Two Applications of Topological Dynamics in Combinatorial Number Theory

The shift system and the Stone-Čech compactification of the non-negative integers

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In Combinatorial Number Theory, various results say that if $\mathbb{Z}^+$ is partitioned into finitely many sets, then one of those sets is large in some sense. Theorems of this type were obtained by Hilbert, Schur, Van der Waerden, Rado and Hindman among others (see [9] and its references). For example, in Van der Waerden's result, 'large' means: containing arithmetic progressions of arbitrary finite length. In this paper, proofs of Van der Waerden's Theorem (2.6 below) and Hindman's Theorem (4.1 below) will be given using methods from Topological Dynamics.

This paper will also illustrate certain techniques from 'abstract' Topological Dynamics, i.e. Topological Dynamics in the tradition of, say, Gottschalk and Hedlund and of Robert Ellis (see [10], [7]). Roughly, Topological Dynamics can be described as the discipline in which one studies asymptotic and recurrence properties of points in a topological space under the action of a group of homeomorphisms or a semigroup of continuous mappings. A historical sketch of this field of mathematical research falls outside the scope of this paper. Let me just say that Topological Dynamics, together with Ergodic Theory, originated from the qualitative theory of differential equations; see for instance [15] or the introductions of [3] or [18] for brief historical sketches. The results that will be presented in this paper are from 'abstract' Topological Dynamics, i.e. there is no direct relationship with the theory of differential equations.

Finally, this paper may be seen as an attempt to persuade the reader to have a look in the beautiful book [8] (although there the main role is played by Ergodic Theory) or at least the paper [9]. Essentially, the present paper is based on [9]; in particular, the proof of Van der Waerden's Theorem is taken from [9]: I have included it just to show how elegantly it follows from an admittedly rather deep dynamical result. The proof of Hindman's theorem is different from the one in [9]; as far as I know, the present proof does not appear in the existing literature. I learned it from a discussion with Dr. B. Balcar (Prague).

The paper is essentially self contained (except for the proof of the Multiple Recurrence Theorem, i.e. Theorem 1.4 below). It can be read by anybody who knows some elementary topology and the basic properties of ultrafilters (summarized in Section 4).

1. Introduction

A *dynamical system* is a pair (X,T) with X a compact Hausdorff space and $T:X\to X$ a continuous mapping. A *homomorphism of dynamical systems* $\phi:(X,T)\to(X',T')$ is a continuous mapping $\phi:X\to X'$ such that $\phi\circ T = T'\circ\phi$. Most of the dynamical properties of a dynamical system are preserved by homomorphisms; the proof that this is so for the notions to be defined below (orbit, orbit-closure, recurrence, almost periodicity) will be left to the reader.

Let (X,T) be a dynamical system. The *orbit* of a point x in X is the set $O(x) := \{T^n x \vdots n \in \mathbf{Z}^+\}$; here T^n is the n'th iterate of T, so $T^n := T\circ T^{n-1}$ $(n = 2,3,...)$, while $T^1 := T$ and $T^0 = id_X$. The *orbit-closure* of x is the closure $\overline{O(x)}$ of the orbit $O(x)$. A point x in X is called *recurrent* whenever it is in the orbit-closure of x. Equivalently, x is recurrent iff for every nbd(= neighbourhood) U of x there exists $n \geqslant 1$ such that $T^n x \in U$. Clearly, if x is a recurrent point, then for each nbd U of x the set of 'return times'

$$R(x,U) := \{n \in \mathbf{Z}^+ \vdots n \geqslant 1 \ \& \ T^n x \in U\}$$

is infinite. It is easy to see that if the space X is metrizable, then a point x is recurrent iff there is an increasing sequence $n_1,n_2,...$ in $\mathbf{Z}^+$ such that $x = \lim_{i\to\infty} T^{n_i} x$.

Recurrent points do exist abundantly. The easiest way to prove this is by invoking the axiom of choice. In fact more will be shown, namely, that uniformly recurrent (terminology of e.g. [2] and [8]) or, as I will call them (terminology of e.g. [7], [10] and [17]), almost periodic points exist. A point x in X is called *almost periodic* whenever for each nbd U of x the set $R(x,U)$ has bounded gaps. This is equivalent to saying that for every nbd U of x there exists $l > 0$ such that

$$\forall n \in \mathbf{Z}^+ \ \exists r \in \{0,...,l\} \vdots \ T^{n+r} x \in U.$$

(It has been remarked that a periodic point returns to itself every hour exactly on the hour; but an almost periodic point returns to a nbd every hour within the hour.) There is a nice characterization of almost periodicity in terms of minimal subsets. A closed subset A of X is called *minimal* whenever $A \neq \varnothing$, A is invariant (that is, $TA \subseteq A$) and A has no closed invariant proper subsets. It is easy to show that a non-empty subset A of x is minimal iff $\overline{O(x)} = A$ for every $x \in A$. (Note that orbits and their closures are invariant.)

1.1. LEMMA. *Let (X,T) be a dynamical system. Every non-empty closed invariant subset of X contains a minimal subset.*

PROOF. Apply Zorn's lemma to the partially ordered (with respect to inclusion) family of all non-empty closed invariant subsets of X. This family is not empty (X belongs to it), and the intersection of a chain in it is non-empty because X is compact. □

Here is the promised characterization of almost periodicity; the result is a form of Birkhoff's Recurrence Theorem:

1.2. PROPOSITION. *Let (X,T) be a dynamical system and $x \in X$. The following statements are equivalent:*
(i) x is an almost periodic point;
(ii) $\overline{O(x)}$ is a minimal set.

PROOF. $(i) \Rightarrow (ii)$: Clearly, $\overline{O(x)}$ is non-empty, closed and invariant. It remains to prove that if $y \in \overline{O(x)}$ then $x \in \overline{O(y)}$. Let U be a nbd of x. Almost periodicity of x implies that there exists $l>0$ such that for every $i \in \mathbf{Z}$, $T^{i+r}x \in U$ for some $r \in \{0,...,l\}$; that is

$$T^i x \in \bigcup_{r=0}^{l} T^{-r}[U].$$

So $O(x) \subseteq \cup_{r=0}^{l} T^{-r}[U]$, hence $\overline{O(x)} \subseteq \cup_{r=0}^{l} T^{-r}[\overline{U}]$. Consequently, if $y \in \overline{O(x)}$, then $T^r y \in \overline{U}$ for some $r \in \{0,...,l\}$ and this means that $\overline{U} \cap O(y) \neq \varnothing$. As every nbd W of x includes the closure of some smaller nbd of x, this implies that $W \cap O(y) \neq \varnothing$ for every nbd W of x. Hence $x \in \overline{O(y)}$.

