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THE CONVERGENCE RATE OF MULTI-LEVEL ALGORITHMS APPLIED TO THE CONVECTION-DIFFUSION EQUATION*

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Abstract. We consider the solution of the convection-diffusion equation in two dimensions by various multi-level algorithms (MLAs). We study the convergence rate of the MLAs and the stability of the coarse-grid operators, depending on the choice of artificial viscosity at the different levels. Four strategies are formulated and examined. A method to determine the convergence rate is described and applied to the MLAs, both in a problem with constant and in one with variable coefficients. As relaxation procedures the 7-point ILU and symmetric point Gauss-Seidel (SGS) methods are used.

Key words. artificial viscosity, convection-diffusion equation, multi-level algorithm, asymptotic stability, Galerkin approximation

1. Introduction. We consider the convection-diffusion equation

$$L_\varepsilon u \equiv -\varepsilon \Delta u + b_1(x, y)\frac{\partial u}{\partial x} + b_2(x, y)\frac{\partial u}{\partial y} = f(x, y) \tag{1.1}$$

for $(x, y) \in \Omega \subset \mathbb{R}^2$, $\varepsilon > 0$, with Dirichlet and Neumann boundary conditions on different parts of $\delta\Omega$.

When the diffusion coefficient ε is small in comparison with the mesh-width h, the stability of discretizations of (1.1) by central differences (CD) or the finite element method (FEM) can be improved by augmenting ε with an artificial viscosity of $O(h)$. This rather crude way of stabilizing the discrete problem may form part of more subtle iterative methods for solving (1.1) with small ε, for instance the mixed defect correction process (cf. Hemker [4]) or the double discretization process (cf. Brandt [3]).

In § 2 we introduce four strategies for choosing the artificial viscosity on the coarse grids in the multi-level algorithm (MLA) (cf. Van Asselt [1]). In § 3 we describe the method which is used to determine the convergence behaviour of the multi-level algorithm for these strategies. In § 4 we compare the convergence rates as measured by the method described in § 3. Finally, some conclusions are formulated in § 5.

2. Artificial viscosity, strategies, stability and asymptotic convergence rate. In this section we derive all theoretical results for the constant coefficient case by local mode analysis neglecting the boundaries. We introduce various strategies for choosing the coarse-grid operators in the MLA. We give a motivation for the choice of these strategies, and analyze their stability (cf. Theorem 2.14, Corollary 2.18, Theorem 2.19, Corollary 2.24). Further we formulate some important properties of the different strategies (cf. Conjectures 2.25–2.27). In the case of FEM discretization we also consider the Galerkin coarse-grid approximation. In this paper we only consider the FEM based on a uniform triangulation of Ω with right-angled triangles.

The trial and test space is spanned by the set of piecewise-linear "hat-functions" ϕ_{ij} which take the value 1 at x_{ij} and 0 at all other vertices of triangles.

We consider the MLA (cf. Hemker [5]) with $l+1$ levels: $0, \cdots, l$ and uniform square meshes on each level with meshwidths h_0 and $h_k = h_{k-1}/2$ for $k = 1, \cdots, l$.

Let $\{L_\varepsilon^{k,l}\}_{k=0,\cdots,l}$ be a sequence of discretizations of L_ε. For the constant-coefficient equation we denote by $\hat{L}_\varepsilon(\omega)$, $\omega \in \mathbb{R}^2$ the symbol (or characteristic form) of the

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continuous operator L_ε. By $\hat{L}_\varepsilon^{k,l}(\omega)$, $\omega \in T_k \equiv [-\pi/h_k, \pi/h_k]^2$, we mean the symbol of the discrete operator $L_\varepsilon^{k,l}$.

When a symbol is small the corresponding operator is unstable in the sense that small changes in the right-hand side cause great changes in the solution. Depending on the boundary conditions the continuous problem can be well posed. Therefore we allow the symbol of the discrete operator to be small only for those frequencies for which the symbol of the continuous operator is small. This idea is formalized in the following definitions.

DEFINITION 2.1. The *ε-asymptotic stability degree of L_ε with respect to the mode* $e^{i\omega x}$ is the quantity $\lim_{\varepsilon\downarrow 0}|\hat{L}_\varepsilon(\omega)|$.

DEFINITION 2.2. The *δ-domain of L_ε* is the set of all $\omega \in \mathbb{R}^2$ for which $\lim_{\varepsilon\downarrow 0}|\hat{L}_\varepsilon(\omega)| > \delta > 0$.

DEFINITION 2.3. The *ε-asymptotic stability degree of $L_\varepsilon^{k,l}$ with respect to the mode* $e^{i\omega x}$ is the quantity $\lim_{\varepsilon\downarrow 0}|L_\varepsilon^{k,l}(\omega)|$.

DEFINITION 2.4. The *δ-domain of $L_\varepsilon^{k,l}$* is the set of all $\omega \in T_k$ for which $\lim_{\varepsilon\downarrow 0}|\hat{L}_\varepsilon^{k,l}(\omega)| > \delta > 0$.

