:

$$(5. 2, 9) \quad \left\{ \begin{array}{l} \text{When } U_{\alpha} = 0, \alpha''_{\pm}(0) = \left\{ \begin{array}{l} + m! n! [(If)^{m-2} (If^3) (If)^{n-1}] (-\infty) + \\ + m! n! [(If)^{m-1} (If^3) (If)^{n-2}] (-\infty) = \\ \\ \frac{m! n! (m+n-2) \cdot \Gamma\left(\frac{m+n-2}{2}\right)}{2^2 \cdot \pi^{\frac{m+n}{2}} \cdot \sqrt{3}} [V_{1^{m-2}, 3, 1^{n-1}} + V_{1^{m-1}, 3, 1^{n-2}}]. \end{array} \right. \end{array} \right.$$

$$(5. 2, 10) \quad \left\{ \begin{array}{l} \text{When } U_{\alpha} = 1, \alpha''_{\pm}(0) = m! n! \{ + 4 [(If)^{m-2} (If^2)^2 (If)^{n-2}] (-\infty) \} + \\ + m! n! \{ + [(If)^{m-3} (If^3) (If)^n] (-\infty) - [(If)^{m-2} (If^3) (If)^{n-1}] (-\infty) \} + \\ + m! n! \{ - [(If)^{m-1} (If^3) (If)^{n-2}] (-\infty) + [(If)^m (If^3) (If)^{n-3}] (-\infty) \} = \\ \\ \frac{m! n! (m+n-2) \cdot \Gamma\left(\frac{m+n-2}{2}\right)}{2^2 \cdot \pi^{\frac{m+n}{2}}} [2 \cdot V_{1^{m-2}, 2, 2, 1^{n-2}} + \frac{1}{\sqrt{3}} \cdot V_{1^{m-3}, 3, 1^{n-1}} - \\ - \frac{1}{\sqrt{3}} \cdot V_{1^{m-2}, 3, 1^{n-1}} - \frac{1}{\sqrt{3}} \cdot V_{1^{m-1}, 3, 1^{n-2}} + \frac{1}{\sqrt{3}} \cdot V_{1^{m, 3, 1^{n-3}}}] . \end{array} \right.$$

Those terms in the last members of (5. 2, 9) and (5. 2, 10) in which a symbolic power has a negative exponent, are to be considered as zero.

From (5. 2, 9) and (5. 2, 10) the following results are obtained by means of (A. 2, 1), (A. 3, 2) and (A. 4, 4):

$m$	$n$	$U_{\alpha}$	$2\alpha$	$\alpha''_{\pm}(0)$
2	2	0	$\frac{1}{3}$	$\frac{2}{\pi^2 \sqrt{3}} [V_{3,1} + V_{1,3}] = \frac{2}{\pi \sqrt{3}} = 0,36755$
2	3	0	$\frac{1}{5}$	$\frac{3 \sqrt{3}}{2 \pi^2} [V_{3,1,1} + V_{1,3,1}] = \frac{\sqrt{3}}{\pi^2} \left[ \arccos \frac{\sqrt{6}}{4} + \arccos \frac{1}{4} \right]$ $= \frac{\sqrt{3}}{\pi^2} \left[ \pi - \arccos \frac{\sqrt{6}}{4} \right] = 0,39132$
2	4	0	$\frac{2}{15}$	$\frac{16 \sqrt{3}}{\pi^3} [V_{3,1,1,1} + V_{1,3,1,1}] =$ $= \frac{2 \sqrt{3}}{\pi^2} \left[ \arcsin \frac{\sqrt{30} - \sqrt{6}}{8} + \arcsin \frac{\sqrt{6}}{4} \right] =$ $= \frac{2 \sqrt{3}}{\pi^2} \arccos \frac{1}{2} = \frac{2 \sqrt{3}}{3\pi} = 0,36755$

