STOCHASTIC GEOMETRY AND IMAGE ANALYSIS*

by

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Summary

We list recent ideas in stochastic geometry which are closely related to image analysis. These include the synthesis of stochastic models of images, techniques for evaluating models and algorithms, general concepts of 'geometrical information' and the theory of random sets, problems of image irregularity and errors in observation, techniques of geometric integration theory, and fractional dimensional irregularity.

1. Introduction

The development of computerized image processing and image analysis already seems to have prompted considerable study of the relations between geometry, probability theory and computer science. Rosenfeld [29, preface] observes that all image processing algorithms must be based explicitly or implicitly on mathematical models of the images to be processed. Some of the newer stochastic image models presented in [29] are based on Markov processes, random fields, random mosaics (tessellations) and stochastic grammars. Apart from image modeling, we imagine other mathematical contributions should include a theoretical background for the comparison of algorithms, and mathematical techniques for the treatment of image models.

Independently of such requirements, many concepts related to image analysis have evolved in other areas, notably in stochastic geometry, stereology and geometric integration theory. *Stochastic geometry* is that part of probability theory dealing with random subsets of a geometrical space, and interactions between probability and geometry. This includes all stochastic image models, at least in principle, but some frequently studied models are: elementary constructions of random lines, circles or triangles; spatial schemes such as random mosaics and random coverings of the plane; and general random processes and random sets. The main body of theory concentrates on *uniformly random* models, for which there are simple explicit solutions. However, the last decade

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has seen the introduction of more flexible techniques and a completely general theoretical foundation for random sets.

This paper summarizes some recent work in stochastic geometry (drawing also on stereology and geometric integration theory) which could be connected with image analysis. Section 2 introduces the range of random image models in stochastic geometry, and outlines the classical theory of uniformly random models. The more recent combinatorial theory (section 3) has an application to problems of image complexity. Section 4 discusses the Kendall-Matheron abstract theory of random sets, which has many similarities to tenets of image analysis. J. Serra's mathematical morphology and image analysis theory is touched upon in Section 5. Recent thoughts about image irregularity and observation errors (Section 6) are developed using geometric integration theory. Finally Section 7 speculates on the usefulness of fractal (fractional dimensional) models of image irregularity.

2. Classical stochastic geometry

Detailed surveys of stochastic geometry can be consulted in the literature [24, 3, 7, 32, 35] and we shall give here a very brief sketch. Probability models available for generating random geometrical objects (hence random image models) can be classified as

(a) elementary constructions;
(b) stochastic processes;
(c) theory of random sets.

(a) Elementary constructions are the simple geometrical figures of Euclid with an added component of randomness, as for example the output of a computer graphics program when the input is a random number generator. Points, lines, triangles, circles and other figures are determined by $n < \infty$ real parameters so that a random figure can be defined as a probability distribution on the n-dimensional parameter space. Of course we may also construct the random line joining two random points, and so on. Using parametrisations of the rotation and translation groups we may generate random positions of an arbitrary object. Typical problems include finding the probability that two random figures (or a random figure and a fixed figure) will intersect; the mean area of length of overlap between figures; and the probability that N random figures will completely cover a specified region.

Even the simplest problems for random figures lead to difficult multiple integrals. An exception to this rule is that *uniformly distributed* random figures often lead to simple explicit solutions. For example, a random two-dimensional point $X = (x_1, x_2)$ is a *uniformly random* (UR) point in the region $A \subset \mathbb{R}^2$ if it has constant probability density $f(x_1, x_2) = K$. The constant must be $K = 1/\text{area}(A)$ since probability integrates to 1. For any measurable subset $B \subset A$ we find the probability

$$P(X \text{ falls in } B) = \frac{\text{area}(B)}{\text{area}(A)}, \tag{1}$$

which is what we understand by a 'simple explicit solution'. Now consider a random circle $C(X,r)$ of fixed radius r obtained by randomizing the centre point X. Let X be a uniformly random point in the disc D_{R+r} of radius $R+r$ and centre 0. Then the circle $C(X,r)$ always intersects D_R, the disc of radius R about 0. We say $C(X,r)$ is a uniformly random circle hitting D_R. For any (fixed) point $x \in D_R$,

$$P(C(X,r) \text{ contains } x) = P(X \text{ falls in } C(x,r)) = \frac{\pi r^2}{\pi (R+r)^2}$$

by (1), which does not depend on x. Furthermore, the mean or expected area of overlap between $C(X,r)$ and D_R is by Fubini's theorem

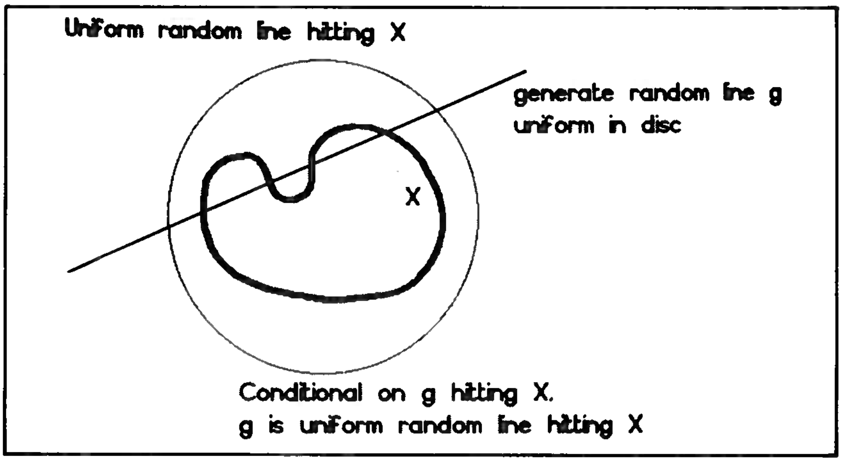
$$\mathbb{E}(\text{area } C(X,r) \cap D_R) = \int_{D_R} P(x \text{ lies in } C(X,r))dx$$
$$= \pi R^2 \frac{r^2}{(R+r)^2},$$

i.e. proportional to the product of areas of $C(X,r)$ and D_R.