DEFINITION 2.5. A *strategy for coarse-grid operators* is a set $\{L_\varepsilon^0, L_\varepsilon^1, \cdots, L_\varepsilon^l, \cdots\}$ with $L_\varepsilon^l \equiv \{L_\varepsilon^{0,l}, \cdots, L_\varepsilon^{l,l}\}$.

DEFINITION 2.6. Let S be a strategy for coarse-grid operators, then S is *ε-asymptotically stable with respect to L_ε* if for every $\delta_0 > 0$ there exists a $\delta_1 > 0$ such that for all $0 \leqq k \leqq l$, we have the δ_1-domain of $L_\varepsilon^{k,l} \supset \delta_0$-domain of $L_\varepsilon \cap T_k$.

Remark 2.7. In order to avoid residual transfers in the MLA that are useless due to oscillating solutions, we require that a strategy is ε-asymptotically stable with respect to L_ε. Moreover we need a relaxation method for which the smoothing factors on all grids are less than 1. We then expect rapid convergence of the MLA.

Another approach would be to admit ε-asymptotically unstable strategies and to require that the relaxation method is such that bad components in the residuals are sufficiently smoothed. This poses very strong demands upon the relaxation method. If a strategy is not ε-asymptotically stable with respect to L_ε, and the relaxation method can not sufficiently damp the oscillations we may expect divergence if the number of levels increases.

By $L_{\varepsilon+\beta_k^l, h_k}$ we denote a discretization of (1.1) with artificial viscosity β_k^l and meshwidth h_k, and for fixed h_0 and $\gamma > 0$ (independent of ε, k and l) we will consider the following four strategies for coarse-grid operators:

Strategy 1 (S_1):

$$L_\varepsilon^{k,l} = L_{\varepsilon+\beta_k^l, h_k} \quad \text{and} \quad \beta_k^l = \gamma h_l, \quad k = 0, \cdots, l. \tag{2.8}$$

Strategy 2 (S_2):

$$L_\varepsilon^{k,l} = L_{\varepsilon+\beta_k^l, h_k}, \qquad \beta_l^l = \gamma h_l, \quad \beta_k^l = \gamma h_{k+1}, \quad k = 0, \cdots, l-1. \tag{2.9}$$

Strategy 3 (S_3):

$$L_\varepsilon^{k,l} = L_{\varepsilon+\beta_k^l, h_k} \quad \text{and } \beta_k^l = \gamma h_k, \quad k = 0, \cdots, l. \tag{2.10}$$

Strategy 4 (S_4):

$$L_\varepsilon^{l,l} \equiv L_{\varepsilon+\beta_l^l, h_l} \quad \text{with } \beta_l^l = \gamma h_l, \qquad L_\varepsilon^{k,l} \equiv R_{k,k+1} L_\varepsilon^{k+1,l} P_{k+1,k}, \quad k = l-1, \cdots, 0. \tag{2.11}$$

($R_{k,k+1}$ and $P_{k+1,k}$ are the restriction and the prolongation which are consistent with the FEM used.)

Remark 2.12. The choice of $L_\varepsilon^{k,l}$ according to S_4 is called Galerkin coarse-grid approximation. If we consider a constant-coefficient problem and neglect the boundaries, then a coarse-grid operator constructed with the FEM according to S_1, is identical with the Galerkin coarse-grid approximation as in S_4. The molecule is given by

$$L_{\varepsilon+\beta_k^l,h_k}=\frac{\varepsilon+\beta_k^l}{h_k^2}\begin{bmatrix} & 0 & -1 & \\ & -1 & 4 & -1 \\ & & -1 & 0 \end{bmatrix}+\frac{b_1}{6h_k}\begin{bmatrix} -1 & 1 & \\ -2 & 0 & 2 \\ & -1 & 1 \end{bmatrix}+\frac{b_2}{6h_k}\begin{bmatrix} 1 & 2 & \\ -1 & 0 & 1 \\ & -2 & -1 \end{bmatrix}.$$

Remark 2.13. It follows from (2.8)-(2.10) that

$$\text{for } S_1\text{: } \lim_{l\to\infty}\beta_0^l/h_k=\lim_{l\to\infty}\gamma/2^l=0,$$

$$\text{for } S_2\text{: } \beta_k^l/h_k\geqq\gamma/2 \text{ uniformly for all } k,\ l,$$

$$\text{for } S_3\text{: } \beta_k^l/h_k=\gamma \text{ uniformly for all } k,\ l.$$

In Theorem 2.14, Corollary 2.18 and Corollary 2.24 we will prove that S_1 and S_4 are not ε-asymptotically stable and S_2 and S_3 are. Further we will point out that the convergence rate of the MLA with S_2 is better than with S_3.

THEOREM 2.14. *Consider the* CD- *or* FEM-*discretizations of* (1.1) *with artificial viscosity* β_k^l *and constant coefficients*; *then* S_1 *is not* ε-*asymptotically stable with respect to* L_ε.