$(ii) \Rightarrow (i)$: Let U be a nbd of x. By minimality, one has for every $z \in \overline{O(x)}$ that $x \in \overline{O(z)}$, hence $U \cap O(z) \neq \varnothing$, i.e. $z \in T^{-r}[U]$ for some $r \in \mathbf{Z}^+$. This shows that $\overline{O(x)} \subseteq \cup_{r=0}^{\infty} T^{-r}[U]$, and compactness implies that $\overline{O(x)}$ can be covered by finitely many of the sets $T^{-r}[U]$, say with $r \in \{0,...,l\}$. In particular, for each $n \in \mathbf{Z}^+$ there exists $r \in \{0,...,l\}$ such that $T^n x \in T^{-r}[U]$, hence $T^{n+r} x \in U$. □

1.3. COROLLARY. *Let (X,T) be a dynamical system. Every non-empty closed invariant subset of X, in particular X itself, contains an almost periodic point.* □

Remark. Since homomorphisms of dynamical systems preserve almost periodicity (this is almost trivial), it follows from 1.2 that homomorphisms preserve minimal sets: if $\phi:(X,T) \rightarrow (X',T')$ is a homomorphism and M is a mimimal subset of X, then $\phi[M]$ is a minimal subset of X'. (Of course, this can also be proved rather easily directly!)

As every almost periodic point is recurrent, it follows from 1.3 that every dynamical system has recurrent points. Other proofs of this fact are possible, e.g. using the existence of an invariant measure (the Poincaré Recurrence Theorem). The following result is much stronger, although its validity is restricted to metrizable spaces (as long as one is not willing to use 'exceptional' axioms like Martin's Axiom). If X is a compact Hausdorff space and if for $j = 1,...,l$ $T_j:X \rightarrow X$ is a continuous mapping, then a point x in X is called *multiply recurrent* (under $T_1,...,T_l$) whenever for each nbd U of x there exists $n \geqslant 1$ such that $T_j^n x \in U$ for $j = 1,...,l$. If X is metrizable, then x is multiply recurrent under $T_1,...,T_l$ iff there exists an increasing sequence $n_1, n_2, ...$ in $\mathbf{Z}^+$ such that $x = \lim_{i \rightarrow \infty} T_j^{n_i} x$ for $j = 1,...,l$.

1.4. THEOREM. *Let X be a compact metrizable space, and for $j = 1,...,l$, let $T_j:X \rightarrow X$ be mutually commuting continuous mappings. Then there exists a mutually recurrent point in X.*

PROOF. See [9], or [8] Chapter 2. □

In the next section, we describe an application of this result due to H. Furstenberg and B. Weiss (see [9]).

2. Symbolic dynamics

An important source of examples are the so-called shift systems and their subsystems. For literature on these systems, see for example [14] and its references. For applications to the investigation of Anosov diffeomorphisms, see [4], and for applications in coding theory, see [1]. We shall use the shift system mainly for illustration of the notions defined in Section 1.

2.1. Let S be a finite set, say $S = \{0,..., s-1\}$ with $s \geq 2$, and let $\Omega := S^{\mathbf{Z}^+}$, the space of all (one-sided) infinite sequences $(\xi(0),\xi(1),\xi(2),...)$ with $\xi(n) \in S$ for $n \geq 0$. With the usual product topology, Ω is a compact Hausdorff space, homeomorphic with the Cantor discontinuum. A basis for the nbd system of a point ξ in Ω is formed by the set of all so-called *cylinders*, i.e. all sets of the form

$$[\xi(0),...,\xi(k)] := \{\eta \in \Omega : \eta(i) = \xi(i) \text{ for } i = 0,...,k\}$$

with $k \in \mathbf{Z}^+$.

The (one-sided) *shift* on Ω is the continuous mapping $\sigma:\Omega \to \Omega$ given by

$$(\sigma\xi)(n) := \xi(n+1) \quad \text{for } n \in \mathbf{Z}^+$$

where $\xi = (\xi(n))_{n \in \mathbf{Z}^+}$ in Ω. Thus, σ shifts sequences ξ one place to the left. Note that σ is an s-to-one mapping: the s possible values for $\xi(0)$ do not affect the value of $\sigma\xi$. (In a similar way, a two-sided shift can be defined on the space $S^{\mathbf{Z}}$, and this is a homeomorphism.) We consider the dynamical system (Ω,σ).

Observe that for $\xi,\eta \in \Omega$ and $k \in \mathbf{Z}^+$ one has $\sigma^r \eta \in [\xi(0),...,\xi(k)]$ for some $r \in \mathbf{Z}^+$ if and only if the finite sequence ('block') $\xi(0),...,\xi(k)$ occurs in η at place r, that is, if and only if $\xi(i) = \eta(r+i)$ for $i = 0,...,k$. Using this, the following observations are easy to prove:

2.2. OBSERVATIONS. (i) *A point ξ in Ω has a dense orbit iff every finite block over S occurs in ξ at some place.*
(ii) *A point ξ in Ω is recurrent iff every block which occurs in ξ occurs infinitely often in ξ.*
(iii) *A point ξ in Ω is almost periodic iff each block B which occurs in ξ occurs with bounded gaps, that is, there exists a number $l > 0$ such that every block of length l in ξ contains a copy of B.*

2.3. In view of the characterizations mentioned in 2.2, the following facts are almost trivial:
(i) *There exists a point in Ω which has a dense orbit:* let $B_1,B_2,...$ be an enumeration of all possible finite blocks. Then the point

$$\xi_0 = B_1 B_2 ...,$$

i.e. the sequence which is obtained from concatenation of all blocks B_i, has a dense orbit by 2.2 (i).

(ii) *The point* ξ_0 *defined in* (i) *is recurrent:* let B be a block which occurs in ξ_0; then B occurs in some beginning block $B' := B_1 ... B_k$ with $k \in \mathbb{N}$. In the sequence of all blocks $B_1, B_2, ...$ the block B' has a place with index $\geq k+1$. Hence the original block B occurs in ξ also at some place $\geq k+1$. Repetition of this argument shows that B occurs infinitely often in ξ_0. So by 2.2(ii) the point ξ_0 is recurrent.

Remark. If η is an arbitrary recurrent point of Ω and B is an arbitrary block from the beginning of η, then by 2.2(ii) there is a block C such that $\eta = BCB ...$. Repeating this procedure, starting with the 1-element block a given by $a = \eta(0)$, we see that

$$\eta = \{(aC_1a)C_2(aC_1a)\}C_3\{(aC_1a)C_2(aC_1a)\}...$$

for certain blocks $C_1, C_2, ...$. Conversely, every sequence η with this structure is recurrent.

(iii) *The set* Ω *is not minimal under the shift:* there are many non-empty closed invariant subsets, e.g. the (finite) orbits of the periodic points, i.e. all sequences which have the following form:

$$\zeta = BBB...B...$$

for some finite block B. (Note that the set of periodic points is dense in Ω: if $\xi \in \Omega$, then the cylindrical nbd $[\xi(0), ..., \xi(k)]$ of ξ contains the periodic point $\zeta = BBB...$ with $B := \xi(0)...\xi(k)$.) So it follows from 1.2 that the point ξ_0 defined in (i) above is *not* almost periodic.

