A percolation process on the binary tree where large finite clusters are frozen

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Abstract

We study a percolation process on the planted binary tree, where clusters freeze as soon as they become larger than some fixed parameter N. We show that as N goes to infinity, the process converges in some sense to the frozen percolation process introduced by Aldous in [1].

In particular, our results show that the asymptotic behaviour differs substantially from that on the square lattice, on which a similar process has been studied recently by van den Berg, de Lima and Nolin [8].

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1 Introduction and statement of results

Aldous [1] introduced a percolation process where clusters are frozen when they get infinite, which can be described as follows. Let $G = (V, E)$ be an arbitrary simple graph with vertex set V, and edge set E. On every edge $e \in E$, there is a clock which rings at a random time τ_e with uniform distribution on $[0, 1]$, these random times τ_e, $e \in E$, being independent of each other. At time 0, all the edges are closed, and then each edge $e = (u, v) \in E$ becomes open at time τ_e if the open clusters of u and v at that time are both finite – otherwise, e stays closed. In other words, an open cluster stops growing as soon as it becomes infinite: it freezes, hence the name *frozen percolation* for this process.

The above description is informal – it is not clear that such a process exists. In [1], Aldous studies the special cases where G is the infinite binary tree (where every vertex has degree three), or the planted binary tree (where one vertex, the root vertex, has degree one, and all other vertices have degree three). He showed that the frozen percolation process exists for these choices of G. However, Benjamini and Schramm [2] showed that for $G = \mathbb{Z}^2$, there is no process satisfying the aforementioned evolution. For more details see Remark (i) after Theorem 1 of [9]. It seems that no simple condition on the graph G is known that guarantees the existence of the frozen percolation process.

To get more insight in the non-existence for $\mathbb{Z}^2$, a modification of the process was studied in [8]. In the modified process, an open cluster freezes as soon as it reaches size at least N, where N (a positive integer) is the parameter of the model. See Definition 2 below for the meaning of 'size'. Formally, the evolution of a frozen percolation process with parameter N is the following.

At time 0, every edge is closed. At time t, an edge $e = (u, v) \in E$ becomes open if $\tau_{(u,v)} = t$ *and* the open clusters of u and v at time t have size strictly smaller than N – otherwise, e stays closed. We call this modified process the N-parameter frozen percolation process. Note that replacing N by ∞ corresponds

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formally to Aldous' infinite frozen percolation process, therefore we sometimes refer to it as the ∞-parameter frozen percolation process.

The N-parameter frozen percolation process does exist on $\mathbb{Z}^2$ (and on many other other graphs including the binary tree), since it can be described as a finite-range interacting particle system. For general existence results of interacting particle systems, see for example Chapter 1 of [6]. Van den Berg, de Lima and Nolin [8] study the distribution of the final cluster size (i.e. the size of the cluster of a given vertex at time 1). They show that, for $\mathbb{Z}^2$, the final cluster size is smaller than N, but still of order of N, with probability bounded away from 0. In the light of the earlier mentioned fundamental difference (the existence versus the non-existence of the ∞-parameter frozen percolation process), it is natural to ask if the N-parameter process for the planted binary behaves, for large N, very differently from that on $\mathbb{Z}^2$. It turns out that this is indeed the case: We show that the N-parameter frozen percolation process for the planted binary tree converges (in some sense, see Theorem 1) to Aldous' process as the parameter goes to infinity. In particular, the probability that the final cluster has size less than N, but of order N, converges to 0 (see (1.1) below).

Before stating our main result, let us give some notation. We distinguish between different frozen percolation processes by using subscripts for the probability measures. We thus use $\mathbb{P}_N$ to denote the probability measure for the N-parameter frozen percolation process where the size of a cluster is measured by its volume, while for the ∞-parameter frozen percolation process, we use the notation $\mathbb{P}_\infty$. We denote the open cluster of the root vertex at time t by $\mathcal{C}_t$. For a connected sub-graph (cluster) C of the graph G, the volume of C, i.e. the number of edges of C, will be denoted by $|C|$. Our main result is the following.

Theorem 1. *For the N-parameter frozen percolation process on the planted binary tree, where the size of a cluster is measured by its volume, we have*

$$\mathbb{P}_N\left(\mathcal{C}_t = C\right) \to \mathbb{P}_\infty\left(\mathcal{C}_t = C\right) \text{ as } N \to \infty$$

for all finite clusters C. Moreover

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mathbb{P}_N\left(k \le |\mathcal{C}_t| < N\right) = 0, \tag{1.1}$$

and hence the probability that the open cluster of the root vertex is frozen also converges:

$$\mathbb{P}_N\left(N \le |\mathcal{C}_t|\right) \to \mathbb{P}_\infty\left(|\mathcal{C}_t| = \infty\right) \text{ as } N \to \infty.$$

The theorem above considers the case where size is measured by the volume. It can be extended to other notions of size. To state our more general result, we need to introduce some additional definitions. We denote the planted binary tree by T, and by $\mathscr{C}$ the set of finite clusters (finite connected components) of T.