Definition of a uniformly random line is less intuitive. Let parameters (p,θ) specify the line

$$\{(x,y): x\cos\theta + y\sin\theta = p\},$$

i.e. $|p|$ is the distance of the line from the origin, and θ determines its direction. A *uniformly random* (UR) line is such that (p,θ) is a uniformly distributed point in some bounded region of $\mathbb{R} \times [0,\pi)$. For example a UR line hitting the disc D_R is obtained when p and θ are independent random variables uniformly distributed over $[-r,+r]$ and $[0,\pi)$ respectively. In general for $X \subset \mathbb{R}^2$ the set of lines intersecting X is some irregular set of (p,θ) points in the allowable region. To generate a UR line hitting X, in practice, find a disc D_R circumscribing X. Generate a UR line L hitting D_R; if $L \cap X = \varnothing$, reject this attempt and generate another line L; until L hits X. Then L is UR hitting X.



Uniform random lines have the invariance property that if L is a UR line hitting X, and if $Y \subset X$, then the probability $P(L \text{ hits } Y)$ does not depend on the position or orientation of Y within X. All parts of X are equally likely to be 'sampled' by L. This fair sampling property, which characterizes the uniform distribution, can be recognised as invariance under the euclidean group of rigid motions. Another nice characterization of UR lines is based on the two-person game where A 'hides' a set Y inside X and player B draws a line L through X to find Y. Optimal strategy for B is to generate a uniform random line.

We state two fundamental results concerning UR lines. Let L be a UR line through X, a bounded measurable plane set. If $A \subset X$ is measurable then

$$\mathbb{E} \text{ length } (L \cap A) = \frac{\pi . \text{area}(A)}{K} \tag{2}$$

where $\mathbb{E}$ again denotes expected (mean) value, and K is a constant depending on X. If $C \subset X$ is a plane curve then

$$\mathbb{E} n(L \cap C) = \frac{2.\text{length}(C)}{K} \tag{3}$$

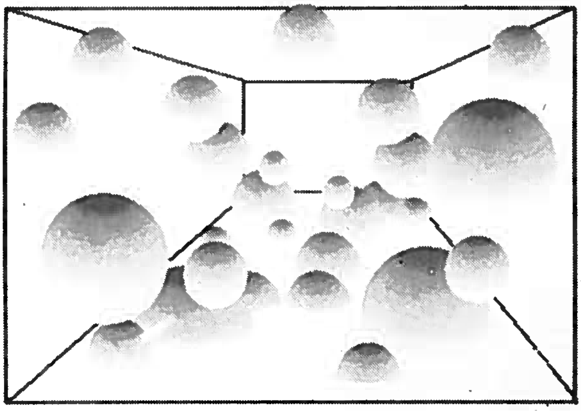
where $n(L \cap C)$ is the number of intersection points between L and C. Thus, the mean amount of overlap between a UR line and a fixed figure is given by (2), (3) *regardless* of the geometrical configuration of the figure. This generality is the basis of the classical theory. Corresponding formulae hold in higher dimensions and noneuclidean spaces [30].

Apart from the obvious application of (2)-(3) to stochastic image models, we can interpret them to give methods for measurement of length and area. If an image consists of several curves, their total length can be statistically estimated by randomly rotating the image, superimposing a grid of parallel lines and counting the number of crossing points.

Statements about image complexity also follow from (2)-(3). Suppose the image consists of curves of total length l, the screen is divided into an $n \times n$ square grid, and we wish to estimate the number of grid squares which contain part of the image. Assuming the image and grid are randomly superimposed, the mean number of grid-image intersections is $4(n-1)l$. For large n this approximates the mean number of *squares* crossed, i.e. the mean complexity.

Stochastic image models may also be based on (b) *stochastic processes.* To generate a random pattern extending over the entire plane, divide $\mathbb{R}^2$ into squares, and place a random number of random points in each square. A random pattern of lines is a random pattern of (p, θ) points in $\mathbb{R} \times [0, \pi)$, and so on. Thus we define a *random point process* in space S as a random locally finite set of points in S, where 'locally finite' means each bounded region of S only contains a finite (random) number of (random) points. *A random line process* 'is' a random point process in $\mathbb{R} \times [0, \pi)$, or more intrinsically, is a random locally finite set of lines in $\mathbb{R}^2$. In calculations one uses the correspondence between a random point process and the system of random variables

$N(A)$ = (number of points in A), $A \subset S$, which constitute a *random measure* $N(\cdot)$ on S. A random line process is a random measure on $\mathbb{R} \times [0,\pi)$, or intrinsically, corresponds to a random capacity function $H(A)$ = (number of lines *intersecting* A), $A \subset \mathbb{R}^2$. See [18,12].