Remark. It follows immediately from Observation 2.2(iii) that a recurrent point of the form

$$\eta = \{(aC_1a)C_2(aC_1a)\}C_3\{(aC_1a)C_2(aC_1a)\}...$$

is almost periodic iff the blocks $C_1, C_2, ...$ have bounded lengths. All kinds of methods have been invented to produce almost periodic points in Ω (i.e. minimal orbit-closures) and points whose orbit closures have other topological-dynamical or ergodic properties. For an important method - substitutions - cf. [19]. A famous almost periodic point is the so-called Morse sequence

$$\underbrace{\underbrace{\underbrace{\overset{01}{B_1}\;\overset{10}{B_1'}}_{B_2}\;\overset{1001}{B_2'}}_{B_3}\;\overset{10010110}{B_3'}}\;\cdots$$

(Here B' denotes the block which is obtained from B by replacing every 0 by a 1 and every 1 by a 0).

(iv) *Although there are many recurrent points (it can be shown that they form a residual subset, i.e. they contain a dense* G_δ*-set; cf. [10], 7.15 and 12.24(2); this* *follows also rather easily from the description given in (ii) above), there are also many non-recurrent points;* e.g. the point 101001000100001··· is not recurrent.

2.4. For the proof of the next theorem, it will be convenient to use the following metric d on Ω:

$$d(\xi,\eta) := \begin{cases} (1+ \min \{n \in \mathbf{Z}^+ : \xi(n) \neq \eta(n)\})^{-1} & \text{if } \xi \neq \eta \\ 0 & \text{if } \xi = \eta. \end{cases}$$

It is straightforward to verify that d is a metric. In addition, for every $\xi \in \Omega$ one has

$$[\xi(0),...,\xi(k)] = \{\eta \in \Omega : d(\xi,\eta) \leqslant \frac{1}{k+2}\}.$$

This implies immediately that d is compatible with the topology of Ω. The following special feature of the metric d will be needed: if $\xi,\eta \in \Omega$ and $d(\xi,\eta) < 1$, then $\xi(0) = \eta(0)$.

2.5. Elements of Ω can be used to describe partitions of $\mathbf{Z}^+$ as follows. Suppose $\mathbf{Z}^+ = B_0 \cup ... \cup B_{q-1}$ with $B_i \cap B_j = \varnothing$ for $i \neq j$. Let $S := \{0,...,q-1\}$ and $\Omega := S^{\mathbf{Z}^+}$, and consider the point $\xi = (\xi(n))_{n \in \mathbf{Z}^+} \in \Omega$ with

$$\xi(n) := i \quad \text{if} \quad n \in B_i \quad (i = 0,...,q-1)$$

Thus, ξ registers the 'colour' i for each $n \in \mathbf{Z}^+$. Conversely, it is clear that every point ξ in Ω defines a partition of $\mathbf{Z}^+$ into q disjoint subsets such that the given point ξ gives the 'colouring' of this partition. This correspondence is the basis of the proof of the following theorem.

2.6. THEOREM (Van der Waerden). *Let* $\mathbf{Z}^+ = B_0 \cup ... \cup B_{q-1}$ *with* $B_i \cap B_j = \varnothing$ *for* $i \neq j$. *Then one of the sets* B_i *contains arithmetic progressions of arbitrary length.*

PROOF. ([9], [10]). It is sufficient to show that for every $l \in \mathbf{Z}^+, l \neq 0$, there exists an index $i(l) \in \{0,...,q-1\}$ and numbers $n, m \in \mathbf{Z}^+$ (also depending on l) such that $m + jn \in B_{i(l)}$ for $j = 0,...,l$. Indeed, since there are only q possible values for $i(l)$, there is $i_0 \in \{0,...,q-1\}$ such that $i(l) = i_0$ for infinitely many values of l in $\mathbf{Z}^+$; then B_{i_0} contains arithmetical progressions of arbitrary length.

Let $\Omega := \{0,...,q-1\}^{\mathbf{Z}^+}$ and form $\xi_0 \in \Omega$ corresponding to the given partition of $\mathbf{Z}^+$ as indicated in 2.5. Let $X := \overline{O(\xi_0)}$ and let for any given integer $l > 0$ and $j = 1,...,l$ the mappings $T_j : X \rightarrow X$ be defined by $T_j := (\sigma|_X)^j$. Then the continuous mappings $T_1,...,T_l$ satisfy the hypothesis of Theorem 1.4, so there exists a point $\eta \in X$ and an increasing sequence $n_1, n_2, ...$ such that

$$T_j^{n_k} \eta \rightarrow \eta \quad \text{for } k \rightarrow \infty \ (j = 1,...,l).$$

Consequently, there exists $n \in \mathbf{N}$ such that the points η, $T_1^n \eta$, $T_2^n \eta$, ..., $T_l^n \eta$, that is, the points

$$\eta, \sigma^n \eta, \sigma^{2n} \eta, ..., \sigma^{ln} \eta$$

have mutual distances $<\frac{1}{2}$. By continuity of the maps σ^{jn}, $j = 0,...,l$, there exists a nbd V of η in Ω such that for each $\eta' \in V$ the points

$$\eta', \sigma^n \eta', \sigma^{2n} \eta', ..., \sigma^{ln} \eta'$$

also have distances $<\frac{1}{2}$. As $\eta \in X = \overline{O(\xi_0)}$, $V \cap O(\xi_0) \neq \emptyset$, hence $\sigma^m \xi_0 \in V$ for some $m \in \mathbf{Z}^+$, and the points

$$\sigma^m \xi_0, \sigma^{m+n} \xi_0, \sigma^{m+2n} \xi_0, ..., \sigma^{m+ln} \xi_0$$

have distances $<\frac{1}{2}$. By the last remark of 2.4, these points have equal zero'th coordinates, that is,

$$\xi_0(m) = \xi_0(m+n) = \xi_0(m+2n) = ... = \xi_0(m+ln).$$

If we denote this value by $i(l)$, then this shows that the set $B_{i(l)}$ contains the arithmetic progression $m, m+n, m+2n, ..., m+ln$ of length $l+1$. □

3. The semigroup $\beta\mathbf{Z}^+$

In this section the basic elements will be discussed of a machinery that is one of the corner stones of abstract topological dynamics. For a thorough discussion in a general context see [7] (or [5]). In this paper, we will develop just enough of this machinery to present a very elegant proof of Hindman's Theorem (Theorem 4.1 below). The key idea is to consider a natural semigroup structure on the Stone-Čech compactification of $\mathbf{Z}^+$ (denoted $\beta\mathbf{Z}^+$) and to relate dynamical properties of a point in a dynamical system with the algebraic structure of $\beta\mathbf{Z}^+$.

3.1. For those who do not use the concept of Stone-Čech compactification in their daily work, a few basic facts will be recalled.