Proof. We give the proof only for the CD-discretizations; the proof for the FEM-discretizations is similar. The CD-discretization of (1.1) with artificial viscosity β_k^l and constant coefficients b_1 and b_2, $b_1^2+b_2^2=1$, reads

$$\begin{aligned} L_{\varepsilon+\beta_k^l,h_k}u\equiv&\left(-\frac{\varepsilon+\beta_k^l}{h_k^2}-\frac{b_2}{2h_k}\right)u_{i,j-1}^{h_k}+\left(-\frac{\varepsilon+\beta_k^l}{h_k^2}+\frac{b_2}{2h_k}\right)u_{i,j+1}^{h_k}\\ &+\left(-\frac{\varepsilon+\beta_k^l}{h_k^2}-\frac{b_1}{2h_k}\right)u_{i-1,j}^{h_k}+\left(-\frac{\varepsilon+\beta_k^l}{h_k^2}+\frac{b_1}{2h_k}\right)u_{i+1,j}^{h_k}\\ &+4\left(\frac{\varepsilon+\beta_k^l}{h_k^2}\right)u_{i,j}^{h_k}=f_{i,j}^{h_k}. \end{aligned}\tag{2.15}$$

Its characteristic form reads

$$\hat{L}_{\varepsilon+\beta_k^l,h_k}(\omega)=-\frac{2(\varepsilon+\beta_k^l)(\cos\omega_1h_k+\cos\omega_2h_k-2)}{h_k^2}+\frac{\underline{i}(b_1\sin\omega_1h_k+b_2\sin\omega_2h_k)}{h_k}.\tag{2.16}$$

The characteristic form of L_ε reads

$$\hat{L}_\varepsilon(\omega)=\varepsilon(\omega_1^2+\omega_2^2)+\underline{i}(b_1\omega_1+b_2\omega_2),\tag{2.17}$$

hence the δ_0-domain of L_ε is the set of all $\omega\in\mathbb{R}^2$ for which $|b_1\omega_2+b_2\omega_2|>\delta_0>0$. We have to show that a $\delta_0>0$ exists such that for all $\delta_1>0$ there exist k, $l\in\mathbb{Z}$, $0\leqq k\leqq l$, such that for an $\tilde{\omega}\in\mathbb{R}^2$ with $\tilde{\omega}\in(\delta_0\text{-domain of } L_\varepsilon)\cap T_k$ we have $\tilde{\omega}\notin\delta_1$-domain of $L_{\varepsilon+\beta_k^l,h_k}$. For that purpose we proceed as follows. Take $\delta_0=0.1\pi/h_0$ and let $\delta_1>0$ be arbitrary. Take $k=0$ and $l>\log_2(4\gamma/h_0\delta_1)$; then for either $\tilde{\omega}=(\pi/h_0,0)\in T_0$ or $\tilde{\omega}=(0,\pi/h_0)\in T_0$ both $|b_1\tilde{\omega}_1+b_2\tilde{\omega}_2|>\delta_0$ and $\lim_{\varepsilon\downarrow0}|\hat{L}_{\varepsilon+\beta_0^l,h_0}(\tilde{\omega})|=4\gamma/(h_02^l)<\delta_1$ hold. Hence S_1 is not ε-asymptotically stable with respect to L_ε.

This leads us to

COROLLARY 2.18. *Consider* L_ε *with constant coefficients* b_1 *and* b_2; *then* S_4 *is not* ε-*asymptotically stable with respect to* L_ε.

Proof. The proof follows immediately from (2.12) and (2.14).

THEOREM 2.19. *Consider the* CD-*discretizations of* (1.1) *with artificial viscosity* β_k^l *and constant coefficients. Let* S *be a strategy with* $\beta_k^l/h_k \geqq C > 0$ *uniformly for all* $k, l (k \leqq l) \in \mathbb{Z}$; *then* S *is* ε-*asymptotically stable.*

Proof. Again we use (2.15)-(2.17). We have to prove:

$$\forall \delta_0 > 0\ \exists \delta_1 > 0\ \forall k, l, 0 \leqq k \leqq l$$
$$\Rightarrow \delta_0\text{-domain of } L_\varepsilon \cap T_k \subset \delta_1\text{-domain of } L_{\varepsilon+\beta_{k}^l, h_k}.$$

Take $\delta_1 \equiv \min(\frac{1}{2}, 2C/5)\delta_0$. In the case $\delta_0 > 2^{1/2}\pi/h_k$ the inclusion is trivially satisfied because δ_0-domain of $L_\varepsilon \cap T_k = \emptyset$. If $0 < \delta_0 \leqq 2^{1/2}\pi/h_k$ then $\omega \in \delta_0$-domain of $L_\varepsilon \cap T_k$ implies

$$\delta_0 h_k < |b_1 \omega_1 h_k + b_2 \omega_2 h_k|.$$

The normalization $b_1^2 + b_2^2 = 1$ and the inequality $|\sin x - x| \leqq |x^3|/4$ for all $x \in \mathbb{R}$ yield

$$\delta_0 h_k < |b_1 \sin \omega_1 h_k + b_2 \sin \omega_2 h_k| + \frac{|\omega_1 h_k|^3}{4} + \frac{|\omega_2 h_k|^3}{4}. \tag{2.20}$$