A finite population of spheres in $\mathbb{R}^3$
or
a realisation of a random sphere process

Explicit calculations are usually unsuccessful except for *uniform Poisson processes,* in which each bounded part of the process consist of independent uniformly random points/lines, and $N(A)$, $N(B)$ are independent when $A \cap B = \emptyset$. Equations (1)-(3) yield the expected values of $N(A)$, $H(A)$, the number of crossings of a fixed curve, the total length of lines overlapping A, and the number of line-line crossings inside A.

General random point processes and line processes have been studied using moments [12,19,32] and Palm probabilities [26]. For a point process the first two moment measures are the intensity measure $\mu(A) = \mathbb{E}[N(A)]$ on $\mathbb{R}^2$, and the second moment measure $\mu^{(2)}$ on $\mathbb{R}^2 \times \mathbb{R}^2$ defined by $\mu^{(2)}(A \times B) = \mathbb{E}[N(A)N(B)]$, which together contain all variance-covariance information. If the process is statistically stationary, then $\mu(A) = \lambda\,\text{area}(A)$ where $\lambda > 0$ is the intensity, while $\mu^{(2)}$ 'disintegrates',

$$d\mu^{(2)}(x,y) = d\gamma(y-x)d\mu(x) \quad x,y \in \mathbb{R}^2$$

and the measure γ on $\mathbb{R}^2$ describes correlations between points in the process. The correlation characteristics can be estimated from observations of the process, furnishing a general empirical approach to point- and line- processes [33]. Second-order statistics characterize many of the visible characteristics of an image or pattern [11], but are not infallible [28,5]. A direct analysis of dependence between points or lines in a process is obtained using the Palm probabilities P^x, essentially the conditional probability distribution of the random process *given* that there is a random point at x.

A random line process or circle process subdivides the plane into a random tessellation. This is a potentially important model of random images [14, 24, 31]. Characteristics of the polygons formed by a *Poisson* line process have been determined by Miles [23], in particular the means and variances of polygon area, perimeter length and number of sides. Another important random tessellation is the Dirichlet or Voronoi tessellation: if $\{x_i, i \in Z\}$ are the points in a point process, let the tile corresponding to x_i be

$$T_i = \{y \in \mathbb{R}^2 : |y - x_i| \leq |y - x_j|, j \neq i\}.$$

The T_i are polygons tessellating $\mathbb{R}^2$. Characteristics of the Voronoi tessellation induced by a Poisson point process are given by Miles [21].

Finally, random image models can be based on (c) *the theory of random sets.* This is discussed in Section 4.

3. Combinatorial theory

More results have recently been obtained for classical problems, by simplifying geometry and applying combinatorial probability methods [1]. We will first prove the curve length formula (3),

$$\mathbb{E}n(L \cap C) = \frac{2 \text{ length } (C)}{K}$$

where C is a plane curve, L is a UR line hitting $X \supset C$, and $n(L \cap C)$ = number of intersection points in $L \cap C$. Suppose C is a *polygonal* curve consisting of line segments $S_1, S_2, \ldots, S_n$. Let $[S_i]$ denote the event $L \cap S_i \neq \emptyset$, that is L hits S_i. Put

$$1_{[S_i]} = \begin{cases} 1 & \text{if } L \cap S_i \neq \emptyset \\ 0 & \text{if } L \cap S_i = \emptyset . \end{cases}$$

Clearly we have

$$n(L \cap C) = \sum_{i=1}^{n} 1_{[S_i]}$$

with probability 1, since $P(L$ contains $S_i) = 0$. But immediately

$$\mathbb{E}n(L \cap C) = \sum_{i=1}^{n} \mathbb{E}\, 1_{[S_i]} = \sum_{i=1}^{n} P([S_i]).$$

It can easily be argued that uniform random lines have $P([S_i])$ proportional to length (S_i);

$$\mathbb{E}n(L \cap C) = \alpha \sum_{i=1}^{n} \text{length } (S_i) = \alpha . \text{ length } (C)$$

which proves (3) up to the constant factor.

The proof reveals importance of *additivity*, meaning both the linearity of the integral $\mathbb{E}$ and the additivity of the counting function $n(L \cap C)$. Together with the natural properties of uniform distributions, this property forms the basis of stochastic geometry.

Suppose now we want the *distribution* of the variable $n(L \cap C)$: computation of $P\{n(L \cap C) = k\}$ is not obvious. Consider two segments S_1, S_2 and evaluate $P([S_1] \cap [S_2])$, the probability that L intersects *both* S_1, S_2. Case 1: if S_1, S_2 have a common point, let T be the third side of the triangle. Then

$$1_{[S_1] \cap [S_2]} = \tfrac{1}{2}(1_{[S_1]} + 1_{[S_2]} - 1_{[T]}) \qquad \text{a.s.}$$

since if L intersects both S_1, S_2 the sum in brackets equals 2, and otherwise is zero. Case 2: if S_1, S_2 have no common point we can derive a similar expression

$$1_{[S_1] \cap [S_2]} = \tfrac{1}{2}(1_{[A_1]} + 1_{[A_2]} - 1_{[B_1]} - 1_{[B_2]}) \qquad \text{a.s.}$$

where A_1, A_2, B_1, B_2 are segments joining the four endpoints of S_1, S_2. But this implies that *every expression* $1_{[S_i] \cap [S_j]} = 1_{[S_i]} \cdot 1_{[S_j]}$ *can be written as a linear combination* of variables $1_{[T_k]}$, where T_k are line segments joining vertices of C.