The space $\beta\mathbf{Z}^+$ is a compact Hausdorff space which contains $\mathbf{Z}^+$ as a dense subspace. It is characterised by the following 'universal' property: every mapping $\psi: \mathbf{Z}^+ \to X$ with X a compact Hausdorff space has a unique continuous extension $\overline{\psi}: \beta\mathbf{Z}^+ \to X$. For a proof that a space $\beta\mathbf{Z}^+$ with these properties exists and is unique up to homeomorphism, the reader is referred to introductory topology textbooks. Concerning a useful model for $\beta\mathbf{Z}^+$ some remarks will be made in Section 4 below.

3.2. The universal property of $\beta\mathbf{Z}^+$ mentioned above implies that for every $n \in \mathbf{Z}^+$ the mapping

$$\omega^n : k \mapsto n+k : \mathbf{Z}^+ \to \mathbf{Z}^+ \subset \beta\mathbf{Z}^+$$

has a continuous extension to $\beta\mathbf{Z}^+$, which will also be denoted by ω^n. The value of $\xi \in \beta\mathbf{Z}^+$ under ω^n will be denoted by $n \oplus \xi$:

$$\omega^n : \xi \mapsto n \oplus \xi : \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+. \qquad (1)$$

This implies that for every $\xi \in \beta\mathbf{Z}^+$ the mapping

$$\omega_\xi : n \mapsto n \oplus \xi : \mathbf{Z}^+ \to \beta\mathbf{Z}^+$$

is well defined. It has a continuous extension, denoted by

$$\omega_\xi : \eta \mapsto \eta \oplus \xi : \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+. \tag{2}$$

Now a mapping $\omega : \beta\mathbf{Z}^+ \times \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+$ can be defined by

$$\omega(\eta,\xi) := \omega_\xi(\eta) = \eta \oplus \xi \quad \text{for } \xi,\eta \in \beta\mathbf{Z}^+.$$

Note that ω extends the addition in $\mathbf{Z}^+$ to all of $\beta\mathbf{Z}^+$. It can be shown that ω is not commutative on $\beta\mathbf{Z}^+$. Related to this is the fact that all right translations $\omega_\xi : \eta \mapsto \eta \oplus \xi : \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+$ for $\xi \in \beta\mathbf{Z}^+$ are continuous (cf. (2) above), but not all left translations $\omega^\eta : \xi \mapsto \eta \oplus \xi : \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+$ for $\omega \in \beta\mathbf{Z}^+$ (notice the place of ξ and η as sub- respectively super-script). In particular, $\omega : \beta\mathbf{Z}^+ \times \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+$ is not continuous. (For each $n \in \mathbf{Z}^+$, ω^n is continuous (cf. (1) above), and no doubt the $\beta\mathbf{Z}^+$-specialists have figured out characterizations for those points $\eta \in \beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ for which ω^η is continuous, but until now I have found none in the literature.) The 'addition' ω in $\beta\mathbf{Z}^+$ is associative. The straightforward proof is as follows: for all $m,n \in \mathbf{Z}^+$ the continuous (!) mappings

$$\left.\begin{array}{l} \omega^{m+n} : \zeta \mapsto (m+n) \oplus \zeta \\ \omega^m \circ \omega^n : \zeta \mapsto m \oplus (n \oplus \zeta) \end{array}\right\} : \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+$$

are equal to each other on the dense subset $\mathbf{Z}^+$ of $\beta\mathbf{Z}^+$ (associativity of addition in $\mathbf{Z}^+$). Hence they are equal on all of $\beta\mathbf{Z}^+$. This means exactly that for all $m \in \mathbf{Z}^+$ and $\zeta \in \beta\mathbf{Z}^+$ the continuous (!) mappings

$$\left.\begin{array}{l} \omega_\zeta \circ \omega^m : \eta \mapsto (m \oplus \eta) \oplus \zeta \\ \omega^m \circ \omega_\zeta : \eta \mapsto m \oplus (\eta \oplus \zeta) \end{array}\right\} : \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+$$

are equal on $\mathbf{Z}^+$. Hence they coincide on all of $\beta\mathbf{Z}^+$. This can be rephrased by saying that the continuous mappings $\omega_\zeta \circ \omega_\eta$ and $\omega_{\eta \oplus \zeta}$ are equal on $\mathbf{Z}^+$. Hence they are equal on $\beta\mathbf{Z}^+$, which means that $(\xi \oplus \eta) \oplus \zeta = \xi \oplus (\eta \oplus \zeta)$ for all ξ,η and ζ in $\beta\mathbf{Z}^+$.

Conclusion: *$(\beta\mathbf{Z}^+,\oplus)$ is an (associative) semigroup in which $\mathbf{Z}^+$ is a dense subsemigroup. All right translations ω_ξ with $\xi \in \beta\mathbf{Z}^+$ are continuous, and so are the left translations ω^n with $n \in \mathbf{Z}^+$.*

3.3. Let $\tilde{T} : \beta\mathbf{Z}^+ \to \beta\mathbf{Z}^+$ be defined by $\tilde{T} := \omega^1$, that is $\tilde{T}\xi := 1 \oplus \xi$ for $\xi \in \beta\mathbf{Z}^+$. Then $(\beta\mathbf{Z}^+,\tilde{T})$ is a dynamical system. The following proposition gives a relationship between dynamical properties of $(\beta\mathbf{Z}^+,\tilde{T})$ and algebraic properties of the semigroup $(\beta\mathbf{Z}^+,\oplus)$. First we need two definitions. An *idempotent* in $\beta\mathbf{Z}^+$ is an element ξ such that $\xi \oplus \xi = \xi$. A *left ideal* in $\beta\mathbf{Z}^+$ is a non-empty subset K such that $\beta\mathbf{Z}^+ \oplus K \subseteq K$ (if P,Q are subsets of $\beta\mathbf{Z}^+$, then of course $P \oplus Q := \{\xi \oplus \eta \,\vdots\, \xi \in P \;\&\; \eta \in Q\}$; similarly, $P \oplus \eta := \{\xi \oplus \eta \,\vdots\, \xi \in P\} = \omega_\eta[P]$, etc.). A *minimal left ideal* is a left ideal which does not properly contain any other left ideal. Since for every left ideal K and every element $\xi \in K$ one has $\beta\mathbf{Z}^+ \oplus \xi = \omega_\xi[\beta\mathbf{Z}^+] \subseteq K$, where $\beta\mathbf{Z}^+ \oplus \xi$ is a left ideal (trivial) which is compact, hence closed in $\beta\mathbf{Z}^+$ (image of $\beta\mathbf{Z}^+$ under the continuous right translation ω_ξ), *every minimal left ideal is closed.* Hence the minimal left ideals are the minimal elements of the partially ordered (under inclusion) set of all closed left ideals, and a traditional Zorn argument shows that every closed left ideal contains a minimal left ideal. However, the existence of minimal left ideals will also follow from 3.4(ii) below in combination with Lemma 1.1. In the proof of the proposition below, we shall use the notion of filter base. Those who are not acquainted with this notion might want to read the definition in Section 4 first.