We distinguish the two complementary cases:

(i) $|\omega_1 h_k|^3 \leqq \delta_0 h_k$ and $|\omega_2 h_k|^3 \leqq \delta_0 h_k$;

(ii) $|\omega_1 h_k|^3 > \delta_0 h_k$ or $|\omega_2 h_k|^3 > \delta_0 h_k$.

Because of (2.16) and (2.20) case (i) implies:

$$\lim_{\varepsilon \downarrow 0} |\hat{L}_{\varepsilon+\beta_k^l, h_k}(\omega)| \geqq \frac{|b_1 \sin \omega_1 h_k + b_2 \sin \omega_2 h_k|}{h_k} > \frac{\delta_0}{2} \geqq \delta_1. \tag{2.21}$$

To complete the proof we now consider case (ii). It follows from (2.16) and $\beta_k^l/h_k \geqq C$ that

$$\lim_{\varepsilon \downarrow 0} |\hat{L}_{\varepsilon+\beta_k^l, h_k}(\omega)| \geqq \frac{2C(1 - \cos \omega_1 h_k + 1 - \cos \omega_2 h_k)}{h_k}, \tag{2.22}$$

and from (ii) and $0 < \delta_0 h_k \leqq 2^{1/2}\pi$ it follows that the right-hand side of (2.22) is greater than or equal to

$$\frac{2C\delta_0(1 - \cos((\delta_0 h_k)^{1/3}))}{\delta_0 h_k},$$

hence

$$\lim_{\varepsilon \downarrow 0} |\hat{L}_{\varepsilon+\beta_k^l, h_k}(\omega)| > \frac{2C\delta_0}{5} \geqq \delta_1 > 0. \tag{2.23}$$

Both (2.21) and (2.23) hold uniformly for all k, l so S is ε-asymptotically stable with respect to L_ε.

Note that the condition of Theorem 2.19 is satisfied by taking on coarser grids the artificial viscosity proportional to the current meshwidth.

COROLLARY 2.24. *Consider the* CD-*discretizations of* (1.1) *with artificial viscosity* β_k^l *and constant coefficients; then* S_2 *and* S_3 *are* ε-*asymptotically stable with respect to* L_ε.

Proof. The proof follows immediately from Remark 2.13 and Theorem 2.19.

It is obvious that the ε-asymptotic stability degree of the individual grid-operators belonging to S_2 is larger than in the case of S_1. Moreover for decreasing γ the smoothing factors for S_1 become worse (cf. Table 2). We formulate this in the following

Conjecture 2.25. For a *fixed* number of levels the set of γ-values for which the MLA with S_2 converges, is larger than that for which the MLA with S_1 converges.

In case of a two-level algorithm (TLA), $l=1$, and a constant-coefficient problem, a two-level analysis shows that the asymptotic rate of convergence for S_1 or S_2, for which the artificial viscosity is equal on both levels is better than for S_3, where the artificial viscosity corresponds to the meshwidth. (cf. Van Asselt [1]). Therefore in S_1 we take an equal artificial viscosity on all levels. For this strategy, however, stability problems may occur on coarser grids (cf. Theorem 2.14). S_3 is ε-asymptotically stable (cf. Corollary 2.24), but the two-level analysis indicates that the convergence rate is slower. S_2 is an intermediate strategy where on levels l and $l-1$ the artificial viscosity is the same, and it is also ε-asymptotically stable (cf. Corollary 2.24). These arguments lead to the following

Conjecture 2.26. S_2 combines the rapid convergence rate of S_1 with the stability of S_3.

At level l the discrete operators $L_{\varepsilon+\beta_l^l,h_l}$ using S_1, S_2, S_3 are equal.

At level $l-1$ the discrete operators $L_{\varepsilon+\beta_{l-1}^l,h_{l-1}}$ using S_1, S_2 are equal (S_3 is not), and the relative order of consistency of the S_1 and S_2 operators on level l and $l-1$ is the same and higher than that of S_3. Furthermore, consider the part of T_l where the smoothing effect of a relaxation method applied to S_2 and S_3 is the same as in the case of S_1 in terms of local mode analysis. For S_2 this part is larger than for S_3 (cf. Fig. 1). For S_4 the same arguments hold as for S_1 (cf. Remark 2.12). This leads us to formulate the following

Conjecture 2.27. For a *finite* number of levels and γ sufficiently large the difference between the asymptotic rate of convergence of the MLAs using S_1 or S_4 and S_2 is smaller than that between S_3 and S_2. The properties stated in Theorem 2.14, Corollary 2.18, Corollary 2.24 and Conjectures 2.25–2.27 will be confirmed by numerical experiments in § 4.