Definition 1. We say that a function h on the set of vertices of T into itself is a *homomorphism* if it maps any edge (s,t), with s closer to the root than t, to an edge $(h(s),h(t))$, with $h(s)$ closer to the root than $h(t)$.

Definition 2. A *good* size function of clusters is a function $s : \mathscr{C} \to \mathbb{N}$, which satisfies the following conditions:

1. *Compatibility with homomorphisms.* For all $C \in \mathscr{C}$ and injective homomorphisms h we have $s(h(C)) = s(C)$.
2. *Finiteness.* For all $N \in \mathbb{N}$ and for any vertex v, the set $\{C \in \mathscr{C} \mid v \in C,\ s(C) \le N\}$ is finite.
3. *Monotonicity.* If $C, C' \in \mathscr{C}$ with $C \subseteq C'$, then $s(C) \le s(C')$.
4. *Boundedness above by the volume.* For all $C \in \mathscr{C}$, we have $s(C) \le |C|$.

The conditions of Definition 2 are satisfied for most of the usual size functions such as the diameter (the length of the longest self-avoiding path in the cluster) or the depth (the length of the longest self-avoiding path starting from the root).

We indicate the dependence on the size function with an additional superscript: $\mathbb{P}_N^{(s)}$ denotes the probability measure for the N-parameter frozen percolation process with size function s. With this notation, the following generalization of Theorem 1 holds.

Theorem 2. *Let s be a good size function for the planted binary tree. Then we have*

$$\mathbb{P}_N^{(s)}\left(\mathcal{C}_t = C\right) \to \mathbb{P}_\infty\left(\mathcal{C}_t = C\right) \text{ as } N \to \infty \tag{1.2}$$

for all finite clusters C. Moreover

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mathbb{P}_N^{(s)}\left(k \le s\left(\mathcal{C}_t\right) < N\right) = 0, \tag{1.3}$$

and hence the probability that the open cluster of the root vertex is frozen also converges:

$$\mathbb{P}_N^{(s)}\left(N \le s\left(\mathcal{C}_t\right)\right) \to \mathbb{P}_\infty\left(\left|\mathcal{C}_t\right| = \infty\right).$$

Remark 1. Equation (1.2) is valid even without condition 4 of Definition 2.

Remark 2. The behaviour described in Theorem 2 is very different from that of the square lattice: In [8] it is showed that for $G = \mathbb{Z}^2$, and for any fixed $a, b \in \mathbb{R}$ with $0 < a < b < 1$

$$\liminf_{N\to\infty} \mathbb{P}_N^{(diam)}\left(aN < diam\left(\mathcal{C}_t\right) < bN\right) > 0, \tag{1.4}$$

where *diam* denotes the diameter, while this probability tends to 0 when G is the planted binary tree, thanks to Eq.(1.3).

Let us finally mention that since Aldous' seminal paper [1], several related questions were studied. For example, Chapter 4 of [4] considers frozen percolation on $\mathbb{Z}$, and variants of that model are investigated in [7] and [3], respectively on the complete graph and on the binary tree.

The paper is organized as follows. In Section 2 we prove Theorem 1. The proof relies on a careful study of the probability that the root edge is closed at time t, which we denote by $\beta_N(t)$. In Sections 2.1 and 2.2 we show that β_N satisfies a first order differential equation which involves the generating function of the Catalan numbers. In Section 2.3, we give an implicit solution of the aforementioned differential equation, and we use this in Sections 2.4 and 2.5 to prove the convergence of β_N as $N \to \infty$. We finish the proof of Theorem 1 in Section 2.6. In Section 3 we point out the changes in the proof of Theorem 1 required to prove Theorem 2.

2 Proof of Theorem 1

2.1 Setting

In this section, we consider the N-parameter frozen percolation process where the size of a cluster is measured by its number of edges – we recall the notation $\mathbb{P}_N$. We denote by $\mathcal{A}_t$ the set of open edges at time t.

Let $e_0 = (v_0, v_1)$ be the root edge, where v_0 is the root vertex. The central quantity of our analysis is the following probability:

$$\beta_N\left(t\right) := \mathbb{P}_N\left(e_0 \notin \mathcal{A}_t\right) = \mathbb{P}_N\left(e_0 \text{ is closed at time } t\right) \tag{2.1}$$

(note that $\beta_N(t) = \mathbb{P}_N\left(\left|\mathcal{C}_t\right| = 0\right)$).

Remark 3. >From the definition, it is easy to see that $\beta_N\left(t\right)$ is decreasing in t. Moreover, from the equality

$$\beta_N\left(t\right) = 1 - t + \mathbb{P}_N\left(\tau_{e_0} < t \text{ but } e_0 \text{ is closed at time } t\right), \tag{2.2}$$

we can see that $\left(\beta_N\left(t\right) - 1 + t\right)$ is increasing in t.