$$\begin{aligned}
m & n & U_\alpha & 2\alpha & \alpha''(0) \\
2 & 4 & 1 & \frac{4}{15} & \frac{6}{\pi^2} \left[ 2 V_{2,2,1,1} - \frac{1}{\sqrt{3}} V_{3,1,1,1} \right] = \\
& & & & = \frac{6}{\pi^2} \left[ 2 \arcsin \frac{\sqrt{20}-\sqrt{2}}{6} - \frac{1}{\sqrt{3}} \arcsin \frac{\sqrt{30}-\sqrt{6}}{8} \right] \\
& & & & = 0,51398. \\
3 & 3 & 0 & \frac{1}{10} & \frac{24\sqrt{3}}{\pi^3} V_{1,3,1,1} = \frac{3\sqrt{3}}{\pi^2} \arcsin \frac{\sqrt{6}}{4} = 0,34698. \\
3 & 3 & 1 & \frac{1}{5} & \frac{72}{\pi^2} \left[ V_{1,2,2,1} + \frac{1}{\sqrt{3}} V_{3,1,1,1} - \frac{1}{\sqrt{3}} V_{1,3,1,1} \right] = \\
& & & & = \frac{9}{\pi^2} \left[ \arcsin \frac{2}{3} + \frac{1}{\sqrt{3}} \arcsin \frac{\sqrt{30}-\sqrt{6}}{8} - \right. \\
& & & & \left. - \frac{1}{\sqrt{3}} \arcsin \frac{\sqrt{6}}{4} \right] = 0,52282.
\end{aligned}$$

Remark 3. For the values of  $\alpha$  cf. MANN and WHITNEY (2) or 3. 2.

Remark 4. Because of theorem 6 (5. 13) it is not necessary to calculate separately the case  $\left\{ \begin{matrix} m=3, & n=2 \\ m=4, & n=2 \end{matrix} \right\}$ .

5. 3. A comparison with  $\alpha''_\pm(0)$  for Student's test for the difference of two means.

The alternatives to the hypothesis tested ( $\mu = 0$ ) are  $\mu \neq 0$  (therefore the notation  $\alpha_\pm$  is used, cf. (3. 3, 3)). Under the same assumptions about  $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n$  as in 4. 3, the critical region for testing, according to STUDENT'S test, the hypothesis  $\mu = 0$  against  $\mu \neq 0$  at the level of significance  $2\alpha$  is given by  $|\mathbf{t}| \geq t_\alpha$ . Here  $t$  and  $t_\alpha$  are defined as in 4. 3. As  $t_\alpha$  for the  $\alpha$ -values needed has been tabulated in 3 decimal places only, or not at all, the quantity  $\eta_\alpha = \frac{N-2}{N-2+t_\alpha^2} (N = m+n)$  was calculated directly from

$$(5. 3, 1) \quad \alpha = \frac{1}{2} I_{\eta_\alpha} \left( \frac{N-2}{2}, \frac{1}{2} \right),$$

where  $I_x(p, q)$  is the incomplete  $B$ -function tabulated by K. PEARSON (6).

The power function of the critical region  $|\mathbf{t}| \geq t_\alpha$  is (cf. P. L. HSU (1)):

$$(5. 3, 2) \quad \alpha_\pm(\mu) = \begin{cases} \frac{e^{-\frac{1}{2} \frac{mn}{N} \mu^2}}{\sqrt{N-2} \cdot \Gamma\left(\frac{N-2}{2}\right)} \cdot \sum_{k=0}^{\infty} \left[ \frac{mn}{2N(N-2)} \right]^k \cdot \\ \cdot \frac{\mu^{2k} \cdot \Gamma\left(\frac{2k+N-1}{2}\right)}{k! \Gamma\left(k+\frac{1}{2}\right)} \int_{t_\alpha^2}^{\infty} x^{k-1} \left(1 + \frac{x}{N-2}\right)^{-k-\frac{N-1}{2}} dx. \end{cases}$$

$$\begin{aligned}
& \text{Hence } \alpha''_\pm(0) = \\
& = \frac{mn \cdot \Gamma\left(\frac{N-1}{2}\right)}{N \sqrt{N-2} \cdot \sqrt{\pi} \cdot \Gamma\left(\frac{N-2}{2}\right)} \cdot \\
& \cdot \left[ \int_{t_\alpha^2}^{\infty} x^{\frac{1}{2}} \left(1 + \frac{x}{N-2}\right)^{-\frac{N+1}{2}} dx - \int_{t_\alpha^2}^{\infty} x^{-\frac{1}{2}} \left(1 + \frac{x}{N-2}\right)^{-\frac{N-1}{2}} dx \right].
\end{aligned}$$