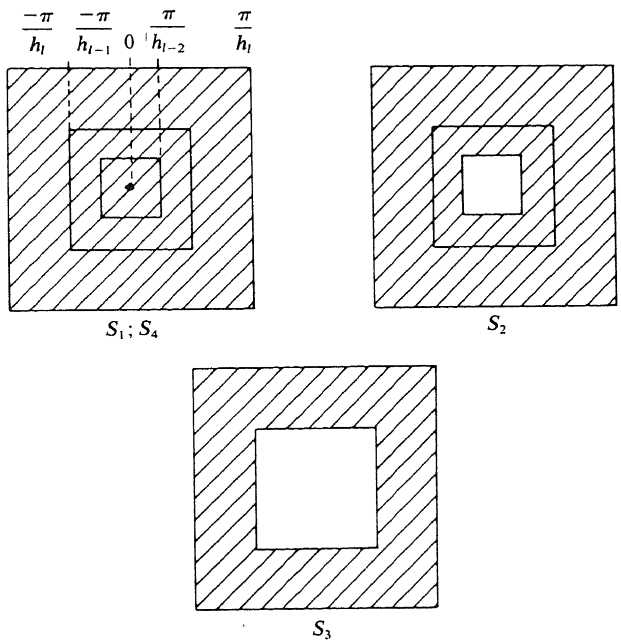

FIG 1. *Parts of T_l where for S_2 and S_3 the smoothing effect is the same as for S_1 and S_4.*

3. Numerical approximation of the convergence rate. In this section we give a description of the method used to determine the asymptotic rate of convergence of the MLA. Let

(3.1) $A_h v_h = f_h$ be a discretization of (1.1).

The MLA used to solve (3.1) can be described as a defect correction process (cf. Hemker [5]):

$$\begin{aligned} &v_h^0 \text{ given start approximation,} \\ &v_h^{i+1} = M_h v_h^i + B_h^{-1} f_h, \quad i = 0, 1, \cdots \end{aligned} \tag{3.2}$$

with amplification matrix $M_h = I_h - B_h^{-1} A_h$. I_h is the identity matrix, and B_h^{-1} is an approximate inverse of A_h, determined by coarse-grid and smoothing operators, prolongation and restriction. We suppose A_h and B_h to be nonsingular. For the error $e_h^i = v_h - v_h^i$, $i = 0, 1, \cdots$ the following relation holds:

$$e_h^{i+1} = M_h e_h^i.$$

The convergence behavior of the MLA is determined by the spectral radius of M_h. This motivates the following:

DEFINITION 3.3. *The asymptotic rate of convergence* of the MLA (3.2) is $-\log_{10} \rho(M_h)$ where $\rho(M_h) \equiv \max_j |\lambda_j|$ is the spectral radius of M_h; λ_j are the eigenvalues of M_h.

THEOREM 3.4.

$$\sup_{x \neq 0} \lim_{k \to \infty} \left(\frac{\|M_h^k x\|}{\|x\|} \right)^{1/k} = \rho(M_h),$$

with $\|\cdot\|$ *an arbitrary norm.*

Proof. See Stoer and Bulirsch [7, (8.2.4)], Varga [8, Thm. (3.2)]. Because of Theorem 3.4 we can compute an approximation $\rho_{m,k}(M_h, e_h^0)$ of $\rho(M_h)$ defined by

$$\rho_{m,k}(M_h, e_h^0) \equiv \left(\frac{\|M_h^{m+k} e_h^0\|_2}{\|M_h^m e_h^0\|_2} \right)^{1/k}, \tag{3.5}$$

where $\|\cdot\|_2$ is the Euclidean norm. Note that

$$\sup_{e_h^0 \neq 0} \lim_{m,k \to \infty} \rho_{m,k}(M_h, e_h^0) = \rho(M_h). \tag{3.6}$$

In numerical computations v_h^j, $j = m, \cdots, m+k$ are obtained by the iterative method under consideration. When for increasing m and k, $\|e_h^j\|_2$ reaches values near the square root of the machine accuracy, we replace e_h^j by $e_{h,\eta}^j$:

$$e_{h,\eta}^{(j)} \equiv \eta e_h^j (\eta \gg 1), \tag{3.7}$$

and replace v_h^j by $v_{h,\eta}^j$:

$$v_{h,\eta}^j \equiv v_h + e_{h,\eta}^j. \tag{3.8}$$

Thus

$$\frac{\|e_{h,\eta}^{j+1}\|_2}{\|e_{h,\eta}^j\|_2} = \frac{\|e_h^{j+1}\|_2}{\|e_h^j\|_2},$$

and as

$$\rho_{m,k}(M_h, e_h^0) = \left(\prod_{j=m}^{m+k-1} \frac{\|e_h^{j+1}\|_2}{\|e_h^j\|_2} \right)^{1/k}, \tag{3.9}$$

in this way values of $\rho_{m,k}(M_h, e_h^0)$ can be computed for large m and k. By this method ultimately the eigenfunctions of M_h corresponding to nondominant eigenvalues will decrease exponentially relative to the dominant eigenfunctions. Note that for small m and k, $\rho_{m,k}$ depends strongly on f_h while ρ does not. There are more refined methods to determine the spectral radius. (cf. Wilkinson [11]). However for our purpose the method described is sufficiently accurate.