For $e \in E$, $e \neq e_0$, $T \setminus \{e\}$ has two connected components, one which contains e_0, and one which does not. Let T_e denote the component which does not contain e_0, together with the edge e: T_e is a subtree of T, isomorphic to T.

For any edge e_1, we define the frozen percolation process on T_{e_1} in the following way. We consider the set of random variables τ_e, $e \in T_{e_1}$, and define the frozen percolation process on T_{e_1} in the same way as we did for T. We denote the set of open edges at time t by $\mathcal{A}_t(e_1)$. Note that the process $\mathcal{A}_t(e_1)$ has the same law as $\mathcal{A}_t$. Moreover, $\mathcal{A}_t(e_1)$ and $\mathcal{A}_t$ are coupled via the random variables τ_e, $e \in T_{e_1}$.

In the following, we think of clusters as sets of edges. The outer boundary of a cluster $C \subseteq E$, denoted by ∂C, is the set of edges in $E \setminus C$ that have a common endpoint with one of the edges of C.

2.2 Differential equation for β_N

Let us denote the kth Catalan number by $c_k = \binom{2k}{k}/(k+1)$, and recall that the generating function of the Catalan numbers is (see for example Section 2.1 of [5])

$$C(x) = \sum_{k=0}^{\infty} c_k x^k = \frac{1-\sqrt{1-4x}}{2x} = \frac{2}{1+\sqrt{1-4x}},$$

which converges for $|x| \leq \frac{1}{4}$. If we denote by C_N the Nth partial sum, that is

$$C_N(x) = \sum_{k=0}^{N} c_k x^k,$$

we have:

Lemma 1. *β_N is differentiable, and its derivative satisfies*

$$\beta_N'(t) = -\frac{\beta_N(t)}{t}\left[C_N(t\beta_N(t)) - 1\right]. \tag{2.3}$$

Remark 4. Since $C_N(x) = 1 + x + \ldots$, Eq.(2.3) is well defined for $t = 0$. In the introduction we pointed out that the model exists, in particular the differential equation (2.3) with initial condition $\beta_N(0) = 1$ has a solution. On the other hand, the general theory of ordinary differential equations provides uniqueness.

Proof. Let us denote the open cluster of v_1 without the edge e_0 at time s by $\tilde{\mathcal{C}}_s$.

We use the defining evolution of the N-parameter frozen percolation process as follows: At time s, if $\tau_{e_0} = s$, then e_0 tries to become open, and it succeeds if and only if $\left|\tilde{\mathcal{C}}_s\right| \leq N-1$. By conditioning on τ_{e_0}, we get that

$$\begin{aligned}\beta_N(t) &= 1 - \int_0^t \mathbb{P}_N\left(\left|\tilde{\mathcal{C}}_s\right| < N | \tau_{e_0} = s\right) ds \\ &= 1 - \int_0^t \sum_{k=0}^{N-1} \mathbb{P}_N\left(\left|\tilde{\mathcal{C}}_s\right| = k \,|\, \tau_{e_0} = s\right) ds.\end{aligned} \tag{2.4}$$

First we compute the probability $\mathbb{P}_N\left(\tilde{\mathcal{C}}_s = C \,|\, \tau_{e_0} = s\right)$ for $|C| \leq N-1$. If $\tilde{\mathcal{C}}_s = C$, $|C| \leq N-1$, then for all $e \in C$, e is open at time s. Moreover, for all $e' \in \partial C \setminus \{e_0\}$, e' is closed at time s. The latter event can happen in two ways: e' is closed at time s in its own frozen percolation process on $T_{e'}$, or there is a big cluster at time s in $T \setminus T_{e'}$ touching e'. Since $|C| < N$, on the event $\left\{\tilde{\mathcal{C}}_s = C, \tau_{e_0} = s\right\}$, the latter cannot happen. Hence

$$\left\{\tilde{\mathcal{C}}_s = C, \tau_{e_0} = s\right\} \subseteq \bigcap_{e' \in \partial C \setminus \{e_0\}} \left\{e' \notin \mathcal{A}_s(e')\right\} =: A.$$

Note that the event A and the random variables τ_e, $e \in C$ are independent. Moreover, conditionally on A, the events $e \in \mathcal{A}_s$, $e \in C$ are independent, and each of them has probability s, so that

$$\mathbb{P}_N\left(\tilde{\mathcal{C}}_s = C \,\middle|\, e' \notin \mathcal{A}_s(e') \text{ for } e' \in \partial C \setminus \{e_0\}, \tau_{e_0} = s\right) = s^{|C|}. \tag{2.5}$$

Recall that the processes $\mathcal{A}_s(e')$, $e' \in \partial C \setminus \{e_0\}$ are independent and have the same law as $\mathcal{A}_s$. Hence the events $e' \notin \mathcal{A}_s(e')$, $e' \in \partial C \setminus \{e_0\}$ are independent, and each of them has probability $\beta_N(s)$. This together with (2.5) gives that

$$\mathbb{P}_N\left(\tilde{\mathcal{C}}_s = C \,|\, \tau_{e_0} = s\right) = s^{|C|}\beta_N(s)^{|\partial C \setminus \{e_0\}|}.$$

