Proportional representation in a regional council

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The members of a regional council are appointed by and from the local councils which participate in the regional cooperation.
The regional council should constitute a fair representation of the local interests and also a fair representation of the political views.
A method is presented to determine the number of members to be appointed by each local council from each political party. Optimal flows in networks solve the problem.

1. Introduction

The region *Gooi en Vechtstreek* is a cooperation of nine municipalities near Amsterdam. The statute of the cooperation stipulates that the members of the regional council are appointed by the participating local councils. Each local council has an odd number of members, a fourth part of which (rounded to the nearest integer) is appointed into the regional council. In this way, the representation of each municipality is proportional to the membership of its council and this should be a fair representation of the local interests in the regional cooperation.

The regional council should also be a fair representation of the political views on the matters to be discussed and decided by that body. Each political party should get a number of seats in the regional council that is close to a fourth part of its total number of seats in the local councils. Thus, if a party has a single seat in each local council then it expects to get two seats in the regional council. However, if the allocation of seats is completely left to the local councils then it may be expected that none of these will give a seat in the regional council to a small minority in their midst, if only for fear that the other councils would do so too. It is clear that the local councils should coordinate the allocation of seats so as to obtain a fair political composition of the regional council.

In *Gooi en Vechtstreek* the allocation of seats is coordinated by the chairman of the cooperation. Based on the outcome of the local elections he calculates the number of seats to be allotted to each party in each local council. The results of his calculations are discussed with the incumbent political leaders of the regional council and then sent, as an advice, to the local councils. In 1982 the method as described by Anthonisse [1] was used. The present paper provides an extended and improved method.

In the Netherlands, cooperation between municipalities is quite common. There are 740 municipalities and over 1250 cooperations. Many cooperations have a very restricted purpose and the political composition of their council is not important. A new law, to become effective 1st January 1985, should improve the surveyability of the cooperations. The country is to be divided into regions of cooperation and, as a rule, cooperation will be restricted to the municipalities in a region. Moreover, the number of cooperations will be reduced by integrating several existing cooperations into a new one. The new law stipulates that the councils are appointed by and from the local councils. The statute of the cooperation must specify the number of members to be appointed by each local council.

It may be expected that, due to these developments, there will be an increased interest in the political composition of the councils of cooperations.

2. Proportional representation as an optimization problem

The problem of proportional representation is to allot the seats in a representative body to the political parties in such a way that the number of seats is proportional to the number of votes for each party. Let v_p denote the number of votes for party p, thus the total number of votes is $V = \sum_p v_p$. Let S denote the number of seats in the body, then party p should obtain $e_p = S \times v_p / V$ seats. In general, however, the numbers e_p are not integers and rounding is required to obtain a feasible allotment. Let s_p denote the number of seats for party p.

Te Riele [2] describes seven methods of proportional representation as methods to solve an optimization problem

$$\text{minimize} \sum_p f(e_p, s_p) \tag{1}$$

subject to

$$\sum_p s_p = S \tag{2}$$

and

$$\text{each } s_p \text{ non-negative integer.} \tag{3}$$

The function $f(e_p, s_p)$ determines the distance between the exact or theoretical allotment e_p and the solution s_p.

The well-known method of the greatest remainders (Roget, Hamilton) corresponds with

$$f(e_p, s_p) = |e_p - s_p|. \tag{4}$$

The equally well-known method of the greatest divisors (D'Hondt, Hagenbach-Bischoff, Jefferson) corresponds with

$$f(e_p, s_p) = (s_p - (e_p - \tfrac{1}{2}))^2 / e_p. \tag{5}$$

Webster's method corresponds with

$$f(e_p,s_p) = (s_p - e_p)^2 / e_p. \tag{6}$$

BALINSKI and YOUNG [3] list a number of properties of the 'ideal' method of proportional representation and show that no such method exists.

The method of the greatest remainders allows the occurrence of the Alabama-paradox: while the v_p remain the same, an s_p may decrease by increasing S. A fair method should not allow this.

The method of the greatest divisors favours the greater parties. This method has a tendency to round e_p downwards as s_p is compared with $e_p - \frac{1}{2}$. However, some e_p must be rounded upwards and this will occur, in general, at the largest e_p.

WEBSTER's method appears to be a very good approximation to the ideal of proportional representation.

3. THE METHOD

Now the problem of allotting the seats in the regional council to the parties in the local councils can be formulated. Throughout $f(\cdot,\cdot)$ denotes any function corresponding with a method of proportional representation. The number of members of party p in the local council of municipality m is denoted by c_{mp}. Here a practical problem occurs, as it is not evident which parties should be distinguished. The municipal elections allot the seats in the local council to, typically, four up to six local parties. Some of these local parties are chapters of national parties, others are coalitions of such chapters and still others are purely local political organizations. Thus it must be decided which combinations of local parties should be considered as regional parties in order to allot the seats in the regional council. A conclusive arrangement is to combine those local parties which, by a joint statement, require to be combined and to consider each remaining local party as a separate one. From these statements the c_{mp} can be determined. The regional strength of a party is $r_p = \sum_m c_{mp}$ and $R = \sum_p r_p$. The local council of m has $s_m = \sum_p c_{mp}$ seats.

