Recent Progress on the Numerical Verification of the Riemann Hypothesis

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It has now been shown that the first 400,000,000 non-trivial zeros of Riemann's zeta function are all simple and lie on the so-called critical line $\sigma=\frac{1}{2}$. This extends previous results described in [1], [2] and [6].

Riemann's zeta function is the meromorphic function $\zeta:\mathbb{C}\setminus\{1\}\rightarrow\mathbb{C}$, which, for $\mathrm{Re}(s)>1$, may be represented explicitly by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad (s = \sigma+it).$$

It is well known (cf. [3], [9]) that

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is an entire function of order 1, satisfying the functional equation

$$\xi(s) = \xi(1-s)$$

so that

$$\Xi(z) := \xi(\tfrac{1}{2}+iz), \quad (z \in \mathbb{C})$$

being an even entire function of order 1, has an infinity of zeros. The Riemann Hypothesis (cf. [3], [7]) is the statement that all zeros of $\Xi(z)$ are real, or, equivalently, that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. Since $\zeta(\bar{s}) = \overline{\zeta(s)}$, we may restrict ourselves to the halfplane $t>0$. To this day, Riemann's Hypothesis has neither been proved nor disproved. Numerical investigations related to this unsolved problem were initiated by Riemann himself (cf. [3]) and later on continued more systematically by the writers listed below (including their progress).

Investigator	Year	The first n complex zeros of $\zeta(s)$ are simple and lie on $\sigma = \frac{1}{2}$
GRAM	1903	$n = 15$
BACKLUND	1914	$n = 79$
HUTCHINSON	1925	$n = 138$
TITCHMARSH	1935/6	$n = 1{,}041$

Those listed above utilized the Euler-Maclaurin summation formula and performed their computations by hand or desk calculator whereas those listed below applied the so-called Riemann-Siegel formula (cf. [3]) in conjunction with electronic computing devices.

LEHMER	1956	n = 25,000
MELLER	1958	n = 35,337
LEHMAN	1966	n = 250,000
ROSSER, YOHE & SCHOENFELD	1968	n = 3,500,000
BRENT	1979	n = 81,000,001
BRENT, van de LUNE, te RIELE & WINTER	1982	n = 200,000,001
van de LUNE & te RIELE	1983	n = 300,000,001

An excellent explanatory account of most of the essentials of these computations may be found in [3].

In practice, the numerical verification of the Riemann Hypothesis in a given range consists of *separating* the zeros of the well-known real function $Z(t)$ (see [3]), or, equivalently, of finding sufficiently many sign changes of $Z(t)$. Our program (aiming at a fast separation of these zeros) is based, essentially, on the modification of Lehmer's [4] method introduced by Rosser et al. [8]. However, we have developed a more efficient strategy of searching for sign changes of $Z(t)$. Brent's average number of Z-evaluations, needed to separate a zero from its predecessor, amounts to about 1.41 (cf. [1]) whereas we have brought this figure down to about 1.19 (cf. [5]). This average number of Z-evaluations could not have been reduced below 1.135 by any program evaluating $Z(t)$ at all Gram points. A complete listing of our FORTRAN/COMPASS program is given in [5]. We note that 98 percent of the running time was spent on Z-evaluations. The program was executed on a CDC CYBER 175-750 and ran about 10 times as fast as the UNIVAC 1100/42 program of Brent. This is roughly what could be expected, given the relative speeds of the different machines. We intend to continue our computions in the near future on a still faster computer.

In order to give an impression of the erratic behaviour of $Z(t)$ we present its graph near the Gram block $B_{61,331,766}$. For more graphs and details see [6].

References

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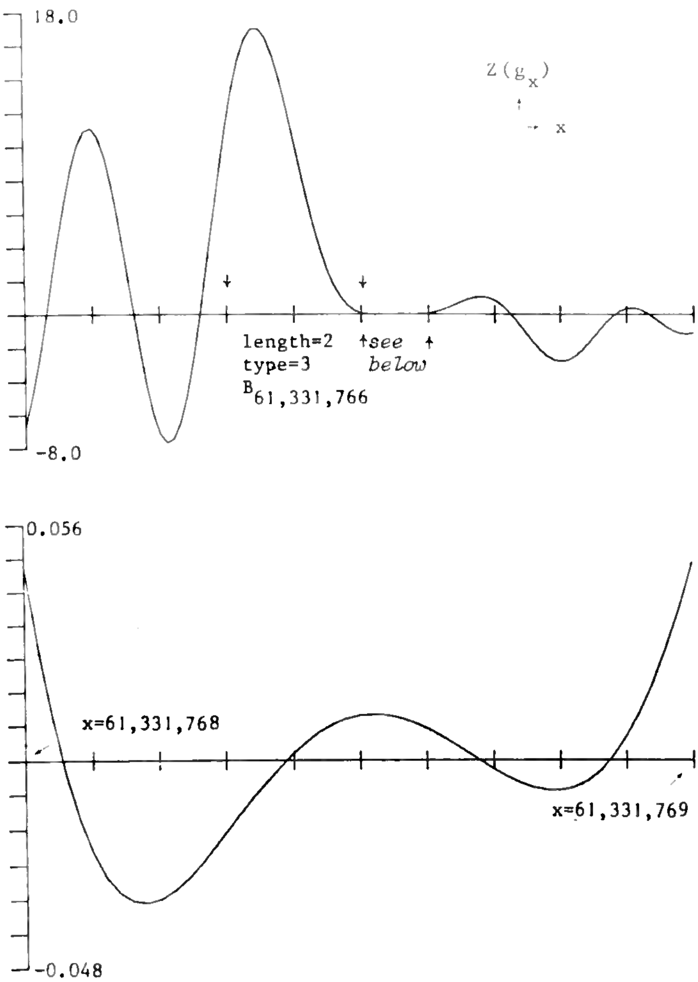

The behaviour of $Z(t)$ near the Gram block $B_{61,331,766}$.