Using that $\left|\partial\tilde{\mathcal{C}}_s \setminus \{e_0\}\right| = \left|\tilde{\mathcal{C}}_s\right| + 2$, we get

$$\mathbb{P}_N\left(\tilde{\mathcal{C}}_s = C \,|\, \tau_{e_0} = s\right) = \beta_N(s)^2\left(s\beta_N(s)\right)^{|C|}. \tag{2.6}$$

It is well known that the number of clusters $C \subseteq T$ having k edges which contain the vertex v_1 but not the edge e_0 is c_{k+1}, the $(k+1)$th Catalan number (see for example Theorem 2.1 of [5]). By this and (2.6) we can rewrite (2.4) as follows:

$$\begin{aligned}\beta_N(t) &= 1 - \int_0^t \beta_N(s)^2 \sum_{k=0}^{N-1} c_{k+1}\left(s\beta_N(s)\right)^k ds.\\ &= 1 - \int_0^t \frac{\beta_N(s)}{s}\left(C_N\left(s\beta_N(s)\right) - 1\right) ds. \end{aligned}\tag{2.7}$$

Recall that $C_N(x) = 1 + x + \ldots$, hence for every fixed positive integer N, the integrand in (2.7) is bounded (since $s, \beta_N(s) \in [0,1]$ and C_N is continuous). Thus we can differentiate Eq.(2.7), which completes the proof of Lemma 1. □

2.3 Implicit formula for β_N

Lemma 2 gives an implicit solution of (2.3) with initial condition $\beta_N(0) = 1$. Before stating and proving the proposition, let us give a heuristic computation to explain where that proposition comes from, without checking if the operations performed are legal or not.

Define the function $\gamma_N(t) = t\beta_N(t)$. It follows from Eq.(2.3) that γ_N satisfies

$$\frac{\gamma_N'(t)}{\gamma_N(t)\left(2 - C_N(\gamma_N(t))\right)} = \frac{1}{t},$$

so

$$\int_a^{\gamma_N(t)} \frac{dx}{x\left(2 - C_N(x)\right)} = \log t + b$$

for some constants a, b. Using $\int_a^{\gamma_N(t)} \frac{dx}{x} = \log t + \log\left(\beta_N(t)/a\right)$, we get

$$\int_a^{\gamma_N(t)} \frac{C_N(x) - 1}{x\left(2 - C_N(x)\right)} dx = -\log \beta_N(t) + b' \tag{2.8}$$

for another constant b'. Finally, by plugging in $\beta_N(0) = 1$ and $\gamma_N(0) = 0$, we can evaluate b', which gives

$$\int_0^{t\beta_N(t)} \frac{C_N(x) - 1}{x\left(2 - C_N(x)\right)} dx = -\log \beta_N(t).$$

This suggests the following lemma.

Lemma 2. *For $t \in [0,1]$, $\beta_N(t)$ is the unique positive solution of the equation in z*

$$\int_0^{tz} \frac{C_N(x) - 1}{x\left(2 - C_N(x)\right)} dx + \log z = 0, \tag{2.9}$$

with the constraint $tz < x_N$, where x_N is the unique positive solution of $C_N(x) - 2 = 0$.

Proof. Let us fix N. First, the polynomial $C_N(x) - 2$ has a positive derivative for $x > 0$, it has thus exactly one non-negative root x_N, and this root has multiplicity one. Note that $x_N > 1/4$, since $C(x) > C_N(x)$ for $x \in (0, 1/4]$, and $C(1/4) = 2$. ($C_N(x)$ and $C(x)$ are close for large N, this also suggests that the root is close to $1/4$ for large N: we will indeed prove that in the following.)

Let us prove that for $t \in [0,1]$, there is exactly one non-negative solution of (2.9) with $tz < x_N$. The integrand in (2.9) is positive, and it is well defined at 0 since $C_N(x) = 1 + x + O(x^2)$ as $x \to 0$ ($N \geq 1$). As $x \nearrow x_N$, this integrand behaves like $\frac{\kappa}{x_N - x}$ for some positive constant κ (using that the positive root x_N of $C_N(x) - 2$ has multiplicity one). Hence,

$$\int_0^{x_N} \frac{C_N(x) - 1}{x(2 - C_N(x))} dx = \infty. \tag{2.10}$$

On the other hand,

$$\int_0^{z} \frac{C_N(x) - 1}{x(2 - C_N(x))} dx < \infty$$

for $z \in [0, x_N)$. This shows that for every $t \in [0,1]$, there is exactly one positive real number $u_N(t)$ which satisfies the equation (2.9), and $tu_N(t) < x_N$.