By the substitution  $1 + \frac{x}{N-2} = y^{-1}$  it easily follows that:

$$(5.3, 3) \left\{ \begin{aligned} \alpha_{\pm}''(0) &= \frac{mn \cdot \Gamma\left(\frac{N-1}{2}\right)}{N \sqrt{\pi} \cdot \Gamma\left(\frac{N-2}{2}\right)} \left[ (N-1) \int_0^{\eta_a} y^{\frac{N-4}{2}} (1-y)^{\frac{1}{2}} dy - \int_0^{\eta_a} y^{\frac{N-4}{2}} (1-y)^{-\frac{1}{2}} dy \right] \\ &= \frac{mn}{N} \cdot \frac{2 \Gamma\left(\frac{N-1}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(\frac{N-2}{2}\right)} \cdot \eta_a^{\frac{N-2}{2}} \cdot (1-\eta_a)^{\frac{1}{2}}. \end{aligned} \right.$$

Here  $N = m + n$ .

In the cases, for which  $\alpha_{\pm}''(0)$  was calculated for WILCOXON'S test (cf. 5.2, at the end) the following results are obtained for STUDENT'S test from (5.3, 3) and (5.3, 1):

$m$	$n$	$2\alpha$	$\eta_a$	$\alpha_{\pm}''(0)$	$\alpha_{\pm}''(0)_{st} - \alpha_{\pm}''(0)_{wi}$
2	2	1/3	5/9	0,37037	0,00282
2	3	1/5	0,527963	0,40270	0,01138
2	4	2/15	0,530988	0,38618	0,01863
2	4	4/15	0,706294	0,54070	0,02672
3	3	1/10	0,468123	0,35959	0,01261
3	3	1/5	0,629850	0,54306	0,02024

In order to facilitate the comparison with  $\alpha_{\pm}''(0)$  for WILCOXON'S test, a column is added containing the difference of  $\alpha_{\pm}''(0)$  for the test of STUDENT and for the test of WILCOXON. This comparison shows that under the conditions which allow the use of STUDENT'S test,<sup>6)</sup> a necessary condition for STUDENT'S test being more powerful than WILCOXON'S one is satisfied (cf. the remark in 2.3), but in the cases investigated the difference is very small.

Remark. The following provisional result for large  $m$  and  $n$  was obtained for the power function of WILCOXON'S test when alternatives are two-sided:

$$\alpha_{\pm}''(0)_{wi} \approx \frac{\sqrt{2}}{\sqrt{\pi}} \cdot e^{-t_a^2} \cdot \zeta_a \cdot \frac{6mn}{2\pi(m+n)},$$

<sup>6)</sup> When  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$  and  $g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$  and a critical region  $|\mathbf{t}| \geq t_a$  is defined as in the beginning of 5.3, then  $\alpha_{\pm}''(\mu)$  depends on  $\sigma$ . Denoting  $\alpha_{\pm}''(0)$  by  $\beta(\sigma^2)$  to make this dependence explicit, it follows from HSU (1) that, when  $m = 2$ ,  $n = 3$ , and  $2\alpha = 1/5$ ,  $\beta'(1) = 0,0447$  for STUDENT'S test (the prime denotes differentiation with respect to  $\sigma^2$ ), whereas calculations showed that the corresponding quantity for WILCOXON'S test is equal to 0,0357. So, in this example at least, WILCOXON'S test is less sensitive to the invalidity of the assumption  $\sigma = 1$  than STUDENT'S test.

except for a relative error of the order  $1/m$  and  $1/n$ . Here  $\zeta_\alpha$  is defined by:

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{\zeta_\alpha}^{\infty} e^{-x^2} dx,$$

where  $2\alpha$  is the level of significance of the test.

For STUDENT'S test it is obtained from (5.3, 3) that:

$$\alpha''_{\pm}(0)_{St} \approx \frac{\sqrt{2}}{\sqrt{\pi}} \cdot e^{-\zeta_\alpha^2} \cdot \zeta_\alpha \cdot \frac{mn}{m+n},$$

(except for a relative error of the order  $1/m$  and  $1/n$ ). Hence  $\alpha''_{\pm}(0)_{Wi}/\alpha''_{\pm}(0)_{St} \approx 3/\pi$ , so that the difference in power between the two tests is not great indeed. And then, WILCOXON'S test is more general in that  $\alpha(H_0)$  has the same value whatever  $F(x)$ , provided  $F(x)$  is continuous.