Theorem. *Let $x_1, \ldots, x_n$ be points in $\mathbb{R}^2$, and s_{ij} the line segment joining x_i with x_j. For a random line L, let $[s_{ij}]$ be the event $L \cap s_{ij} \neq \emptyset$. Let $\mathcal{Q}$ be the ring of events generated (through unions, intersections, set differences) by $[s_{ij}]$, $1 \leq i < j \leq n$. Then for any $A \in \mathcal{Q}$ there exist constants $c_{ij}(A)$ such that*

$$1_A = \sum_{i<j} c_{ij}(A) 1_{[s_{ij}]} \tag{4}$$

holds except when L contains a vertex x_i.

If L is uniformly distributed we take mean values in (4) to get

$$P(A) = 2/K \sum_{i<j} c_{ij}(A) \| x_i - x_j \| \tag{5}$$

i.e. all combinatorial probabilities for UR lines are expressible as sums of lengths of segments s_{ij}. For example, the distribution of $n(L \cap C)$ is expressible in terms of the distances between each pair of vertices of C. This is a great advance, in principle, on the classical theory which was restricted to mean values. An algorithm for the $c_{ij}(A)$ is known, and practicable for small n.

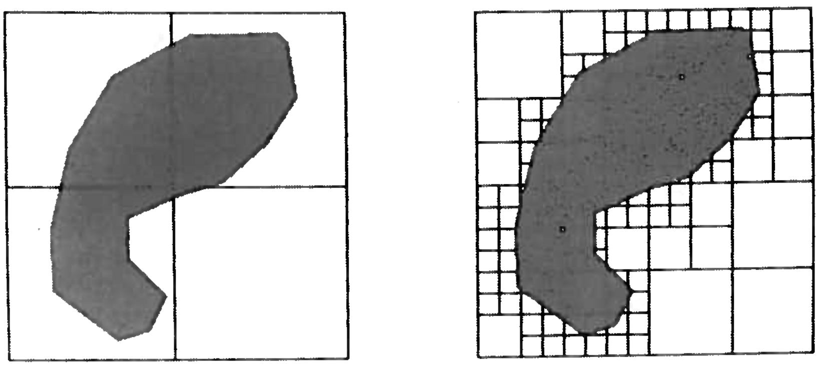
One can also take non-uniform random lines in (4), say with probability distribution Q, to obtain

$$Q(A) = \sum_{i<j} c_{ij}(A) Q[s_{ij}] \tag{6}$$

and note the coefficients $c_{ij}(A)$ are the same as above. The quantity $Q[s_{ij}]$ serves as a generalized length of s_{ij}. Thus, again in principle, nonuniform random lines are no more computationally difficult than UR lines.

Finally we present another application to image complexity, concerning the quad-tree representation of images. An image can be recorded or transmitted as tree structure, as follows. Divide the image field into four equal squares and note which squares, if any, consist of a single colour. The remaining, multicoloured squares are subdivided again into four, and the process repeats until a predetermined level of subdivision is reached. The record of subdivisions and

Quad-tree representation of images



colours forms the *quad tree.* Important questions include the average complexity (number of nodes) of the quad tree, and estimating the increase in complexity if a deeper level (finer subdivision) is added. Both problems depend on the image, but it is reasonable to suppose that in a sufficiently small square, the image boundary can be regarded as a uniformly random line. Consider a UR line hitting a square subdivided into $k \times k$ equal squares. According to (3) the mean number of subsquares crossed equals k. Furthermore using (5) we can compute the distribution of the number N of subsquares crossed. In the interesting case $k = 2$, we have $P(N = 1) = \frac{1}{2}(\sqrt{2}-1)$, $P(N = 2) = 2-\sqrt{2}$, $P(N = 3) = \frac{1}{2}(\sqrt{2}-1)$. Thus the cost of adding one extra level of subdivision is to double the number of terminal nodes, on average. One fifth of the new branches will be triple.

4. Random set theory

In addition to the constructive examples of random geometry in Section 2, one can propose others such as the zero-set (or contours) of a random function. Foundations of a general theory of random sets were laid by G. Matheron [18] and D.G. Kendall [13]. Matheron's theory of random closed sets was expressly developed as a mathematical background to image analysis as well as stochastic geometry. Kendall's theory takes an abstract view of the construction of probability spaces of random sets, emphasising the variety of structures which can be chosen. The two approaches are complementary [27] and both make use of Choquet's capacity theorem.

To introduce the theory we generalize the random events $[S_i]$ which played a formative role in section 3. For the Matheron approach, let $\mathfrak{F}$ be the class of all closed sets in $\mathbb{R}^n$. If $T \subset \mathbb{R}^n$ define the *hitting set*

$$[T] = \{F \in \mathfrak{F}: F \cap T \neq \varnothing\}.$$

Endow $\mathfrak{F}$ with the (weakest) topology such that $[U]$ is an open subset of $\mathfrak{F}$ for all open sets $U \subset \mathbb{R}^n$, and $[K]$ is closed for all compact $K \cap \mathbb{R}^n$ (see [20]). Then $\mathfrak{F}$ becomes a Polish space. Define a *random closed set* as a random element of $\mathfrak{F}$ with the Borel σ-algebra. Under this structure the *events* $[T]$, $T \subset \mathbb{R}^n$ are measurable when T is open, closed or indeed Borel. Intersections and unions of random closed sets are random closed sets. Area, length (where defined) and number of points (where finite) are random variables.