To complete the proof of Lemma 2, it is enough to show that u_N is differentiable, that

$$u_N'(t) = -\frac{u_N(t)}{t}\left[C_N(tu_N(t)) - 1\right] \tag{2.11}$$

for $t \in [0,1]$, and that $u_N(0) = 1$. Indeed, as already noted in Remark 4, the differential equation (2.11) has a unique solution. A substitution into (2.9) shows that $u_N(0) = 1$. It is easy to check the conditions of the implicit function theorem, and get that $u_N(t)$ is a differentiable function with derivative satisfying

$$(tu_N'(t) + u_N(t)) \frac{C_N(tu_N(t)) - 1}{tu_N(t)(2 - C_N(tu_N(t)))} = -\frac{u_N'(t)}{u_N(t)},$$

from which simple computations give (2.11). This completes the proof of Lemma 2. □

2.4 Bounds on β_N

We now compare β_N with the corresponding function in Aldous' paper [1], where clusters are frozen as soon as they become infinite. In Aldous' model, one has

$$\beta_\infty(t) := \mathbb{P}_\infty(e_0 \text{ is closed at time } t) = \begin{cases} 1 - t & \text{if } t \in [0, 1/2], \\ \frac{1}{4t} & \text{if } t \in [1/2, 1]. \end{cases}$$

The following bounds hold true:

Lemma 3. *We have*

$$0 \leq \beta_N(t) - \beta_\infty(t) \leq 2(x_N - 1/4) \quad \textit{for all } t \in [0,1],$$

where x_N ($> 1/4$) is the unique positive root of the polynomial $C_N(x) - 2$.

Proof. >From Lemma 2, we know that $t\beta_N(t) < x_N$, which gives the desired upper bound for $t \in \left[\frac{1}{2}, 1\right]$. We also know (Remark 3) that $\beta_N(t) - 1 + t$ is non-negative and increasing. Hence,

$$0 \leq \beta_N(t) - 1 + t \leq \beta_N(1/2) - 1/2 \leq 2(x_N - 1/4) \tag{2.12}$$

for $t \in \left[0, \frac{1}{2}\right]$, by using also the previously proven upper bound at $t = \frac{1}{2}$. We have thus established the desired lower and upper bounds for $t \in \left[0, \frac{1}{2}\right]$. In particular, for $t = \frac{1}{2}$, we obtain that $\beta_N(1/2) \geq 1/2$.

Now, let us note that $t\beta_N(t)$ is increasing: this is an easy consequence of two facts, that $\beta_N(t)$ is decreasing and that the integrand in the left hand-side of (2.9) is positive. Combined with the bound $\beta_N(1/2) \geq 1/2$, we get

$$\frac{1}{4} \leq \frac{1}{2}\beta_N(1/2) \leq t\beta_N(t),$$

from which the desired lower bound for $t \in \left[\frac{1}{2}, 1\right]$ follows readily. This completes the proof of Lemma 3. □

2.5 Convergence to β_∞

It follows from Lemma 3 that in order to prove uniform convergence of the functions β_N to β_∞, it is enough to prove that $x_N \to 1/4$ as $N \to \infty$. We prove a bit more, namely we give an upper bound on the rate of convergence.

Proposition 1. *There exists a constant K such that $x_N - \frac{1}{4} < \frac{K}{N}$. In particular,*

$$0 \le \beta_N(t) - \beta_\infty(t) \le \frac{2K}{N} \quad \text{for all } t \in [0,1],$$

so that $\beta_N \to \beta_\infty$ uniformly on $[0,1]$.

Proposition 1 follows from the following lemma.

Lemma 4. *The functions $\sqrt{N}\left(C_N\left(\frac{1}{4}+\frac{x}{4N}\right)-2\right)$ converge locally uniformly in $x \in \mathbb{R}$ as $N \to \infty$ to the function*

$$F(x) = \frac{2}{\sqrt{\pi}}\left(\sqrt{x}\int_0^x \frac{e^y}{\sqrt{y}}dy - e^x\right).$$

Proof of Proposition 1. Let us take $K \in \mathbb{R}$, $K > 0$ such that $F(K) > 1$ (such a K exists, since $F(x) \sim \frac{1}{\sqrt{\pi}}\frac{e^{\frac{x}{2}}}{x} \to \infty$ as $x \to \infty$). Then by Lemma 4, we have that for large N,

$$\sqrt{N}\left(C_N\left(\frac{1}{4}+\frac{K}{4N}\right)-2\right) \ge F(K) - \frac{1}{2} > 1 - \frac{1}{2} > \frac{1}{2},$$

and so

$$C_N\left(\frac{1}{4}+\frac{K}{4N}\right) - 2 > 0.$$

For any fixed N, the function $x \mapsto C_N\left(\frac{1}{4}+\frac{x}{4N}\right)-2$ is increasing on $[0,\infty)$. Hence, $\frac{1}{4}+\frac{K}{4N} > x_N$, that is $x_N - \frac{1}{4} < \frac{K}{4N}$. ☐