4. Numerical results. In this section we give the results of numerical experiments to compare the strategies S_1, S_2, S_3 and S_4 and to verify the properties stated in Theorem 2.14 Corollaries 2.18 and 2.24 and Conjectures 2.25-2.27. We take three test problems. Test problem 1 with constant coefficients closely resembles the problem analysed by two-level analysis in Van Asselt [1]. Test problem 2 has variable coefficients. Although a strict application of Fourier analysis arguments does not hold for these variable coefficient problems, the experiments for the latter test problem show that globally the same properties hold as for the constant-coefficient case. For the second problem we also show to what extent the strategies $S_1, \cdots, S_4$ are better than relaxation alone (i.e., without coarse-grid correction). Test problem 3 differs from Test problem 1 by discretization (FEM), relaxation (ILU) and number of levels.

Test problem 1. We consider the following convection-diffusion equation (see Fig. 2)

$$\begin{aligned} &-(\varepsilon+\gamma h)\Delta u + \frac{\partial}{\partial y} u = 0 \quad \text{on } \Omega = [0,1]\times[-1,1], \\ &\varepsilon = 10^{-6}, \qquad h = \tfrac{1}{16}. \end{aligned} \tag{4.1}$$

The boundary conditions are:

$$u|_{\delta_1\Omega} = \begin{cases} 1, & 0 \leqq x < \frac{1}{2} - 10^{-6}, \\ -10^6(x-\frac{1}{2}), & \frac{1}{2} - 10^{-6} \leqq x \leqq \frac{1}{2} + 10^{-6}, \\ -1, & \frac{1}{2} + 10^{-6} < x \leqq 1, \end{cases} \tag{4.2}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\delta_2\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\delta_3\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\delta_4\Omega} = 0,$$

with $\delta_1\Omega, \cdots, \delta_4\Omega$ in Fig. 2.

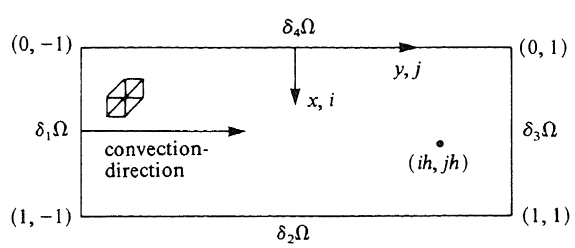

FIG. 2. *The domain* Ω.

Equation (4.1) is discretized by CD on levels $k = 0, \cdots, l = 3$ with meshsize $h_k = 1/2^{k+1}$. The boundary conditions are not substituted. The Dirichlet boundary conditions are implemented with a large number on the main diagonal to avoid unwanted coarse-grid corrections at the boundary. The Neumann boundary conditions are discretized as follows:

$$\delta_2\Omega:\ u(1,y)-u(1-h_k,y)=0,\qquad -1<y\leqq 1,$$
$$\delta_3\Omega:\ u(x,1)-u(x,1-h_k)=0,\qquad 0<x<1,$$
$$\delta_4\Omega:\ u(0,y)-u(h_k,y)=0,\qquad -1<y\leqq 1,\quad k=0,\cdots,l=3.$$

For various values of γ the discretized equation is solved with the W-cycle MLA (i.e., the application of 2 multi-level-iteration steps to approximate the solution of the coarse-grid equation).

We perform one pre- and one post-relaxation step consisting of symmetric point Gauss–Seidel relaxation (SGS) in the y-direction. We use 7-point prolongation and 7-point restriction (cf. Hemker [6], Wesseling [9]). On the coarsest level we solve exactly. A random initial approximation of the solution is used. The values for m and k in (3.9) are 30 and 10 respectively.

Test problem 2. We consider the following convection-diffusion equation (see Fig. 3)

$$(4.3)\qquad \begin{gathered} -(\varepsilon+\gamma h)\Delta u+b_1\frac{\partial}{\partial x}u+b_2\frac{\partial}{\partial y}u=0\quad \text{on}\ \Omega=[0,1]\times[-1,1],\\ \varepsilon=10^{-6},\quad h=\tfrac{1}{16},\quad b_1=y(1-x^2),\quad b_2=-x(1-y^2).\end{gathered}$$

The boundary conditions are

$$(4.4)\qquad \begin{gathered} u|_{\delta_1\Omega}=1+\tanh(10+20x),\qquad -1\leqq x\leqq 0,\\ \left.\frac{\partial u}{\partial n}\right|_{\delta_2\Omega}=\left.\frac{\partial u}{\partial n}\right|_{\delta_3\Omega}=\left.\frac{\partial u}{\partial n}\right|_{\delta_4\Omega}=\left.\frac{\partial u}{\partial n}\right|_{\delta_5\Omega}=0.\end{gathered}$$