Kendall's approach emphasises that the definition of a random set depends on the geometrical information which is assumed to be observable. Its basic constituents are the random events $[T] = \{X \cap T \neq \varnothing\}$ where X is the random set and T is a fixed set called a 'trap'. The associated random variable

$$h(T) = \begin{cases} 1 & \text{if} \quad X \cap T \neq \varnothing \\ 0 & \text{if} \quad \text{not} \end{cases} \tag{7}$$

corresponds to a 'bit' or 'flag' indicating whether X was detected by the trap T. From the observer's point of view, the random set X is characterized by the information $\{h(T), T \in \mathfrak{T}\}$ where $\mathfrak{T}$ is the class of all traps available to the observer. Define a *trapping system* $\mathfrak{T}$ on a space S to be a class of nonempty subsets of S, which cover S, satisfying certain properties analogous to separability and local compactness. A *random* $\mathfrak{T}$*-set* in S is a random function

$$h: \mathfrak{T} \to \{0,1\}$$

i.e. a stochastic process of 0-1 variables $h(T)$, $T \in \mathfrak{T}$, subject to a consistency condition which enables h to be interpreted in the form (7). Note the probability structure depends completely on the choice of trapping-system. If $S = \mathbb{R}^n$ and $\mathfrak{T}$= open sets, a random $\mathfrak{T}$-set is a random closed set in Matheron's sense. Smaller trapping-systems may be inadequate to distinguish all closed sets. A set X is indistinguishable (to the observer) from its $\mathfrak{T}$*-closure*,

$$\text{clos}\,(X, \mathfrak{T}) = \left[\bigcup_{X \cap T = \varnothing} T\right]^c = \bigcap_{X \cap T = \varnothing} T^c$$

(c denotes complement) and we need only consider $\mathfrak{T}$-*closed* sets $X = \mathrm{clos}(X,\mathfrak{T})$. For example if $\mathfrak{T} = \{$open halfplanes of $\mathbb{R}^2\}$ the $\mathfrak{T}$-closed sets are the convex sets of $\mathbb{R}^2$. Thus random $\mathfrak{T}$-sets in this case 'are' random convex sets; and the customary representation of convex sets by support functions can be derived from $h(T)$.

Random set theory provides solid foundations for investigating both stochastic geometry and the observation and processing of images. For example, convergence of random sets is a natural concept in the general theory which has been applied to assess errors committed in digitizing an image, approximation of one image by another, and the stability of image processing transformations [31, Chapter VII] and to derive the statistically important laws of large numbers and a central limit theorem for repeated observations of images [2,36]. The general setting also permits more involved discussion of the probabilistic properties of image models, such as infinite divisibility and the semi-Markov property [18,19]. It is a basic result that the probability distribution of a random set X is determined by its *avoidance function*

$$Q(A) = \mathrm{Prob}\,(X \cap A = \varnothing), \quad A = \bigcup_{i=1}^{n} T_i, \quad T_i \in \mathfrak{T}$$

and the introduction of Q makes for a coherent approach to image models [31,18].

The strongest link between image analysis and random set theory is surely the trapping system. Any image is given to us through an array of detectors (and perhaps subjected to edge detection processing, etc.) which can be formalised as a trapping system. Further, the relationships between various forms of image information (e.g. digitized versions on different lattices; grey tones) can be studied by varying $\mathfrak{T}$ in the stochastic model. The author feels that the great potential of this method is yet unexplored.

5. Mathematical morphology

The work of J. Serra [31] establishes a coherent methodology for image analysis which avoids the fragmentary character of most other approaches. Mathematical morphology developed in parallel with random set theory, beginning with Matheron's [17] geostatistical work and Serra's invention of the 'texture analyzer' image processing devices. The result is a combination of sound theoretical criteria with practical experience. We can only convey the flavour of the subject here.

Transformations of sets arise in many stochastic geometry problems. Consider the probability distribution of the random distance $d(x, A)$ from a fixed set $A \subset \mathbb{R}^2$ to a random point $x \notin A$. Clearly $P\{d(x, A) \leq r\}$ equals the probability that X falls in the region $A_{(r)} = \{x \in \mathbb{R}^2 : d(x, A) \leq r\}$ which we dub the *r-envelope* of A. Equivalently $A_{(r)}$ is the set formed by placing a disc of radius r around every point $a \in A$. The envelope transformation $A \rightarrow A_{(r)}$ is the simplest example of a set transformation. If $A = D_R$ is a disc then $A_{(r)} = D_{R+r}$, while in general the shape of $A_{(r)}$ is more rounded (with smaller holes) than that of A. It is argued that the function $f_A(r) = \text{area}(A_{(r)})$ reflects essential characteristics of the geometry of A. If A is convex then $f_A(r) = \pi r^2 + r.\text{length}(\partial A) + \text{area}(A)$, while if A is a finite set of points then f_A is piecewise quadratic with a behaviour reflecting the sizes of gaps between the points. A series of images $A_1, \ldots, A_n$ could be differentiated or discriminated using the derived functions $f_{A_1}(r), \ldots, f_{A_n}(r)$.

The envelope operation can be performed on a discrete grid of points. A simple algorithm is to scan the entire grid and, for each point x whose digital neighbourhood includes a point of the current image A, we mark x for inclusion in the new image $A_{(r)}$. Furthermore we can watch this process of expansion for increasing r by repeating the algorithm, since $(A_{(r)})_{(s)} = A_{(r+s)}$. This is done by texture analyzers.