Proof of Lemma 4. Using that

$$2 = C(1/4) = \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{k+1}4^{-k},$$

we get

$$\begin{aligned}
&\sqrt{N}\left(C_N\left(\frac{1}{4}+\frac{x}{4N}\right)-2\right)\\
&= \sqrt{N}\sum_{k=0}^{N}\frac{\binom{2k}{k}}{k+1}4^{-k}\left((1+x/N)^k-1\right) - \sqrt{N}\sum_{k=N+1}^{\infty}\frac{\binom{2k}{k}}{k+1}4^{-k}\\
&=: (A) - (B).
\end{aligned} \tag{2.13}$$

We will use the following version of Stirling's formula:

$$k! = \sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\lambda_k} \quad \text{with } \frac{1}{12k+1} < \lambda_k < \frac{1}{12k}. \tag{2.14}$$

Using this formula, we obtain that

$$(B) = \frac{1}{\sqrt{\pi}}\sum_{k=N+1}^{\infty}\frac{\sqrt{N}}{\sqrt{k}\,(k+1)}e^{\lambda_{2k}-2\lambda_k} = \frac{1}{\sqrt{\pi}}\frac{1}{N}\sum_{k=N+1}^{\infty}\frac{1}{\sqrt{\frac{k}{N}}\frac{k+1}{N}}e^{\lambda_{2k}-2\lambda_k},$$

and thus

$$
\begin{aligned}
(B) &= \frac{1}{\sqrt{\pi}}\left(1+O\left(N^{-1}\right)\right)\frac{1}{N}\sum_{k=N+1}^{\infty}\frac{1}{\sqrt{\frac{k}{N}\frac{k}{N}}} \\
&= \frac{1}{\sqrt{\pi}}\left(1+O\left(N^{-1}\right)\right)\left(\int_{1}^{\infty} y^{-3/2}dy + O\left(N^{-3/2}\right)\right) = \frac{2}{\sqrt{\pi}} + O\left(N^{-1}\right). \qquad (2.15)
\end{aligned}
$$

We now divide (A) into two parts. On the one hand, using Eq.(2.14), we get that for some universal constants C, C',

$$
\begin{aligned}
&\sqrt{N}\left|\sum_{k=0}^{\lfloor\sqrt{N}\rfloor}\frac{\binom{2k}{k}}{k+1}4^{-k}\left((1+x/N)^k-1\right)\right| \\
&\quad\le \sqrt{N}C\sum_{k=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{\sqrt{k}\,(k+1)}\left|\left(1+\frac{x}{N}\right)^k-1\right| \\
&\quad\le \frac{|x|}{\sqrt{N}}C\sum_{k=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{\sqrt{k}\,(k+1)}\left(1+\left(1+\frac{|x|}{N}\right)+\ldots+\left(1+\frac{|x|}{N}\right)^{k-1}\right) \\
&\quad\le \frac{|x|}{\sqrt{N}}C\sum_{k=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{\sqrt{k}}\left(1+\frac{|x|}{N}\right)^{k-1} \\
&\quad\le \frac{|x|}{\sqrt{N}}C\left(1+\frac{|x|}{N}\right)^{\sqrt{N}}\sum_{k=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{\sqrt{k}} \\
&\quad\le C'\,|x|\,e^{|x|}N^{-1/4}. \qquad (2.16)
\end{aligned}
$$

On the other hand, using again Eq.(2.14),

$$
\begin{aligned}
&\sqrt{N}\sum_{k=\lfloor\sqrt{N}\rfloor+1}^{N}\frac{\binom{2k}{k}}{k+1}4^{-k}\left((1+x/N)^k-1\right) \\
&\quad= \sqrt{\frac{N}{\pi}}\sum_{k=\lfloor\sqrt{N}\rfloor+1}^{N}\frac{1}{\sqrt{k}\,(k+1)}e^{\lambda_{2k}-2\lambda_k}\left(\left(1+\frac{x}{N}\right)^k-1\right) \\
&\quad= \frac{1}{\sqrt{\pi}}\left(1+O\left(N^{-1/2}\right)\right)\frac{1}{N}\sum_{k=\lfloor\sqrt{N}\rfloor+1}^{N}\frac{1}{\sqrt{\frac{k}{N}\frac{k+1}{N}}}\left(\left(1+\frac{x}{N}\right)^{N(k/N)}-1\right) \\
&\quad= \frac{1}{\sqrt{\pi}}\left(1+O\left(N^{-1/2}\right)\right)\left(\int_{1/\sqrt{N}}^{1} y^{-3/2}\left(e^{xy}-1\right)dy + e^{|x|}O\left(N^{-3/2}\right)\right) \\
&\quad= \frac{1}{\sqrt{\pi}}\int_0^1 y^{-3/2}\left(e^{xy}-1\right)dy + e^{|x|}O\left(N^{-1/4}\right). \qquad (2.17)
\end{aligned}
$$