Let a_m denote the number of members of the regional council to be appointed by and from the local council of municipality m. Thus $A = \sum_m a_m$ is the number of seats in the regional council. The numbers a_m are found by consulting the statute of the cooperation.

Now the allotment can be found by solving two problems. First, the political composition of the regional council is found by computing the number of seats b_p for each party p. Secondly, the numbers x_{mp} are computed, which denote the number of members of party p to be appointed from the local council of municipality m.

The first problem is to find b_p:

$$\text{minimize} \sum_p f(e_p,b_p) \tag{7}$$

subject to

$$\sum_p x_{mp} = a_m \quad (m = 1,2,...) \tag{8}$$

$$\sum_m x_{mp} = b_p \quad (p=1,2,...) \tag{9}$$

$$x_{mp} \leqslant c_{mp} \quad (m=1,2,...;\ p=1,2,...) \tag{10}$$

and

$$\text{each } b_p, x_{mp} \text{ non-negative integer,} \tag{11}$$

where e_p denotes a theoretical allotment of the A seats, e.g. $e_p = A \times r_p / R$.

The second problem is to find x_{mp}:

$$\text{minimize} \sum_m \sum_p f(e_{mp}, x_{mp}) \tag{12}$$

subject to

$$\sum_p x_{mp} = a_m \quad (m=1,2,...) \tag{13}$$

$$\sum_m x_{mp} = b_p \quad (p=1,2,...) \tag{14}$$

$$x_{mp} \leqslant c_{mp} \quad (m=1,2,...;\ p=1,2,...) \tag{15}$$

and

$$\text{each } x_{mp} \text{ non-negative integer,} \tag{16}$$

where b_p denotes the solution of the first problem and e_{mp} denotes a theoretical allotment of the seats, e.g. $e_{mp} = A \times c_{mp} / R$.

The first problem obviously has a feasible solution, provided $a_m \leqslant s_m$. The latter condition is certainly satisfied, thus any sample of a_m from the s_m members satisfies (8) and (10). Constraint (9) merely defines b_p.

It is clear that the second problem, with the b_p which solve the first one, also has a feasible solution. Constraints (8)-(11) are identical to constraints (13)-(16).

It is tempting to replace the first problem by a simpler one. The b_p might be computed by allotting the A seats proportional to r_p. This amounts to a relaxation of the constraints (8)-(11) into

$$\left.\begin{array}{l} \sum_p b_p = A \\ b_p \text{ non-negative integer.} \end{array}\right. \tag{17}$$

However, this may yield a nonfeasible second problem, as the following example shows.

c_{mp}	$p=$ 1	2	3	4	5	6	a_m
$m=1$	7						2
2	7						2
3	7						2
4	6	1					2
5		30	15				11
6			16	29			11
7				2	31	12	11
8						19	5
b_p	6	8	8	8	8	8	46 $=A$

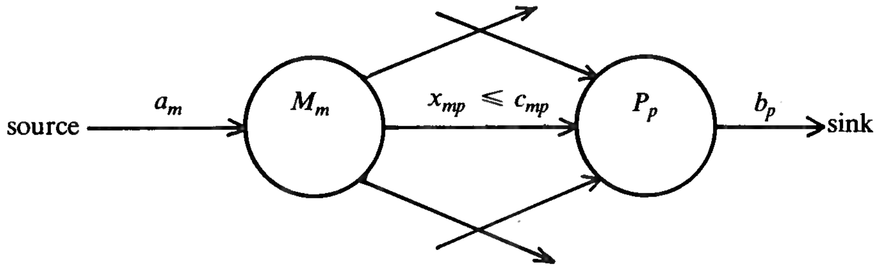
The method of the greatest remainders was used here to determine the b_p proportional to (27,31,31,31,31,31). This results in 6 seats for party 1, but this party will get at least 7 seats from the first four municipalities. Similar examples have been constructed for other methods of proportional representation.

Thus the first problem must be solved to ensure the feasibility of the second problem.

In practice, however, the simpler method (7), (17) to compute b_p may be tried first. If, with these b_p, the second problem is feasible then these b_p constitute an optimal solution of the first problem. In the opposite case the problem (7)-(11) must be solved to obtain the correct b_p.

4. Flows in networks

Both the first and the second problem as defined in the previous section can be formulated as problems of finding an optimal flow in a network. In both cases the network contains nodes M_m corresponding with the municipalities and P_p corresponding with the parties. Node M_m receives a fixed flow of a_m units from a source. The arc from M_m to P_p has a capacity of c_{mp} units, the flow in this arc is denoted by x_{mp}. There is a flow of b_p units from node P_p into a sink. At each node, the incoming flow equals the outgoing flow.



In the first problem, $f(e_p, b_p)$ can be interpreted as the cost of sending b_p units of flow from node P_p towards the sink. The problem is to find a feasible flow which minimizes these costs. Most algorithms to solve such problems assume that the cost is a linear function of the flow through an arc. Due to the convexity of $f(e_p, b_p)$ the problem is easily put into this form.