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#### 6. Summary.

Some properties of the critical region and the power function of WILCOXON'S non-parametric solution of the problem of two samples are studied. Under the conditions which allow the use of STUDENT'S test the difference in power between the two tests is investigated, as well for one-sided as for two-sided sets of alternative hypotheses, when the sum of the sample-sizes is  $\leq 5$  and  $\leq 6$  respectively. In these cases the difference is rather small. Indications are that for large sample sizes, too, the difference in power is not great. In an appendix the relation of the power function of WILCOXON'S test with the volume of a spherical simplex is exposed, which shows the limitation of the sample sizes (to 5 and 6 resp.) to be relevant. In an introduction the concepts of critical region and of power function in general are exposed.

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## A. APPENDIX

### CALCULATION OF A CERTAIN MULTIPLE INTEGRAL

A. 1. *The relation of the integral with a spherical simplex.*

Let

$$(A. 1, 1) \quad J = \left[ \prod_{i=1}^{\tau} (I f^{k_i}) \right] (-\infty) = \int_G \dots \int \prod_{i=1}^{\tau} \{f^{k_i}(x_i) dx_i\}$$

where

$$k_i > 0, \sum_{i=1}^{\tau} k_i = N, f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}$$

and the region  $G$  is defined by  $-\infty < x_1 < x_2 < \dots < x_{\tau} < +\infty$ .

Then

$$(A. 1, 1a) \quad J = \frac{1}{(2\pi)^{iN}} \int_G \dots \int e^{-\sum_{i=1}^{\tau} k_i x_i^2} \prod_{i=1}^{\tau} dx_i$$

Be  $\cos \theta_j$  denoted with  $c_j$  and  $\sin \theta_j$  with  $s_j$  ( $j = 1, \dots, \tau - 1$ ). Make the substitutions:

$$(A. 1, 2) \quad x_i = \frac{1}{\sqrt{k_i}} \cdot R \cdot s_{\tau-i} \cdot \prod_{j=\tau-i+1}^{\tau-1} c_j \quad (i = 1, \dots, \tau)$$

with  $s_0 = 1$ ;  $0 \leq \theta_1 \leq 2\pi$ ;  $0 \leq \theta_j \leq \pi$  ( $j = 2, \dots, \tau - 1$ ).

Then

$$\frac{\partial (x_1, x_2, \dots, x_{\tau-1}, x_{\tau})}{\partial (R, \theta_1, \dots, \theta_{\tau-2}, \theta_{\tau-1})} = \frac{1}{\sqrt{\prod_{i=1}^{\tau} k_i}} R^{\tau-1} \cdot \prod_{j=1}^{\tau-1} c_j^{j-1}.$$

Hence

$$(A. 1, 3) \quad J = \frac{1}{(2\pi)^{iN} \cdot \sqrt{\prod_{i=1}^{\tau} k_i}} \int_0^{\infty} e^{-R^2} \cdot R^{\tau-1} dR \cdot \mathcal{L} = \frac{\Gamma\left(\frac{\tau}{2}\right)}{2^{i(N-\tau-2)} \pi^{iN} \sqrt{\prod_{i=1}^{\tau} k_i}} \cdot \mathcal{L}$$

Here

$$(A. 1, 4) \quad \mathcal{L} = \int_{G'} \dots \int \prod_{j=1}^{\tau-1} \{\cos^{j-1} \theta_j, d\theta_j\},$$

where the region  $G'$  is defined by:

$$-\frac{1}{\sqrt{k_1}} < \frac{1}{\sqrt{k_1}} s_{\tau-1} < \frac{1}{\sqrt{k_2}} s_{\tau-2} c_{\tau-1} < \dots < \frac{1}{\sqrt{k_\tau}} \prod_{j=1}^{\tau-1} c_j < + \frac{1}{\sqrt{k_\tau}}.$$

Now the ( $\tau$ -dimensional) volume of that part of the  $\tau$ -dimensional<sup>7)</sup> unit sphere, the orthogonal Cartesian coordinates  $z_i$  ( $i=1, \dots, \tau$ ) of which satisfy the inequalities (defining the region  $G''$ )  $\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \dots < \frac{z_\tau}{\sqrt{k_\tau}}$  is given by

$$(A. 1, 5) \quad \int_{G''} \dots \int \prod_{i=1}^{\tau} dz_i = V_{k_1, k_2, \dots, k_\tau} \quad (\text{say})$$