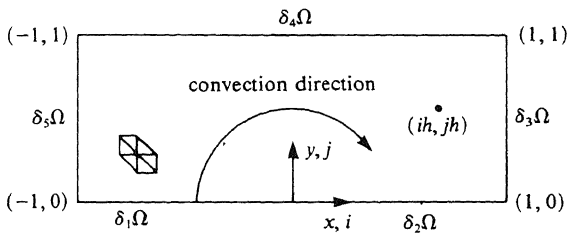

FIG. 3. *The domain* Ω.

Equations (4.3) and (4.4) are discretized by the FEM on levels $k=0,\cdots,l=4$ with mesh-size $h_k=(1/2)^k$. The boundary conditions are not substituted and the Dirichlet boundary conditions are implemented with a large number on the main diagonal. For different values of γ, and S_1–S_4 the discretized equation is solved with the W-cycle MLA. We perform one pre- and one post-relaxation step by means of 7-point ILU relaxation, (cf. Wesseling and Sonneveld [10]). The ILU-decomposition is ordered lexicographically (cf. Fig. 3). On the coarsest level we solve exactly. Again we use 7-point prolongation and 7-point restriction (that are consistent with the FEM discretization), and a random initial approximation. In (3.9) m and k are again 30 and 10.

Test problem 3. For $l=4, 5, 6$, we consider (4.1) with different h, and (4.2) discretized by the FEM on levels $k=0,\cdots,l$, with mesh size $h_k=(\frac{1}{2})^{k+1}$, $\gamma=\frac{1}{2}$. The boundary conditions are not substituted and the Dirichlet boundary conditions are implemented with a large number on the main diagonal.

The discretized equation is solved with the W-cycle MLA. We perform one pre- and post-relaxation step by means of 7-point-ILU relaxation (on the coarsest level we do not solve directly, but perform 2 relaxation sweeps). The ILU-decomposition is ordered lexicographically (cf. Fig. 3). We use 7-point prolongation and 7-point restriction. A random initial approximation of the solution is used. The values for m and k in (3.9) are 20 and 10 respectively.

Figures 4 and 5 show the properties in Conjectures (2.25)-(2.27) for Test problems 1 and 2, respectively. Figure 5 also shows that all strategies S_1-S_4 are better than relaxations without coarse-grid corrections. In Table 1 for S_1, S_2 and S_3 the smoothing factors of SGS are given at different levels and for different γ. We notice that for $\log_2\gamma > 0$ the big difference in the asymptotic rate of convergence of S_2 and S_3 (cf. Fig. 4) is mainly caused by the order of consistency and to a small extent by the relaxation method, because the smoothing factors are almost the same.

In order to demonstrate Theorem 2.14, Corollaries 2.18 and 2.24 in connection with Remark 2.7 we take Test problem 3. Table 2 shows the convergence rates as measured (cf. Definition 3.3). Note that S_1 and S_4 show similar stability and convergence behavior (cf. Remark 2.12).

Remark 4.5. With respect to Remark 2.7 we notice that in many cases a decreasing stability coincides with a worsening smoothing factor (cf. Table 1).

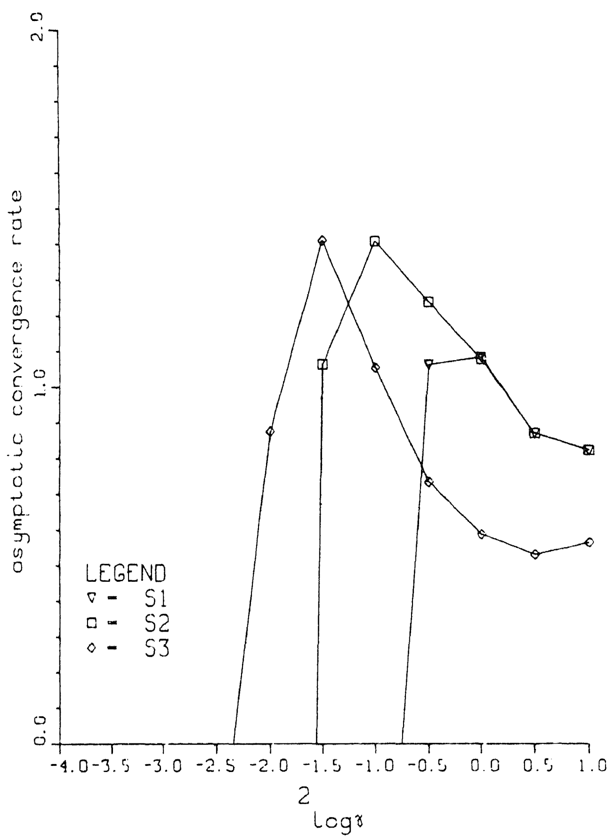

FIG. 4. *Asymptotic convergence rates for Test problem* 1. *Only the part of the figure with positive asymptotic convergence rate is drawn.* ($l=3$, $h_l=\frac{1}{16}$).