The *Minkowski sum* of two sets $A, B \subset \mathbb{R}^2$ is defined as

$$A \oplus B = \{a + b : a \in A, b \in B\}$$

in the sense of vector addition. If B is the disc D_r then $A \oplus D_r = A_{(r)}$, the r-envelope. More generally $A \oplus B$ is the superposition of translated copies of B centred on each of the points of A, if we take the origin 0 as the 'centre' of B. Shifted copies of A are obtained when B is a single point, $A \oplus \{b\} = \{a + b : a \in A\}$. Defining $\check{B} = \{-b : b \in B\}$ one can interpret $A \oplus B = \{x \in \mathbb{R}^2 : (x \oplus \check{B}) \cap A = \varnothing\}$, the set of all 'centres' of shifted copies of $\check{B}$which intersect A. Hence the transformation $A \rightarrow A \oplus B$ also has a clear interpretation in stochastic geometry, and can be claimed to reflect important characteristics of the geometry of A. This and other set transformations can be implemented on a discrete grid by including or removing points x according to the state of the entire digital neighbourhood of x.

Minkowski subtraction of $A, B \subset \mathbb{R}^2$ is defined by

$$A \ominus B = (A^c \oplus B)^c$$

i.e. the complement A^c is enlarged by B. For example, if $B = D_r$ is a disc, $A \ominus D_r = \{x \in A : d(x, A^c) \geqslant r\}$ is the *inner parallel set.* In general $A \ominus B = \{x \in A : x \oplus \check{B} \subset A\}$ is the set of all centres of copies of $\check{B}$ *contained* in A. This has a natural interpretation and the function $g(r) = \text{area}(A \ominus D_r)$ is claimed to contain essential information about the geometry of A. Define two further set transformations, the closure

$$A^B = (A \oplus \check{B}) \ominus B$$

and opening

$$A_B = (A \ominus \check{B}) \oplus B.$$

Thus A_B is the union of all copies of B contained in A; and A^B is the result of a similar operation on A^c. A set is B-closed, $A^B = A$, iff it is $\mathfrak{T}$-closed in the sense of Section 4 where $\mathfrak{T}$ is the class of all translated copies of B. Apart from their natural interpretation in stochastic geometry, A^B and A_B can be used to develop a rigorous definition of size and size distribution for images [18,31].

The mathematical morphology approach to an image processing problem is to select an image transformation (built from $\oplus, \ominus$, A^B, A_B etc.) suitable to the application, and make numerical analyses of the transformed images. One chooses transformations either by experience, intuition about the scientific problem, or by setting down criteria which the transformation must satisfy.

Some limitations of mathematical morphology as it currently stands call for brief comments. The texture analyser is designed on a hexagonal point lattice for the digitized image. Naturally the theory is strongly dependent on this choice of instrumentation, and probably does not answer all questions about random image models that are required in different applications. Associated with the choice of instrumentation is the adoption [31, pp 8-15] of a list of theoretical principles which notably excludes *rotational* stability. A hexagonal grid has only three basic directions and there have been difficulties with the analysis of image orientation or directionality. There may also be practical reasons for employing a rectangular grid or other system of image detection - for example, satellite data may already be in this form. Another problem with all image analysis based on stochastic geometry is that images are not sharply divided black and white sets, but grey tone functions. This is a drawback to the widespread use of texture analyzers. Mathematical morphology for grey-tone functions is under development [31, Chapter XII].

The author suspects one can be led astray by excessive analysis of a single image, when this image is to be representative of a larger population. This applies particularly in stereology, where the planar image is a random plane section $X \cap E$ of a three-dimensional body X which is the real object of interest. It is then important that the sampling procedure used to generate $X \cap E$ should be known, and appropriate. Statistical inferences depend on the sampling method used. It is not quite sufficient to base image analysis on considerations of the trapping-system and other geometrical structures, without incorporating statistical models for the origins of data.

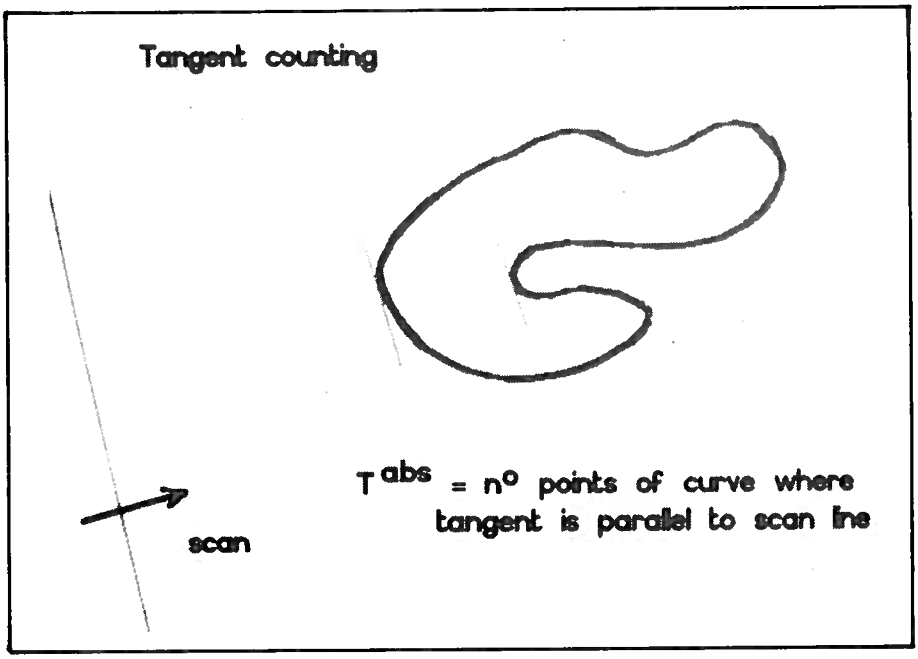
6. Image irregularity, observation errors and geometric measure theory

Elementary formulae from stochastic geometry (see (1)-(3) in Section 2) are widely used in stereology for measuring curve lengths, estimating surface areas and so on. Yet these results were derived for ideal smooth curves and it is a priori doubtful whether they apply to irregular images or images observed under error.