(cf. Remark 1). Bij the substitutions  $z_i = -z'_i$  ( $i=1, \dots, \tau$ ) it is easily seen that

$$(A. 1, 6) \quad V_{k_1, k_2, \dots, k_{\tau-1}, k_\tau} = V_{k_\tau, k_{\tau-1}, \dots, k_2, k_1}.$$

Moreover by the substitutions:

$$(A. 1, 7) \quad z_i = R \cdot s_{\tau-i} \cdot \prod_{j=i+1}^{\tau-1} c_j \quad (i=1, \dots, \tau)$$

with  $s_0 = 1$ ;  $0 \leq \theta_1 \leq 2\pi$ ;  $0 \leq \theta_j \leq \pi$  ( $j=2, \dots, \tau-1$ ) it is seen that

$$(A. 1, 8) \quad V_{k_1, \dots, k_\tau} = \int_0^1 R^{\tau-1} dR \cdot \mathcal{L} = (1/n) \cdot \mathcal{L}.$$

Thus by (A. 1, 3) and (A. 1, 8):

$$(A. 1, 9) \quad J = \frac{\tau \cdot \Gamma\left(\frac{\tau}{2}\right)}{2^{\frac{1}{2}(N-\tau+2)} \pi^{\frac{1}{2}N} \sqrt{\prod_{i=1}^{\tau} k_i}} \cdot V_{k_1, \dots, k_\tau}.$$

Remark 1. If in  $V_{k_1, \dots, k_\tau}$   $k_{i+1} = k_{i+2} = \dots = k_{i+h} = k$ , the sequence of suffices  $k_{i+1}, \dots, k_{i+h}$  will be denoted shortly with the symbolic power  $k^h$ .

Remark 2.  $V_{k_1, k_2, \dots, k_\tau}$  will be calculated below for  $\tau = 2, 3, 4$ .

#### A. 2. The case $\tau = 2$ .

$V_{k_1, k_2}$  is the area of that part of the unit circle the coordinates  $z_1, z_2$  of which satisfy  $\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}}$ . Clearly this is just half of the total area. So

$$(A. 2, 1) \quad V_{k_1, k_2} = \pi/2 \text{ for every } k_1, k_2.$$

<sup>7)</sup>  $\tau$  here is the number of dimensions of the underlying space.

A. 3. *The case  $\tau = 3$ .*

$V_{k_1, k_2, k_3}$  is the volume of that part of the 3-dimensional unit sphere, the coordinates  $z_1, z_2, z_3$  of which satisfy the inequalities

$$\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \frac{z_3}{\sqrt{k_3}}.$$

This region is bounded by the surface of the sphere and by two planes, the equations of which are given by

$$(A. 3, 1) \quad \frac{z_1}{\sqrt{k_1}} - \frac{z_2}{\sqrt{k_2}} = 0 \quad \text{and} \quad \frac{z_2}{\sqrt{k_2}} - \frac{z_3}{\sqrt{k_3}} = 0.$$

The volume  $V_{k_1, k_2, k_3}$  is built up out of points which make both the first members of the equations (A. 3, 1) negative.

The corresponding angle between the two planes is given by  $\pi - \varphi$ , where  $\varphi$  is the angle between the positive normals.

$$\varphi = \arccos \frac{-\frac{1}{\sqrt{k_2}} \cdot \frac{1}{\sqrt{k_2}}}{\sqrt{\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \left(\frac{1}{k_2} + \frac{1}{k_3}\right)}} = \pi - \arccos \frac{\sqrt{k_1 k_3}}{\sqrt{(k_1 + k_2)(k_2 + k_3)}}.$$

So

$$(A. 3, 2) \quad V_{k_1, k_2, k_3} = \frac{\pi - \varphi}{2\pi} \cdot \frac{4}{3} \pi = \frac{2}{3} \arccos \frac{\sqrt{k_1 k_3}}{\sqrt{(k_1 + k_2)(k_2 + k_3)}}.$$