Substituting (2.15), (2.16) and (2.17) into (2.13), we get

$$
\sqrt{N}\left(C_N\left(\frac{1}{4}+\frac{x}{4N}\right)-2\right) = \frac{1}{\sqrt{\pi}}\int_0^1 y^{-3/2}\left(e^{xy}-1\right)dy - \frac{2}{\sqrt{\pi}} + (1+|x|)\,e^{|x|}O\left(N^{-1/4}\right). \qquad (2.18)
$$

Finally, an integration by parts gives

$$\int_0^1 y^{-3/2}\left(e^{xy}-1\right)dy = \left[\frac{y^{-1/2}}{-1/2}\left(e^{xy}-1\right)\right]_{y=0}^{y=1} - \int_0^1 \frac{y^{-1/2}}{-1/2} x e^{xy} dy$$
$$= -2\left(e^x - 1\right) + 2x\int_0^1 \frac{e^{xy}}{\sqrt{y}} dy, \tag{2.19}$$

and combining Eqs.(2.18) and (2.19) (and a change of variable) completes the proof of Lemma 4. □

2.6 Completion of the proof of Theorem 1

Recall the notation $\mathcal{C}_t$. Let $|C| < N$ be a fixed cluster of the root vertex. By similar arguments as in the proof of Lemma 1, we have

$$\mathbb{P}_N\left(\mathcal{C}_t = C\right) = t^{|C|}\beta_N\left(t\right)^{|\partial C|} = \beta_N\left(t\right)\left(t\beta_N\left(t\right)\right)^{|C|}. \tag{2.20}$$

(since $|\partial C| = |C| + 1$). Hence for any fixed finite cluster C, we have, as $N \to \infty$,

$$\mathbb{P}_N\left(\mathcal{C}_t = C\right) = \beta_N\left(t\right)\left(t\beta_N\left(t\right)\right)^{|C|} \to \beta_\infty\left(t\right)\left(t\beta_\infty\left(t\right)\right)^{|C|} = \mathbb{P}_\infty\left(\mathcal{C}_t = C\right), \tag{2.21}$$

which gives the first part of Theorem 1.

An argument similar to the beginning of the proof of Lemma 1 gives that

$$\mathbb{P}_N\left(k \le |\mathcal{C}_t| < N\right) = \sum_{n=k}^{N-1} c_n \beta_N\left(t\right)\left(t\beta_N\left(t\right)\right)^n.$$

Lemma 2 and Proposition 1 then imply that $t\beta_N\left(t\right) < x_N \le \frac{1}{4} + \frac{K'}{4N}$, hence (using again Eq.(2.14))

$$\begin{aligned}
\mathbb{P}_N\left(k \le |\mathcal{C}_t| < N\right) &\le \beta_N\left(t\right)\sum_{n=k}^{N-1}\frac{\binom{2n}{n}}{n+1}\left(t\beta_N\left(t\right)\right)^n \\
&\le K_1 \sum_{n=k}^{N-1}\frac{1}{\sqrt{n}}\frac{1}{n+1}\left(1+\frac{K'}{N}\right)^n \\
&\le K_2 e^{K'}\sum_{n=k}^{\infty}\frac{1}{\sqrt{n}}\frac{1}{n+1} \\
&\le K_3\int_k^\infty \frac{dx}{x^{3/2}} = \frac{K_4}{\sqrt{k}}.
\end{aligned}$$

It follows that

$$\lim_{k\to\infty}\limsup_{N\to\infty}\mathbb{P}_N\left(k \le |\mathcal{C}_t| < N\right) = 0, \tag{2.22}$$

which completes the second part of Theorem 1.

Now, using the trivial upper bound $\mathbb{P}_N\left(N \le |\mathcal{C}_t|\right) \le \mathbb{P}_N\left(k \le |\mathcal{C}_t|\right)$ for $k \le N$, we get

$$\limsup_{N\to\infty}\mathbb{P}_N\left(N \le |\mathcal{C}_t|\right) \le \lim_{k\to\infty}\limsup_{N\to\infty}\mathbb{P}_N\left(k \le |\mathcal{C}_t|\right) = \lim_{k\to\infty}\mathbb{P}_\infty\left(k \le |\mathcal{C}_t|\right) = \mathbb{P}_\infty\left(|\mathcal{C}_t| = \infty\right), \tag{2.23}$$

where we used (2.21) for the first equality.

On the other hand, for all $k \in \mathbb{N}$, $k \le N$, we have

$$\mathbb{P}_N\left(N \le |\mathcal{C}_t|\right) = \mathbb{P}_N\left(k \le |\mathcal{C}_t|\right) - \mathbb{P}_N\left(k \le |\mathcal{C}_t| < N\right). \tag{2.24}$$

Hence, taking first the limit infimum as $N \to \infty$, and then the limit as $k \to \infty$, we get

$$\begin{aligned}\liminf_{N\to\infty} \mathbb{P}_N \left(N \leq |\mathcal{C}_t|\right) &\geq \lim_{k\to\infty} \liminf_{N\to\infty} \mathbb{P}_N \left(k \leq |\mathcal{C}_t|\right) - \lim_{k\to\infty} \limsup_{N\to\infty} \mathbb{P}_N \left(k \leq |\mathcal{C}_t| < N\right) \\ &= \mathbb{P}_\infty \left(|\mathcal{C}_t| = \infty\right) - 0,\end{aligned} \tag{2.25}$$

where for the last equality we used, respectively, (2.21) – as in (2.23) – and (2.22).