An extreme example is *tangent counting*. Let C be a twice differentaible plane curve, $\theta \in [0,\pi)$ and $T^{abs}(\theta)$ = number of tangents to C parallel to direction θ. This would be found by scanning a straight line across the image (parallel to θ) and counting the positions where the image is tangent to C. We have

$$\int_0^\pi T^{abs}(\theta)d\theta = \int_C |\kappa(s)|ds \qquad (8)$$

where $\kappa(s)$ is the curvature of C at point s. If the scan direction θ is generated at random (uniformly), $\pi T^{abs}(\theta)$ is a statistically unbiased estimator of the total absolute curvature of C. Additionally if C is itself a random plane section of a curved surface, then T^{abs} yields an estimate of the total 'absolute' surface curvature.



Even assuming that real images are differentiable, the tangent count is unstable in the sense that small perturbations (kinks, ripples) in C may cause large changes in T^{abs} and κ. More realistically if C is the boundary of a finite union of convex compact sets (hence, almost everywhere differentiable) T^{abs} does not share the properties usually required of a good statistic. Serra [31, p.141 ff] nevertheless shows that a precise and useful interpretation can be given to the tangent count or 'convexity number' of such curves, and that this can be approximately determined from a digitized image.

Practical stereologists and image analysts follow procedures for counting 'tangents' to image curves, even when these are irregular, thick or fuzzy, broken or digitized. A tangent counting algorithm may be built into the image analyzing device. Mathematicians should be discussing the performance of such algorithms, their relation to real geometry, and the effects of observation errors.

Standard proofs of (1)-(3) and (8) do not accommodate a discussion of perturbations or errors, being applications of Fubini's theorem to simple geometrical models. We need the more powerful methods of geometric measure theory [6], principally the *coarea formula*. Briefly, let M, N be m- and n-dimensional domains (rectifiable surfaces), $m \geqslant n$, and let $p: M \to N$ be a Lipschitz-continuous map. For almost every $x \in N$, $p^{-1}\{x\} = \{z \in M : p(z) = x\}$ is an $m-n$ dimensional rectifiable set. If $m = n$, then $p^{-1}\{x\}$ is a finite set. There is a function $J^n p$ defined on M called the approximate Jacobian of p, such that the *coarea formula*

$$\int_M f(z)(J^n p)(z) d\mathcal{H}^m z = \int_N \int_{p^{-1}\{x\}} f(z) d\mathcal{H}^{m-n} z \, d\mathcal{H}^n x \tag{9}$$

holds for any $\mathcal{H}^m$-integrable function $f: M \to \mathbb{R}$, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure ('k-dimensional volume integration', see Section 7).

Thus (9) is a kind of generalization of Fubini's theorem which incorporates the Jacobian for a change of variables.

To prove (8), for example, let C be a twice differentiable curve, and introduce

$$C^* = \{(s,l): s \in C, \ l \text{ is the tangent to } C \text{ at } s\}.$$

This is a one-parameter set of points in $\mathbb{R}^2 \times \mathbb{R} \times [0,\pi)$. Apply the coarea formula (9) to the map

$$p: C^* \to C, \quad p(s,l) = s.$$

This has $(J^1 p)(s,l) = (1+\kappa^2)^{-\frac{1}{2}}$ where $\kappa = \kappa(s)$ is the curvature of C, and since $p^{-1}\{s\}$ is a single point (s,l) we get

$$\int_{C^*} f(s,l)(1+\kappa^2)^{-\frac{1}{2}} d\mathcal{H}^1(s,l) = \int_C f(s,l) ds$$

for any function f. Similarly, for the map

$$q: C^* \to [0,\pi), \quad q(s,l) = \text{direction of line } l,$$

we have $(J^1 q)(s,l) = (\kappa^2/1+\kappa^2)^{\frac{1}{2}}$. Since $q^{-1}\{\theta\}$ consists of all pairs (s,l) where l is parallel to θ, we get

$$\int_{C^*} \tilde{f}(s,l)(\kappa^2/1+\kappa^2)^{\frac{1}{2}} d\mathcal{H}^1(s,l) = \int_0^\pi \sum_{q^{-1}\{\theta\}} \tilde{f}(s_i,l_i) d\theta.$$

If $\tilde{f} = 1$, the sum on the right hand side above is $T^{abs}(\theta)$. Choosing $f(s,l) = |\kappa|$ so that the two left hand sides agree, we get equation (8).