3.4. PROPOSITION. *Let M be a non-empty closed subset of $\beta\mathbf{Z}^+$ and let $\xi\in\beta\mathbf{Z}^+$. Then the following equivalences are valid:*
(i) *M is invariant under $\tilde{T} \Leftrightarrow M$ is a left ideal;*
(ii) *M is minimal under $\tilde{T} \Leftrightarrow M$ is a minimal left ideal;*
(iii) *ξ is recurrent under $\tilde{T} \Leftrightarrow \exists\eta\in\beta\mathbf{Z}^+\setminus\mathbf{Z}^+ : \eta\oplus\xi = \xi$;*
(iv) *ξ is almost periodic under $\tilde{T} \Leftrightarrow \exists$ minimal left ideal $M : \xi\in M$.*
Moreover, the element η in (iii) can always be assumed to be an idempotent.

PROOF. (i): M is invariant under $\tilde{T}$ iff $\mathbf{Z}^+\oplus M\subseteq M$, that is, $\omega_\xi[\mathbf{Z}^+]\subseteq M$ for every $\xi\in M$. Since ω_ξ is continuous and M is closed, this is equivalent to $\omega_\xi[\beta\mathbf{Z}^+]\subseteq M$ for every $\xi\in M$, that is, $\beta\mathbf{Z}^+\oplus M\subseteq M$.
(ii): Clear from (i).
(iii): '$\Rightarrow$'. Let $\xi\in\Omega$ be recurrent. Then the family $\{R(\xi,U) : U$ a nbd of $\xi\}$ is a filter base in $\mathbf{Z}^+$, which has an adherence point $\eta\in\beta\mathbf{Z}^+$, i.e. $\eta\in\cap\{\overline{R(\xi,U)} : U$ a nbd of $\xi\}$ (the bar denotes closure in $\beta\mathbf{Z}^+$). We may assume that $\eta\notin\mathbf{Z}^+$. Indeed, if ξ is not periodic, then this filter base is not fixed, that is, no $n\in\mathbf{Z}^+$ belongs to all sets $R(\xi,U)$. Consequently, the point η does not belong to $\mathbf{Z}^+$. If ξ is periodic, say with period n, then the sequence $\{kn\}_{k\in\mathbf{N}}$ has an adherence point $\eta\in\beta\mathbf{Z}^+\setminus\mathbf{Z}^+$. Clearly, η is in $\overline{R(\xi,U)}$ for every nbd U of ξ, that is, η is an adherence point of the filter base. So in all cases we have an adherence point $\eta\in\beta\mathbf{Z}^+\setminus\mathbf{Z}^+$. As the mapping ω_ξ is continuous, $\omega_\xi(\eta) = \eta\oplus\xi$ is an adherence point of the filter basis $\{R(\xi,U)\oplus\xi : U$ a nbd of $\xi\}$. However, $R(\xi,U)\oplus\xi\subseteq U$, hence $\eta\oplus\xi\in\overline{U}$ for every nbd U of ξ. As $\beta\mathbf{Z}^+$ is a Hausdorff space, this implies that $\eta\oplus\xi = \xi$.
'$\Leftarrow$' Let U be a nbd of ξ. As $\omega_\xi(\eta) = \xi$, continuity of ω_ξ implies that there is a nbd V of η such that $\omega_\xi[V]\subseteq U$, that is, $V\oplus\xi\subseteq U$. As $\mathbf{Z}^+$ is dense in $\beta\mathbf{Z}^+$ and $\eta\neq 0$, V may be chosen so that $0\notin V\cap\mathbf{Z}^+\varnothing$. Hence there exists $n\in\mathbf{Z}^+$, $n\neq 0$ with $\tilde{T}^n\xi\in U$ (any $n\in V$ suffices).
(iv) Clear from (ii) and 1.2.
In the case that ξ is a recurrent point, (iii) shows that the subset $\{\eta\in\beta\mathbf{Z}^+\setminus\mathbf{Z}^+ : \eta\oplus\xi = \xi\}$ is not empty. It is easily seen that this is a subsemigroup of $\beta\mathbf{Z}^+$. Finally, since it is the intersection of the closed sets $\beta\mathbf{Z}^+\setminus\mathbf{Z}^+$ and $\omega_\xi^{-1}[\xi]$ (again, by continuity of ω_ξ), it is a non-empty, closed subsemigroup of $\beta\mathbf{Z}^+$. By Lemma 3.6 below, it contains an idempotent. □

3.5. COROLLARY. *For every idempotent $\eta\in\beta\mathbf{Z}^+\setminus\mathbf{Z}$ and every element $\xi\in\beta\mathbf{Z}^+$, $\eta\oplus\xi$ is recurrent under $\tilde{T}$, and in particular, η is recurrent under $\tilde{T}$. If η is an idempotent in $\beta\mathbf{Z}^+\setminus\mathbf{Z}^+$, then η is almost periodic iff η belongs to a* *minimal left ideal.* □

Remark. It can be shown that $\beta\mathbf{Z}^+$ contains at least 2^c different minimal left ideals ($c :=$ cardinality of $\mathbb{R}$). Note that each minimal left ideal is included in $\beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$. (Otherwise, its intersection with $\beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ would be a proper closed subideal; that $\beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ is a left ideal follows from the fact that ω^n maps $\beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ into $\beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ for each $n \in \mathbf{Z}^+$. Cf. [12], 6.11.)

Since every minimal left ideal is a closed subsemigroup of $\beta\mathbf{Z}^+$, the lemma below implies that every minimal left ideal contains at least one idempotent. More about the structure of minimal left ideals can be found in [20]. At this point it is instructive to observe that any minimal left ideal, as a minimal subset of the dynamical system $(\beta\mathbf{Z}^+,\tilde{T})$, is the orbit closure of each of its points: in particular, it is the orbit closure of an idempotent. Thus, the almost periodic points of $(\beta\mathbf{Z}_+,\tilde{T})$ are just all points which are in the orbit closures of idempotents $\eta \in \beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ which are situated in a minimal left ideal of $\beta\mathbf{Z}^+$. In 3.8 it will be shown that there exist idempotents in $\beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ that do not belong to any minimal left ideal.