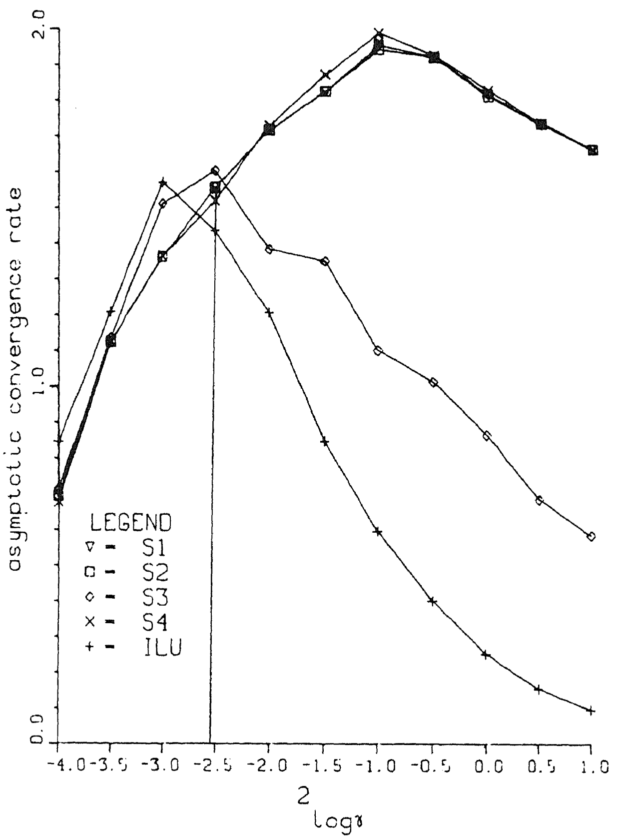

FIG. 5. *Asymptotic convergence rates for Test problem 2. The graph depicted by "+" represents two* ILU *relaxation sweeps in one iteration step without coarse-grid correction. Only the part of the figure with positive asymptotic convergence rate is drawn.* ($l=4$, $h_l=\frac{1}{16}$).

5. Conclusions. In order to solve the convection-diffusion equation in two dimensions by a multi-level algorithm (MLA), we consider 4 strategies for coarse-grid operators:

S_1: on each coarse grid the same artificial viscosity as on the finest grid;

S_3: on each coarse grid the artificial viscosity corresponding to the mesh width;

S_2: an intermediate choice, with the same artificial viscosity on the two finest grids;

S_4: Galerkin approximation for the coarse-grid operators.

For S_1 and S_4 the artificial viscosity may become too small on coarse grids, and hence stability problems and bad smoothing-factors may occur. S_1 and S_4 are not ε-asymptotically stable, S_2 and S_3 are. (cf. Definition 2.6, Theorem 2.14, Corollaries 2.18, 2.24 and Table 1).

If the finest-grid artificial viscosity is sufficiently large, the asymptotic rate of convergence of the MLA according to S_2 is far better than that of S_3 (cf. Conjecture 2.26 and Figs. 4, 5).

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TABLE 1
Smoothing-factors for one SGS sweep, Test problem 1, different γ, levels and strategies (local mode analysis, cf. Brandt [2]).

k \ S	S_1	S_2	S_3
3	0.36	0.36	0.36
2	4.84	4.84	0.36
1	186	4.84	0.36

$\log_2 \gamma = -1.5$

k \ S	S_1	S_2	S_3
3	0.24	0.24	0.24
2	0.80	0.80	0.24
1	15625	0.80	0.24

$\log_2 \gamma = -1.0$

k \ S	S_1	S_2	S_3
3	0.23	0.23	0.23
2	0.36	0.36	0.23
1	4.84	0.36	0.23

$\log_2 \gamma = -0.5$

k \ S	S_1	S_2	S_3
3	0.24	0.24	0.24
2	0.24	0.24	0.24
1	0.80	0.24	0.24

$\log_2 \gamma = 0.0$

k \ S	S_1	S_2	S_3
3	0.24	0.24	0.24
2	0.23	0.23	0.24
1	0.36	0.23	0.24

$\log_2 \gamma = 0.5$

k \ S	S_1	S_2	S_3
3	0.25	0.25	0.25
2	0.24	0.24	0.25
1	0.24	0.24	0.25

$\log_2 \gamma = 1.0$

TABLE 2
Convergence rates for Test problem 3, S_1-S_4, and increasing l.

		strategy			
level l	h_l	S_1	S_2	S_3	S_4
4	1/32	2.01	1.78	1.61	2.01
5	1/64	$\ll 0$	1.70	1.33	$\ll 0$
6	1/128	$\ll 0$	1.17	0.87	$\ll 0$

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