Let $\hat{b}_p$ denote an integer value of b_p which minimizes $f(e_p, b_p)$. Then

$$b_p = \hat{b}_p + (b_{p1}^+ + b_{p2}^+ + \cdots) - (b_{p1}^- + b_{p2}^- + \cdots)$$

where

$$b_{pi}^+ \in \{0,1\} \quad \text{and} \quad b_{pi}^- \in \{0,1\}.$$

Then

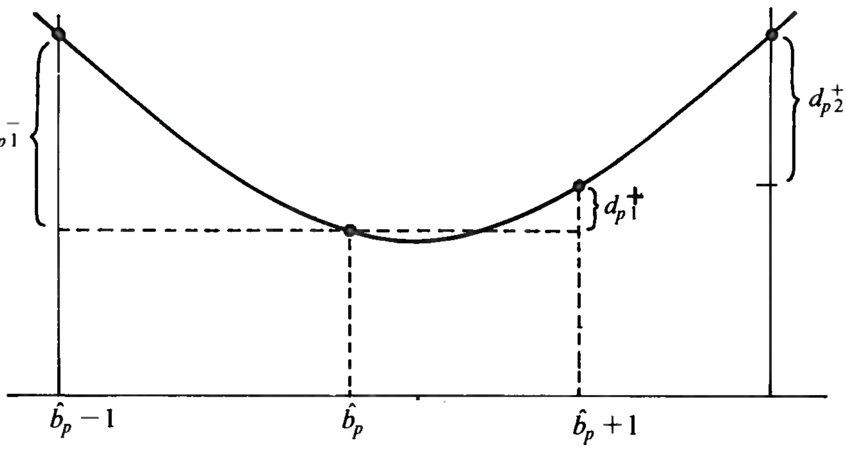
$$f(e_p, b_p) = f(e_p, \hat{b}_p) + \sum_i d_{pi}^+ b_{pi}^+ + \sum_i d_{pi}^- b_{pi}^-$$

where

$$d_{pi}^+ = f(e_p, \hat{b}_p + i) - f(e_p, \hat{b}_p + i - 1)$$

and

$$d_{pi}^- = f(e_p, \hat{b}_p - i) - f(e_p, \hat{b}_p - i + 1).$$



Now the arc from P_p towards the sink having a nonlinear cost can be replaced by a number of parallel arcs. One contains a fixed flow of $\hat{b}_p$ units. The other arcs have unit capacity. Some of the arcs are directed from P_p towards the sink. These correspond with flows b_{pi}^{+} at cost d_{pi}^{+}. Other arcs are directed from the sink towards P_p. These correspond with flows b_{pi}^{+} at cost d_{pi}^{-}. The mincost flow in this network corresponds with the solution of the first problem.

In the second problem, the flows from P_p towards the sink are fixed at b_p. Now the mincost flow between the nodes M_m and P_p is to be found. Again, the problem can be linearized by replacing the arcs from M_m to P_p by parallel arcs in both directions, with the appropriate capacities and costs.

With the above linearizations the two problems are easily solved, either by a network flow algorithm or by a general algorithm for linear programming. The first complete treatment of flows in networks was given by FORD and FULKERSON [4]. The book by KENNINGTON and HELGASON [5] contains computer programs of flow algorithms. These programs are available in OPERAL, the CWI library of Operations Research Algorithms.

5. CONCLUDING REMARKS

The two problems defined above can be solved simultaneously as

$$\text{minimize } W_1 \sum_p f(e_p, b_p) + \sum_m \sum_p f(e_{mp}, x_{mp})$$

subject to (8)-(11), where W_1 denotes a sufficiently large value. For $W_1 = 0$ the optimal x_{mp} are completely defined by the local preferences and can be found by solving a proportional representation problem for each m separately. An increase of W_1 means that more weight is attached to the regional preferences.

It might occur that the statute specifies A only and that the a_m are to be found by solving a proportional representation problem. Now the three problems may be solved simultaneously

$$\text{minimize } W_0 \sum_m f(A \times s_M / R, a_m) + W_1 \sum_p f(e_p, b_p) + \sum_m \sum_p f(e_{mp}, x_{mp})$$

subject to (8)-(11) and

$$a_m \text{ non-negative integer and } \sum_m a_m = A,$$

where both W_0 and W_1 are sufficiently large.

As mentioned above, there is a wide choice of functions $f(\cdot,\cdot)$ that could be used in the above method.

Moreover, another definition of e_{mp} could be used, e.g. $e_{mp} = a_m \times c_{mp} / s_m$. These e_{mp} reflect the local preferences. By using these e_{mp} the composition of the regional council would not deviate more from the sum of the local preferences than is necessary to reflect the regional strength of the parties.

Also, another definition of e_p could be used, e.g., $e_p = \sum_m a_m \times c_{mp} / s_m$. This, however, introduces a bias because the a_m are not an exact proportional distribution of the A seats over the municipalities.

In the above description, the apportionment is based on the number of representatives in the local councils. They could also be based on the number of votes for a party.

Similar optimization models to the ones described above can be formulated to solve the other problems which may arise when the new law becomes effective.

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