3.6. LEMMA. *Let S be a compact Hausdorff space which has a (multiplicatively written) semigroup structure such that all right translations $p \mapsto pq : S \to S$ with $q \in S$ are continuous. Then S contains an idempotent.*

PROOF ([7], [20]): By Zorn's lemma, S contains a minimal closed nonempty subsemigroup E. Let $q \in E$. Then Eq is a closed (continuous image of E under right translation) subsemigroup of E, so by minimality, $Eq = E$. In particular, $q \in Eq$, so $W := \{p \in E : pq = q\} \neq \varnothing$. Clearly, W is a subsemigroup of E and, again by continuity of right translation over q, W is closed. Now minimality of E implies $W = E$. In particular, $q \in W$, that is, $qq = q$. □

Next, consider an arbitrary dynamical system (X,T). For every $x \in X$, the mapping $\delta_x : n \mapsto T^n x : \mathbf{Z}^+ \to X$ has a continuous extension $\bar{\delta}^x : \beta\mathbf{Z}^+ \to X$. Put $T^\xi x := \bar{\delta}_x(\xi)$ for $\xi \in \beta\mathbf{Z}^+$. This can be done for every $x \in X$, so we obtain a mapping $T^\xi : X \to X$ for every $\xi \in \beta\mathbf{Z}^+$. (It can be shown that T^ξ need not be continuous for every $\xi \in \beta\mathbf{Z}^+$.) By an argument very similar to the proof that the operation $\oplus$ in $\beta\mathbf{Z}^+$ is associative (see 3.2 above), it can be shown that $T^{\eta \oplus \xi} x = T^\eta(T^\xi x)$ for all $x \in X$ and $\xi,\eta \in \beta\mathbf{Z}^+$. Thus,

$$\forall \xi,\eta \in \beta\mathbf{Z}^+ : T^{\eta \oplus \xi} = T^\eta \circ T^\xi.$$

This can be restated as follows: the given action of $\mathbf{Z}^+$ on X (by means of $(n,x) \mapsto T^n x$) can be extended to an action of the semigroup $\beta\mathbf{Z}^+$ on X. In this situation, there are also relations between algebraic properties of the semigroup $(\beta\mathbf{Z}^+,\oplus)$ and the dynamical properties of (X,T).

3.7. PROPOSITION. *Let (X,T) be a dynamical system and let $x \in X$. The following equivalences are valid:*

(i) *x is recurrent* $\Leftrightarrow \exists \eta \in \beta\mathbf{Z}^+ \setminus \mathbf{Z}^+ : T^\eta x = x;$

(ii) *x is almost periodic $\Leftrightarrow\forall$ minimal left ideal M in $\beta\mathbf{Z}^+$ there is an idempotent $\eta\in M$ with $T^\eta x = x$;*

(iii) *x is almost periodic $\Leftrightarrow\exists$ minimal left ideal M in $\beta\mathbf{Z}^+$ such that $T^\eta x = x$ for some $\eta\in M$;*

Also, in (i) *the element η may be assumed to be an idempotent.*

PROOF. (i): Completely similar to 3.4(i).
(ii) Assume that x is almost periodic, i.e. that $\overline{O(x)}$ is minimal. Let M be an arbitrary minimal left ideal in $\beta\mathbf{Z}^+$ i.e. a minimal subset of the dynamical system $(\beta\mathbf{Z}^+,\tilde{T})$ (cf. 3.4). It is easy to check that the continuous function $\bar{\delta}_x:\beta\mathbf{Z}^+\rightarrow X$ defined above satisfies $\bar{\delta}_x\circ\tilde{T} = T\circ\bar{\delta}_x$. So $\bar{\delta}_x$ is a homomorphism of dynamical systems from $(\beta\mathbf{Z}^+,\tilde{T})$ into (X,T). It follows that $\bar{\delta}_x[M]$ is a closed invariant subset of X. However,

$$\bar{\delta}_x[M]\subseteq\bar{\delta}_x[\beta\mathbf{Z}^+] = \bar{\delta}_x[\overline{\mathbf{Z}^+}]\subseteq\overline{\delta_x[\mathbf{Z}^+]} = \overline{O(x)},$$

and by minimality of $\overline{O(x)}$, $\bar{\delta}_x[M] = \overline{O(x)}$. In particular, $x\in\bar{\delta}_x[M]$. Stated differently, the set $\{\eta\in M: T^\eta x = x\}$ is not empty. Obviously, it is a subsemigroup of M, and as $\bar{\delta}_k\,|_M:\eta\mapsto T^\eta x$ is continuous, it is closed. So by 3.6 above it contains an idempotent. This proves one implication. The other implication will follow from the proof of (iii).
(iii) If x is almost periodic, existence of M as wanted follows from the implication proved in (ii). Conversely, assume that $T^\eta x = x$ for some element η of some minimal left ideal. Then $x\in\bar{\delta}_x[M]$. Since M is a minimal subset of $(\beta\mathbf{Z}^+,\tilde{T})$ and $\bar{\delta}_x:(\beta\mathbf{Z}^+,\tilde{T})\rightarrow(X,T)$ is a homomorphism, the remark following 1.3 implies that $\bar{\delta}_x[M]$ is a minimal subset of X. Since $x\in\bar{\delta}_x[M]$, 1.2 implies that x is almost periodic. □

3.8. Remark. In the shift system (Ω,σ) we have observed the existence of a point which is recurrent but not almost periodic. It follows immediately from 3.7 that this implies the existence of an idempotent in $\beta\mathbf{Z}^+\setminus\mathbf{Z}^+$ which is not contained in any minimal left ideal. Consequently, the dynamical system $(\beta\mathbf{Z}^+,\tilde{T})$ also has a point which is recurrent but not almost periodic (cf. 3.5).

4. A characterization of idempotents in $\beta\mathbf{Z}^+\setminus\mathbf{Z}^+$

In this section, we study a particular model of $\beta\mathbf{Z}^+$ in more detail. This will enable us to give a nice description of idempotents in $\beta\mathbf{Z}^+$. Using this, a very short proof of Hindman's theorem is possible. Our proof is different from those in [8] and [9]. Although in [11] a proof is also given using the Stone-Čech compactification, that proof is different too. The theorem is as follows:

4.1. THEOREM (Hindman). Let $\mathbf{Z}^+ = A_1\cup...\cup A_q$ *with* $A_i\cap A_j = \emptyset$ *for* $i\neq j$. *Then one of the sets* A_i *is an IP-set.*

A set $A\subseteq\mathbf{Z}^+$ is called an *IP-set* whenever there exists a sequence $x_0\leqslant x_1\leqslant x_2\leqslant...$ in A such that $x_{n_1}+...+x_{n_k}\in A$ for every finite subset $\{n_1,..., n_k\}$ of different elements of $\mathbf{Z}^+$. So an IP-set consists (at least) of a non-decreasing sequence in $\mathbf{Z}^+$ together with all sums of finitely many elements of the sequence (no repetitions allowed).