A. 4. *The case  $\tau = 4$ .*

$V_{k_1, k_2, k_3, k_4}$  is the volume of that part of the 4-dimensional unit sphere, the coordinates  $z_1, z_2, z_3, z_4$  of which satisfy the inequalities

$$\frac{z_1}{\sqrt{k_1}} < \frac{z_2}{\sqrt{k_2}} < \frac{z_3}{\sqrt{k_3}} < \frac{z_4}{\sqrt{k_4}}.$$

This region is bounded by the "surface" of the sphere and by three hyperplanes, the equations of which are given by

$$(A. 4, 1) \quad \left\{ \begin{array}{l} \frac{z_1}{\sqrt{k_1}} - \frac{z_2}{\sqrt{k_2}} = 0 \quad (I) \\ \frac{z_2}{\sqrt{k_2}} - \frac{z_3}{\sqrt{k_3}} = 0 \quad (II) \\ \frac{z_3}{\sqrt{k_3}} - \frac{z_4}{\sqrt{k_4}} = 0 \quad (III) \end{array} \right.$$

The hyperplane  $P \equiv \sum_{i=1}^4 z_i \sqrt{k_i} = 0$  is perpendicular to the line

$$\frac{z_1}{\sqrt{k_1}} = \frac{z_2}{\sqrt{k_2}} = \frac{z_3}{\sqrt{k_3}} = \frac{z_4}{\sqrt{k_4}},$$

which is contained in each of the three bounding hyperplanes. The points,  $P$  has in common with  $V_{k_1, k_2, k_3, k_4}$ , fill up that part of the

3-dimensional unit sphere in the hyperplane  $P$  with  $(0, 0, 0, 0)$  as a centre, which is common to the point sets for which

$$P = 0; \frac{z_1}{\sqrt{k_1}} - \frac{z_2}{\sqrt{k_2}} < 0; \frac{z_2}{\sqrt{k_2}} - \frac{z_3}{\sqrt{k_3}} < 0; \frac{z_3}{\sqrt{k_3}} - \frac{z_4}{\sqrt{k_4}} < 0.$$

Be  $v$  the volume of this part of the 3-dimensional unit sphere in the hyperplane  $P$ . Then:

$$(A. 4, 2) \quad V_{k_1, k_2, k_3, k_4} = v \cdot \int_{-1}^{+1} (1 - q^2)^{1/2} dq = \frac{3\pi}{8} v.$$

As to  $v$ , the following equality holds good:

$v/\frac{4}{3}\pi = \Phi/4\pi$ , where  $\Phi$  is the spherical excess of the (spherical) triangle defined by the three hyperplanes mentioned above;  $\frac{4}{3}\pi$  is the volume and  $4\pi$  the surface of the 3-dimensional unit sphere. So

$$(A. 4, 3) \quad v = \frac{1}{3} \Phi = \frac{1}{3} [(I, II) + (II, III) + (III, I) - \pi]$$

where  $(I, II)$  is the angle between the hyperplanes I and II corresponding to the inequalities defining  $v$  and the same holds for  $(II, III)$  and  $(III, I)$ .  $(I, II)$  is the supplement of the angle between the positive normals on I and II respectively; etc. So  $(III, I) = \pi/2$ , whereas

$$(I, II) = \arccos \frac{\sqrt{k_1 k_3}}{\sqrt{(k_1 + k_2)(k_2 + k_3)}}; (II, III) = \arccos \frac{\sqrt{k_2 k_4}}{\sqrt{(k_2 + k_3)(k_3 + k_4)}}.$$

Hence

$$\Phi = \arcsin \frac{\sqrt{k_2 k_3} [\sqrt{(k_1 + k_2 + k_3)(k_2 + k_3 + k_4)} - \sqrt{k_1 k_4}]}{(k_2 + k_3) \sqrt{(k_1 + k_2)(k_3 + k_4)}}$$

and, by (A. 4, 3) and (A. 4, 2) it follows that

$$(A. 4, 4) \quad V_{k_1, k_2, k_3, k_4} = \frac{\pi}{8} \frac{\sqrt{k_2 k_3} [\sqrt{(k_1 + k_2 + k_3)(k_2 + k_3 + k_4)} - \sqrt{k_1 k_4}]}{(k_2 + k_3) \sqrt{(k_1 + k_2)(k_3 + k_4)}}.$$