Now suppose that C is nondifferentiable, and that the experimenter has some algorithm for counting or detecting apparent tangents to C. Let

$$\tilde{C} = \{(s,l): s \in \mathbb{R}^2, l \text{ is a line; the algorithm counts } l \text{ as a tangent to at } s\}.$$

Then under suitable conditions we may replace C^* above by $\tilde{C}$ and perform the same calculations to get

$$\int_0^\pi \tilde{T}^{abs}(\theta)d\theta = \int_\Gamma \tilde{\kappa}(s)ds,$$

where $\tilde{T}^{abs}$ is the experimentally observed tangent count, $\Gamma = p(\tilde{C})$ is the set of points at which tangents are detected, and $\tilde{\kappa} = J^1q/J^1p$ is a kind of generalized curvature. For example, let C be an irregular curve $C(t) = A(t)+\epsilon(t)$, $0 \leq t \leq 1$, where curve A is smooth and $\|\epsilon(t)\| < r$. If the tangent algorithm is such that $s \in A$ and l is tangent to $A \oplus D_r$, then $\Gamma = A$, and $\tilde{\kappa}$ is a function of r and the curvature of A. Thus Γ is a rectified version of C. Secondly, if C is smooth, but a tangent where $\kappa(s)$ is small may not be observed, we get

$$\mathbb{E}(\pi \tilde{T}^{abs}) = \int_C |\kappa(s)| u(\kappa(s))ds$$

where $u(\kappa)$ = probability of detecting a given tangent at curvature κ. Further examples are explored in [4].

Thus we still have a geometrical interpretation of the image analysis algorithm when it is applied to non-ideal images. This is achieved by concentrating on intrinsic behaviour of the algorithm or observation method, encapsulated in the projection maps p,q. More generally we can regard an image analysis algorithm as an *operator* on images in the sense of generalized functions, and the mathematical prerequisites for such an approach already exist [6].

7. Fractals

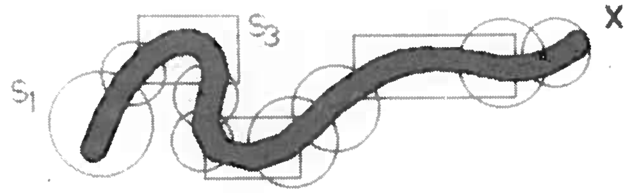
Mandelbrot [15,16] explored the concept of fractal (fractional dimensional) sets initiated by Besicovitch, which have wide mathematical associations and seem to be useful models for real images. The simplest kind of fractal set is *self-similar:* if X can be divided into k disjoint sets each of which is congruent to X after magnification by a factor α, then $\Delta = \log\alpha / \log k$ is the similarity dimension of X. For curves $\Delta = 1$; for a disc $\Delta = 2$; but for the Cantor set, $k = 2$, $\alpha = 3$, $\Delta = \log 3/\log 2$ is fractional. When X is magnified, its content increases by a fractional power of the magnification. This extreme form of fractal behaviour is not generally required (except in the limit of small scale). Define for each real $t \geq 0$ the t-dimensional Hausdorff measure $\mathcal{H}^t$,

$$\mathcal{H}^t(X) = \lim_{\epsilon \downarrow 0} c_t.\inf\{\sum_{i=1}^N (\text{diam } S_i)^t : S_1, \ldots, S_N \text{ cover } X, \text{diam } S_i < \epsilon\}$$

where the infimum ranges over (say) all families of compact sets S_i with diameters less than ϵ. The limit may be infinite. Define the Hausdorff-Besicovitch dimension of X as

$$D(X) = \sup\{t \geq 0 : \mathcal{H}^t(X) < \infty\} = \inf\{t \geq 0 : \mathcal{H}^t(X) = \infty\}.$$

Hausdorff measure H^t



$$H^t(X) = c_t \lim_{\delta \downarrow 0} \inf \left\{ \sum (\text{diam } S_i)^t : S_1, \ldots, S_n \text{ cover } X, \ \text{diam } S_i < \delta \right\}$$

Then X is fractal if $D(X)$ is not an integer. If X is a random closed set (Section 4), the topology of $\mathcal{F}$ is such that $D(X)$ is a random variable.

Other examples of fractals include the graphs and zero sets of random continuous functions (the graph of Brownian motion is *statistically* self similar) and limit sets of iterations of quadratic maps in the complex plane. The viewer's impression of a fractal curve is one of sharp irregularity and unbounded oscillation.

Real objects and images do often behave non-linearly with magnification. Coastlines are the best-known example. Given a picture of a fractal curve ($1 < D < 2$) we could estimate D as the slope of the regression line relating $\log L(\alpha)$ to $\log\alpha$, where $L(\alpha)$ is an estimate of length obtained at magnification α. Applied to coastlines this has produced a range of fractional dimensions, which seem to reflect degrees of irregularity. A more serious application concerns the measurement of lung membrane surface area [34, p. 156] from plane section curves. Conflicting estimates based on different magnifications have been reconciled and a consistent estimate of D obtained.

Many real phenomena and images have been described as 'fractal' and their empirical values of D determined. The theory of ideal fractals has not kept pace with this development of approximate fractals. Any empirical value of D is a partial description of the image, at certain scales only, and over different scales the 'dimension' may vary. This should not be an objection to the use of fractals as a geometrical model (naturally any model is confined to a chosen scale), but the meaning of a fractal approximation needs to be clarified [10]. We have already observed that Hausdorff dimension fits into the general theory of random closed sets, and indeed $D(X)$ represents an asymptotic index of the frequency of intersections between X and small *traps* $D_r, r \to 0$. It seems to the author that fractional dimensional irregularity could be better understood from the empirical and statistical viewpoint of stochastic geometry.

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