The following model of $\beta\mathbf{Z}^+$ will be useful (for details, see e.g [12]). First, recall that a *filter* in a set S is a family $\mathfrak{F}$ of subsets of S such that
(i) $\varnothing \notin \mathfrak{F}$;
(ii) if $A, B \in \mathfrak{F}$, then $A \cap B \in \mathfrak{F}$;
(iii) if $A \in \mathfrak{F}$ and B is a subset of S such that $A \subseteq B$, then $B \in \mathfrak{F}$.
A family B of subsets such that $\varnothing \notin \mathfrak{B}$ and for all $A, B \in \mathfrak{B}$ there exists $C \in \mathfrak{B}$ with $C \subseteq A \cap B$ is called a *filter base*. Note that if $\mathfrak{B}$ is a filter base, then $\{A \subseteq S : \exists B \in \mathfrak{B} \text{ with } A \supseteq B\}$ is a filter. An *ultrafilter* in S is a filter which is not properly contained in any other filter. The following characterization of ultrafilters is rather easy to prove: if $\mathfrak{F}$ is a filter in S, then $\mathfrak{F}$ is an ultrafilter iff for every subset A of S either $A \in \mathfrak{F}$ or $S \setminus A \in \mathfrak{F}$. Although we shall not need it explicitly, it is essential for the proofs of the statements about $\beta\mathbf{Z}^+$ below that ultrafilters do exist: by Zorn's lemma, every filter can be extended to an ultrafilter. Examples of ultrafilters are e.g. the fixed ultrafilters, i.e. the filters of the form $\{A \subseteq S : s \in A\}$, where $s \in S$.
Points of $\beta\mathbf{Z}^+$: ultrafilters in $\mathbf{Z}^+$. The points of $\mathbf{Z}^+$ will be identified with the fixed ultrafilters; to be precise, the point $n \in \mathbf{Z}^+$ is identified with the ultrafilter

$$h(n) := \{B \subseteq \mathbf{Z}^+ : n \in B\}.$$

Topology of $\beta\mathbf{Z}^+$: a basis for the topology is given by the family of all subsets of $\beta\mathbf{Z}^+$ of the form

$$h(A) := \{\xi \in \beta\mathbf{Z}^+ : A \in \xi\}$$

with A a subset of $\mathbf{Z}^+$. It can be shown that the sets $h(A)$ for $A \subseteq \mathbf{Z}^+$ are both open and closed in $\beta\mathbf{Z}^+$.
It follows from the definitions of $h(A)$ for $A \subseteq \mathbf{Z}^+$ and $h(n)$ for $n \in \mathbf{Z}^+$ that $h(A) \cap \mathbf{Z}^+$ is the set of all fixed ultrafilters $h(n)$ with $A \in h(n)$, i.e. $n \in A$. Thus, if we identify n with $h(n)$, then $h(A) \cap \mathbf{Z}^+ = A$.

If $\xi \in \beta\mathbf{Z}^+$, then a nbd base of ξ is given by the set of all open sets $h(A)$ with $A \subseteq \mathbf{Z}^+$ such that $\xi \in h(A)$, i.e. the set

$$\mathfrak{B}_\xi := \{h(A) : A \subseteq \mathbf{Z}^+ \,\&\, A \in \xi\} \tag{1}$$

The trace $\mathfrak{B}_\xi \cap \mathbf{Z}^+$ of $\mathfrak{B}_\xi$ in $\mathbf{Z}^+$ (i.e. the set of all intersections $h(A) \cap \mathbf{Z}^+$ with $h(A) \in \mathfrak{B}_\xi$) apparently consists of all sets A with $A \in \xi$. This means that

$$\mathfrak{B}_\xi \cap \mathbf{Z}^+ = \xi. \tag{2}$$

Next, consider a mapping $\psi : \mathbf{Z}^+ \to X$, X a compact Hausdorff space. Let $\bar{\psi} : \beta\mathbf{Z}^+ \to X$ be its continuous extension. Consider $\xi \in \beta\mathbf{Z}^+$ and put $x := \bar{\psi}(\xi)$. For every open nbd U of x, $\bar{\psi}^{-1}[U]$ includes an element of $\mathfrak{B}_\xi$, hence $\bar{\psi}^{-1}[U] \cap \mathbf{Z}^+$ includes an element of ξ. As ξ is a filter, $\bar{\psi}^{-1}[U] \cap \mathbf{Z}^+$ itself belongs to ξ. This shows that

$$\forall U : U \text{ a nbd of } \bar{\psi}(\xi) \text{ in } X \Rightarrow \{m \in \mathbf{Z}^+ : \psi(m) \in U\} \in \xi \tag{3}$$

Using this, the operation $\oplus$ in $\beta\mathbf{Z}^+$ can be described as follows.

If $n \in \mathbf{Z}^+$ and $\xi \in \beta\mathbf{Z}^+$, then application of (3) to the mapping $\omega^n : m \mapsto n+m : \mathbf{Z}^+ \to \mathbf{Z}^+ \subset \beta\mathbf{Z}^+$ gives, in view of (1) and (2):

$$\forall A \in n \oplus \xi : \{m \in \mathbf{Z}^+ \,\vdots\, n+m \in A\} \in \xi.$$

For convenience, write $A \dot{-} n := \{m \in \mathbf{Z}^+ \,\vdots\, n+m \in A\} = (A-n) \cap \mathbf{Z}^+$. Then the above can be rewritten as: for all $A \subseteq \mathbf{Z}^+$:

$$A \in n \oplus \xi \Rightarrow A \dot{-} n \in \xi \qquad (4)$$

(The converse can also be proved, but this will not be needed in the sequel.) Next, apply (3) to the mapping ω_η: $m \mapsto m \oplus \eta$: $\mathbf{Z}^+ \to \beta\mathbf{Z}^+$, with $\eta \in \beta\mathbf{Z}^+$, and find that for every nbd U of the image $\xi \oplus \eta$ of $\xi \in \beta\mathbf{Z}^+$ under the extended mapping, the set $\{m \in \mathbf{Z}^+ \,\vdots\, m \oplus \eta \in U\}$ belongs to ξ. So by (1) and (2), if $A \subseteq \mathbf{Z}^+$, then

$$A \in \xi \oplus \eta \Rightarrow \{m \in \mathbf{Z}^+ \,\vdots\, A \in m \oplus \eta\} \in \xi. \qquad (5)$$

Using (4), it follows that

$$\{m \in \mathbf{Z}^+ \,\vdots\, A \in m \oplus \eta\} \subseteq \{m \in \mathbf{Z}^+ \,\vdots\, A \dot{-} m \in \eta\}.$$

As ξ is a filter, it follows from (5) that

$$\forall A \subseteq \mathbf{Z}^+ : A \in \xi \oplus \eta \Rightarrow \{m \in \mathbf{Z}^+ \,\vdots\, A \dot{-} m \in \eta\} \in \xi. \qquad (6)$$

Finally, we need one more (trivial) observation: if $\xi \in \beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ and $A \in \xi$, then A must be an infinite subset of $\mathbf{Z}^+$. Indeed, as ξ is not a fixed ultrafilter, for every $n \in \mathbf{Z}^+$ there is $B_n \in \xi$ such that $n \notin B_n$. In particular, $A \cap (\bigcap_{n \in A} B_n) = \varnothing$. If A were finite, this would contradict the filter property of ξ.