Combining (2.23) and (2.25) provides the final part of Theorem 1.

□

3 Proof of Theorem 2

In this section we give a brief outline of the changes required to deduce Theorem 2 from the arguments in Section 2.

First, for any good size function s, the corresponding N-parameter frozen percolation process does exist. Indeed, conditions 1 and 2 of Definition 2 ensure that the process is still a finite-range interacting particle system, and the general theory of such systems [6] provides existence, as in the case of volume.

In that previous case, the function $\frac{C_N(x)-1}{x}$ played an important role. It is the generating function of the number of clusters of v_1 which do not contain the edge e_0 and have volume at most $N-1$. For other good size functions s, the following generating function plays the role of $\frac{C_N(x)-1}{x}$:

$$G_N^{(s)}\left(x\right) = \sum_{k=0}^{\infty} a_{k,N-1}^{(s)} x^k,$$

where $a_{k,N-1}^{(s)}$ denotes the number of clusters C of v_1 for which $e_0 \notin C$, $|C| = k$ and $s(C) \leq N-1$.

Keeping this in mind, one can easily modify the proof of Theorem 1. We define the function $\beta_N^{(s)} : [0,1] \to \mathbb{R}$ as

$$\beta_N^{(s)}\left(t\right) := \mathbb{P}_N^{(s)}\left(e_0 \notin \mathcal{A}_t\right).$$

Using the conditions 1, 2 and 3 of Definition 2, by simple adjustments of the proof of Lemma 1 we deduce that $\beta_N^{(s)}$ is differentiable, and that its derivative satisfies

$$(\beta_N^{(s)})'\left(t\right) = -\big(\beta_N^{(s)}\left(t\right)\big)^2 G_N^{(s)}\left(t\beta_N^{(s)}\left(t\right)\right).$$

Moreover, it follows from the definition of $\beta_N^{(s)}$ that $\beta_N^{(s)}\left(0\right) = 1$.

Recall that x_N, the unique positive root of $C_N(x) = 2$, was another important quantity. Since in our present general setup $G_N^{(s)}\left(x\right)$ plays the role of $\frac{C_N(x)-1}{x}$, the analogue of x_N is the unique positive root of the equation $xG_N^{(s)}(x) = 1$, which we denote by $x_N^{(s)}$. Using the arguments of Section 2.3, we deduce that for each fixed t, $\beta_N^{(s)}\left(t\right)$ is equal to the unique positive root of the equation in z

$$\int_0^{tz} \frac{G_N^{(s)}\left(x\right)}{1 - xG_N^{(s)}\left(x\right)} dx + \log z = 0$$

with the constraint $tz < x_N^{(s)}$.

By simple modifications of Section 2.4, we get that $0 \leq \beta_N^{(s)}\left(t\right) - \beta_\infty\left(t\right) \leq 2\left(x_N^{(s)} - \frac{1}{4}\right)$ for all $t \in [0,1]$, which is the analogue of Lemma 3 in this general setting. By condition 3 of Definition 2, $a_{k,N-1}^{(s)}$ is an increasing function of N for each fixed k. Moreover, since $s(C)$ is finite for all finite clusters C, $a_{k,N-1}^{(s)} \uparrow c_{k+1}$ as $N \to \infty$. Hence $G_N^{(s)}\left(x\right) \uparrow \frac{C(x)-1}{x}$ for all $x \in \left[0, \frac{1}{4}\right]$, and $G_N^{(s)}\left(x\right) \uparrow \infty$ for $x > \frac{1}{4}$. Thus $x_N^{(s)} \to \frac{1}{4}$ as

$N \to \infty$. By the aforementioned analogue of Lemma 3, we get that $\beta_N^{(s)} \to \beta_\infty$ point-wise. This concludes the proof of the first part (Eq.(1.2)) of Theorem 2.

Note that up to now we did not use that s satisfies Condition 4 of Definition 2. We use this condition to prove a rate of convergence for $x_N^{(s)}$, which was the key ingredient in the proof of (1.1). Condition 4 implies that $a_{k,N-1}^{(s)} \geq c_{k+1}$ for $k \leq N-1$, hence

$$G_N^{(s)}(x) \geq \frac{C_N(x) - 1}{x} \text{ for } x \geq 0,$$

and thus $\frac{1}{4} \leq x_N^{(s)} \leq x_N = x_N^{(|.|)}$. Proposition 1 then implies that $0 \leq x_N^{(s)} - \frac{1}{4} \leq \frac{K}{N}$, from which a computation similar to Section 2.6 completes the proof of Theorem 2.

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