4.2. LEMMA. *Let $\eta \in \beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ be an idempotent. Then every $A \in \eta$ is an IP-set.*

PROOF. By (6) with $\xi = \eta$ and therefore $\xi \oplus \eta = \eta \oplus \eta = \eta$:

$$\forall B \subseteq \mathbf{Z}^+ : B \in \eta \Rightarrow \{m \in \mathbf{Z}^+ \,\vdots\, B \dot{-} m \in \eta\} \in \eta.$$

So in particular, as η is a filter:

$$\forall B \in \eta : B \cap \{m \in \mathbf{Z}^+ \,\vdots\, B \dot{-} m \in \eta\} \in \eta.$$

In view of the remark just before the lemma, this implies that for every $B \in \eta$ there are infinitely many elements $m \in \mathbf{Z}^+$ such that $m \in B$ and $B \dot{-} m \in \eta$. Using this, one easily defines by induction for a given element $A \in \eta$ a sequence $x_0, x_1, x_2, \ldots$ in A and elements $A_n \in \eta$ ($n \in \mathbf{Z}^+$) such that

(i) $A_0 := A$;
(ii) $x_n \in A_n$, $A_n \dot{-} x_n \in \eta$ and $x_n > x_{n-1}$;
(iii) $A_{n+1} := A_n \cap (A_n \dot{-} x_n)$.

Note that by the filter property of η, if $A_n \in \eta$ and x_n satisfies (ii), then $A_{n+1} \in \eta$. Also, as $x_n \in A_n$ and the sequence $\{A_n\}_{n \in \mathbf{Z}^+}$ is decreasing, $x_n \in A_0 = A$ for all $n \in \mathbf{Z}^+$.

Now consider $n_1 < n_2 < ... < n_k$ in $\mathbb{Z}^+$. Then $n_{i+1} \geqslant n_i + 1$ for $i = 1,...,k-1$ hence

$$A_{n_{i+1}} \subseteq A_{n_i+1} \subseteq A_{n_i} \dot{-} x_{n_i}$$

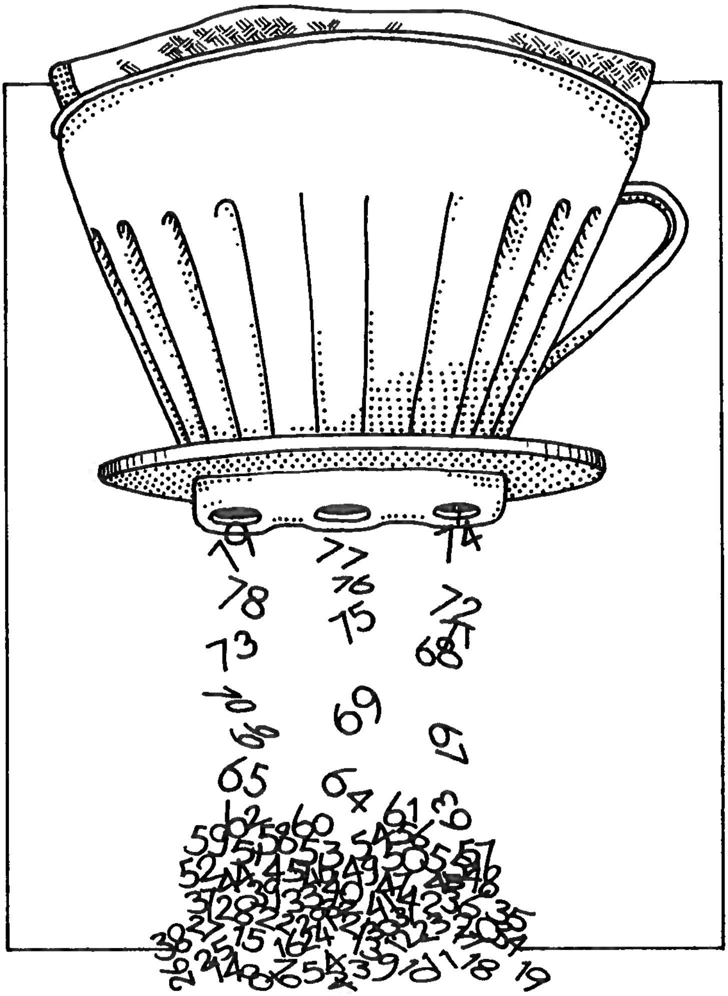
by (iii). This implies that

$$x_{n_i} + A_{n_{i+1}} \subseteq A_{n_i}.$$

By induction it follows that

$$x_{n_1} + ... + x_{n_{k-1}} + x_{n_k} \subseteq A_{n_1} \subseteq A_0 = A.$$

So the sequence $\{x_n\}_{n \in \mathbb{Z}^+}$ in A has the required property with respect to A.
□



4.3. PROOF OF THEOREM 4.1. By 3.6, there exists an idempotent η in the closed semigroup $\beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ of $\beta\mathbf{Z}^+$. Since η is an *ultra*filter in $\mathbf{Z}^+$, there is $i \in \{1,...,q\}$ such that $B_i \in \eta$ (this follows with induction from the property that for every subset B of $\mathbf{Z}^+$, either $B \in \eta$ or $\mathbf{Z}^+ \setminus B \in \eta$). Now apply 4.2. □

4.4. COROLLARY ([9]). *If (X,T) is a dynamical system and $x \in X$ is a recurrent point, then for every nbd U of x the set $R(x,U)$ is an IP-set.*

PROOF. In 3.7 we have observed that there exists an idempotent η in $\beta\mathbf{Z}^+ \setminus \mathbf{Z}^+$ such that $T^\eta x = x$. Since $T^\eta x = \bar{\delta}_x(\eta)$, where $\bar{\delta}_x$ is the continuous extension to $\beta\mathbf{Z}^+$ of the evaluation mapping $\delta_x : n \mapsto T_n x : \mathbf{Z}^+ \to X$, it follows from formula (3) above that for every nbd U of x in X:

$$\{n \in \mathbf{Z}^+ : \delta_x(n) \in U\} \in \eta.$$

This means exactly that $R(x,U) \in \eta$. Now apply 4.2. □

4.5. Remark. As an application of 4.4. one sees that if ξ is a recurrent point in a shift dynamical system (Ω,σ), then for any block B which occurs in ξ the set of numbers $k \in \mathbf{Z}^+$ such that B occurs in ξ at place k is an IP-set.

5. Epilogue

The results and methods described above form only a minor part of 'Abstract Topological Dynamics', by which I mean the part of the theory that was initiated by Gottschalk and Hedlund (cf. [10]). More of this theory can be found in [6], [7], [8] and [5]. In [17] and the final chapters of [5] relations between the abstract theory and differential equations are still visible. The theory in books like [2], [3], [13], [16] and [18] is in much closer contact with the classical qualitative theory of differential equations.

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